Meson mass splittings in the nonrelativistic model

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Mass splittings between isodoublet meson pairs and between 0^- and 1^- mesons of the same valence quark content are computed in a detailed nonrelativistic model. The field-theoretic expressions for such splittings are shown to reduce to kinematic and Breit-Fermi terms in the nonrelativistic limit. Algebraic results thus obtained are applied to the specific case of the linear-plus-Coulomb potential, with resultant numbers compared to experiment.

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I. INTRODUCTION

The splitting of the masses of mesons in an isospin doublet, sometimes called electromagnetic splitting, has traditionally been attributed primarily to explicit isospin breaking (i.e., $m_u \neq m_d$) and differences between the charges of the valence quark-antiquark pairs ($Q_u \neq Q_d$), with hyperfine, spin-orbit, and other effects neglected in comparison. Such a model serves to explain the observed splittings $K^0 - K^+ = 4.024 \pm 0.032$ MeV and $D^+ - D^0 = 4.77 \pm 0.27$ MeV, but has failed in light of the surprisingly small $B^0 - B^+ = 0.1 \pm 0.8$ MeV.

It is precisely this mass difference which has led to the proposal of a variety of models. Some of these [1–4] are based on the nonrelativistic model of hadron masses put forth by De Rújula, Georgi, and Glashow [5] soon after the development of QCD. Such models have the unfortunate tendency to predict numbers no smaller than $B^0 - B^+ \simeq 2$ MeV, well outside the current experimental limits. Using more phenomenological models [6, 7], one can obtain a smaller splitting in closer agreement with experiment. Nevertheless, it may seem odd that the usual nonrelativistic model, which works well for the D and even the K mesons, should fail in the case of the B, which boasts an even heavier quark.

The primary conclusions of this work are that it *is* possible to explain the mass splittings of heavy mesons (D and B, but not K) in an ordinary nonrelativistic model, as long as we take into account *all* corrections to consistent orders of magnitude, that expectation values of the mesonic wave functions in general have mass dependence, and that the running of the strong-coupling constant is not negligible.

In this spirit, the paper is organized as follows: In Sec. II we consider the problem of computing mesonic mass contributions in field theory. Then, in Sec. III, we demonstrate that the nonrelativistic limit of the field-theoretic result leads to kinematic terms and the Breit-Fermi interaction, exactly as stated in De Rújula, Georgi, and Glashow. This is followed in Sec. IV by an exhibition of the full mass splitting relations for isodoublet 0^- and 1^- meson pairs, as well as (0^-, 1^-) pairs with the same valence quarks. Section V discusses the application of quantum-mechanical theorems, including a very useful generalized virial theorem, to the problem of reducing the number of independent expectation values in the splitting formulas. These theorems are applied to the popular choice of a linear-plus-Coulomb potential in Sec. VI, with numerical results presented in Sec. VII.

II. MASS COMPUTATION IN FIELD THEORY

Typically, the computation of mesonic mass splittings in a nonrelativistic model is accomplished by starting with the Breit-Fermi interaction ([8], Secs. 38–42)

$$\begin{aligned} H_{\rm BF} = \sum_{i>j} (\alpha Q_i Q_j + k\alpha_s) \Bigg\{ & \frac{1}{|\mathbf{r}_{ij}|} - \frac{1}{2m_i m_j |\mathbf{r}_{ij}|} [\mathbf{p}_i \cdot \mathbf{p}_j + \hat{\mathbf{r}}_{ij} \cdot (\hat{\mathbf{r}}_{ij} \cdot \mathbf{p}_i) \mathbf{p}_j] \\ & - \frac{\pi}{2} \delta^3(\mathbf{r}_{ij}) \left\{ \frac{1}{m_i^2} + \frac{1}{m_j^2} + \frac{4}{m_i m_j} \left[\frac{4}{3} \mathbf{s}_i \cdot \mathbf{s}_j + \left(\frac{3}{4} + \mathbf{s}_i \cdot \mathbf{s}_j \right) \delta_{q_i \bar{q}_j} \right] \right\} \\ & - \frac{1}{2|\mathbf{r}_{ij}|^3} \left[\frac{1}{m_i^2} \mathbf{r}_{ij} \times \mathbf{p}_i \cdot \mathbf{s}_i - \frac{1}{m_j^2} \mathbf{r}_{ij} \times \mathbf{p}_j \cdot \mathbf{s}_j \right. \\ & \left. + \frac{2}{m_i m_j} (\mathbf{r}_{ij} \times \mathbf{p}_i \cdot \mathbf{s}_j - \mathbf{r}_{ij} \times \mathbf{p}_j \cdot \mathbf{s}_i + 3(\mathbf{s}_i \cdot \hat{\mathbf{r}}_{ij})(\mathbf{s}_j \cdot \hat{\mathbf{r}}_{ij}) - \mathbf{s}_i \cdot \mathbf{s}_j) \right] \Bigg\}, \end{aligned} \tag{1}$$

