Fermions in one-loop quantum cosmology

A. Yu. Kamenshchik* and I. V. Mishakov*
Nuclear Safety Institute, Russian Academy of Sciences, 52 Bolshaya Tulskaya, Moscow, 113191, Russia
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In recent papers by D'Eath and Esposito two kinds of boundary conditions, local and nonlocal (spectral), were used to study the contribution of fermions to the one-loop prefactor in the Hartle-Hawking wave function of the Universe. Using the ζ-function technique they found that for the case of massless Majorana fermions on a flat background bounded by a three-sphere the values $\zeta(0)$ coincide for the two kinds of boundary conditions mentioned above. Implementing our version of ζ regularization elaborated earlier, we calculate $\zeta(0)$ for both the massive and massless fermions on the background representing the part of the four-dimensional de Sitter sphere bounded by a three-sphere. For the massless fermions our results coincide with results for a flat background and, consequently, the results for both types of boundary conditions are the same. However, for massive fermions the values $\zeta(0)$ for local and spectral boundary conditions differ on the de Sitter background and on the flat one as well.
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I. INTRODUCTION

In the last few years a lot of papers were devoted to calculations in the one-loop approximation in quantum cosmology [1–14]. Some of these papers [1–3,5–7,9–14] deal with calculation of the prefactor of the Hartle-Hawking wave function of the Universe [15,16] using the asymptotic expansion of the heat kernel for fields of various spins on manifolds with boundaries. It is worth noticing that calculation of the Schwinger-DeWitt coefficients [17] on manifolds with boundaries is rather an old mathematical task which was considered also in Refs. [18–24]. In the case of bosonic fields it seems quite natural to choose the Dirichlet boundary condition provided we work only with physical degrees of freedom. Such a choice of boundary conditions corresponds to a fixation of prescribed values of physical fields on the boundary. In the case of fermionic fields satisfying the first-order Dirac equation we cannot require that all the spinor components obey the Dirichlet boundary conditions, because then the boundary data will be overdetermined. Thus we have to choose some boundary conditions resorting to certain physical and mathematical considerations.

There are two natural options of boundary conditions for fermions: spectral and local. Both were considered in recent works by D'Eath and Esposito [10,11]. The spectral boundary conditions for fermions in quantum cosmology were initially considered by D'Eath and Halliwell in Ref. [25]. They considered a quantum cosmological model in which the Dirac field is regarded as a perturbation around a Friedmann background gravitational model containing a family of three-spheres of radius $a(t)$. Using two-component spinor notation, the unprimed spin-$\frac{1}{2}$ field ψ_A on a given three-sphere may be split into a sum $\psi_A^{(+)} + \psi_A^{(-)}$, where $\psi_A^{(+)}$ is a sum of harmonics having positive eigenvalues for the three-dimensional Dirac operator ${}_e n_{AA'} e^{BA'j}\, {}^{(3)}D_j$ on the S^3, and $\psi_A^{(-)}$ is a sum of harmonics having negative eigenvalues. Here ${}_e n_{AA'}$ is the spinor version of the unit Euclidean normal to the three-sphere, $e^{BA'j}$ is the spinor version of the orthonormal spatial triad on the three-sphere, and ${}^{(3)}D_j$ is the three-dimensional covariant derivative ($j = 1, 2, 3$) [25,26]. A similar decomposition may be applied to the primed field $\tilde{\psi}_{A'}$. Boundary conditions for invesigating the Hartle-Hawking quantum state may be found by asking for data on a three-sphere bounding a compact region with a Riemannian metric, such that the classical Dirac equation is well posed. For a massless field, if one uses spectral boundary conditions, one is forced to specify $\psi_A^{(+)}$ and $\tilde{\psi}_{A'}^{(+)}$ on the boundary, and not $\psi_A^{(-)}$ and $\tilde{\psi}_{A'}^{(-)}$. In addition to these physical arguments, spectral boundary conditions are of considerable mathematical interest and their foundations lie in the theory of elliptic equations and in the index theory for the Dirac operator [27].

One can also choose local boundary conditions for fermionic fields. For a spin-$\frac{1}{2}$ Majorana field $(\psi_A, \tilde{\psi}_{A'})$ in Riemannian space, these conditions are

$$\sqrt{2}{}_e n_A^{A'} \psi^A = \epsilon \tilde{\psi}^{A'}, \tag{1.1}$$

on the bounding surface. Here ϵ will be taken to be either $+1$ or -1. These boundary conditions were introduced by Breitenlohner, Freedmann, and Hawking [28,29] for gauge supergravity theories in anti–de Sitter space, which can be seen as the maximally supersymmetric solution of the O(N) gauge supergravity theories. Another way of introducing of local boundary conditions was proposed in papers by Luckock and Moss [30] and Moss and Poletti [5]. These conditions are formulated in terms of four-component spinor fields and can be written as

$$P\Psi = 0 \tag{1.2}$$

*Electronic address: grg@ibrae.ins.su

on the boundary, where P is a projector, which looks like $P = (1-\epsilon\gamma_5\gamma^\mu n_\mu)/2$. For the case of spin-$\frac{1}{2}$ fields boundary conditions (1.2) coincide with ones from Eq. (1.1). It is important to stress that boundary conditions (1.2) are self-consistent for both Majorana and Dirac spinors and for massive fermions as well as for massless ones. Having made the transformation from four-component spinors to two-component ones we see that the boundary conditions (1.1) survive in the massive case and in the case of Dirac fermions are substituted by

$$\sqrt{2}\,{}_e n_A^{\ A'}\psi^A = \epsilon\tilde{\chi}^{A'}, \tag{1.3}$$

$$\sqrt{2}\,{}_e n_A^{\ A'}\chi^A = \epsilon\tilde{\psi}^{A'}, \tag{1.4}$$

where χ^A is the second two-component spinor compounding Dirac field $(\psi^A, \tilde{\chi}_{A'})$.

In Refs. [10,11] the one-loop prefactors in the Hartle-Hawking wave function were studied using the generalized Riemann ζ function formed from the squared eigenvalues of the four-dimensional fermionic operators. For a massless Majorana spin-$\frac{1}{2}$ field, the values $\zeta(0)$ describing scaling properties if the wave function were calculated on the flat background bounded by a three-sphere for the cases of local and spectral boundary conditions. Remarkably, the same value $\zeta(0) = \frac{11}{360}$ was found for both the cases. The natural question arises as to whether the equality of the local and spectral values for $\zeta(0)$ is a feature peculiar to the highly symmetrical example of a three-sphere surrounding a region of flat four-space, or whether there is an extension of this result to a more general context.

This question is under consideration in the present paper. In our preceding papers [12,13] a new version of the ζ-function technique [31,32] was elaborated. This version allows us to reduce the calculation of $\zeta(0)$ to a comparatively simple manipulation with asymptotic expansions of basis functions of corresponding eigenvalue equations. In Refs. [13,14] the value $\zeta(0)$ for a massive spin-$\frac{1}{2}$ field on the background representing the part of a four-dimensional de Sitter sphere bounded by three-sphere was calculated for the case of spectral boundary conditions. After taking the corresponding limit our result coincides with one from [11]. Here we calculate $\zeta(0)$ for a massive spin-$\frac{1}{2}$ field on the de Sitter background at the local boundary conditions and compare our results with the our preceding ones and with those by D'Eath and Esposito [10,11]. We see that for the massless case our results coincide with the corresponding ones for a flat background and, hence, each other. However, for the case of a massive spin-$\frac{1}{2}$ field the results for the problems with different boundary conditions differ not only on the de Sitter background, but also on the flat one.

