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Einstein equation with quantum corrections reduced to second order

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We consider the Einstein equation with first-order (semiclassical) quantum corrections. Although the quantum corrections contain up to fourth-order derivatives of the metric, the solutions which are physically relevant satisfy reduced equations which contain derivatives no higher than second order. We obtain the reduced equations for a range of stress-energy tensors. These reduced equations are suitable for a numerical solution, are expected to contain fewer numerical instabilities than the original fourth-order equations, and yield only physically relevant solutions. We give analytic and numerical solutions or reduced equations for particular examples, including Friedmann-Lemaître universes with a cosmological constant, a spherical body of constant density, and more general conformally flat metrics.

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I. INTRODUCTION

Quantum corrections to general relativity are expected to be important in regimes where the curvature is near the Planck scale ($l_{\rm Pl}=\sqrt{G\hbar/c^3}\approx 1.6\times 10^{-33}$ cm). In a regime where the curvature approaches but always remains (significantly) less than the Planck scale, a semiclassical approximation to the full theory of quantum gravity should be sufficient. Examples of this regime include small evaporating black holes, when still much larger than the Planck mass ($m_{\rm Pl}=\sqrt{\hbar c/G}\approx 2.2\times 10^{-5}$ g), and the early Universe after it has reached a size of many Planck lengths. In the standard semiclassical approximation, the gravitational field itself is treated classically, but is driven by the expectation value of quantum matter stress energy.

The form of the semiclassical corrections to Einstein's field equations is known for many important cases [1]. For example, for conformally flat classical backgrounds (in four dimensions), when the quantum state is constructed from the conformal vacuum, the corrections are completely determined by local geometry (the metric, the curvature, the covariant derivatives of the curvature) [2]:

$$\kappa\langle T_{ab}\rangle = R_{ab}-\tfrac{1}{2}Rg_{ab}+\Lambda g_{ab}+\alpha_1\hbar(\tfrac{1}{2}R^2 g_{ab}-2RR_{ab}-2\Box R g_{ab}+2\nabla_a\nabla_b R)$$
$$+\alpha_2\hbar(\tfrac{1}{2}R^{cd}R_{cd}g_{ab}-\Box R_{ab}-\tfrac{1}{2}\Box R g_{ab}+\nabla_a\nabla_b R-R^{cd}R_{cadb})+\alpha_3\hbar(-\tfrac{1}{12}R^2 g_{ab}+R^{cd}R_{cadb})+O(\hbar^2)\ . \quad (1.1)$$

The parameters α_1, α_2, and α_3 depend on the particular form of matter and regularization scheme, so we do not assume specific values or signs. Factors of $\hbar$ have been made explicit. Because the corrections are purely geometric, it is common to consider them not as matter source terms but as metric field terms (despite their matter origin). Nonconformally flat backgrounds can have more quantum corrections than Eq. (1.1). Of the examples above, the state-independent terms of Eq. (1.1) do not contribute in the case of the black hole, where the exterior Ricci curvature vanishes, but they do contribute in the case of cosmological solutions. Several cosmological models are examined below. Because the new terms contain fourth derivatives in the metric, the new terms qualitatively change the field equations from a system of second-order equations to a system of fourth-order equations.

The new fourth-order theory contains whole new classes of solutions unavailable to the classical theory. Many of these solutions have been examined [3]. One set of these solutions is particularly disturbing however. Solutions to the linearized theory around a flat background strongly indicate that flat space is unstable to ultraviolet fluctuations [4,5]. Using a $1/N$ approximation, Hartle and Horowitz showed that the ultraviolet instability can be made to occur at frequencies arbitrarily far below the Planck frequency, indicating that the instabilities cannot be easily fixed by calling the full quantum theory of gravity to the rescue [5]. Additional instabilities have also been found by Suen [6]. This strongly indicates that semiclassical gravity, if all its solutions are considered physical, is not a good description of the near classical limit of quantum gravity.

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It was shown in earlier work [7] that it is possible, and indeed desirable, to modify semiclassical gravity in a way suggested by and consistent with the perturbative nature of its derivation. The effective action and field equations of semiclassical gravity are perturbative expansions (formally, asymptotic expansions) in powers of $\hbar$, truncated at first order in $\hbar$. All behavior higher order and nonperturbative in $\hbar$ has already been lost in the process of deriving the (approximate) effective action and field equations. Self-consistency then requires that only the solutions that are also asymptotic expansions in powers of $\hbar$, truncated to first order, will be approximations to solutions of the full, nonperturbative effective action. Solutions not in this form are likely to be unphysical and should be excluded. A simple model, presented below, will demonstrate that retaining nonperturbative solutions to a perturbatively derived higher derivative action results in false predictions. The nonperturbatively expandable solutions are spurious artifacts arising from the higher derivatives appearing in the perturbative correction, and will be referred to as spurious. For convenience, perturbatively expandable solutions will sometimes be referred to as physical, since only they correspond to predictions of the self-consistent semiclassical theory. For semiclassical gravity, it has been shown that the physical solutions show no signs of any instability of flat space (to first order in $\hbar$) [7].

The easiest way of implementing the self-consistent method in semiclassical gravity is by reducing the fourth-order equation, which has both physical and nonphysical solutions, to a second-order equation, which has only physical solutions (with one caveat described below). This iterative reduction has been demonstrated in a similar context by Bel and Sirousse-Zia for the case $\alpha_3=0$ [8]. Much of the reduction (though not always all) can be done covariantly. It is clearly more efficient to find solutions to the reduced second-order equations, almost all of which are physical, rather than finding all solutions to the full fourth-order equations, most of which are spurious, and only using those which are physical.

The aim of this work is to apply the reduction of order to a wide variety of gravitational systems. These include computing the reduced semiclassical equations for Friedmann cosmologies (homogeneous isotropic solutions with perfect fluid matter), Friedmann-Lemaître cosmologies (Friedmann cosmologies with cosmological constant), an interior Schwarzschild solution, and the general, conformally flat metric in terms of its conformal factor. Examples of analytic and numerical methods are employed. In particular, we find the exact semiclassical solutions for spatially flat, radiation-filled Friedmann cosmologies, and exact and numerical semiclassical "bounce" solutions for radiation-filled Friedmann-Lemaître cosmologies. As expected, the semiclassical corrections usually play only a small role in most systems far from the Planck scale. There are exceptions to this rule of thumb, however, which we demonstrate by analyzing the semiclassical corrections to the (unstable) eternal Einstein universe. Here the corrections can cause large deviations from the classical solutions and yet remain within the domain of reliability. We do not explicitly account for effects of particle creation (except in conformally flat spacetimes), only for "state-independent" contributions to the stress energy.

II. REVIEW OF SEMICLASSICAL CORRECTIONS

The semiclassical field equations of general relativity (including cosmological constant) take the form

$$R_{ab}-\tfrac{1}{2}Rg_{ab}+\Lambda g_{ab}=\kappa\langle T_{ab}\rangle\ , \tag{2.1}$$

where $\langle T_{ab}\rangle=O(\hbar)$ is the expectation value or transition amplitude of the matter stress-energy tensor. For convenience, we consider only massless, conformally coupled fields (of arbitrary spin). We may reasonably restrict the form of $\langle T_{ab}\rangle$ to obey Wald's physical axioms [9]: (1) covariant conservation; (2) causality; (3) standard results for "off-diagonal" matrix elements; (4) standard results in Minkowski space. Wald showed that any $\langle T_{ab}\rangle$ that obeys the first three axioms is unique up to the addition of a local, conserved tensor. Furthermore, any local, conserved tensor can reasonably be considered part of the geometrical dynamics and so be written on the left-hand side of the field equations. We shall do so, rewriting Eq. (2.1) as

$$R_{ab}-\tfrac{1}{2}Rg_{ab}+\Lambda g_{ab}+\Omega_{ab}=\kappa\langle T_{ab}\rangle\ , \tag{2.2}$$

where Ω_{ab} is conserved and purely local; i.e., it is constructed purely from the metric, the curvature, and a (finite number of) its covariant derivatives.

Only terms in Ω_{ab} that are first order in $\hbar$ will be considered, since the semiclassical approximation already neglects higher-order contributions [10]. Any term contributing to Ω_{ab} with a constant coefficient proportional to $\hbar$ must have dimensions of $(\text{length})^{-4}$, since the only length scale is the Planck length $l_{\rm Pl}$ and $\hbar=l_{\rm Pl}^2$ in units where $G=1$. This restricts the form of Ω_{ab} for general spacetimes in four dimensions to linear combinations of two possible contributing terms [11]:

$$\begin{aligned}{}^{(1)}H_{ab}&=\frac{1}{\sqrt{g}}\frac{\delta}{\delta g^{ab}}\int d^4x\sqrt{g}R^2\\&=\tfrac{1}{2}R^2g_{ab}-2RR_{ab}-2\Box Rg_{ab}+2\nabla_a\nabla_bR\ ,\end{aligned} \tag{2.3}$$

$$\begin{aligned}{}^{(2)}H_{ab}&=\frac{1}{\sqrt{g}}\frac{\delta}{\delta g^{ab}}\int d^4x\sqrt{g}R^{cd}R_{cd}\\&=\tfrac{1}{2}R^{cd}R_{cd}g_{ab}-\Box R_{ab}-\tfrac{1}{2}Rg_{ab}\\&\quad+\nabla_a\nabla_bR-R^{cd}R_{cadb}\ .\end{aligned} \tag{2.4}$$

These two expressions are automatically conserved from their variational definitions. They are also fourth order in time derivatives of the metric. In conformally flat, four-dimensional space-times (where the Weyl tensor $C^c_{\ \rm def}$ vanishes), ${}^{(1)}H_{ab}$ and ${}^{(2)}H_{ab}$ no longer remain linearly independent (in this case ${}^{(1)}H_{ab}=3\,{}^{(2)}H_{ab}$). However, a new quantity appears,

$$\begin{aligned}{}^{(3)}H_{ab}&=-\tfrac{1}{12}R^2g_{ab}+R^{cd}R_{cadb}\\&=-R^c_aR_{cb}+\tfrac{2}{3}RR_{ab}+\tfrac{1}{2}R_{cd}R^{cd}g_{ab}-\tfrac{1}{4}R^2g_{ab}\ ,\end{aligned} \tag{2.5}$$

which is conserved only in conformally flat space-times, but not as a result of a variational derivation, nor as the limit of a conserved quantity in nonconformally flat space-times [12]. It is second order in derivatives of the metric, unlike ${}^{(1)}H_{ab}$ and ${}^{(2)}H_{ab}$. Nevertheless, it is allowed by Wald's axioms, and, in general, contributes to the conformal anomaly. The most general expression for Ω_{ab} is then

$$\Omega_{ab}=\alpha_1\hbar\,{}^{(1)}H_{ab}+\alpha_2\hbar\,{}^{(2)}H_{ab}+\alpha_3\hbar\,{}^{(3)}H_{ab}+O(\hbar^2)\ , \tag{2.6}$$

where it should be understood that the ${}^{(3)}H_{ab}$ term is only present when ${}^{(1)}H_{ab}=3\,{}^{(2)}H_{ab}$. Values of α_1, α_2, and α_3 are predicted by specific matter couplings and regularization schemes, but we will treat them as free parameters. Factors of $\hbar$ have been made explicit. Inserting Eqs. (2.3)–(2.6) into Eq. (2.2) produces Eq. (1.1).

As pointed out above, the new fourth-order theory contains whole new classes of solutions unavailable to the classical, second-order theory. It is the new solutions that would indicate the instability of flat space. Since flat space is experimentally stable (or at least very metastable), this strongly indicates that semiclassical gravity, if all its solutions are considered physical, is not a good description of the near classical limit of quantum gravity. However, the derivation of the theory is well founded, and it seems likely that some of the solutions do correspond to what we expect from quantum corrections to classical theory. It is necessary to break up the solutions to the semiclassical field equations into those we do not consider part of the theory (spurious or "unphysical") and the rest of the solutions ("physical"), which contain all the important information of the theory.