where $\mathbf{r}_i$, $\mathbf{p}_i$, m_i, $\mathbf{s}_i$, and Q_i denote the coordinate, momentum, (constituent) mass, spin, and charge (in units of the protonic charge) of the ith quark, respectively; $\mathbf{r}_{ij} \equiv \mathbf{r}_i - \mathbf{r}_j$; α and α_s are the (running) QED and QCD coupling constants; and $k = -\frac{4}{3}$ $(-\frac{2}{3})$ is a color binding factor for mesons (baryons). This expression includes an annihilation term if $q_i = \bar{q}_j$ are in a relative $j = 1$ state. From this, one chooses the terms that are considered significant and then calculates the appropriate quantum-mechanical expectation values. We will pursue this course of action in the next section; however, this author feels that it would be worthwhile to consider first the derivation of this interaction for the mesonic system from the more fundamental field theories of QED and QCD, since this approach entails greater generality and may provide impetus for work beyond the scope of this paper.

We first consider the question of the mass of a composite system from the point of view of the S matrix and interaction-picture perturbation theory. The mass of a system, defined as the expectation value of the total Hamiltonian in the center-of-momentum frame of the constituents, receives contributions from both the noninteracting and interacting pieces of the Hamiltonian; the former gives rise to the masses and kinetic energies of the constituents, and the latter produces the interaction energy. Technically, the matrix element of the noninteracting piece *in the interaction picture* produces terms which contribute to interactions between *renormalized* constituents. Thus one may think of interactions between "dressed" constituents, a topic to which we return momentarily.

Let us follow the method of Gupta [9] to derive the interaction potential from the field-theoretical interaction Hamiltonian. We begin by writing the S matrix in the Cayley form

$$S = \frac{1 - \frac{1}{2}iK}{1 + \frac{1}{2}iK}, \tag{2}$$

and expand the Hermitian operator K:

$$K = \sum_n K_n. \tag{3}$$

The purpose of this expansion, rather than expanding S directly, is to preserve unitarity in each partial sum of S. The physical effect of this parametrization is to eliminate diagrams with real intermediate states from the S-matrix expansion.

Computing the terms K_n, one finds

$$K_1 = \int_{-\infty}^{+\infty} dt\, H^I_{\text{int}}(t), \tag{4}$$

where I indicates the interaction picture. Now observe that we may invent an effective Hamiltonian H^I_{eff} such that its first-order contribution is equivalent to the contribution from H^I_{int} *to all orders.* Thus

$$K = \int_{-\infty}^{+\infty} dt\, H^I_{\text{eff}}(t). \tag{5}$$

The interaction energy is then

$$\Delta E = \frac{\langle f^I |\, H^I_{\text{eff}}(0)\, | i^I \rangle}{\langle f^I \,|\, i^I \rangle}, \tag{6}$$

with $|i^I\rangle$ and $|f^I\rangle$ actually the same state since the system is stable.