In Sec. II we introduce notation for spinor harmonincs and write down basis functions for the Dirac equation and the equations determining eigenvalues for local and spectral boundary conditions. The notation and logic of exposition in this section coincide with those used in the papers by D'Eath and Esposito [10,11] but their approach is generalized for the case of massive fermions, both Majorana and Dirac. Section III contains a sketch of the technique of the generalized ζ function on a compact Euclidean manifold with a boundary. In Sec. IV we apply this technique to the spinor field and discuss the obtained results.

II. EIGENVALUE EQUATIONS FOR SPINOR FIELDS WITH LOCAL AND SPECTRAL BOUNDARY CONDITIONS

In this section we shall use the notation of Refs. [25,10,11]. The action of the Dirac field in the curved background with the Lorentzian signature has the form

$$S = -\frac{i}{2}\int d^4x e(\tilde{\phi}^{A'} e^\mu_{AA'} D_\mu \phi^A + \tilde{\chi}^{A'} e^\mu_{AA'} D_\mu \chi^A) + \text{H.c.} - \frac{m}{\sqrt{2}}\int d^4x e(\chi_A\phi^A + \tilde{\phi}^{A'}\tilde{\chi}_{A'}) + \text{boundary terms}. \tag{2.1}$$

The gravitational field is described by the tetrad e^a_μ, where $a, b, \ldots = 0,1,2,3$ are tetrad indices and $\mu, \nu, \ldots = 0,1,2,3$ are world indices, or equivalently by the Hermitian spinor-valued forms

$$e^\mu_{AA'} = e^{\mu a}\sigma_a^{AA'},$$

where

$$\sigma_0 = -\frac{I}{\sqrt{2}}, \quad \sigma_i = \frac{\Sigma_i}{\sqrt{2}}, \quad i = 1,2,3,$$

where Σ_i are the Pauli matrices. It is necessary also to introduce the unit timelike future-directed normal to the boundary surface of our manifold n^μ and its spinor version $n^{AA'}$ which obeys relations $n_{AA'}e_i^{AA'} = 0$, $n_{AA'}n^{AA'} = 1$. The transition to the Euclidean signature can be achieved by rotating the basis $e^0_\mu \to -ie^0_\mu$, while still using Lorentzian conventions for the tetrad metric η_{ab} and for spinors: the space-time metric $g_{\mu\nu}$ then becomes positive definite. It is convenient to define the Euclidean normal spinor ${}_e n^{AA'} = -in^{AA'}$, which corresponds to a unit spacelike normal vector ${}_e n^\mu$.

The action (2.1) with corresponding boundary term leads to the Dirac equations

$$e^\mu_{AA'}D_\mu\phi^A = i\frac{m}{\sqrt{2}}\tilde{\chi}_{A'}, \quad e^\mu_{AA'}D_\mu\chi^A = i\frac{m}{\sqrt{2}}\tilde{\phi}_{A'},$$
$$e^\mu_{AA'}D_\mu\tilde{\phi}^{A'} = -i\frac{m}{\sqrt{2}}\chi_A, \quad e^\mu_{AA'}D_\mu\tilde{\chi}^{A'} = -i\frac{m}{\sqrt{2}}\phi_A. \tag{2.2}$$

Before consideration of the eigenvalue problem for the Dirac operator it is necessary to introduce a complete set of harmonics for the expansion of any spinor field on the three-sphere. These harmonics were described in Ref. [25]. In terms of these harmonics the expansion for the Dirac field on the Friedmann-Robertson-Walker background with the metric

$$ds^2 = -dt^2 + a^2(t)d^2\Omega_3 \tag{2.3}$$

looks like

$$\phi_A = \frac{a^{-3/2}}{2\pi} \sum_{npq} \alpha_n^{pq}[m_{np}(t)\rho_A^{nq}(\mathbf{x}) + \tilde{r}_{np}(t)\bar{\sigma}_A^{nq}(\mathbf{x})], \tag{2.4}$$

$$\tilde{\phi}_{A'} = \frac{a^{-3/2}}{2\pi} \sum_{npq} \alpha_n^{pq}[\tilde{m}_{np}(t)\bar{\rho}_{A'}^{nq}(\mathbf{x}) + r_{np}(t)\sigma_{A'}^{nq}(\mathbf{x})], \tag{2.5}$$

$$\chi_A = \frac{a^{-3/2}}{2\pi} \sum_{npq} \beta_n^{pq}[s_{np}(t)\rho_A^{nq}(\mathbf{x}) + \tilde{t}_{np}(t)\bar{\sigma}_A^{nq}(\mathbf{x})], \tag{2.6}$$

$$\tilde{\chi}_{A'} = \frac{a^{-3/2}}{2\pi} \sum_{npq} \beta_n^{pq}[\tilde{s}_{np}(t)\bar{\rho}_{A'}^{nq}(\mathbf{x}) + t_{np}(t)\sigma_{A'}^{nq}(\mathbf{x})]. \tag{2.7}$$

Here the harmonics ρ_A^{nq} and $\sigma_{A'}^{nq}$ have positive eigenvalues $\frac{1}{2}(n+\frac{3}{2})$ of the intrinsic three-dimensional Dirac operator ${}_e n_{AA'} e^{BA'j}\,{}^{(3)}D_j$ on the three-sphere, while the harmonics $\bar{\rho}_{A'}^{nq}$ and $\bar{\sigma}_A^{nq}$ have negative eigenvalues $-\frac{1}{2}(n+\frac{3}{2})$; ${}^3D_j$ is the three-dimensional covariant derivative, n runs from 0 to infinity, p and q from 1 to $(n+1)(n+2)$ [$(n+1)(n+2)$ is the multiplicity of the corresponding eigenvalue of the three-dimensional Dirac operator]. α_n^{pq} and β_n^{pq} are a collection of matrices introduced for convenience, to avoid couplings between different values of p in the expansion of the action, where, for each n, α_n^{pq} is block diagonal in the indices pq, with blocks

$$\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix},$$

β_n^{pq} is block diagonal with blocks

$$\begin{pmatrix} 1 & -1 \\ -1 & -1 \end{pmatrix},$$

and the time-dependent coefficients $m_{np}, r_{np}, t_{np}, s_{np}$, and their complex conjugates are taken to be odd elements of Grassmann algebra.