That a theory should contain unphysical solutions should not be surprising to anyone who has examined Dirac's theory of charged particles including electromagnetic back reaction [13]. The electromagnetic back-reaction problem shares several features with the quantum back reaction described above. Its most important features are the following: (1) a small correction term to the equations of motion changes the order of the equations of motion (from second order to third order) and (2) not all solutions to the new equations of motion are physical; some must be excluded by external criteria. Dirac's equation of motion is

$$\ddot{x}^{\mu}-\frac{2}{3}\frac{e^2}{m}(\dddot{x}^{\mu}-\ddot{x}_{\nu}\ddot{x}^{\nu}\dot{x}^{\mu})=\frac{e}{m}F_{\nu}{}^{\mu}\dot{x}^{\nu}\ . \tag{2.7}$$

The classic example of a nonphysical solution has $F_{\nu}{}^{\mu}=0$ but exponentially increasing acceleration:

$$\dot{x}^{\mu}=\left\{\cosh\left[\exp\left[\frac{3mc^2}{2e^2}\tau\right]\right],\sinh\left[\exp\left[\frac{3mc^2}{2e^2}\tau\right]\right],0,0\right\}. \tag{2.8}$$

If the theory is to make useful predictions, unphysical solutions such as this must be excluded. In fact, this so-called "runaway" solution shares much in common with the particular solutions to linearized semiclassical gravity which contribute to the instability of flat space. Other theories with higher derivative corrections, such as cosmic strings with rigidity corrections, can give unphysical solutions with negative kinetic energy [14–16].

Similar problems occur even for semiclassical quantum electrodynamics (QED). The running of the electron charge coupling results in an effective action and Lagrangian density with higher-order corrections [17]:

$$\mathcal{L}_{\rm eff}=-\tfrac{1}{4}F^{\mu\nu}\left[1-\frac{\hbar e_L^2}{60\pi^2m^2}\Box\right]F_{\mu\nu}+\cdots+\text{matter}\ , \tag{2.9}$$

where e_L is the low-energy electron charge. In this case it is clear what should be done. The higher derivatives in Eq. (2.9) do not correspond to new degrees of freedom for the electromagnetic field, but rather to the running of the charge. Since the effective action is a perturbative expansion in $\hbar$, one may first solve the lower-order equations of motion, and then solve higher orders iteratively. Treating Eq. (2.9) as a true fourth-order equation would result in a theory very different from classical electrodynamics, where the electromagnetic field possesses negative-energy modes. This is clearly undesirable from a physical point of view (though the resulting theory is mathematically well defined).

One technique of distinguishing and excluding unphysical solutions of higher derivative theories is called the self-consistent method [15,18], and was first applied in the case of the classical Dirac electron by Bhabha [19]. In the case of semiclassical QED shown above, it is equivalent to the obvious method of constructing higher-order solutions iteratively. It can be applied to any theory for which higher derivative terms in the field equations are perturbative corrections to a lower-order theory, and is most naturally applied to theories derived from an effective action with a small parameter that has been expanded in a power series. One expects extrema of that expanded action to be (perturbative) approximations to the extrema of the full effective action. For ordinary actions, whose variations give second-order differential equations, this is usually true. For actions which possess higher derivative expansions, however, the opposite is true: most solutions of the perturbatively expanded field equations *do not even have* a perturbative expansion (i.e., they are not analytic in the expansion parameter as the parameter approaches zero).

An obvious cure for this behavior is to only rely on those solutions that *are* perturbatively expandable (in the same sense as their effective action) to be physical. All other solutions are treated as spurious by-products of the higher derivatives, not to be considered part of the theory (e.g., the runaway Dirac electron).

The self-consistent approach is extremely powerful. It removes all runaway and negative-energy solutions. It can be applied to semiclassical gravity as easily as to Dirac's classical electron. The amount of initial data required to specify a physical solution is the same as for the original uncorrected theory [20]. In the case of cosmic strings, where the full action is known exactly, any method not equivalent to the self-consistent approach simply gives the wrong results. The self-consistent approach seems clearly applicable to the case of semiclassical gravity.

The solutions of semiclassical gravity are solutions to a fourth-order differential equation. The amount of initial data required to specify a *physical solution,* however, is the same as for classical gravity, which is given by solutions to a second-order differential equation. One can often find a second-order differential equation which contains all the physical solutions to semiclassical gravity. This reduction of order greatly simplifies the process of finding physical solutions since most unphysical solutions are completely bypassed. The reduction is performed iteratively, using lowest- (perturbative) order results to simplify the higher-order semiclassical corrections. Reductions for corrections containing ${}^{(1)}H_{ab}$ and ${}^{(2)}H_{ab}$ have been calculated for several cases by Bel and Sirousse-Zia [8]. The nonlinearity of general relativity makes reduction of order awkward for the most general solutions unless the stress-energy tensor has an extremely simple dependence on the metric. Just as in the case of classical gravity, however, the presence of symmetries can make soluble an otherwise intractable problem. We begin with the example of semiclassical corrections in Friedmann-Lemaître universes.

There is a small but important caveat regarding removing spurious solutions by the reduction of order. All the physical solutions to the higher-order field equations are also solutions to the reduced equations, but, still, not all solutions to the reduced equations may be physical solutions. If the corrections are nonlinear in the field variable, there may remain a much smaller number of spurious that must still be expunged. On reduction, however, it is often easy to identify the unphysical solutions and remove them.

Reduction of order (and the above caveat) are well demonstrated by a simple example. The example uses a toy "semiclassical" equation

$$\ddot{x} = -\omega^2 x + \alpha\hbar \ddddot{x}^{\,2} + O(\hbar^2) , \tag{2.10}$$

which is nonlinear in the fourth derivative correction. This is straightforwardly reduced with the substitution

$$\hbar \ddddot{x} = -\hbar\omega^2 \ddot{x} + O(\hbar^2) , \tag{2.11}$$

resulting in the reduced equation

$$\ddot{x} = -\omega^2 x + \alpha\hbar\omega^4 \ddot{x}^2 + O(\hbar^2) . \tag{2.12}$$

This second-order equation still contains all the physical solutions (perturbatively expandable in $\hbar$), but because of the quadratic nature of the equation, contains solutions nonperturbative in $\hbar$. Solving the quadratic equation gives

$$\ddot{x} = \begin{cases} \dfrac{-2\omega^2 x}{1+\sqrt{1+4\alpha\hbar\omega^6 x}} = -\omega^2 x + \alpha\hbar\omega^4 x^2 + O(\hbar^2) , \\[2ex] \dfrac{1+\sqrt{1+4\alpha\hbar\omega^6 x}}{2\alpha\hbar\omega^4} = \dfrac{1}{\alpha\hbar\omega^4} + \omega^2 x + \cdots . \end{cases} \tag{2.13}$$

The first of these manifestly produces only solutions perturbative in $\hbar$ and so contains only physical solutions. The second set is found only after treating $\hbar$ nonperturbatively: it would not be found using strictly perturbative methods and thus describes nonphysical solutions. The point is that even though Eq. (2.12) is reduced to second order, some care must still be exercised until the equation has been put into form of the first line of Eq. (2.13). In practice, we will often use reduced equations in the form of Eq. (2.12), but if numerical methods are used, the form of Eq. (2.12) may not be adequate and one may need analogues of Eq. (2.13).

III. FRIEDMANN-LEMAÎTRE MODELS WITH QUANTUM CORRECTIONS

In this section, we consider general Robertson-Walker metrics of the form

$$ds^2 = -dt^2 + a(t)^2 \left[\frac{dr^2}{1-kr^2} + r^2 d\theta^2 + r^2 \sin^2\theta \, d\phi^2 \right] , \tag{3.1}$$

where k takes the values ± 1 or 0. The matter consists of radiation and the cosmological constant need not be zero. The (t,t) component of the Einstein equation with first-order quantum corrections is of third order in time derivatives of the scale factor $a(t)$. We first reduce the equation to first order (the same order as the corresponding classical Einstein equation), which the physically relevant solutions satisfy. For various values of the spatial curvature k and cosmological constant Λ, we obtain analytic and numerical solutions. For some solutions within this class of Friedmann-Lemaître universes we find that there are models for which the first-order quantum corrections remain small at all times. In some cases, the effect of small quantum corrections can cause a large deviation from the corresponding classical solution over a long period of time.

In this conformally flat class of metrics, the Einstein equations with quantum corrections are of the form

$$R_{ab} - \tfrac{1}{2} g_{ab} R + \Lambda g_{ab} + \hbar\alpha_1 \, {}^{(1)}H_{ab} + \hbar\alpha_3 \, {}^{(3)}H_{ab} + O(\hbar^2) = \kappa T_{ab} . \tag{3.2}$$

Here Λ is the cosmological constant, and T_{ab} includes classical matter contributions and the lowest-order expectation value of quantum matter, and ${}^{(1)}H_{ab}$ and ${}^{(3)}H_{ab}$ are defined by Eqs. (2.3) and (2.5). The state-independent local quantum corrections of order $\hbar$ are included in the H terms on the left-hand side.

With the metric of Eq. (3.1),

$${}^{(1)}H_{tt} = \frac{-18k^2}{a^4} + \frac{36k\dot{a}^2}{a^4} + \frac{54\dot{a}^4}{a^4} - \frac{36\dot{a}^2\ddot{a}}{a^3} + \frac{18\ddot{a}^2}{a^2} - \frac{36\dot{a}a^{(3)}}{a^2} \tag{3.3}$$

and

$${}^{(3)}H_{tt} = \frac{3k^2}{a^4} + \frac{6k\dot{a}^2}{a^4} + \frac{3\dot{a}^4}{a^4} . \tag{3.4}$$

The (t,t) component of the generalized Einstein equation (3.2), with matter consisting of radiation, is

$$0=-\Lambda-\frac{\kappa\rho_0 a_0^4}{a^4}+\frac{3k+3\dot{a}^2}{a^2}+\alpha_1\hbar\frac{-18k^2+36k\dot{a}^2+54\dot{a}^4-36a\dot{a}^2\ddot{a}+18a^2\ddot{a}^2-36a^2\dot{a}a^{(3)}}{a^4}$$
$$+\alpha_3\hbar\frac{3k^2+6k\dot{a}^2+3\dot{a}^4}{a^4}+O(\hbar^2)\ . \qquad (3.5)$$

Here, ρ_0 is the radiation density at the time when the scale factor is a_0. The a^{-4} dependence of the ρ_0 term follows directly from conservation of the stress energy and the fact that the expectation value of the trace, T_a^a, vanishes to lowest order. The order-$\hbar$ quantum corrections to this trace give rise to the α_1 and α_3 terms. The values of the α's depend on which massless fields make up the radiation. In the spatially closed universe ($k=1$), as a result of the nonlocal Casimir vacuum energy, the value of ρ_0 will have added to it a small constant value proportional to $\hbar$ [21,22]. There is no additional nonlocal contribution to the stress energy because we are dealing with conformally invariant free radiation fields in Robertson-Walker metrics, which are all conformally flat. Before attempting to solve this equation, we reduce it to lower order in time derivatives of $a(t)$, so that all solutions are assured to be of physical significance.