In our case, in which H^I_{eff} is composed of the interaction terms of QED and QCD, the lowest-order contribution is K_2, corresponding to two interaction vertices: the exchange of one vector boson. It is easily shown that

$$\begin{aligned} K_2 = iS_2 &= (2\pi)^4 \delta^4(P_f - P_i) \mathcal{M}^{(2)}_{fi} \\ &= \left\langle f^I \middle| \int dt\, H^{I\,(2)}_{\text{eff}}(t) \middle| i^I \right\rangle, \end{aligned} \tag{7}$$

where the superscript (2) indicates second order in the coupling constant, and $\mathcal{M}$ is the usual invariant amplitude for the process. Eliminating the δ functions that arise in the rightmost expressions, we find

$$\Delta E^{(2)} = \frac{\langle f^I |\, H^{I\,(2)}_{\text{eff}}(0)\, | i^I \rangle}{\langle f^I \,|\, i^I \rangle} = \mathcal{M}^{(2)}_{fi}. \tag{8}$$

Beyond second order the relation between the interaction energy and invariant amplitude becomes less trivial, but nevertheless Gupta has shown that it can be done. However, we do not continue to fourth order in this work, and henceforth suppress the (2) in the following.

In general, $\mathcal{M}_{fi}$ at any given order is represented by diagrams of the form indicated in Fig. 1. The composite state is formed by superposition of the constituent particle wave functions in such a way that the desired overall quantum numbers for the composite state are obtained. For the mesonic system, $\mathcal{M}_{fi}$ is represented by the diagram in Fig. 2, where the lowest-order interaction is the exchange of a single gauge boson. This class of diagrams allows for only the valence quark and antiquark (no sea $q\bar{q}$ pairs or glue), and thus would be a poor model if we chose these to be current quarks. Instead, the quarks in our diagrams will be constituent quarks, and the gauge couplings will assume their running values. In this way we can model the hadronic cloud, as well as renormalizations of the lines and vertices of our diagram, so that its particles are "dressed" in two senses. There is also an annihilation diagram if the quark and antiquark are of the same flavor. In this work we consider only the exchange diagram, since the mesons of greatest interest to us are those with one heavy and one light quark.

The next step is to obtain the amplitude $\mathcal{M}_{fi}$, in which

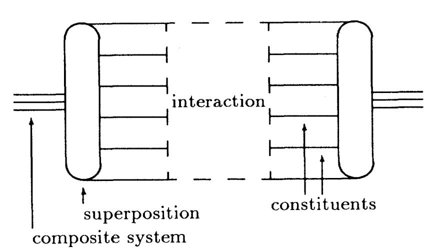

FIG. 1. Diagrammatical representation of $\mathcal{M}_{fi}$.

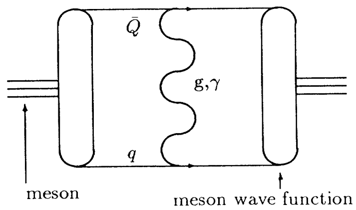

FIG. 2. Diagram for $\mathcal{M}_{fi}$ in the mesonic system.

the constituent legs are bound in the composite system, from the Feynman amplitude $\mathcal{M}$ (Fig. 3) for the same interaction with *free* external constituent legs. To do this, we need only constrain the free external legs in a way which reflects the wave function and rotational properties of the meson state. In general, if the variables z_n are the degrees of freedom of the meson state $|\Phi\rangle$, then we may write

$$|\Phi\rangle = \sum\!\!\!\!\!\!\int dz_n\, \phi(z_n)\, \mathcal{O}(z_n)\, |0\rangle\,. \tag{9}$$

The function ϕ is an amplitude in the variables z_n, i.e., a wave function, and $\mathcal{O}$ is a collection of Fock-space operators which specifies the rotational properties of $|\Phi\rangle$. The integral-sum symbol indicates summation over both continuous and discrete z_n. In this notation, we obtain the result

$$\Delta E = \sum\!\!\!\!\!\!\int dz_f\, dz_i\, \phi^*(z_f)\phi(z_i) f(z_i, z_f) \mathcal{M}(z_i, z_f), \tag{10}$$

where $f(z_i, z_f) \equiv \langle 0|\, \mathcal{O}^\dagger(z_f)\mathcal{O}(z_i)\, |0\rangle$ is a constraint function. We have written the energy contribution in this very general way in order to demonstrate the power of the technique.