Now denoting m_{np}, t_{np}, by x and s_{np} and r_{np} by y and substituting (2.3)–(2.7) into (2.2) we can have the following field equations after the transition to Euclidean space-time:

$$\frac{dx}{d\tau} - \frac{(n+\frac{3}{2})x}{a(\tau)} + m\tilde{y} = 0, \tag{2.8}$$

$$\frac{d\tilde{x}}{d\tau} + \frac{(n+\frac{3}{2})\tilde{x}}{a(\tau)} - my = 0, \tag{2.9}$$

$$\frac{dy}{d\tau} - \frac{(n+\frac{3}{2})y}{a(\tau)} - m\tilde{x} = 0, \tag{2.10}$$

$$\frac{d\tilde{y}}{d\tau} + \frac{(n+\frac{3}{2})\tilde{y}}{a(\tau)} + mx = 0. \tag{2.11}$$

Further, we can receive from (2.8)–(2.11) the following set of Euclidean second-order equations:

$$\frac{d^2x}{d\tau^2} + \frac{(n+\frac{3}{2})}{a^2(\tau)}\frac{da}{d\tau}x - \frac{(n+\frac{3}{2})^2}{a^2(\tau)}x - m^2x = 0, \tag{2.12}$$

$$\frac{d^2y}{d\tau^2} + \frac{(n+\frac{3}{2})}{a^2(\tau)}\frac{da}{d\tau}y - \frac{(n+\frac{3}{2})^2}{a^2(\tau)}y - m^2y = 0, \tag{2.13}$$

$$\frac{d^2\tilde{x}}{d\tau^2} - \frac{(n+\frac{3}{2})}{a^2(\tau)}\frac{da}{d\tau}\tilde{x} - \frac{(n+\frac{3}{2})^2}{a^2(\tau)}\tilde{x} - m^2\tilde{x} = 0, \tag{2.14}$$

$$\frac{d^2\tilde{y}}{d\tau^2} - \frac{(n+\frac{3}{2})}{a^2(\tau)}\frac{da}{d\tau}\tilde{y} - \frac{(n+\frac{3}{2})^2}{a^2(\tau)}\tilde{y} - m^2\tilde{y} = 0. \tag{2.15}$$

Now, one can show that the eigenvalue problem for Eqs. (2.12) and (2.15) is equivalent to one for a couple of entangled equations (2.8) and (2.11). In fact, let us suppose that the function x is an eigenfunction of the second-order operator from Eq. (2.12) with eigenvalue Λ. Then we can assume that there is some function $\tilde{y}$ which in combination with x gives a solution of the eigenvalue problem for the couple of equations (2.8) and (2.11) with the eigenvalue λ. This function $\tilde{y}$ can be found from Eq. (2.8):

$$\tilde{y} = \left(\frac{dx}{d\tau} - \frac{(n+\frac{3}{2})x}{a(\tau)}\right)(\lambda - m)^{-1}. \tag{2.16}$$

At the same time $\tilde{y}$ must obey Eq. (2.11) with $\lambda\tilde{y}$ on the right-hand side. Substituting (2.16) into (2.11) we can see that this equation is equivalent to Eq. (2.12) provided

$$(\lambda - m)^2 = \Lambda + m^2. \tag{2.17}$$

Further if x and $\tilde{y}$ satisfy the pair of equations (2.8) and (2.11) with λ obeying (2.17) that not only (2.12) but also (2.15) is satisfied with the eigenvalue Λ.

Now, we can find the connection between the determinant of the first-order operator from Eqs. (2.8), (2.11), and one of the second-order operator from Eq. (2.12). One can see from the square equation (2.17) that each eigenvalue Λ of the second-order operator corresponds to two eigenvalues λ of the first-order operator from Eqs. (2.8), (2.11), and the product of these eigenvalues λ_1 and λ_2 is equal to the Λ. Thus, we can say that the determinant of this first-order operator is equal to the determinant of the second-order operator. We can show in a similar way that the determinant of the first-order operator from the second couple of equations (2.9) and (2.10) is equal to the determinant of the second-order operator from Eq. (2.12). Naturally, all that is written above is correct provided consistent boundary conditions for x, y, $\tilde{x}$, and $\tilde{y}$ are chosen.

Before specifying these boundary conditions we shall find the basis functions of our equations which will be used for the calculation of determinants of the corresponding operators by with the help of the ζ-function technique. Firstly, we fix the background as a Euclidean de Sitter space with the metric

$$ds^2 = d\tau^2 + a^2(\tau)d^2\Omega_3 = R^2(d\theta^2 + \sin^2\theta\, d^2\Omega_3), \tag{2.18}$$

where R is the radius and θ is the "latitude" angle on the de Sitter four-sphere.

Because Eqs. (2.13) and (2.15) coincide with (2.12) and (2.14) correspondingly, we shall consider only equations for x and $\tilde{x}$. On the background (2.18) these equations with eigenvalue terms on the right-hand side turn into

$$\frac{d^2x}{d\theta^2} + \frac{n+\frac{3}{2}}{\sin^2\theta}\cos\theta\, x - \frac{(n+\frac{3}{2})^2}{\sin^2\theta}x - (m^2R^2 + \Lambda R^2)x = 0, \tag{2.19}$$

$$\frac{d^2\tilde{x}}{d\theta^2} - \frac{n+\frac{3}{2}}{\sin^2\theta}\cos\theta\tilde{x} - \frac{(n+\frac{3}{2})^2}{\sin^2\theta}\tilde{x} - (m^2R^2 + \Lambda R^2)\tilde{x} = 0. \tag{2.20}$$

We are interested in such solutions which are regular on the part of the de Sitter sphere bounded by the three-sphere, parametrized by the latitude angle θ_+. After the corresponding substitutions we find that x and $\tilde{x}$ can be expressed through hypergeometric functions

$$x_n(\theta) = N_1 \tan^{n+3/2}\frac{\theta}{2}\, {}_2F_1\left(i\sqrt{m^2+\Lambda}R, -i\sqrt{m^2+\Lambda}R; n+2; \sin^2\frac{\theta}{2}\right) \tag{2.21}$$

and

$$\tilde{x}_n(\theta) = N_2 \sin^{n+5/2}\frac{\theta}{2}\cos^{-n-1/2}\frac{\theta}{2}\, {}_2F_1\left(1+i\sqrt{m^2+\Lambda}R, 1-i\sqrt{m^2+\Lambda}R; n+3; \sin^2\frac{\theta}{2}\right), \tag{2.22}$$

where N_1 and N_2 are some normalization constants.

Now, we must write down boundary conditions in terms of obtained basis functions. In the case of spectral boundary conditions the situation is very simple. We should fix on the boundary those harmonics that correspond to positive eigenvalues of a three-dimensional sphere; i.e., we have to require that functions m, s, r, and t obey the Dirichlet boundary conditions. In other words the basis function (2.21) must be equal to zero on the boundary or

$$x_n(\theta_+) = N_1 \tan^{n+3/2}\frac{\theta_+}{2}\, {}_2F_1\left(i\sqrt{m^2+\Lambda}R, -i\sqrt{m^2+\Lambda}R; n+2; \sin^2\frac{\theta_+}{2}\right) = 0. \tag{2.23}$$

Equation (2.23) defines eigenvalues Λ and will be used for the calculation $\zeta(0)$ for Dirac operator at spectral boundary conditions. The result, which will be obtained for the second-order operator (2.19), must be multiplied by two for the Majorana spinor and by four for the Dirac one.

The situation by local boundary conditions is more complicated. We have to consider the Dirac and Majorana cases separately.