A. Reduction of equation for scale factor

Multiplying Eq. (3.5) by $\hbar$ and working to first order in $\hbar$ gives

$$\hbar\dot{a}^2=-\hbar k+\hbar\frac{\kappa\rho_0 a_0^4}{3a^2}+\hbar\frac{\Lambda a^2}{3}+O(\hbar^2)\ . \qquad (3.6)$$

This expression can be used twice to find that

$$\hbar\dot{a}^4=\hbar k^2+\hbar\frac{2\Lambda\kappa\rho_0 a_0^4}{9}+\hbar\frac{\kappa^2\rho_0^2 a_0^8}{9a^4}-\hbar\frac{2k\kappa\rho_0 a_0^4}{3a^2}$$
$$-\hbar\frac{2\Lambda k a^2}{3}+\hbar\frac{\Lambda^2 a^4}{9}+O(\hbar^2)\ . \qquad (3.7)$$

Differentiating Eq. (3.6) leads to

$$\hbar\ddot{a}=-\hbar\frac{\kappa\rho_0 a_0^4}{3a^3}+\hbar\frac{\Lambda a}{3}+O(\hbar^2)\ . \qquad (3.8)$$

Here we have divided by $\dot{a}$, assuming that it is not zero in the time interval of interest. Making use of this result twice gives

$$\hbar\ddot{a}^2=\hbar\frac{\kappa^2\rho_0^2 a_0^8}{9a^6}-\hbar\frac{2\Lambda\kappa\rho_0 a_0^4}{9a^2}+\hbar\frac{\Lambda^2 a^2}{9}+O(\hbar^2)\ . \qquad (3.9)$$

We also need the third derivative $\hbar a^{(3)}$, which is obtained by differentiating Eq. (3.8):

$$\hbar a^{(3)}=\hbar\frac{\Lambda\dot{a}}{3}+\hbar\frac{\kappa\rho_0 a_0^4\dot{a}}{a^4}+O(\hbar^2)\ . \qquad (3.10)$$

Using the above results repeatedly in Eq. (3.5) reduces it to the first-order differential equation:

$$0=-\Lambda+\frac{3k}{a^2}+\frac{3\dot{a}^2}{a^2}-\frac{\kappa\rho_0 a_0^4}{a^4}-\alpha_1\hbar\frac{8\Lambda\kappa\rho_0 a_0^4}{a^4}$$
$$+\alpha_3\hbar\left[\frac{\Lambda^2}{3}+\frac{2\Lambda\kappa\rho_0 a_0^4}{3a^4}+\frac{\kappa^2\rho_0^2 a_0^8}{3a^8}\right]+O(\hbar^2)\ . \qquad (3.11)$$

Except for the correction term involving a^{-8}, all of the terms of order $\hbar$ can be absorbed by a renormalization of the constants Λ and κ. We make use of this renormalization when we discuss below models with radiation, spatial curvature, and a nonzero cosmological constant.

B. Spatially flat model with radiation and $\Lambda=0$

A simple but illuminating solution to the reduced Friedmann-Lemaître-Einstein equation (2.11) is the spatially flat ($k=0$) case with zero cosmological constant ($\Lambda=0$) and pure radiation [$\rho=\rho_0(a_0/a)^4$]. This should be a good approximation to our Universe for one part of its history, after any inflationary epochs which smoothed out inhomogeneities but before massive fields cooled to nonrelativistic temperatures. The classical solution, which has the scale factor grow as the square root of cosmological time, begins with a curvature singularity, and expands forever, becoming more and more flat. Because the physical semiclassical solutions are corrections to the classical solutions in powers of curvature, we expect that, at late times, when the classical solution is nearly flat, the semiclassical corrections will be small. At early times, when the curvature is below the Planck scale but not above it, we expect the semiclassical corrections to be significant. At very early times, however, when the classical curvature is near or above the Planck scale, we expect that the semiclassical approximation will break down because neglected higher-order corrections would dominate. We shall see how these effects manifest themselves below.

For $\Lambda=0$, $k=0$, and $\rho=\rho_0(a_0/a)^4$, Eq. (2.11) reduces to

$$\dot{a}^2=\frac{\kappa\rho a_0^4}{3a^2}-\alpha_3\hbar\frac{\kappa^2\rho^2 a_0^8}{9a^6}+O(\hbar^2)\ ,$$

which can be simplified by redefining the scale factor, cosmological time, and correction constant α_3, in dimensionless units

$$\tilde{a}\equiv a\left[\frac{4\kappa\rho_0 a_0^4}{3}\right]^{-1/2},$$

$$\tilde{t}\equiv t\left[\frac{4\kappa\rho_0 a_0^4}{3}\right]^{-1/2},$$

$$\tilde{\alpha}_3\equiv\alpha_3\left[\frac{64\kappa\rho_0 a_0^4}{3}\right]^{-1},$$

giving

$$\dot{\tilde{a}}^2=\frac{1}{4\tilde{a}^2}-\hbar\tilde{\alpha}_3\frac{1}{\tilde{a}^6}+O(\hbar^2)\ , \tag{3.12}$$

where $\dot{\tilde{a}}=d\tilde{a}/dt$. This can be solved iteratively with the ansatz

$$\tilde{a}(\tilde{t})=\tilde{a}_0(\tilde{t})+\hbar\tilde{a}_1(\tilde{t})+O(\hbar^2)\ . \tag{3.13}$$

Inserting Eq. (3.13) into Eq. (3.12) and expanding in powers of $\hbar$ gives

$$\begin{aligned}\hbar^0:&\ \dot{\tilde{a}}_0^2=\tfrac{1}{4}\tilde{a}_0^{-2}\ ,\\ \hbar^1:&\ 2\dot{\tilde{a}}_0\dot{\tilde{a}}_1=-\tfrac{1}{2}\tilde{a}_1\tilde{a}_0^{-3}+\tilde{\alpha}_3\tilde{a}_0^{-6}\ .\end{aligned} \tag{3.14}$$

The first equation gives the one-parameter family of (classical) solutions

$$\tilde{a}_0(\tilde{t})=(\tilde{t}-\tau_0)^{1/2}\ ,$$

where τ_0 is a constant of integration. When this is substituted into the second differential equation in (3.14), it gives the first-order, linear, inhomogeneous equation

$$\dot{\tilde{a}}_1+\frac{1}{2(\tilde{t}-\tau_0)}\tilde{a}_1=\tilde{\alpha}_3(\tilde{t}-\tau_0)^{-5/2}\ ,$$

which, for a given τ_0, has a one-parameter family of solutions,

$$\tilde{a}_1(\tilde{t})=-\tilde{\alpha}_3(\tilde{t}-\tau_0)^{-3/2}-\tfrac{1}{2}\tau_1(\tilde{t}-\tau_0)^{-1/2}\ ,$$

given by τ_1, another constant of integration. The full solution is

$$\tilde{a}(\tilde{t})=(\tilde{t}-\tau_0)^{1/2}-\hbar\tilde{\alpha}_3(\tilde{t}-\tau_0)^{-3/2}-\tfrac{1}{2}\hbar\tau_1(\tilde{t}-\tau_0)^{-1/2}+O(\hbar^2)\ . \tag{3.15}$$

The apparently extra one-parameter family of solutions (the freedom to specify τ_1 as well as τ_0) is a result of the ambiguity in what it means to specify a solution. The freedom to specify both τ_0 and τ_1 is equivalent to the freedom to specify one constant of integration, τ, to lowest and first order in $\hbar$. This can be seen by expanding τ in powers of $\hbar$, $\tau=\tau_0+\hbar\tau_1+O(\hbar^2)$, and using it to write solution (3.15) as

$$\tilde{a}(\tilde{t})=(\tilde{t}-\tau)^{1/2}-\hbar\tilde{\alpha}_3(\tilde{t}-\tau)^{-3/2}+O(\hbar^2)\ .$$

This freedom is related to the freedom of choosing different asymptotic expansions for the solutions, and is discussed in more detail in Appendix B.

Another solution to Eq. (3.12), to the same order in $\hbar$, is

$$\tilde{A}(\tilde{t})=[(\tilde{t}-\tau)-2\hbar\tilde{\alpha}_3(\tilde{t}-\tau)^{-1}]^{1/2}+O(\hbar^2)\ .$$

$\tilde{a}$ and $\tilde{A}$ differ only by terms higher order in $\hbar$, so either solution is as good as the other. To the extent the two solutions disagree, the semiclassical approximation cannot predict which solution is more accurate.

Figure 1 shows plots of $\tilde{a}$, $\tilde{A}$, and $\tilde{a}_0$ as functions of $\tilde{t}$, for two values of $\tilde{\alpha}_3$ (or, equivalently, α_3). The three regimes referred to above can be observed. When the Universe is nearly flat (for $\tilde{t}\gtrsim 1$), the semiclassical corrections are small. For intermediate scales ($0.25\lesssim\tilde{t}\lesssim 0.75$, for the particular $\tilde{\alpha}_3$ plotted, $\tilde{\alpha}_3=\pm 0.01$), the corrections are more substantial and cause noticeable deviations from the classical solution. For very early times ($\tilde{t}\lesssim 0.25$), the corrections dominate the classical solution. This, unfortunately, is the regime in which they cannot be trusted. This is made most obvious by noting the substantial

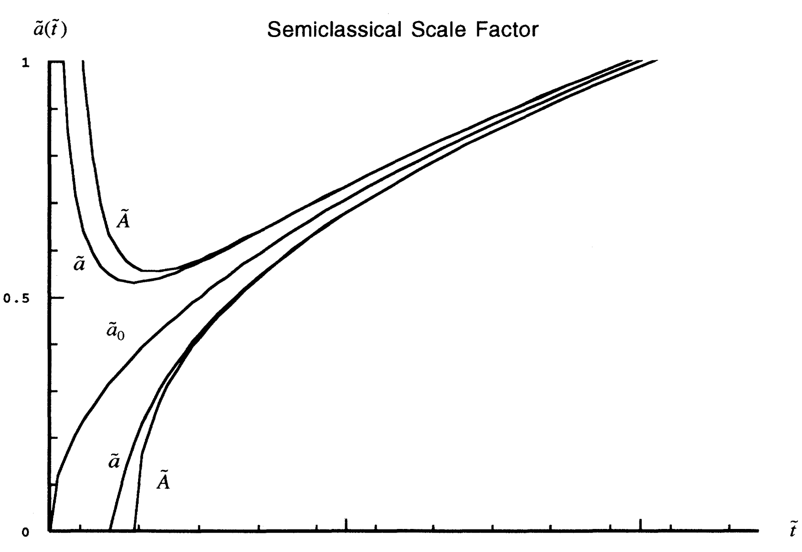

FIG. 1. Plots of the scale factor $\tilde{a}$, $\tilde{A}$, and $\tilde{a}_0$ as functions of $\tilde{t}$, for a zero cosmological constant and zero spatial curvature. The top two plots are for $\tilde{\alpha}_3=+0.01$, the bottom two are for $\tilde{\alpha}_3=-0.01$, and the central plot, $\tilde{a}_0$, is for $\tilde{\alpha}_3=0$ (classical solution). $\tilde{a}$ and $\tilde{A}$ are equally legitimate solutions to the semiclassical equations, differing only at order $O(\hbar^2)$.

differences (along the $\tilde{a}$ scale) between the solutions $\tilde{a}$ and $\tilde{A}$ for small $\tilde{t}$. Since they are both equally valid solutions to the reduced semiclassical equations, to the extent they differ, the predictions of either are not meaningful. For positive (negative) $\tilde{\alpha}_3$ the effect of the corrections is to make the Universe larger (smaller) than it would have been at small times, but the most dramatic predictions indicated by the solutions (and their plots) are in regimes where the solutions should not be trusted. All one can say is that at small times and large curvature, semiclassical corrections are important, and that at very small times and very large curvature, the contribution from higher-order corrections of quantum gravity are necessary in order to make any meaningful predictions. The values chosen for $\tilde{\alpha}_3$ in Fig. 1 (± 0.01) have been chosen extraordinarily large (though still well within the perturbative regime) to demonstrate the qualitative effects of the quantum corrections. If one were to choose α_3 of the Planck scale and a standard cosmological value for $\rho_0 a_0^4$, $\tilde{\alpha}_3$ might be as small as 10^{-100}.