Now we apply this prescription to the usual case of Feynman rules. Then z_n are quark momenta, ϕ is the mesonic momentum-space wave function, and f specifies the spin of the meson, as we shall see below. The energy contribution is evaluated in the quark center-of-momentum frame (i.e., the meson rest frame), in which the relative momenta of the quark-antiquark pair, intitially and finally, are denoted by $\mathbf{p}$ and $\mathbf{p}'$, respectively. Fourier transformation of the wave functions from momentum space to position space yields

$$\Delta E_{\text{c.m.}} = \int d^3\mathbf{x}_f \int d^3\mathbf{x}_i\, \psi^*(\mathbf{x}_f) K(\mathbf{x}_f, \mathbf{x}_i) \psi(\mathbf{x}_i),$$

where

$$K(\mathbf{x}_f, \mathbf{x}_i) = \int d^3\mathbf{p}' \int d^3\mathbf{p}\, \exp[i(\mathbf{p}'\cdot\mathbf{x}_f - \mathbf{p}\cdot\mathbf{x}_i)] \times \sum_{\text{spins}} f(\text{spins}) \mathcal{M}(\mathbf{p}', \mathbf{p}, \text{spins}), \tag{11}$$

$$\int d^3\mathbf{x}\, \psi^*(\mathbf{x})\psi(\mathbf{x}) = 1.$$

As a technical point of fact, it is necessary to keep track of the normalization conventions used for wave functions, Fourier transforms, and Feynman rules in order to obtain the true convention-independent ΔE. As it stands, Eq. (11) locks us into a particular set of Feynman-rule normalizations, which should be made clear in the following expression. The kinematic conventions are established in Fig. 4. Then the Feynman amplitude for free external quark legs and a virtual photon is

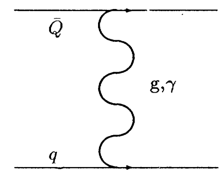

FIG. 3. Free-quark Feynman amplitude $\mathcal{M}$.

$$\mathcal{M} = i \left[\frac{1}{(2\pi)^{3/2}}\right]^4 \sqrt{\frac{M}{E_f}}\sqrt{\frac{M}{E_i}}\sqrt{\frac{m}{\varepsilon_f}}\sqrt{\frac{m}{\varepsilon_i}} \times \left[\bar{v}_{H_i}(\mathbf{P}_i)(-iQe\gamma_\mu)v_{H_f}(\mathbf{P_f})\right]\left(\frac{-ig^{\mu\nu}}{k^2}\right) \times \left[\bar{u}_{h_f}(\mathbf{p}_f)(-iqe\gamma_\mu)u_{h_i}(\mathbf{p}_i)\right], \tag{12}$$

with Qqe^2 replaced by g_s^2 for the gluon-mediated diagram. Note the use of helicity rather than spin eigenstate spinors, which is done in order to implement a relativistic description of the mesons. In a nonrelativistic picture in which the meson spin originates solely from the spin of the quarks (s waves), spin-0(1) mesons have spin-space wave functions described by the usual singlet and triplet quark wave-function $\bar{Q}q$ combinations:

$$\frac{\bar{Q}_\uparrow q_\downarrow \pm \bar{Q}_\downarrow q_\uparrow}{\sqrt{2}},\ \uparrow, \downarrow \text{ spins.} \tag{13}$$

The above expression remains true in a relativistic picture if we take the initial and final spin-quantization axes to coincide with the axes of relative momenta $\mathbf{p}$ and $\mathbf{p}'$, respectively, and then take $\uparrow$, $\downarrow$ as *helicity* eigenstates. This is nothing more than the simplest nontrivial case of the Jacob-Wick formalism [10]. It is then a simple matter to write the constraint function for singlet (triplet) mesons:

$$f(\text{helicities}) = \frac{1}{\sqrt{2}}(\delta_{h_i\uparrow}\delta_{H_i\downarrow} \pm \delta_{h_i\downarrow}\delta_{H_i\uparrow}) \times \frac{1}{\sqrt{2}}(\delta_{h_f\uparrow}\delta_{H_f\downarrow} \pm \delta_{h_f\downarrow}\delta_{H_f\uparrow}), \tag{14}$$

and so the object of interest is the constrained matrix element $\mathcal{M}_{\text{sing}}$ or $\mathcal{M}_{\text{trip}}$, which is the Feynman amplitude multiplied by the constraint function and summed over spins (or helicities). This is the object that is Fourier transformed in Eq. (11).