Let us begin by considering local boundary conditions for Dirac spinors. Substituting expansions (2.4) and (2.7) for spinors ϕ_A and $\tilde{\chi}_{A'}$ correspondingly into Eq. (1.3) and using the relations from Ref. [25] which permit us to express the harmonics $\bar{\rho}^{np}_{A'}$ through ρ^{nq}_A and $\bar{\sigma}^{np}_A$ through $\sigma^{nq}_{A'}$,

$$\bar{\rho}^{np}_{A'} = -2n^A_{A'}\sum_q \rho^{nq}_A (A_n^{-1}H_n)^{qp},$$

$$\bar{\sigma}^{np}_A = -2n^{A'}_A\sum_q \sigma^{nq}_{A'} (A_n^{-1}H_n)^{qp},$$

where $(A_n^{-1}H_n)$ is the block-diagonal matrix with blocks $\begin{pmatrix} 0 & \sqrt{2} \\ -\sqrt{2} & 0 \end{pmatrix}$, we can obtain the relations

$$-im_{np}(\theta_+) = \epsilon\tilde{s}_{np}(\theta_+), \tag{2.24}$$

$$it_{np}(\theta_+) = \epsilon\tilde{r}_{np}(\theta_+). \tag{2.25}$$

In quite a similar way, after substituting the expansions (2.5) and (2.6) into the second local boundary condition (1.4) we shall get another pair of relations

$$is_{np}(\theta_+) = \epsilon\tilde{m}_{np}(\theta_+), \tag{2.26}$$

$$-ir_{np}(\theta_+) = \epsilon\tilde{t}_{np}(\theta_+). \tag{2.27}$$

It will be convenient to rewrite our relations (2.24)–(2.28) through x, y, $\tilde{x}$, and $\tilde{y}$. In this case Eqs. (2.24) and (2.25) turn into

$$-ix_n(\theta_+) = \epsilon\tilde{y}_n(\theta_+), \tag{2.28}$$

$$ix_n(\theta_+) = \epsilon\tilde{y}_n(\theta_+), \tag{2.29}$$

and Eqs. (2.26) and (2.27) turn into

$$iy_n(\theta_+) = \epsilon\tilde{x}_n(\theta_+), \tag{2.30}$$

$$-iy_n(\theta_+) = \epsilon\tilde{x}_n(\theta_+). \tag{2.31}$$

To obtain the equation determining the eigenvalues for the Dirac equation at local boundary conditions, just like Eq. (2.23) defines such eigenvalues for the problem with spectral boundary conditions, we have to consider simultaneously the Dirac equations (2.8)–(2.11) and boundary conditions (2.28)–(2.31). To begin with, let us consider Eqs. (2.28) and (2.8). Remembering that $x_n(\theta)$ and $y_n(\theta)$ are defined as (2.21) and (2.22) we can have from (2.28) the following relation between normalization constants N_1 and N_2:

$$\frac{N_1}{N_2} = \frac{i\epsilon \sin\theta_+ \; {}_2F_1(1+i\sqrt{m^2+\Lambda}R, 1-i\sqrt{m^2+\Lambda}R; n+3; \sin^2\frac{\theta_+}{2})}{2\; {}_2F_1(i\sqrt{m^2+\Lambda}R, -i\sqrt{m^2+\Lambda}R; n+2; \sin^2\frac{\theta_+}{2})}. \tag{2.32}$$

We can also obtain from (2.8), by using well-known formula [33]

$$\frac{dF(a,b;c;z)}{dz} = \frac{ab}{c}F(a+1,b+1,c+1;z),$$

the relation

$$\frac{N_1}{N_2} = \frac{(\lambda-m)R(n+2)}{m^2R^2+\Lambda R^2}. \tag{2.33}$$

Equating (2.32) and (2.33) we have the following condition on the eigenvalues:

$${}_2F_1\left(i\sqrt{m^2+\Lambda}R, -i\sqrt{m^2+\Lambda}R; n+2; \sin^2\frac{\theta_+}{2}\right) - \frac{i\epsilon(m^2+\Lambda)R\sin\theta_+}{2(n+2)(\lambda-m)} \times {}_2F_1\left(1+i\sqrt{m^2+\Lambda}R, 1-i\sqrt{m^2+\Lambda}R; n+3; \sin^2\frac{\theta_+}{2}\right) = 0. \tag{2.34}$$

Analogously, from Eqs. (2.29) and (2.8) we can obtain the following condition on the eigenvalues

$${}_2F_1\left(i\sqrt{m^2+\Lambda}R, -i\sqrt{m^2+\Lambda}R; n+2; \sin^2\frac{\theta_+}{2}\right) + \frac{i\epsilon(m^2+\Lambda)R\sin\theta_+}{2(n+2)(\lambda-m)} \times {}_2F_1\left(1+i\sqrt{m^2+\Lambda}R, 1-i\sqrt{m^2+\Lambda}R; n+3; \sin^2\frac{\theta_+}{2}\right) = 0. \tag{2.35}$$

Multiplying (2.34) by (2.35) and taking into account that $\epsilon^2 = 1$ and $(\lambda-m)^2 = m^2+\Lambda$ [see Eqs. (2.17)], we have the condition

$$\left[{}_2F_1\left(i\sqrt{m^2+\Lambda}R, -i\sqrt{m^2+\Lambda}R; n+2; \sin^2\frac{\theta_+}{2}\right)\right]^2 + \frac{(m^2+\Lambda)R^2\sin^2\theta_+}{4(n+2)^2} \times \left[{}_2F_1\left(1+i\sqrt{m^2+\Lambda}R, 1-i\sqrt{m^2+\Lambda}R; n+3; \sin^2\frac{\theta_+}{2}\right)\right]^2 = 0. \tag{2.36}$$

It is possible to show that the combination of the boundary conditions (2.30) and (2.31) with the equation of motion (2.10) gives us the eigenvalue condition (2.36) again.

Now we can consider the local boundary conditions for Majorana spinors. Our exposition will be very close to that of Ref. [10]. In the case of Majorana spinors local boundary conditions (1.1) will entangle the functions m_{np} and r_{np} with different values of p in contrast with the case of Dirac spinors. To be precise we can say that functions with adjacent values of index p, $2k+1$ and $2k+2$, where $k = 0, 1, \ldots, \frac{n}{2}(n+3)$, are entangled. It will be enough to consider as a typical case a pair of indices: $p=1$ and $p=2$. Let us introduce the notation

$$x \equiv m_{n1},\; X \equiv m_{n2},\; \tilde{x} \equiv \tilde{m}_{n1},\; \tilde{X} \equiv \tilde{m}_{n2},$$
$$y \equiv r_{n1},\; Y \equiv r_{n2},\; \tilde{y} \equiv \tilde{r}_{n1},\; \tilde{Y} \equiv \tilde{r}_{n2}. \tag{2.37}$$

Now, substituting the expansions (2.4) and (2.5) into local boundary conditions for Majorana spinors (1.1) we can get the following conditions for functions (2.37):

$$-ix(\theta_+) = \epsilon\tilde{X}(\theta_+), \tag{2.38}$$

$$iX(\theta_+) = \epsilon\tilde{x}(\theta_+), \tag{2.39}$$

$$iY(\theta_+) = \epsilon\tilde{y}(\theta_+), \tag{2.40}$$

$$-iy(\theta_+) = \epsilon\tilde{Y}(\theta_+). \tag{2.41}$$

The Dirac eigenvalue equation has the following form for Majorana spinors:

$$\frac{dx}{d\tau} - \frac{(n+\frac{3}{2})x}{a(\tau)} + (m-\lambda)\tilde{x} = 0, \tag{2.42}$$

$$\frac{d\tilde{x}}{d\tau} + \frac{(n+\frac{3}{2})\tilde{x}}{a(\tau)} + (m-\lambda)x = 0, \tag{2.43}$$

$$\frac{dy}{d\tau} - \frac{(n+\frac{3}{2})y}{a(\tau)} - (m-\lambda)\tilde{y} = 0, \tag{2.44}$$

$$\frac{d\tilde{y}}{d\tau} + \frac{(n+\frac{3}{2})\tilde{x}}{a(\tau)} - (m-\lambda)y = 0. \tag{2.45}$$