C. Models with radiation, spatial curvature, and $\Lambda \neq 0$

More general models with general spatial curvature and cosmological constant and radiation, are governed to first order in $\hbar$ by Eq. (3.11). This can be rewritten in the form

$$0=\frac{3\dot{a}^2}{a^2}+\frac{3k}{a^2}-\Lambda_r-\frac{\kappa_r\rho_0 a_0^4}{a^4}+\alpha_3\hbar\frac{\kappa_r^2\rho_0^2 a_0^8}{3a^8}+O(\hbar^2)\ , \tag{3.16}$$

where constant terms first order in $\hbar$ have been absorbed into a renormalization of the gravitational and cosmological constants:

$$\Lambda_r \equiv \Lambda(1-\tfrac{1}{3}\alpha_3\hbar\Lambda) \tag{3.17}$$

and

$$\kappa_r \equiv \kappa(1-\tfrac{2}{3}\alpha_3\hbar\Lambda_r+8\alpha_1\hbar\Lambda_r)\ . \tag{3.18}$$

With the exception of the final term, Eq. (3.16) has the same form as in the corresponding classical Einstein equation, but with κ and Λ having constant corrections of order $\hbar$.

Before numerically integrating Eq. (3.16), it is advantageous to carry out some analytic simplification. Define the constants

$$A \equiv \tfrac{4}{3}\Lambda_r\ , \tag{3.19}$$

$$B \equiv \tfrac{4}{3}\kappa_r\rho_0 a_0^4\ , \tag{3.20}$$

$$C \equiv BA^{-1}-4k^2A^{-2}\ . \tag{3.21}$$

Also define a dimensionless independent variable

$$s \equiv A^{1/2}t\ , \tag{3.22}$$

and a function $g(s)$ by

$$g \equiv a^2-2kA^{-1}\ . \tag{3.23}$$

Then Eq. (3.16) can be written as

$$\left[\frac{d}{ds}g\right]^2-g^2-C+\hbar\alpha_3\frac{A^{-1}B^2}{4(g+2kA^{-1})^2}=0\ . \tag{3.24}$$

Notice that C has the same value for $k=\pm 1$, so that the solution of this equation without the final term of order $\hbar$ is the same for positive or negative spatial curvature, and is the same as the classical solution with renormalized gravitational and cosmological constants.

Probably the most interesting case to consider in more detail is that of positive spatial curvature, $k=1$, with positive cosmological constant, $\Lambda>0$. This case includes both the "hesitation" and "turn-around" models of Lemaître.

The hesitation model spends a long time near the configuration of the Einstein static universe, during which density perturbations can grow rapidly. Close relatives of this model, which do not require a cosmological constant, have been recently considered in connection with the growth of perturbations and galaxy formation [23]. Such models may be affected by quantum corrections in a manner similar to that considered here.

The turn-around model with no singularity is of interest because the perturbative approximation for the quantum correction is valid during the entire evolution of the model. Over a long period of time this quantum correction can cause a significant deviation of the radius of the Universe from that of the corresponding classical model.

The hesitation solution occurs when $C>0$, and has zeroth-order solution of Eq. (3.24):

$$g=C^{1/2}\sinh(s)\ . \tag{3.25}$$

The turn-around solution occurs when $C<0$, and has the zeroth-order solution

$$g=|C|^{1/2}\cosh(s)\ . \tag{3.26}$$

In these solutions s could be replaced by $s-s_0$, with s_0 constant. Also, since the zeroth-order equation (for g) does not depend on the sign of k, these classical solutions remain valid when $k=-1$. However, the quantum correction does depend on the sign of k; and we will only consider the case of positive spatial curvature. We next consider the corrections to the turn-around model, and then the corrections to the hesitation model.

1. *Turn-around or bounce universe*

Dividing Eq. (3.24) by $|C|$, and defining the dimensionless function

$$G(s)=|C|^{-1/2}g(s)\ , \tag{3.27}$$

and the dimensionless constants

$$u=A^{-1}|C|^{-1/2}\ , \tag{3.28}$$

which regulates the abundance of radiation relative to the cosmological constant, and

$$v=\tfrac{1}{4}\hbar\alpha_3 B^2/(AC^2)\ , \tag{3.29}$$

which regulates the quantum corrections, we obtain in the case of the turn-around solution ($C<0$):

$$\left[\frac{d}{ds}G(s)\right]^2-G(s)^2+1+v[G(s)+2u]^{-2}=0 \ . \tag{3.30}$$

(The hesitation model obeys the same equation with $+1$ replaced by -1.) The initial conditions for a turn-around solution are that $G>1$ and $(d/ds)G<0$, so that G will approach a minimum value near 1 and then increase. The perturbative solution will remain valid at all times for which

$$v[G(s)+2u]^{-2}\ll 1 \ . \tag{3.31}$$

The relationship between a and G can be written as

$$(A^{1/2}a)^2=(u^{-1}G+2)^{1/2} \ .$$

The upper curves in Fig. 2 show the radius a as a function of time t, with both a and t measured in units of $A^{-1/2}=(4\Lambda_r/3)^{-1/2}$. Both curves have $u=0.5$. The lower curve has $v=0$. The upper curve includes the quantum corrections coming from α_3, and has $v=0.4$. The boundary condition for both curves was that $G(1)=\cosh(1)$. Again, as in Fig. 1, the numerical value of the quantum correction, in this case $v=0.4$, has been chosen to be much larger than one would expect from Planck scale contributions, in order to emphasize the qualitative effects of the corrections.

The curves were obtained by numerical integration of the equation for $(d/ds)G(s)$. In solving Eq. (3.30) for $(d/ds)G(s)$, the positive square root was used in the range from $s=1$ to 0.092 33 at which $G(s)$ has its minimum value of 1.0467. The negative square root was used for smaller values of s, since $G(s)$ is a decreasing function of s in that range. The correction term in Eq. (3.30) remains smaller than 0.1 and has its largest value when G goes through its minimum. Thus, the perturbative solution is valid for all values of s.

The lower curve is the same as the classical solution with $\Lambda=\Lambda_r$, $\kappa=\kappa_r$. It corresponds to the solution given in Eq. (3.26), and coincides with the quantum corrected solution at $t=1$ (in units of $A^{-1/2}$). It is clear that as a result of the deviation introduced near the minimum radius by the quantum correction, the two models have increasingly different radii as one goes back into the past.

2. *Semiclassical hesitation universe*

From the previous examples, one can see that it is difficult to find examples of semiclassical corrections that are not overwhelmingly small. The last example used the exponential growth of the scale factor to magnify small correction terms. One instance semiclassical corrections might have a macroscopic impact is when the classical field equations describe an unstable system. Even initially small quantum effects could dominate the behavior at late times.

One important example of an unstable cosmological solution to the classical Einstein equations is the unstable Einstein static universe. Historically, Einstein looked for a static cosmological solution to describe an eternal universe, leading him to modify his equations with a cosmological constant. This solution is unstable, however, and small changes in initial data cause the Universe to expand forever at an exponential rate (due to the cosmological constant) or to collapse to a curvature singularity. One would expect quantum corrections to have similar, drastic effects.

The simplest Einstein static universe is spatially closed ($k=1$), contains a radiation fluid, with density $\rho=\rho_0(a_0/a)^4$, and cosmological constant, Λ, both carefully chosen such that $\Lambda=9/(4\kappa\rho_0 a_0^4)$. Defining a natural length scale, L_0, such that $L_0=\sqrt{3/\Lambda}=\sqrt{4\kappa\rho_0 a_0^4/3}$, we can write the classical Einstein equation as

$$\begin{aligned} O(\hbar)&=\dot a^2-\frac{\Lambda}{3}a^2+k-\frac{\kappa\rho_0 a_0^4}{3a^2}\\ &=\dot a^2-L_0^{-2}a^2+1-\frac{1}{4L_0^{-2}a^2} \ . \end{aligned} \tag{3.32}$$

This has the solutions

$$a^2=\begin{Bmatrix} L_0^2/2 \\ (L_0^2/2)\{1-\exp[\mp 2(t-t_0)/L_0]\} \\ (L_0^2/2)\{1+\exp[\pm 2(t-t_0')/L_0]\} \end{Bmatrix}+O(\hbar) \ , \tag{3.33}$$

plotted in Fig. 3. The first solution is the Einstein static universe. The second begins at a singularity and asymp-

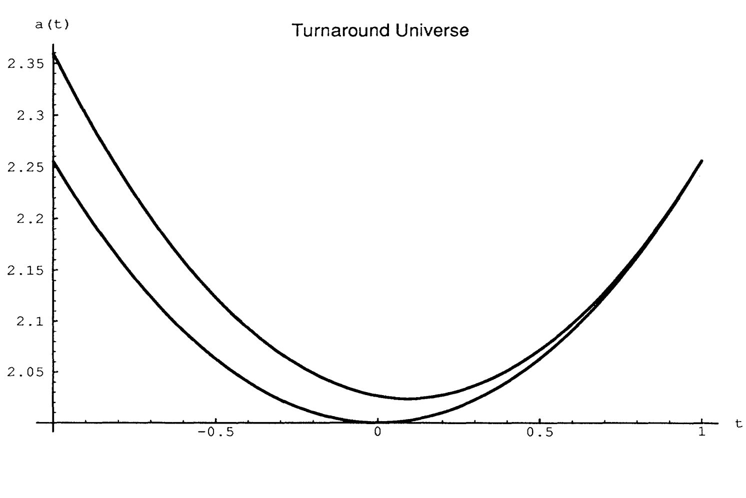

FIG. 2. Plot of a as a function of t, each in units of $(4\Lambda_r/3)^{-1/2}$. The lower curve is the classical solution ($v=0$) with $u=0.5$ and $\Lambda=\Lambda_r$, $\kappa=\kappa_r$. The upper curve includes quantum corrections, with $v=0.4$ and other parameters unchanged.

totically approaches the Einstein static universe at late times (the positive exponential is the time-reversed solution). The third spends an infinite time near the Einstein static universe, but pulls away and ends in an infinite inflationary epoch (the negative exponential is the time-reversed solution). The first and second solutions (and the time-reversed third) are unstable.

The behavior of the semiclassical corrections should reflect the instability of the classical solutions. The semiclassical Einstein equation of Eq. (2.11) becomes

$$O(\hbar)=\dot{a}^2-L_0^{-2}a^2+1-\frac{1}{4L_0^{-2}a^2}-6\alpha_1\hbar\frac{1}{a^2}+\alpha_3\hbar\frac{1}{a^2}\left[L_0^{-2}a^2+\frac{1}{4L_0^{-2}a^2}\right]^2 . \tag{3.34a}$$

This can be solved iteratively, as above, by expanding the solutions as a series in $\hbar$:

$$a=a_{\mathrm{cl}}+\hbar a_1+O(\hbar^2) , \tag{3.34b}$$

where a_{cl} is one of the classical solutions in Eq. (3.33). Only the second and third solutions of Eq. (3.33) generate solutions to the semiclassical equations in general. [It should not be surprising that the constant solution has no semiclassical counterpart. The effect of the semiclassical corrections in Eq. (3.34a), if viewed as a Newtonian energy equation, is to change the potential term without changing the total energy. The energy corresponding to a static configuration without the corrections will not, in general, correspond to a static configuration when the potential is changed; to regain a static solution one must change the initial matter density appropriately.] The equation of motion for a_1, obtained by inserting Eq. (3.34b) into Eq. (3.34a), is a linear, first-order equation, and can be solved in closed form.