In summary, mass contributions due to a binding in-

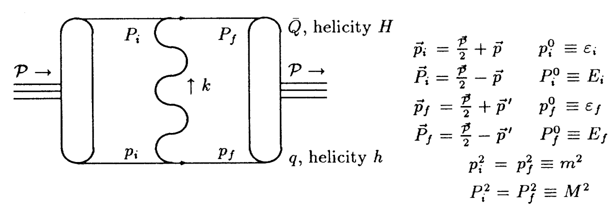

FIG. 4. Notation and conventions for the mesonic system.

teraction in a system of particles may be computed by writing down the Feynman amplitude induced by the interaction Hamiltonian, constraining the component particles to satisfy the symmetry properties of the system, and convolving with the appropriate system wave function. The specific implementation of this technique to spin-0 and spin-1 mesons with constituent quarks in a relative $\ell = 0$ state is described by Eqs. (11), (12), and (14).

III. THE NONRELATIVISTIC LIMIT

With the method for computing mass contributions in hand, we find ourselves with two possible courses of action. The first is to compute $\mathcal{M}_{\text{sing}}$ or $\mathcal{M}_{\text{trip}}$ in a fully relativistic manner, and then Fourier transform the result to obtain $\Delta E_{\text{c.m.}}$. The second is to immediately reduce the spinor bilinears via Pauli approximants, thus producing a nonrelativistic expansion. Let us explore both directions for the pseudoscalar case; the vector case is not much different.

The relativistic result is noncovariant, because the energy contribution is evaluated specifically in the c.m. frame of the quarks. We see this reflected in the computation of the matrix element. For example, it is convenient to eliminate spinors from the calculation by means of relations such as

$$\sum_h u_h(\mathbf{p}_A)\bar{u}_h(\mathbf{p}_B) = \frac{(m_A + \not{p}_A)}{\sqrt{2m_A(E_A + m_A)}}\frac{(1+\gamma_0)}{2} \times \frac{(m_B + \not{p}_B)}{\sqrt{2m_B(E_B + m_B)}}, \tag{15}$$

and the explicit γ_0 is a signal of the noncovariance. Once the spinor reductions and the resultant trace are performed, we find the expression

$$\mathcal{M}_{\text{sing}} = -(Qqe^2 + g_s^2)\mathcal{N}\mathcal{T}\frac{1}{k^2}, \tag{16}$$

where $\mathcal{N}$ results from the normalization factors, and $\mathcal{T}$ is the γ-matrix trace. They are given by

$$\mathcal{N} = \frac{1}{(2\pi)^6}\frac{1}{2^5}\left[E_i(E_i + M)E_f(E_f + M)\varepsilon_i(\varepsilon_i + m)\varepsilon_f(\varepsilon_f + m)\right]^{-1/2}$$

and

$$\begin{aligned}\mathcal{T} = 8\{ & (p_i \cdot P_i)[2\varepsilon_f E_f + 3(mE_f + M\varepsilon_f + mM)] \\ & +(p_f \cdot P_f)[2\varepsilon_i E_i + 3(mE_i + M\varepsilon_i + mM)] + (p_i \cdot P_i)(p_f \cdot P_f) \\ & -(p_i \cdot p_f)[2E_iE_f + M(E_i + E_f + M)] - (P_i \cdot P_f)[2\varepsilon_i\varepsilon_f + m(\varepsilon_i + \varepsilon_f + m)] + (p_i \cdot p_f)(P_i \cdot P_f) \\ & -(p_i \cdot P_f)[mE_i + M\varepsilon_f + mM] - (P_i \cdot p_f)[mE_f + M\varepsilon_i + mM] - (p_i \cdot P_f)(P_i \cdot p_f) \\ & +[-2mM(E_i - E_f)(\varepsilon_i - \varepsilon_f) + mM(m(E_i + E_f) + M(\varepsilon_i + \varepsilon_f) + mM) \\ & \quad +2m^2E_iE_f + 2M^2\varepsilon_i\varepsilon_f]\}. \end{aligned} \tag{17}$$