A set of equations for X, Y, $\tilde{X}$, and $\tilde{Y}$ is quite similar to the system (2.42) and (2.45). Because the functions x and y in Eqs. (2.38)–(2.45) are disentangled we can consider for the beginning a pair of the conditions (2.38) and (2.39) together with Eqs. (2.42) and (2.43). That pair of equations gives the second-order equations which coincide with (2.12) and (2.14). Thus we can introduce the following system of basis functions:

$$\tilde{x} = C_1 \sin^{n+\frac{5}{2}}\frac{\theta}{2}\cos^{-n-\frac{1}{2}}\frac{\theta}{2}\ {}_2F_1\left(1+i\sqrt{m^2+\Lambda}R, 1-i\sqrt{m^2+\Lambda}R; n+3; \sin^2\frac{\theta}{2}\right), \tag{2.46}$$

$$\tilde{X} = C_2 \sin^{n+\frac{5}{2}}\frac{\theta}{2}\cos^{-n-\frac{1}{2}}\frac{\theta}{2}\ {}_2F_1\left(1+i\sqrt{m^2+\Lambda}R, 1-i\sqrt{m^2+\Lambda}R; n+3; \sin^2\frac{\theta}{2}\right), \tag{2.47}$$

$$x = C_3 \tan^{n+\frac{3}{2}}\frac{\theta}{2}\ {}_2F_1\left(i\sqrt{m^2+\Lambda}R, -i\sqrt{m^2+\Lambda}R; n+2; \sin^2\frac{\theta}{2}\right), \tag{2.48}$$

$$X = C_4 \tan^{n+\frac{3}{2}}\frac{\theta}{2}\ {}_2F_1\left(i\sqrt{m^2+\Lambda}R, -i\sqrt{m^2+\Lambda}R; n+2; \sin^2\frac{\theta}{2}\right). \tag{2.49}$$

Substituting (2.48) and (2.47) into (2.38) we have

$$\frac{C_3}{C_2} = i\epsilon\frac{\sin\theta_+}{2}\frac{{}_2F_1\left(1+i\sqrt{m^2+\Lambda}R, 1-i\sqrt{m^2+\Lambda}R; n+3; \sin^2\frac{\theta_+}{2}\right)}{{}_2F_1\left(i\sqrt{m^2+\Lambda}R, -i\sqrt{m^2+\Lambda}R; n+2; \sin^2\frac{\theta_+}{2}\right)}. \tag{2.50}$$

Substituting (2.48) and (2.49) into (2.38) we obtain

$$\frac{C_4}{C_1} = -i\epsilon\frac{\sin\theta_+}{2}\frac{{}_2F_1(1+i\sqrt{m^2+\Lambda}R, 1-i\sqrt{m^2+\Lambda}R; n+3; \sin^2\frac{\theta_+}{2})}{{}_2F_1(i\sqrt{m^2+\Lambda}R, -i\sqrt{m^2+\Lambda}R; n+2; \sin^2\frac{\theta_+}{2})}. \tag{2.51}$$

From Eq. (2.42) we obtain the relation

$$\frac{C_3}{C_1} = \frac{(\lambda-m)R(n+2)}{(m^2+\Lambda)R^2}. \tag{2.52}$$

Analogously we can also obtain

$$\frac{C_4}{C_2} = \frac{(\lambda-m)R(n+2)}{(m^2+\Lambda)R^2}. \tag{2.53}$$

Now combining together Eqs. (2.50)–(2.53) we have

$$\begin{aligned} 1 &= \frac{C_3}{C_2}\frac{C_4}{C_1}\frac{C_1}{C_3}\frac{C_4}{C_2} \\ &= \frac{\sin^2\theta_+}{4}\frac{(m^2+\Lambda)R^2}{(n+2)}\frac{[{}_2F_1(1+i\sqrt{m^2+\Lambda}R, 1-i\sqrt{m^2+\Lambda}R; n+3; \sin^2\frac{\theta_+}{2})]^2}{[{}_2F_1(i\sqrt{m^2+\Lambda}R, -i\sqrt{m^2+\Lambda}R; n+2; \sin^2\frac{\theta_+}{2})]^2} \end{aligned} \tag{2.54}$$

or

$$\left[{}_2F_1\left(i\sqrt{m^2+\Lambda}R, -i\sqrt{m^2+\Lambda}R; n+2; \sin^2\frac{\theta_+}{2}\right)\right]^2$$

$$-\frac{(m^2+\Lambda)R^2}{4(n+2)}\sin^2\theta_+\left[{}_2F_1\left(1+i\sqrt{m^2+\Lambda}R, 1-i\sqrt{m^2+\Lambda}R; n+3; \sin^2\frac{\theta_+}{2}\right)\right]^2 = 0. \tag{2.55}$$

We can also obtain the eigenvalue condition (2.55) considering harmonics denoted by y, $\tilde{y}$, Y, $\tilde{Y}$.

Thus, we have obtained the eigenvalue condition for Majorana spinors provided local boundary conditions are chosen. It is worthwhile to note that if in the case of spectral boundary conditions the eigenvalue equation (2.23) is the same for Majorana and Dirac spinors, in the case of local boundary conditions the corresponding equations (2.55) and (2.36) are different.

III. THE TECHNIQUE OF THE GENERALIZED ζ FUNCTION ON A COMPACT EUCLIDEAN MANIFOLD WITH A BOUNDARY

It is known that the Hartle-Hawking wave function [15,16] in the one-loop approximation can be represented in the form

$$\Psi(q_+) = \exp(-I[q_0] - W_{1\ \mathrm{loop}}), \tag{3.1}$$

where q_+ are the values of all (gravitational and matter) fields on the boundary ∂M of the Euclidean manifold M, q_0 are the solutions of classical equations of motion, satisfying the corresponding boundary conditions, and $W_{1\ \mathrm{loop}}$ are one-loop contributions to (3.1) which equals

$$W_{1\ \mathrm{loop}} = \frac{1}{2}\mathrm{Tr}\ln F, \tag{3.2}$$

where the second-order differential operator F is

$$F = \frac{\delta^2 I}{\delta\xi\delta\xi} \tag{3.3}$$

and ξ are physical degrees of freedom. Introducing generalized Riemannian ζ function

$$\zeta(s) = \sum_\lambda \frac{1}{\lambda^s}, \tag{3.4}$$

where λ are eigenvalues of F, it is possible to show [31,32] that

$$W_{1\ \mathrm{loop}} = -\frac{1}{2}\zeta'(0) - \frac{1}{2}\zeta(0)\ln\mu^2. \tag{3.5}$$

Here μ^2 is a renormalization-mass parameter, reflecting the renormalization ambiguity of the theory.