The semiclassical solution generated by the initially singular classical solution is

$$\begin{aligned}a^2=\frac{L_0^2}{2}(1-\exp[-2(t-\tau)L_0^{-1}])\Bigg[1&+3\alpha_1\hbar L_0^{-2}\frac{\exp[2(t-\tau)L_0^{-1}]}{1-\exp[-2(t-\tau)L_0^{-1}]}\\&-\frac{\alpha_3}{4}\hbar L_0^{-2}\Bigg\{2\frac{\exp[2(t-\tau)L_0^{-1}]}{1-\exp[-2(t-\tau)L_0^{-1}]}-\frac{\exp[-2(t-\tau)L_0^{-1}]}{(1-\exp[-2(t-\tau)L_0^{-1}])^2}\\&\qquad+8(t-\tau)L_0^{-1}\frac{\exp[-2(t-\tau)L_0^{-1}]}{1-\exp[-2(t-\tau)L_0^{-1}]}\\&\qquad+3\frac{\exp[-2(t-\tau)L_0^{-1}]}{1-\exp[-2(t-\tau)L_0^{-1}]}\ln(1-\exp[-2(t-\tau)L_0^{-1}])\Bigg\}\Bigg]+O(\hbar^2) ,\end{aligned} \tag{3.35}$$

where $\tau=t_0+\hbar t_1+O(\hbar^2)$. As shown in Appendix A, the form of solution (3.35) is ambiguous up to the addition of $O(\hbar)$ terms proportional to $\dot{a}_{\mathrm{cl}}$, arising from shifts $O(\hbar)$ in the initial time ($t_0\to\tau=t_0+\hbar t_1$). One may always choose a t_1 such that the coefficient of the ambiguous term is zero. This solution is plotted in Fig. 4 for $\alpha_1\hbar L_0^{-2}=0.0001$ and $\alpha_3\hbar L_0^{-2}=-0.0001$ (though the qualitative behavior is independent of those values, as long as $6\alpha_1-\alpha_3>0$). As in the case of the spatially flat, radiation-filled universe above, the semiclassical approximation breaks down when too close to the initial singularity. At later times, where the semiclassical approximation is good, the effect of the corrections is to pull away from the Einstein static solution and begin an inflationary epoch. At very late times, when the correction terms dominate the classical solution completely, the corrections are untrustworthy, but at these late times the other (late-time de Sitter) semiclassical solution exhibits the same inflationary behavior, and in a trustworthy regime. One can hope to match the two semiclassical solutions in the intermediate regime, where both are valid. We do this below.

The semiclassical counterpart to the late-time de Sitter classical solutions of Eq. (3.33) takes the form

$$\begin{aligned}a^2=\frac{L_0^2}{2}(1+\exp[-2(t-\tau')L_0^{-1}])\Bigg[1&-3\alpha_1\hbar L_0^{-2}\frac{\exp[2(t-\tau')L_0^{-1}]}{1+\exp[-2(t-\tau')L_0^{-1}]}\\&+\frac{\alpha_3}{4}\hbar L_0^{-2}\Bigg\{2\frac{\exp[2(t-\tau')L_0^{-1}]}{1+\exp[-2(t-\tau')L_0^{-1}]}-\frac{\exp[-2(t-\tau')L_0^{-1}]}{(1+\exp[-2(t-\tau')L_0^{-1}])^2}\\&\qquad-8(t-\tau')L_0^{-1}\frac{\exp[2(t-\tau')L_0^{-1}]}{1+\exp[2(t-\tau')L_0^{-1}]}\\&\qquad+3\frac{\exp[2(t-\tau')L_0^{-1}]}{1+\exp[2(t-\tau')L_0^{-1}]}\ln(1+\exp[-2(t-\tau')L_0^{-1}])\Bigg\}\Bigg]+O(\hbar^2) ,\end{aligned} \tag{3.36}$$

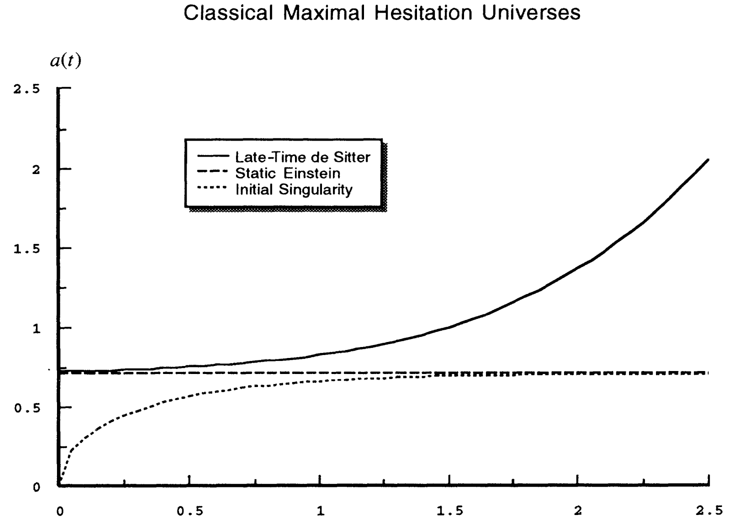

FIG. 3. Solutions to the classical maximal hesitation equations. The upper plot is a maximal hesitation solution which spends an infinite time near the Einstein static universe, but pulls away and ends in an infinite inflationary epoch. The constant solution is the Einstein static universe. The lower plot, also a maximal hesitation solution, begins at a singularity and asymptotically approaches the Einstein static universe at late times.

where $\tau' = t_0' + \hbar t_1' + O(\hbar^2)$ and t_1' is chosen analogously to t_1 above. Equation (3.36) is plotted in Fig. 4 for two values of α_1 and α_3. At late times, the corrections are very small compared to the classical solution. At intermediate times, the corrections are small but non-negligible, and at early times the corrections are so large as to be untrustworthy ($\hbar a_1 / a_{\rm cl} \gtrsim 1$).

Because the semiclassical solutions of Eqs. (3.35) and (3.36) are valid in different regimes, it is important to ask if there is any overlap of the regimes where both solutions are valid. Furthermore, if there is such a regime, perhaps the solutions can be smoothly joined, corresponding to a universe beginning at large curvature near a singularity, flattening off at nearly constant scale factor for an extended period of time, and then proceeding to inflate in a de Sitter-like phase. This would correspond to a classical "hesitation" universe in which the matter density (or cosmological constant) is slightly greater than necessary for the Einstein static universe.

For $6\alpha_1 - \alpha_3 > 0$ [the parameter range for which the solution of Eq. (3.35) is expanding at late times] there is an overlap region in which we can match the solution of Eq. (3.35) to the solution of Eq. (3.36), as shown in Fig. 4 by using the freedom to set the base times (τ and τ') of each solution individually. The matching can always be done smoothly, since the curves of a cross for all values of $\tau - \tau'$, and we can adjust $\tau - \tau'$ such that $\dot{a}$ is continuous (sufficient for matching solutions of a first-order equation).

Furthermore, $\ddot{a}$ is discontinuous only by terms $O(\hbar^2)$. The matching can be done in regions where $\hbar a_1 / a_{\rm cl} \ll 1$ for both solutions for a wide variety of parameters (such that $6\alpha_1 - \alpha_3 > 0$). We may naturally interpret this joining of matched solutions as a unique solution to the semiclassical equation that is everywhere perturbatively valid (except the region near the initial singularity). The time of hesitation $t_h \approx \tau' - \tau$ determined by the matching conditions, is logarithmically related to the coefficients of the

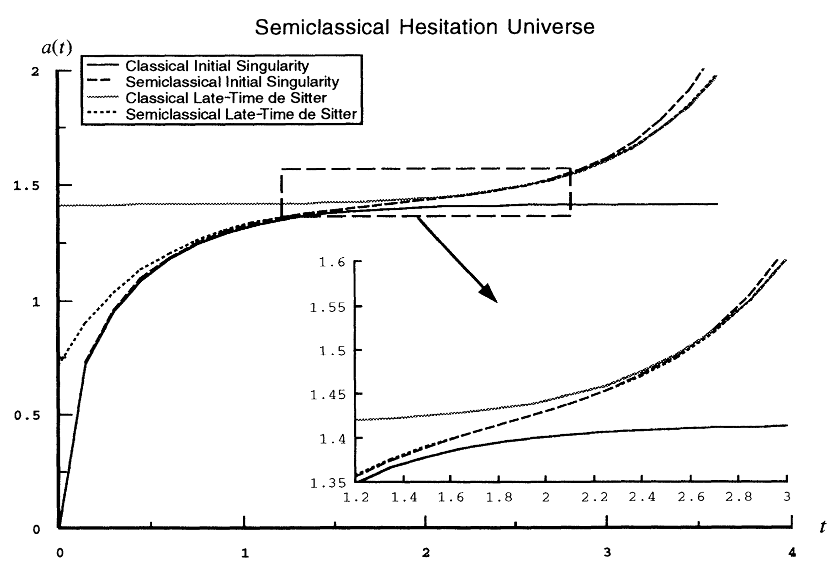

FIG. 4. Solutions of quantum corrections to the maximal hesitation equations. Two classical solutions and their semiclassical counterparts are shown for $\alpha_1 \hbar L_0^{-2} = -\alpha_3 \hbar L_0^{-2} = 0.0001$. Both the classical and semiclassical initial singularity solutions begin at small scale factor, but the classical solution asymptotically approaches the static Einstein solution while the semiclassical solution diverges exponentially from the classical solution (departing from the perturbative regime). The classical and semiclassical late-time de Sitter solutions both agree at late times, but at early times the classical solution asymptotically nears the static Einstein solution, and the semiclassical solution diverges exponentially from the classical solution (departing from the perturbative regime). The two semiclassical solutions may be smoothly matched, as seen in the inset, resulting in a nonmaximal hesitation solution that is always perturbatively valid.

semiclassical corrections: $t_h \approx L_0 \ln[(6\alpha_1-\alpha_3)\hbar L_0^{-4}]$, due to the exponential time dependence of the semiclassical solutions.

The only potential obstacle to this interpretation is that, although $\hbar a_1/a_{\rm cl} \ll 1$ where the joining is done, $\hbar \dot a_1/\dot a_{\rm cl} \approx 1$. We feel that this is no reason to doubt the validity of the joining, however, since $\hbar \dot a_1/\dot a_{\rm cl}$ becomes large due to $\dot a_{\rm cl}$ vanishing, not due to $\hbar \dot a_1$ becoming large.

IV. CONFORMALLY FLAT CORRECTIONS

A conformally flat background space-time [for which Eq. (3.2) gives the general form of the local corrections to the stress energy] has a metric tensor related to the Minkowski metric tensor by

$$g_{ab}^{\rm cl} = e^{2S}\eta_{ab} \; , \tag{4.1}$$

where e^{2S} is the conformal factor (a general scalar function), and η_{ab} is the Minkowski tensor (note that the conformal transformation is not, in general, a diffeomorphism). Conformally flat metrics are special in this sense—the entire metric (a symmetric tensor with six independent components) is completely specified by a single scalar function on the space-time.

As Eq. (4.1) is written, it is a tensor equation on the space-time with classical metric. Here the Minkowski tensor η_{ab} is a tensor function defined on the physical space-time. Since η_{ab} is also the metric of flat Minkowski space-time (or a piece of Minkowski space-time), Eq. (4.1) implies that there is a map from the physical space-time with metric $g_{ab}^{\rm cl}$ to an unphysical flat space-time with metric η_{ab}, and the conformal factor e^{2S} can be viewed as a scalar function on either space-time. If the conformal factor is known on the unphysical flat space-time, this knowledge can be exploited to simplify the calculation of the semiclassical corrections, i.e., to do all the calculations on the unphysical flat background.