Also,

$$k^2 = (p_i - p_f)^2 = (\varepsilon_i - \varepsilon_f)^2 - (\mathbf{p} - \mathbf{p}')^2. \tag{18}$$

It is, in principle, possible to Fourier transform the product $\mathcal{M}_{\text{sing}}$ of these unwieldy functions to obtain the full relativistic result for $\Delta E_{\text{c.m.}}$; this has not yet been performed. We can also perform the expansion of the energy factors in powers of $\frac{p}{m}$, where all such momentum-over-mass quotients that occur are taken to be of the same order.

However, this is unnecessary work, for if we require only a nonrelativistic expansion, there is a much faster way, namely, expansion of the spinor bilinears via the Pauli approximants

$$\begin{aligned}\bar{u}(\mathbf{p}')\gamma u(\mathbf{p}) &= \langle\chi'|\,\frac{(\mathbf{p}+\mathbf{p}')}{2m} + i\frac{\boldsymbol{\sigma}\times(\mathbf{p}'-\mathbf{p})}{2m}\,|\chi\rangle + o\left[\left(\frac{p}{m}\right)^3\right], \\ \bar{u}(\mathbf{p}')\gamma^0 u(\mathbf{p}) &= \langle\chi'|\,1 + \frac{(\mathbf{p}+\mathbf{p}')^2}{8m^2} + i\frac{\boldsymbol{\sigma}\cdot(\mathbf{p}'\times\mathbf{p})}{4m^2}\,|\chi\rangle + o\left[\left(\frac{p}{m}\right)^4\right].\end{aligned} \tag{19}$$

Using these expansions in Eq. (12) and taking $|\chi\rangle\,,|\chi'\rangle$ in helicity basis, we quickly find

$$\mathcal{M}_{\text{sing}} \overset{\text{NR}}{\Rightarrow} \frac{Qqe^2 + g_s^2}{(2\pi)^6(\mathbf{p}-\mathbf{p}')^2}\left\{1 + \frac{(\mathbf{p}+\mathbf{p}')^2}{4mM} - \frac{(\mathbf{p}-\mathbf{p}')^2}{8}\left[\frac{1}{m^2} - \frac{4}{mM} + \frac{1}{M^2}\right] + o\left[\left(\frac{p}{m}\right)^4\right]\right\}. \tag{20}$$

The gluon diagram has the additional physical constraint that the initial and final $q\bar{q}$ pairs are combined into a color singlet; this introduces an additional factor of $-\frac{4}{3}$. Then Fourier transformation of this result produces

$$\Delta E_{\text{c.m.,sing}} = \left(\alpha Q q - \frac{4}{3}\alpha_s\right)\left\{\left\langle\frac{1}{r}\right\rangle + \frac{1}{2mM}\left\langle\frac{1}{r}[\mathbf{p}^2 + \hat{\mathbf{r}}\cdot(\hat{\mathbf{r}}\cdot\mathbf{p})\mathbf{p}]\right\rangle - \frac{\pi}{2}\left(\frac{1}{m^2} - \frac{4}{mM} + \frac{1}{M^2}\right)\langle\delta^3(\mathbf{r})\rangle\right\} + \cdots . \tag{21}$$

In comparison, the energy contribution from the Breit-Fermi interaction [Eq. (1)] for a quark-antiquark pair of masses m, M in the c.m. reduces to

$$\langle H_{\text{BF}}\rangle = \left(\alpha Q q - \frac{4}{3}\alpha_s\right)\left\{\left\langle\frac{1}{r}\right\rangle + \frac{1}{2mM}\left\langle\frac{1}{r}[\mathbf{p}^2 + \hat{\mathbf{r}}\cdot(\hat{\mathbf{r}}\cdot\mathbf{p})\mathbf{p}]\right\rangle - \frac{\pi}{2}\langle\delta^3(\mathbf{r})\rangle\left[\frac{1}{m^2} + \frac{1}{M^2} + \frac{4}{mM}(\mathcal{G} + \delta_{S,1}\delta_{q\text{flavors}})\right] - \frac{1}{2}\left\langle\frac{1}{r^3}\right\rangle\left\langle\mathbf{L}\cdot\left(\frac{\mathbf{s}_q}{m^2} + \frac{\mathbf{s}_{\bar{Q}}}{M^2} + \frac{2\mathbf{S}}{mM}\right) + \frac{S_{12}}{2mM}\right\rangle\right\}, \tag{22}$$