The main problem is connected with the necessity to calculate $\zeta(0)$ and $\zeta'(0)$ without explicit knowledge of the spectrum (3.3). It is possible to do using the methods of the theory of functions of complex variables. Let us suppose that we have a full set of basis functions $u(\tau|m^2)$ of operator $(F+m^2)$, i.e.,

$$(F+m^2)u(\tau|m^2) = 0, \tag{3.6}$$

where τ is Euclidean time. Then the eigenvalues λ of $F+m^2$ with Dirichlet zero boundary conditions satisfy the equation

$$u(\tau_+|m^2-\lambda) = 0, \tag{3.7}$$

where τ_+ is the value of Euclidean time parametrizing the boundary ∂M. For example, in the case of spectral boundary conditions for spinors Eq. (2.23) plays the role of condition (3.7) defining the eigenvalues of the corresponding second-order differential operator (2.19). At the same time in the cases when we have more complicated boundary conditions such as in Eqs. (2.36) and (2.55), in Eqs. (3.7) it is necessary to use corresponding combinations of basis functions. In further exposition we shall write simply $u(\tau_+|m^2-\lambda)$ keeping in mind that instead of u some combinations of basis functions or (in the case of Robin boundary conditions) even a combination including first derivatives of basis functions might stay.

Now using the Cauchy formula, we can write $\zeta(s)$ as an integral

$$\zeta(s) = \frac{1}{2\pi i}\int_C \frac{dz}{z^s}\frac{d}{dz}\ln u(\tau_+|m^2-z) \tag{3.8}$$

over the contour C in the complex plane of z, which encircles all the roots of (3.7) (see Fig. 1). Using the properties of basis functions [12] we can continuously deform the original contour of integration C to the new one $\tilde{C}$, which encircles the cut in the complex plane of the function z^{-s}, coinciding with the negative real axis (see Fig. 1). After this Eq. (3.8) can be written as

$$\zeta(s) = \frac{\sin\pi s}{s}\int_\varepsilon^\infty \frac{dM^2}{M^{2s}}\frac{d}{dM^2}\ln u(\tau_+|m^2+M^2) + \frac{1}{2\pi i}\int_{C_\varepsilon}\frac{dz}{z^s}\frac{d}{dz}\ln u(\tau_+|m^2-z), \tag{3.9}$$

where the first term is a jump of the integrand of (3.8) on the cut of the function z^{-s}, integrated along this cut $z=-M^2$, and C_ε is a circle around the point $z=0$ of

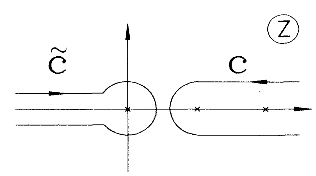

FIG. 1. Integration contours C and $\tilde{C}$ in the complex plane of the auxiliary mass parameter z.

some small radius ε.

Let us transform Eq. (3.9) by the following sequence of operations. First analytically continue both terms into the neighborhood of $s = 0$ and then go to the limit $\varepsilon = 0$. The integral along C_ε will vanish because of the regularity of $u(\tau_+|m^2 - z)$ at $z = 0$. Thus, expression (3.9) is boiled down to the integral along the cut.

For the case of the field theories with an infinite number of modes we have instead of (3.8) the expression

$$\zeta(s) = \frac{1}{2\pi i}\int_{\bar{C}} \frac{dz}{z^s}\frac{d}{dz}\sum_A \ln u(\tau_+|m^2 - z), \qquad (3.10)$$

where collective index A enumerates field harmonics. It is necessary to use ζ regularization in a such way to provide also regularization of ultraviolet divergences. Because of the properties of so-called uniform asymptotic expansions [34,35,12] we can carry this out by making the change of integration variable $z \to n^2 z$, where n is an integer parametrizing index A. Making this change of integration variable one can represent the ζ function in the form

$$\zeta(s) = \frac{1}{2\pi i}\int_{\bar{C}} \frac{dz}{z^s}\frac{d}{dz} I(-z, s), \qquad (3.11)$$

where $I(-z, s)$ is a manifestly regularized infinite sum:

$$I(-z, s) = \sum_A \frac{1}{n^{2s}} \ln u_A(\tau_+|m^2 - z). \qquad (3.12)$$

The series (3.12) analytically continued from its convergence domain generally has a pole at $s = 0$:

$$I(M^2, s) = \frac{I^{\text{pole}}(M^2)}{s} + I^R(M^2) + O(s), \qquad (3.13)$$

where $I^R(M^2)$ is a regular at $s \to 0$.

After some calculations [12] it is possible to show that

$$\zeta(s) = (I^R)_{\log} + I^{\text{pole}}(\infty) - I^{\text{pole}}(0) + s\left[I^R(\infty) - I^R(0) - \int_0^\infty dM^2 \ln M^2 \frac{dI^{\text{pole}}(M^2)}{dM^2}\right] + O(s^2), \qquad (3.14)$$

where $(I^R)_{\log}$ denotes the coefficient at the $\ln M^2$ in the expansion of I^R at $M^2 \to \infty$, $I^R(\infty)$ is the finite part of I^R at $M^2 \to \infty$. Thus we have

$$\zeta(0) = (I^R)_{\log} + I^{\text{pole}}(\infty) - I^{\text{pole}}(0), \qquad (3.15)$$

$$\zeta'(0) = I^R(\infty) - I^R(0) - \int_0^\infty dM^2 \ln M^2 \frac{dI^{\text{pole}}(M^2)}{dM^2}. \qquad (3.16)$$

It is necessary to notice that the terms, including I^{pole} do not have analogous ones in a theory with a finite number of modes. These terms are responsible for the nontrivial renormalization of the ultraviolet divergences performed by the ζ function regularization.

The proposed method of calculating $\zeta(0)$ and $\zeta'(0)$ requires knowledge of $(I^R)_{\log}$, $(I^R)(\infty)$, $(I^R)(0)$, $(I^{\text{pole}})(M^2)$. These quantities can be obtained from the uniform asymptotic expansion for basis functions in the case where these functions and their expansions are known. But the computation of these values is a tedious task. However, it is not necessary to use all this information for calculating only $\zeta(0)$. It is possible to simplify the technique for the calculation of $\zeta(0)$ by using a convenient choice of normalization of basis functions. As a matter of fact we can choose the normalization for these functions in a such way as to obtain the equality $(I^{\text{pole}})(M^2)|_{M^2\to\infty} = 0$. In this case (3.15) turns into

$$\zeta(0) = (I^R)_{\log} - I^{\text{pole}}(0). \qquad (3.17)$$

Let us consider in what way the term $[I^{\text{pole}}(M^2)]/s$ appears in the decomposition (3.13) of a regularized sum (3.12). We can rewrite the series $I(-z, s)$ in the form

$$I(M^2, s) = \sum_{n=0}^{\infty} \frac{\dim A_n}{n^{2s}} \ln u_n, \qquad (3.18)$$

where $\dim A_n$ denotes the degeneracy for harmonics corresponding to n (for two-component spinors $\dim A_n = (n+1)(n+2)$, see Ref. [25]), u_n is a hypergeometric function from (2.23) for the case of spectral boundary conditions or a combination of hypergeometric functions from formulas (2.36) or (2.55) for local boundary conditions. Expanding u_n in inverse degrees of n we can reduce (3.18) in the convergence domain $2s > 4$ to the sum of usual Riemann ζ functions. Then we investigate the expression obtained at the limit $s \to 0$. It was shown in Ref. [12] that in the general case $I(M^2, s)$ can be represented as