We expand the general semiclassical Einstein equations for a conformally flat background

$$G_{ab}(g_{cd})+\alpha_1\hbar\,{}^{(1)}H_{ab}(g_{cd})+\alpha_3\hbar\,{}^{(3)}H_{ab}(g_{cd}) -\kappa T_{ab}(g_{cd}) = O(\hbar^2) \; , \tag{4.2}$$

where

$$g_{cd} = g_{cd}^{\rm cl} + \hbar h_{cd} + O(\hbar^2) \; . \tag{4.3}$$

This has the classical lowest-order expansion

$$G_{ab}(g_{ab}^{\rm cl}) - \kappa T_{ab}(g_{ab}^{\rm cl}) = O(\hbar) \; , \tag{4.4}$$

which is already known if S is known. The equation first order in $\hbar$ is

$$\hbar \frac{\delta G_{ab}}{\delta g_{ef}}(g_{cd}^{\rm cl})h_{ef} + \alpha_1 \hbar\,{}^{(1)}H_{ab}(g_{cd}^{\rm cl}) + \alpha_3\hbar\,{}^{(3)}H_{ab}(g_{cd}^{\rm cl}) - \hbar\kappa \frac{\delta T_{ab}}{\delta g_{ef}}(g_{cd}^{\rm cl})h_{ef} = O(\hbar^2) \; , \tag{4.5}$$

where

$$\frac{\delta G_{ab}}{\delta g_{ef}} h_{ef} \equiv -\tfrac{1}{2}\nabla_a\nabla_b h - \tfrac{1}{2}\Box h_{ab} + \nabla_c \nabla_{(a} h_{b)}{}^c - \tfrac{1}{2}R^{\rm cl} h_{ab} - \tfrac{1}{2} g_{ab}^{\rm cl}[-\Box h + \nabla_c\nabla_d h^{cd} + R_{cd}^{\rm cl} h^{cd}] \tag{4.6}$$

and all derivatives and raising of indices are with respect to the classical, physical metric. We have not explicitly expanded the last term of Eq. (4.5), the functional derivative of the stress-energy tensor, since its functional dependence on the metric depends on the particular form of matter present. If the functional dependence is known (as is often the case) then it is straightforward to calculate.

This is a set of second-order, linear, inhomogeneous equations for h_{ab}. The second-order equation produces a two-parameter family of solutions (just as for the classical equation). The freedom to choose two additional free parameters in the semiclassical solutions arises from the freedom to specify the two parameters of the solution to the full semiclassical equations at both classical and semiclassical order independently.

The power of Eq. (4.1) is most apparent when the conformal factor e^{2S} is known as a function of the fictitious flat space-time. All semiclassical calculations can be performed on the flat space-time instead of the physical space-time.

Then Eq. (4.6) becomes

$$\begin{aligned}\frac{\delta G_{ab}}{\delta g_{ef}} h_{ef} = \tfrac{1}{2} e^{-2S}[& 2h_{c(a;b)}{}^c - h_{;ab} - h_{ab;c}{}^c + 8S_{;c}{}^c h_{ab} + 4S_{;c(a}h_{b)}{}^c \\ & -4S_{;(a}h_{b)c;}{}^c + 4S_{;(a}h_{;b)} + 2S_{;c}h_{ab;}{}^c + 2S_{;a}S_{;b}h + S_{;c}S_{;}{}^c h_{ab} - 8S_{;c}S_{;(a}h_{b)}{}^c \\ & + \eta_{ab}(-h_{cd;}{}^{cd} + h_{;c}{}^c + 2S_{;c}h_{;}{}^c + 2S_{;c}S_{;}{}^c h)] \; ,\end{aligned} \tag{4.7}$$

and the inhomogeneous terms are

$$\begin{aligned}{}^{(1)}H_{ab}(g_{cd}^{\rm cl}) &= {}^{(1)}H_{ab}(e^{2S}\eta_{cd}) \\ &= 6e^{-2S}[-12S_{;c}S_{;a}S_{;b}S_{;}{}^c - 12S_{;a}S_{;b}S_{;c}{}^c + 12S_{;c}S_{;b}S_{;a}{}^c \\ &\quad + 12S_{;c}S_{;a}S_{;b}{}^c - 4S_{;ca}S_{;b}{}^c + 6S_{;b}S_{;c}{}^c{}_a + 6S_{;a}S_{;c}{}^c{}_b - 4S_{;c}S_{;a}{}^c{}_b \\ &\quad - 2S_{;c}{}^c{}_{ab} + \eta_{ab}(3S_{;c}S_{;d}S_{;}{}^cS_{;}{}^d - 3S_{;c}{}^cS_{;d}{}^d - 12S_{;c}S_{;d}S_{;}{}^{cd} + 4S_{;cd}S_{;}{}^{cd} - 2S_{;c}S_{;}{}^c{}_d{}^d + 2S_{;c}{}^c{}_d{}^d)] \; ,\end{aligned} \tag{4.8}$$

$$
\begin{aligned}
{}^{(3)}H_{ab}(g^{\rm cl}_{cd}) &= {}^{(3)}H_{ab}(e^{2S}\eta_{cd}) \\
&= e^{-2S}[-4S_{;c}S_{;a}S_{;b}S_{;}{}^{c}-4S_{;a}S_{;b}S_{;c}{}^{c}+4S_{;c}{}^{c}S_{;ab}+4S_{;c}S_{;b}S_{;a}{}^{c}+4S_{;c}S_{;a}S_{;b}{}^{c} \\
&\quad -4S_{;ca}S_{;b}{}^{c}+\eta_{ab}(S_{;c}S_{;d}S_{;}{}^{c}S_{;}{}^{d}-2S_{;c}{}^{c}S_{;d}{}^{d}-4S_{;c}S_{;d}S_{;}{}^{cd}+2S_{;cd}S_{;}{}^{cd})] ,
\end{aligned} \tag{4.9}
$$

where semicolons refer to derivatives covariant with respect to the unphysical flat metric and all raising and lowering of indices in Eqs. (4.7)–(4.9) are with η_{ab}. For a given stress energy, it is straightforward to put the last term of Eq. (4.6) into a similar form. By putting Eq. (4.6) in this form, the problem has now been simplified from solving a second-order partial differential equation in curved space-time to solving a second-order partial differential equation in flat space-time.

Useful formulas analogous to Eqs. (4.8) and (4.9), but where the covariant derivatives and raising and lowering of indices are performed with the physical background metric, have also been calculated [24]. These would be helpful in the case that the map from the physical space-time to the unphysical space-time implied by (4.1) were not known explicitly.

V. GENERAL CASE

The previous examples involved Friedmann-Robertson-Walker, Lemaître, and other conformally flat space-times. We now turn to general space-times. The perturbative constraints in the general case can be used to express the curvature tensors appearing in the first-order quantum corrections (i.e., the H's) in terms of the lowest-order (classical) stress-energy tensor T_{ab} of the matter. Usually this lowest-order T_{ab} involves fewer derivatives of the metric than do the curvature tensors, so that this procedure results in an equation with fewer derivatives than the original. For example, for a classical fluid or a minimally coupled scalar field, the resulting equation contains no more than second derivatives of the metric.

The reduction process must be modified when the stress-energy tensor contains explicit curvature terms, as for a conformally coupled scalar field. One way to deal with such a case is to evaluate the curvature tensor appearing in T_{ab} using the lowest-order classical solution, since only that will contribute to the correction terms of order $\hbar$.

For simplicity, we will suppose that a cosmological constant term, if present in the Einstein equation, is included in the definition of T_{ab}. Such a term in T_{ab} involves no derivatives of the metric.

The perturbatively constrained equations should not have the instabilities exhibited when one tries to integrate numerically higher derivative equations. Such instabilities can be produced by the tendency for the growing or runaway solutions in the enlarged solution space to dominate the nearly classical perturbative part of solutions. The runaway solutions will not be present in the solution space after the reduction to lower derivatives.

The first-order perturbative constraint coming from the Einstein equations with quantum corrections, Eq. (1.1), is

$$\hbar G_{ab}=\hbar\kappa T_{ab}+O(\hbar^2) , \tag{5.1}$$

where T_{ab} is the zero-order contribution of the stress-energy tensor. It follows that

$$\hbar R=-\hbar\kappa T+O(\hbar^2) , \tag{5.2}$$

where $T=T^a_a$. Also,

$$\hbar R_{ab}=\hbar\kappa(T_{ab}-\tfrac{1}{2}g_{ab}T)+O(\hbar^2) . \tag{5.3}$$

Substituting these perturbative constraints into the expressions for ${}^{(i)}H_{ab}$ gives (in four dimensions)

$$
\begin{aligned}
{}^{(1)}H_{ab} &= \kappa(-2T_{;ab}-\tfrac{1}{2}g_{ab}\kappa T^2+2g_{ab}T_{;p}{}^{p}+2\kappa T T_{ab}) \\
&\quad +O(\hbar) ,
\end{aligned} \tag{5.4}
$$

$$
\begin{aligned}
{}^{(2)}H_{ab} &= \kappa(-T_{;ab}+T_{ab;p}{}^{p}+2T_{a}{}^{p}{}_{;bp}-\tfrac{1}{2}\kappa g_{ab}T^2 \\
&\quad +2\kappa T T_{ab}-2\kappa T_{pb}T_{a}{}^{p}+\tfrac{1}{2}\kappa g_{ab}T_{pq}T^{pq})+O(\hbar) ,
\end{aligned} \tag{5.5}
$$

and

$$
\begin{aligned}
{}^{(3)}H_{ab} &= \kappa(-\tfrac{1}{6}g_{ab}\kappa T^2+\tfrac{1}{3}\kappa T T_{ab} \\
&\quad -\kappa T_{pb}T_{a}{}^{p}+\tfrac{1}{2}\kappa g_{ab}T_{pq}T^{pq})+O(\hbar) .
\end{aligned} \tag{5.6}
$$

Then the perturbatively constrained Einstein equation with quantum corrections in a general space-time is given by Eqs. (2.2)–(2.6) with these values for ${}^{(1)}H_{ab}$ and ${}^{(2)}H_{ab}$ (with $\Lambda=0$). Only the lowest order or classical T_{ab} appears in these quantum correction terms, and if there is a classical cosmological constant present, then it is included in the definition of the lowest order T_{ab}. In general, the stress tensor contribution on the right-hand side of the semiclassical Einstein equation will include nonlocal state-dependent contributions, such as those coming from gravitationally induced particle creation and other effects. Local state-independent quantum corrections to the stress-energy tensor are, of course, already included with the local correction terms on the left-hand side. If the zeroth-order stress tensor has vanishing trace T, as for radiation or massless particles, then the equations simplify considerably.

In a conformally flat space-time, ${}^{(2)}H_{ab}$ is replaced by ${}^{(3)}H_{ab}$, and further simplification may occur if the field is a massless conformally invariant field. The way in which this occurs was already discussed in the introductory section. Similar expressions for ${}^{(1)}H_{ab}$ and ${}^{(2)}H_{ab}$ were also obtained by Bel and Sirousse-Zia [8].

A. Spherical body with quantum corrections

We next make use of the previous expressions for the correction terms to write the equations governing the quantum corrections to the gravitational field of a static spherical body. These equations are in a form suitable for numerical integration.

We first calculate the local state-independent quantum corrections which enter into the Einstein equations for the most general spherically symmetric space-time, which has the line element

$$ds^2 = -B(r)dt^2 + A(r)dr^2 + r^2 d\theta^2 + r^2\sin^2(\theta)d\phi^2 . \tag{5.7}$$

Here A and B will consist of a classical part and a quantum correction of order $\hbar$.