$$\sum_{k=-3}^{\infty}\sum_{n=n_0}^{\infty} n^{-k-2s} I_k(m^2), \qquad (3.19)$$

where n_0 is a different number for fields with different spins. We can calculate (3.19) term by term using the ζ regularization technique and obtaining the usual Riemann ζ functions. Taking into account that the only simple pole of $\zeta_R(z)$ is $z = 1$,

$$\zeta_R(z) = \frac{1}{z-1} + O((z-1)^0),$$

one easily finds from (2.37) that $I^{\text{pole}}(M^2) = I_{-1}(M^2)/2$. Now to provide the right choice of normalization constants for u_n we can use the asymptotic formula for hypergeometric function [36]: for $|\lambda| \to \infty$

$$
\begin{aligned}
{}_2F_1\left(a+\lambda, b+\lambda; c; \frac{1-z}{2}\right) &= \frac{\Gamma(1-b+\lambda)\Gamma(c)}{\Gamma(1/2)\Gamma(c-b-\lambda)} 2^{a+b-1}(1-e^{-\xi})^{-c+1/2}(1+e^{-\xi})^{c-a-b-1/2}\lambda^{-1/2} \\
&\quad \times [e^{(\lambda-b)\xi} + e^{\pm i\pi(c-1/2)}e^{-(\lambda+a)\xi}][1+O(|\lambda^{-1}|)], \\
& z \pm (z^2-1)^{1/2} = e^{\pm\xi}.
\end{aligned}
\tag{3.20}
$$

It is clear from (3.20) that only a term of expansion of $\frac{1}{2}(n+1)(n+2)\ln {}_2F_1(a+\lambda, b+\lambda; c; \frac{1-z}{2})$, which gives a contribution into $I^{\rm pole}(M^2) = 0$, at $M^2 \to \infty$, originates from $\frac{1}{2}(n+1)(n+2)\ln\Gamma(c)$. Thus, our strategy will be the following: in the case of spectral boundary conditions we must use the hypergeometric function (2.23) divided by $\Gamma(n+2)$ and in the case of local boundary conditions we must use expressions (2.36) and (2.55) for Dirac and Majorana spinors correspondingly divided by $[\Gamma(n+2)]^2$. It will be convenient to shift our summation index from n to $n-1$. In this case the summation will begin from $n=1$, and the degeneracy for two-component spinors will be

$$\dim A_n = n(n+1).$$

Now we can write down the expressions for $I(M^2, s)$ which we shall use calculating $\zeta(0)$ according to formula (3.17). For Majorana spinors at spectral boundary conditions,

$$I(M^2,s)_{M\,\rm spect} = \sum_{n=1}^{\infty} \frac{2n(n+1)}{n^{2s}} \ln \frac{{}_2F_1(i\sqrt{m^2+M^2}R, -i\sqrt{m^2+M^2}R; n+1; \sin^2\frac{\theta_+}{2})}{\Gamma(n+1)}. \tag{3.21}$$

For Dirac spinors at spectral conditions we must take expression (3.21) multiplied by a factor 2.

For Dirac spinors at local boundary conditions we have

$$
\begin{aligned}
I(M^2,s)_{D\,\rm local} = \sum_{n=1}^{\infty} \frac{2n(n+1)}{n^{2s}} \ln \frac{1}{[\Gamma(n+1)]^2} \\
\times \Big\{ [{}_2F_1(i\sqrt{m^2+M^2}R, -i\sqrt{m^2+M^2}R; n+1; \sin^2\tfrac{\theta_+}{2})]^2 \\
+ \frac{\sin^2\theta_+}{4(n+1)^2} [{}_2F_1(1+i\sqrt{m^2+M^2}R, 1-i\sqrt{m^2+M^2}R; n+2; \sin^2\tfrac{\theta_+}{2})]^2 \Big\},
\end{aligned}
\tag{3.22}
$$

and for Majorana spinors at local boundary conditions we have

$$
\begin{aligned}
I(M^2,s)_{M\,\rm local} = \sum_{n=1}^{\infty} \frac{n(n+1)}{n^{2s}} \ln \frac{1}{[\Gamma(n+1)]^2} \\
\times \Big\{ [{}_2F_1(i\sqrt{m^2+M^2}R, -i\sqrt{m^2+M^2}R; n+1; \sin^2\tfrac{\theta_+}{2})]^2 \\
- \frac{\sin^2\theta_+}{4(n+1)^2} [{}_2F_1(1+i\sqrt{m^2+M^2}R, 1-i\sqrt{m^2+M^2}R; n+2; \sin^2\tfrac{\theta_+}{2})]^2 \Big\}.
\end{aligned}
\tag{3.23}
$$

IV. RESULTS AND DISCUSSION

Now we are in position to calculate $\zeta(0)$ for spinors at different boundary conditions. To do it we must use formulas (3.21)–(3.23) for calculating $(I^R)_{\log}$ and $I^{\rm pole}(0)$ and then substitute obtained results into formula (3.17).

Let us begin with the calculation of $\zeta(0)$ for Majorana spinors at spectral boundary conditions. Firstly, we calculate $(I^R)_{\log}$. Using the asymptotic formula (3.20) we can extract from (3.21) the coefficient at $\ln M^2$ which looks like

$$(I^R)_{\log} = \lim_{s\to 0} \sum_{n=1}^{\infty} \frac{n(n+1)}{n^{2s}} \left(-n - \frac{1}{2}\right) = -\zeta_R(-3) - \frac{3}{2}\zeta_R(-2) - \frac{1}{2}\zeta_R(-1).$$

Remembering that $\zeta_R(-3) = \frac{1}{120}$, $\zeta_R(-2) = 0$, $\zeta_R(-1) = -\frac{1}{12}$, we have

$$(I^R)_{\log\,M\,\rm spect} = \frac{1}{30}. \tag{4.1}$$

For calculation of $I^{\rm pole}(0)$ we can equal M^2 in the expression (3.21) to zero. After that for an extraction term behaving like $\frac{1}{n}$ from (3.21) it is enough to use the usual expansion of the hypergeometric function into series [33] and the asymptotic formula for $\ln\Gamma(n)$ at $n \to \infty$. As a result of this procedure we have

$$I^{\rm pole}(0)_{M\,{\rm spect}} = \frac{1}{360} + m^2R^2\left(-\sin^4\frac{\theta_+}{2} + \frac{2}{3}\sin^6\frac{\theta_+}{2}\right) + m^4R^4\left(-\frac{1}{2}\sin^4\frac{\theta_+}{2} + \frac{1}{3}\sin^6\frac{\theta_+}{2}\right). \tag{4.2}$$

Now substituting (4.1) and (4.2) into (3.17) we obtain

$$\zeta(0)_{M\,{\rm spect}} = \frac{11}{360} + m^2R^2\left(\sin^4\frac{\theta_+}{2} - \frac{2}{3}\sin^6\frac{\theta_+}{2}\right) + m^4R^4\left(\frac{1}{2}\sin^4\frac{\theta_+}{2} - \frac{1}{3}\sin^6\frac{\theta_+}{2}\right). \tag{4.3}$$

$\zeta(0)_{D\,{\rm spect}}$ for the Dirac spinor field at spectral boundary conditions can be obtained by multiplying (4.3) by two. It is easy to see that in the flat-space limit ($R\sin\theta_+ = a_+, \theta_+ \to 0$ where a_+ is the radius of three-sphere restricting the part of flat space) the expression (4.3) turns into

$$\zeta(0)_{M\,{\rm spect\ flat}} = \frac{11}{360} + \frac{m^4a_+^4}{32} \tag{4.4}$$

and in the massless case coincides with the result from Ref. [11]. Moreover, at $m = 0$ the dependence on θ_+ dissappears from (4.3) and $\zeta(0)$ for the de Sitter case coincides with one for the flat-space case. One can suppose that this coincidence is naturally connected with the well-known fact that the massless spinor field has the property of conformal invariance. (It is worth adding that the analogous coincidence of results for the de Sitter background and for a flat one were found also for a conformally coupled massless field and for an electromagnetic field [13,14].)

Now we can go to the calculation of $\zeta(0)$ for the case of local boundary conditions. It is easy to see, looking at the asymptotic formula (3.20), that the large-M factor in the hypergeometric function with the third parameter $c = n+1$ behaves like $M^{-n-1/2}$ while the analogous factor in the hypergeometric function with the third parameter $c = n+2$ behaves like $M^{-n-3/2}$. Therefore, in expressions (3.22) and (3.23) the terms including functions ${}_2F_1(1+i\sqrt{m^2+M^2}R, 1-i\sqrt{m^2+M^2}R; n+2; \sin^2\frac{\theta_+}{2})$ do not give a contribution to the coefficient at $\ln M^2$, i.e., to $(I^R)_{\log}$. Thus we can see that $(I^R)_{\log M\,{\rm local}}$ coincides with $(I^R)_{\log M\,{\rm spect}}$ formula (4.1) and the corresponding value for Dirac spinors differs from (4.1) only by factor 2. In addition, calculating $(I^{\rm pole})(0)$ in the massless case $m = 0$ we find that at $M = 0$ and $m = 0$ terms including ${}_2F_1(1+i\sqrt{m^2+M^2}R, 1-i\sqrt{m^2+M^2}R; n+2; \sin^2\frac{\theta_+}{2})$ disappear from expressions (3.22) and (3.23) and they coincide with (3.21) precisely as (3.23) and up to factor 2 in the case of Dirac spinors (3.22). Thus, we have seen the way in which, in the case of massless spinors, results for $\zeta(0)$ calculated at spectral and local boundary conditions coincide. However, in the case of massive spinors we must calculate $I^{\rm pole}(0)$ using expansions for combinations of hypergeometrical functions which are arguments of ln in formulas (3.22) and (3.23). After carrying out a straightforward but a bit tedious calculations we obtain

$$I^{\rm pole}(0)_{D\ {\rm local}} = \frac{1}{180} + m^2R^2\left(-\sin^2\frac{\theta_+}{2} + \sin^4\frac{\theta_+}{2} - \frac{2}{3}\sin^6\frac{\theta_+}{2}\right) - m^4R^4\left(-\sin^4\frac{\theta_+}{2} + \frac{2}{3}\sin^6\frac{\theta_+}{2}\right), \tag{4.5}$$

$$I^{\rm pole}(0)_{M\,{\rm local}} = \frac{1}{360} + m^2R^2\left(\frac{1}{2}\sin^4\frac{\theta_+}{2} - \frac{5}{2}\sin^4\frac{\theta_+}{2} + \frac{5}{3}\sin^6\frac{\theta_+}{2}\right) + m^4R^4\left(-\frac{1}{2}\sin^4\frac{\theta_+}{2} + \frac{1}{3}\sin^6\frac{\theta_+}{2}\right). \tag{4.6}$$

Now, substituting the obtained results into (3.17) we have

$$\zeta(0)_{D\,{\rm local}} = \frac{11}{180} + m^2R^2\left(\sin^2\frac{\theta_+}{2} - \sin^4\frac{\theta_+}{2} + \frac{2}{3}\sin^6\frac{\theta_+}{2}\right) + m^4R^4\left(-\sin^4\frac{\theta_+}{2} + \frac{2}{3}\sin^6\frac{\theta_+}{2}\right), \tag{4.7}$$

$$\zeta(0)_{M\,{\rm local}} = \frac{11}{360} + m^2R^2\left(-\frac{1}{2}\sin^4\frac{\theta_+}{2} + \frac{5}{2}\sin^4\frac{\theta_+}{2} - \frac{5}{3}\sin^6\frac{\theta_+}{2}\right) + m^4R^4\left(+\frac{1}{2}\sin^4\frac{\theta_+}{2} - \frac{1}{3}\sin^6\frac{\theta_+}{2}\right). \tag{4.8}$$

It is easy to see from formulas (4.7) and (4.8) that in the massless case $\zeta(0)_{D\,{\rm local}}$ and $\zeta(0)_{M\,{\rm local}}$ coincide with the corresponding result for the case of spectral boundary conditions and all of them coincide with the corresponding results for a flat background. At the same time for the massive case the results for different kinds of boundary conditions differ; moreover, there is a nontrivial difference between the values of $\zeta(0)_{D\,{\rm local}}$ and $\zeta(0)_{M\ {\rm local}}$ in contrast with the case of spectral boundary conditions where $\zeta(0)_{D\,{\rm spect}}$ is simply $\zeta(0)_{M\,{\rm spect}} \times 2$. We can also write down the expressions (4.7) and (4.8) for the case of massive spinors on the flat background. We shall have

$$\zeta(0)_{D\,{\rm local\ flat}} = \frac{11}{180} + \frac{m^2a_+^2}{4} + \frac{m^4a_+^4}{16}, \tag{4.9}$$

$$\zeta(0)_{M\,{\rm local\ flat}} = \frac{11}{360} - \frac{m^2a_+^2}{8} + \frac{m^4a_+^4}{32}. \tag{4.10}$$

Thus, we can see that the difference between values of $\zeta(0)$ at local and spectral boundary conditions is connected with the nonzero mass of fermions and survives not only on the de Sitter background but also on a flat one.

In our previous papers [13,14] $\zeta(0)$ was also calculated for massless gravitinos on the de Sitter background at spectral boundary conditions. Our results coincide with one from Ref. [11]:

$$\zeta(0)_{\text{gravitino}} = -\tfrac{289}{360}. \tag{4.11}$$

Repeating the consideration of structure of $\zeta(0)$ represented in the present paper it is easy to see that due to the masslessness of the gravitino the value of $\zeta(0)_{\text{gravitino}}$ on the de Sitter background at local boundary conditions will also coincide with (4.11).

In conclusion we have to recognize that one rather intricate question is left beyond the scope of this paper. We mean the problem of the discrepancy between the results of covariant calculations for fields with higher spins (see Refs. [23,5,7]) and those obtained by working with physical degrees of freedom (see Refs. [1–4,10–14]). It is interesting that this problem appears not only in the consideration of manifolds with a boundary but also in the case of a compact manifold without boundary (see Ref. [4]). Perhaps the origin of this discrepancy is connected with subtle problems of quantization of higher-spin fields on the nontrivial background. In any case, it is worthy of further investigations.

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