We will assume that the matter is described by a perfect fluid energy-momentum tensor,

$$T^{ab} = pg^{ab} + (p+\rho)u^a u^b , \tag{5.8}$$

where u^a is the four-velocity $dx^a/d\tau$ of the fluid volume element, and the proper pressure p and ρ are functions only of the radial coordinate r. Since the fluid is static, we have $u^r = u^\theta = u^\phi = 0$, and $u^t = B(r)^{-1/2}$.

From the previous section, we have the expressions for ${}^{(1)}H_{ab}$, ${}^{(2)}H_{ab}$, and ${}^{(3)}H_{ab}$ in terms of the classical fluid T_{ab}. Here we give the result of the calculation of those expressions (this is a lengthy calculation, but is simpler than calculating the quadratic curvature tensor expressions directly; in addition, because the perturbative constraints have been used, no higher than second derivatives appear in the result).

One finds, for the first correction term,

$$ {}^{(1)}H_{rr} = \kappa \left[-\tfrac{1}{2}\kappa\rho^2 A + \kappa\rho Ap + \tfrac{3}{2}\kappa Ap^2 + \frac{12p'}{r} + \frac{3p'B'}{B} - \frac{4\rho'}{r} - \frac{3\rho'B'}{B} \right] + O(\hbar) , \tag{5.9}$$

$$ {}^{(1)}H_{\theta\theta} = \kappa \left[-\tfrac{1}{2}\kappa r^2\rho^2 + \kappa r^2\rho p + \tfrac{3}{2}\kappa r^2 p^2 + \frac{6rp'}{A} - \frac{3r^2p'A'}{A^2} + \frac{3r^2p'B'}{AB} + \frac{6r^2p''}{A} - \frac{r^2\rho'B'}{AB} - \frac{2r^2\rho''}{A} - \frac{2r\rho'}{A} + \frac{r^2\rho'A'}{A^2} \right] + O(\hbar) , \tag{5.10}$$

$$ {}^{(1)}H_{\phi\phi} = \sin^2\theta \, {}^{(1)}H_{\theta\theta} , \tag{5.11}$$

and

$$ {}^{(1)}H_{tt} = \left[\frac{3\kappa B}{2rA^2} \right] [-\kappa r\rho^2 A^2 + 2\kappa r\rho p A^2 + 3\kappa r p^2 A^2 - 8p'A + 2rp'A' - 4rp''A + 4\rho'A - r\rho'A' + 2r\rho''A] + O(\hbar) . \tag{5.12}$$

The second correction term is found to be

$$ {}^{(2)}H_{rr} = \kappa \left[\kappa p^2 A + \kappa\rho Ap - \frac{pB'^2}{B^2} - \frac{\rho B'^2}{B^2} + \frac{2p'}{r} + \frac{(3/2)p'B'}{B} - \frac{(1/2)\rho'A'}{A} + \frac{\rho'B'}{B} + \rho'' \right] + O(\hbar) , \tag{5.13}$$

$$ {}^{(2)}H_{\theta\theta} = \kappa \left[\kappa r^2 p^2 + \kappa r^2 \rho p + \frac{rp'}{A} - \frac{(1/2)r^2p'A'}{A^2} + \frac{(1/2)r^2p'B'}{AB} + \frac{r\rho'}{A} + \frac{r^2p''}{A} \right] + O(\hbar) , \tag{5.14}$$

$$ {}^{(2)}H_{\phi\phi} = \sin^2\theta \, {}^{(2)}H_{\theta\theta} , \tag{5.15}$$

and

$$ {}^{(2)}H_{tt} = \kappa \left[3\kappa Bp^2 + 3\kappa Bp\rho - \frac{2pB'}{(rA)} - \frac{2\rho B'}{(rA)} + \frac{(1/2)pA'B'}{A^2} + \frac{(1/2)\rho A'B'}{A^2} - \frac{(1/2)pB'^2}{(AB)} - \frac{(1/2)\rho B'^2}{(AB)} + \frac{(1/2)p'B'}{A} + \frac{2\rho'B}{(rA)} - \frac{(1/2)\rho'BA'}{A^2} - \frac{pB''}{A} - \frac{\rho B''}{A} + \frac{\rho''B}{A} \right] + O(\hbar) . \tag{5.16}$$

For the third correction term, which appears in place of ${}^{(2)}H_{ab}$ in a conformally flat space-time, the final result is much simpler:

$$ {}^{(3)}H_{rr} = \tfrac{1}{3}\kappa^2\rho(\rho+2p)A + O(\hbar) , \tag{5.17}$$

$$ {}^{(3)}H_{\theta\theta} = (r^2/A)\, {}^{(3)}H_{rr} + O(\hbar) , \tag{5.18}$$

$$ {}^{(3)}H_{\phi\phi} = \sin^2\theta \, {}^{(3)}H_{\theta\theta} , \tag{5.19}$$

and

$$ {}^{(3)}H_{tt} = \tfrac{1}{3}\kappa^2\rho^2 B + O(\hbar) . \tag{5.20}$$

It is interesting that ${}^{(3)}H_{ab}$ is in the form of a perfect fluid energy-momentum tensor. It can be absorbed into the change, $T_{ab} \to T_{ab} + \Delta T_{ab}$, with

$$\Delta T_{ab} = \Delta p g_{ab} + (\Delta p + \Delta\rho)u_a u_b + O(\hbar^2) , \tag{5.21}$$

where u_a is as before,

$$\Delta p = -\tfrac{1}{3}\alpha_3 \hbar\kappa\rho(\rho+2p) , \tag{5.22}$$

and

$$\Delta\rho = -\tfrac{1}{3}\alpha_3 \hbar\kappa\rho^2 . \tag{5.23}$$

Therefore, in the case of a conformally flat metric, the $\alpha_3 \hbar^{(3)}H_{ab}$ quantum correction term in the Einstein equation can be absorbed into a redefinition of the pressure and density:

$$p \to p - \tfrac{1}{3}\alpha_3 \hbar\kappa\rho(\rho+2p) + O(\hbar^2) , \tag{5.24}$$

and

$$\rho \to \rho - \tfrac{1}{3}\alpha_3 \hbar\kappa\rho^2 + O(\hbar^2) . \tag{5.25}$$

The semiclassical Einstein equations with quantum corrections in general have the form

$$R_{ab} - \tfrac{1}{2}g_{ab}R + \alpha_1 \hbar^{(1)}H_{ab} + \alpha_2 \hbar^{(2)}H_{ab} + \alpha_3 \hbar^{(3)}H_{ab} + O(\hbar^2) = \kappa \mathcal{T}_{ab} , \tag{5.26}$$

where $\mathcal{T}_{ab}$ includes classical matter contributions and the lowest-order state-dependent part of the expectation value of quantum matter fields. The state-independent local quantum corrections of order $\hbar$ are included in the H terms on the left-hand side. It is understood that we may set $\alpha_3=0$, except when the metric is conformally flat. In the latter case, it is understood that $\alpha_2=0$, since the first two corrections are then proportional to one another. In the formally flat case, the α_3 term arises from the state-dependent part of the quantum stress energy.

With the expressions given above for the H's, the Einstein equations are now easily written down for the general spherically symmetric metric. Only the state-dependent part of the expectation value of the quantum stress-energy tensor requires further work to calculate, but this will not increase the order of the highest metric derivative in most cases, so that the perturbative constraints have succeeded in reducing the semiclassical Einstein equations with quantum corrections to second-order equations having the standard initial data. These equations are thus in suitable form for numerical integration. We will not carry that out here, but plan to return to it in a later paper. However, one spherically symmetric case where further simplification occurs will be discussed briefly in the next section.

B. Fluid sphere of constant proper classical density

Consider a fluid sphere which at the classical level has constant proper density. Let us suppose that, in addition to the classical fluid, only massless conformally invariant-free fields, such as the photon and massless neutrino, are present.

The classical interior solution for a fluid sphere of uniform proper density was found by Schwarzschild in 1916 [25]. It is known that the Weyl tensor of this metric is zero, so that it is conformally flat.

Because this space-time is static and has no event horizons, we may suppose that the quantum fields are in a well-defined vacuum state. It has been shown that for conformally invariant massless free fields in conformally flat space-times, the vacuum stress-energy tensor is determined by the trace anomaly [26,22]. The vacuum stress-energy tensor of these fields is a linear combination of $\hbar^{(1)}H_{ab}$ and $\hbar^{(3)}H_{ab}$. We will suppose that there is no additional Casimir energy contribution in this space-time.

Therefore, the state-dependent part of the vacuum expectation value of the quantum stress tensor is zero for massless conformally invariant free fields propagating on this interior metric. The only effect of the quantum fields in their vacuum state is to give rise, through the conformal trace anomaly, to the α_1 and α_3 state-independent correction terms in the Einstein equations. Thus, in the interior of the fluid sphere, one has the *complete* equations which must be integrated.

The classical interior Schwarzschild solution has the form of Eq. (5.7) with (for $r<R$)

$$A_{\rm cl}(r) = (1-2GMr^2/R^3)^{-1} \tag{5.27}$$

and

$$B_{\rm cl}(r) = \tfrac{1}{4}[3(1-2GM/R)^{1/2} - (1-2GMr^2/R^3)^{1/2}] . \tag{5.28}$$

The classical energy-momentum tensor corresponding to this solution is that of Eq. (5.8), with a constant proper density:

$$\rho = \frac{3M}{4\pi R^3} . \tag{5.29}$$

The pressure p is

$$p(r) = \frac{3M}{4\pi R^3}\left[\frac{(1-2GM/R)^{1/2}-(1-2GMr^2/R^3)^{1/2}}{(1-2GMr^2/R^3)^{1/2}-3(1-2GM/R)^{1/2}}\right] . \tag{5.30}$$

For $r>R$, this interior metric joins with the classical Schwarzschild exterior solution of mass M, and zero density and pressure. It is known that of all stable fluid spheres having a given mass M and radius R, the Schwarzschild uniform density sphere has the smallest central pressure. For the pressure not to become infinite somewhere inside the object, it is necessary that $GM<\frac{4}{9}R$. This means that the radius of the static fluid sphere must be larger than the corresponding Schwarzschild black hole radius. Quantum corrections may possibly change the relationship between these two radii for sufficiently small fluid spheres.

From the previous section, we have the expressions for ${}^{(1)}H_{ab}$ and ${}^{(3)}H_{ab}$ in terms of the classical fluids ρ and p. These are given for the constant density sphere by Eqs. (5.29) and (5.30). The resulting semiclassical Einstein equations are of second order and are ready for numerical or analytic solution. We will carry this further in a later paper.

VI. CONCLUSION

We have considered first-order semiclassical quantum corrections to a variety of classical solutions to the Einstein gravitational field equations. We have used perturbative constraints to obtain the reduced semiclassical

Einstein equations for Friedmann-Robertson-Walker cosmologies, for Friedmann-Lemaître cosmologies, for the gravitational fields of static spherically symmetric fluid bodies, and for the general, conformally flat metric in terms of its conformal factor. The reduced equations we obtained do not contain higher than second derivatives, and do not exhibit runaway solutions or instabilities of the original fourth-order equations. They have the same physical content as the fourth-order equations, but yield only physically relevant solutions. Analytic and numerical solutions to these semiclassical equations were found in the cosmological cases. Although in most cases the semiclassical corrections play only a small role far from the Planck scale, there are some examples in which semiclassical quantum corrections cause significant deviation, or even qualitatively different behavior, from the classical solution.

In the case of spatially flat, radiation-dominated Friedmann-Robertson-Walker solutions, the corrections either strengthen or weaken the singular behavior at early times, in a regime where the perturbative corrections are valid (the perturbative validity does break down, however, before the time of the classical singularity itself can be reached). The corrections at late times become vanishingly small. In the Friedmann-Lemaître "bounce" or "turn-around" solutions, quantum corrections can cause classical and semiclassical models which have the same initial conditions to have significantly different radii at late times. In the case of the "maximal hesitation" Einstein universe, the semiclassical corrections can cause large deviations and even qualitatively different behavior from the corresponding classical solution.

For fluid spheres, we have given the explicit first-order quantum corrections as reduced field equations for the general case, and have discussed the constant classical density fluid in further detail. In future work, we intend to study solutions of these equations. For small fluid spheres the corrections may significantly alter fundamental relations, such as the classical theorem which requires the radius of a static sphere of fluid to be larger than the radius of the Schwarzschild black hole having the same mass.

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APPENDIX A: INTEGRATION FACTOR

In several instances in this paper, the semiclassical solutions are calculated iteratively, from an equation of motion of the form

$$\dot{a}(t)=f_0(a(t))+\hbar f_1(a(t))+O(\hbar^2)\ , \tag{A1}$$

and an ansatz of

$$a=a_0+\hbar a_1+O(\hbar^2)\ . \tag{A2}$$

Inserting Eq. (A2) into Eq. (A1) and expanding in powers of $\hbar$ produces

$$\dot{a}_0+\hbar\dot{a}_1=f_0(a_0)+\hbar a_1 f_0'(a_0)+\hbar f_1(a_0)+O(\hbar^2)\ ,$$

which leads to the series of equations

$$h^0:\ \dot{a}_0(t)=f_0(a_0(t))\ ,$$

$$h^1:\ \dot{a}_1(t)=a_1(t)f_0'(a_0(t))+f_1(a_0(t))\ .$$

The first equation is typically a nonlinear equation which might be solved in a variety of ways. The second is a first-order *linear* inhomogeneous equation in $a_1(t)$ (once the classical solution a_0 has been determined), for which a general solution can be always found in the form

$$a_1(t)=\frac{1}{c\mu(t)}+\frac{1}{\mu(t)}\int^t dt'\mu(t')f_1(a_0(t'))\ , \tag{A3}$$

where c is an arbitrary constant of integration and $\mu(t)$ is an integrating factor given by

$$\mu(t)=\exp\left[-\int^t dt' f_0'(a_0(t'))\right]\ . \tag{A4}$$

Because Eq. (A1) has no explicit dependence on t, we know that there is a one-parameter family of solutions, $a_{t_0}(t)=a(t-t_0)$, to Eq. (A1), parametrized by the initial time t_0. The freedom to choose the constant c in Eq. (A3) must correspond to the freedom to change this initial time by $t_0\rightarrow\tau=t_0+\hbar t_1$. By making this shift, and expanding in powers of $\hbar$, we can determine the integrating factor without the need of integrating Eq. (A4) explicitly:

$$\begin{aligned}a_0(t-t_0)&\rightarrow a_0(t-t_0-\hbar t_1)\\&=a_0(t-t_0)-\hbar t_1\dot{a}_0(t-t_0)\ .\end{aligned} \tag{A5}$$

Comparing this to Eq. (A3) reveals that $[c\mu(t)]^{-1}=-t_1\dot{a}_0(t-t_0)$, permitting us to rewrite Eq. (A3) as

$$a_1(t)=\dot{a}_0(t)\int^t dt'\frac{f_1(a_0(t'))}{\dot{a}_0(t')}-t_1\dot{a}_0(t)+O(\hbar)\ , \tag{A6}$$

where t_1 is an arbitrary parameter with dimensions of time, and the shift of initial time in Eq. (A5) induces only a higher-order $[O(\hbar)]$ change in the integrand in Eq. (A6).

APPENDIX B: UNIQUENESS OF PERTURBATIVE SOLUTIONS

For a general perturbative field equation of the form

$$F_0(q,\dot{q},\ddot{q})+\epsilon F_1(q,\dot{q},\ddot{q},\dddot{q},q^{iv})+\cdots=O(\epsilon^{n+1})\ , \tag{B1}$$

where ϵ is the formal perturbative expansion parameter ($\epsilon=\hbar$ for semiclassical gravity) and q represents all the configuration space variables, there is some ambiguity in the way a perturbative solution

$$q=q_0(t)+\epsilon q_1(t)+\cdots+\epsilon^n q_n(t)+O(\epsilon^{n+1}) \tag{B2}$$

may be expanded in the same expansion parameter. For example, defining q_0 as the quantity that satisfies the lowest-order field equation,

$$F_0(q_0,\dot{q}_0,\ddot{q}_0)=O(\epsilon)\ , \tag{B3}$$

does not unambiguously determine q_0, because we can shift by any quantity $O(\epsilon)$, i.e., $q_0\rightarrow q_0+\epsilon\delta q$, and the new quantity will still satisfy Eq. (B3), only requiring an accompanying shift in the higher-order terms in the expansion of the solution, i.e., $q_1 \to q_1 - \delta q$. Similar ambiguity exists for the field equations of higher-order terms:

$$\epsilon\left[\frac{\partial F_0(q,\dot q,\ddot q)}{\partial q}q_1+\frac{\partial F_0(q,\dot q,\ddot q)}{\partial \dot q}\ddot q_1\right]+\epsilon F_1(q,\dot q,\ddot q,\dddot q,q^{iv})=O(\epsilon^2)\ ,\quad \text{etc.} \tag{B3a}$$

Despite this ambiguity in breaking up the solution into terms that solve the field equations order by order, there is no ambiguity in the sum of all such terms. This can be seen by positing an additional requirement that the individual terms of the solution be explicitly independent of the perturbative expansion parameter:

$$\frac{\partial q_i}{\partial \epsilon}=O(\epsilon^{n+1-i})\ ,\quad i=0,\ldots,n\ . \tag{B4}$$

That this requirement can always be met can be easily seen as follows. Instead of solving Eq. (B3), solve the related equation

$$F_0(q_0,\dot q_0,\ddot q_0)=O(\epsilon^{n+1})\ . \tag{B5}$$

Any solution to (B5) is also a solution to (B3), but there is no ambiguity to $O(\epsilon^n)$. Similarly solve the analogs of Eq. (B3a) to the highest order allowed

$$\epsilon\left[\frac{\partial F_0(q,\dot q,\ddot q)}{\partial q}q_1+\frac{\partial F_0(q,\dot q,\ddot q)}{\partial \dot q}\dot q_1+\frac{\partial f_0(q,\dot q,\ddot q)}{\partial \ddot q}\ddot q_1\right]+\epsilon F_1(q,\dot q,\ddot q,\dddot q,q^{iv})=O(\epsilon^{n+1})\ ,\quad \text{etc.} \tag{B6}$$

This process uniquely defines each of the terms in Eq. (B2), and therefore also the sum [to $O(\epsilon^n)$].

Despite the ability to fix the expansion in this manner, it is often to our advantage to use the freedom to make order-by-order shifts in the terms q_i of the solution, as done in Appendix A. Solutions with different ϵ dependencies can be obtained by adding ϵ-dependent terms:

$$\begin{aligned} q_0 &\to q_0+\epsilon\delta_1 q_0+\epsilon^2\delta_2 q_0+\cdots+\epsilon^n\delta_n q_0\ ,\\ q_1 &\to q_1-\delta_1 q_0-\cdots-\epsilon^{n-1}\delta_n q_0\\ &\quad +\epsilon\delta_1 q_1+\cdots+\epsilon^{n-1}\delta_{n-1}q_1\ ,\\ q_2 &\to q_2-\delta_1 q_1-\cdots-\epsilon^{n-1}\delta_n q_1\\ &\quad +\epsilon\delta_1 q_2+\cdots+\epsilon^{n-1}\delta_{n-2}q_2\ ,\quad \text{etc.}\ , \end{aligned} \tag{B7}$$

where the $\delta_1 q_j$ are arbitrary $O(\epsilon^0)$ functions to be chosen at one's convenience.

APPENDIX C: ANALYTIC SEMICLASSICAL SOLUTION FOR TURN-AROUND OR BOUNCE UNIVERSE

The semiclassical corrections to the turn-around or bounce universe were calculated numerically in Sec. III C 1, but they can also be calculated analytically. The classical turn-around solution is a solution to Eq. (3.11), for positive cosmological constant ($\Lambda > 0$), spatially closed slicing ($k=1$), and sufficiently small radiation density:

$$0<\frac{4\Lambda\kappa\rho a_0^4}{9}<1\ . \tag{C1}$$

The classical solution is

$$a_{\rm cl}^2=\frac{1}{2}\frac{3}{\Lambda}(q\cosh[2\sqrt{\Lambda/3}t]+1)\ , \tag{C2}$$

where

$$q\equiv\left[1-\frac{4\Lambda\kappa\rho a_0^4}{9}\right]^{1/2}\ , \tag{C3}$$

and $0<q<1$.

The ansatz $a=a_{\rm cl}+\hbar a_1+O(\hbar^2)$, when inserted into Eq. (3.11) and expanded in powers of $\hbar$, gives, as the order-$\hbar$ equation,

$$\begin{aligned} O(\hbar)&=2\dot a_1\dot a_{\rm cl}-2\frac{\Lambda}{3}a_1a_{\rm cl}+\frac{2\kappa\rho a_0^4}{3}a_1a_{\rm cl}^{-3}\\ &\quad-\frac{8\alpha_1\Lambda\kappa\rho a_0^4}{3}a_{\rm cl}^{-2}\\ &\quad+\alpha_3\left[\frac{\Lambda}{3}a_{\rm cl}^2+\frac{\kappa\rho a_0^4}{3}a_{\rm cl}^{-2}\right]^2a_{\rm cl}^{-2}\ . \end{aligned} \tag{C4}$$

This is a first-order, linear, inhomogeneous equation in $a_1(t)$, and it may be solved by standard methods shown in Appendix A. The integrations are tedious, but easily within the grasp of a good symbolic integration software package. The result, up to the initial time ambiguity dealt with in Appendix A, is

$$\begin{aligned} a_1&=\alpha_1\sqrt{6\Lambda}(q-q^{-1})\cosh[2\sqrt{\Lambda/3}t](q\cosh[2\sqrt{\Lambda/3}t]+1)^{-1/2}\\ &\quad+\alpha_3\frac{\sqrt{3\Lambda}}{24}\left[\frac{3\sinh[2\sqrt{\Lambda/3}t]}{\sqrt{1-q^2}}\ln\left[\frac{\sqrt{1-q^2}\sinh[2\sqrt{\Lambda/3}t]-\cosh[2\sqrt{\Lambda/3}t]-q}{q\cosh[2\sqrt{\Lambda/3}t]+1}\right]\right.\\ &\qquad\left.+2\sqrt{\Lambda/3}t\sinh[2\sqrt{\Lambda/3}t]+\frac{1}{q}(5q\cosh[2\sqrt{\Lambda/3}t]-1)-\frac{1}{q}\frac{1-q^2}{q\cosh[2\sqrt{\Lambda/3}t]+1}\right]\\ &\quad\times(q\cosh[2\sqrt{\Lambda/3}t]+1)^{-1/2}+O(\hbar)\ . \end{aligned} \tag{C5}$$

The final result for $a(t)$ is given by inserting Eqs. (C2) and (C5) into the perturbative ansatz, $a=a_{\rm cl}+\hbar a_1+O(\hbar^2)$.

[1] See, e.g., N. D. Birrell and P. C. W. Davies, *Quantum Fields in Curved Space* (Cambridge University Press, Cambridge, England, 1982); S. A. Fulling, *Aspects of Quantum Field Theory in Curved Space-Time* (Cambridge University Press, Cambridge, England, 1989), and references in both.

[2] We use the conventions $c=1$, $\eta_{ab}=\mathrm{diag}(-1,1,1,1)$, $R^{l}_{man}=\partial_a\Gamma^{l}_{mn}+\cdots$, and $R^{l}_{mln}=R_{mn}$.

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