where $\mathcal{G} \equiv \frac{4}{3}\langle\mathbf{s}_q\cdot\mathbf{s}_{\bar{Q}}\rangle$, which is -1 $\left(\frac{1}{3}\right)$ for $S = 0$ (1). Also, $\mathbf{S} \equiv \mathbf{s}_q + \mathbf{s}_{\bar{Q}}$, and S_{12} is the $\Delta L = 2$ tensor operator

$$S_{12} \equiv 3(\boldsymbol{\sigma}_1\cdot\hat{\mathbf{r}})(\boldsymbol{\sigma}_2\cdot\hat{\mathbf{r}}) - \boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2. \tag{23}$$

For mesons with differently flavored quarks in a relative $\ell = 0$ state, many of the terms drop out. Let us define

$$B \equiv \left\langle\frac{1}{r}\right\rangle, \quad C \equiv \left\langle\frac{1}{r}[\mathbf{p}^2 + \hat{\mathbf{r}}\cdot(\hat{\mathbf{r}}\cdot\mathbf{p})\mathbf{p}]\right\rangle, \quad D \equiv \langle\delta^3(\mathbf{r})\rangle . \tag{24}$$

Then Eq. (22) becomes

$$\langle H_{\text{BF}}\rangle = \left(\alpha Q q - \frac{4}{3}\alpha_s\right)\left[B + \frac{1}{2mM}C - \frac{\pi}{2}\left(\frac{1}{m^2} + \frac{1}{M^2} + \frac{4\mathcal{G}}{mM}\right)D\right], \tag{25}$$

and this is exactly Eq. (21) where $\mathcal{G} = -1$.

We have been up to now considering only the contributions to the mass originating from the binding interaction due to one-gluon and one-photon exchanges; there are, of course, also contributions from the kinetic energy (K) of the quarks. Were we calculating these quantities in a relativistic theory, we would simply compute $K = \left\langle\sqrt{m^2 + \mathbf{p}^2}\right\rangle$. The square root may be formally expanded in nonrelativistic quantum mechanics (NRQM) as well, resulting in an alternating series in $\langle\mathbf{p}^{2n}\rangle$. However, for large enough n in NRQM, these expectation values tend to diverge. For example, in the hydrogen atom, divergence occurs for s waves at $n = 3$. Furthermore, if the system is not highly nonrelativistic, the inclusion of the $\langle\mathbf{p}^4\rangle$ may cause us to grossly underestimate the true value of the kinetic energy. The problem is that there is no positive $\langle\mathbf{p}^6\rangle$ term to balance the large negative $\langle\mathbf{p}^4\rangle$ term. For these reasons, we incorporate the alternating nature of the series in a computationally simple way by making the *Ansatz*

$$K = \sqrt{m^2 + \langle\mathbf{p}^2\rangle}. \tag{26}$$

In order to evaluate the expectation values in the above equations, we will need to choose a potential. In the meantime, let us simply denote it with $U(r)$. Then at last we have the mass formula

$$M_{\text{meson}} = \sqrt{M^2 + \langle\mathbf{p}^2\rangle} + \sqrt{m^2 + \langle\mathbf{p}^2\rangle} + \langle U(r)\rangle + \langle H_{\text{BF}}\rangle . \tag{27}$$

The static potential $U(r)$ takes the place of L, the universal quark binding function, in Eq. (1) of Ref. [5].

IV. MASS SPLITTING FORMULAS

The static potential in which the quarks interact determines the form of the NRQM wave function. The strong Coulombic term gives the largest energy contribution of terms within the Breit-Fermi interaction, and therefore would also be expected to substantially alter the wave function in perturbation theory. Therefore, we include the strong Coulombic term in the static potential:

$$V(r) \equiv U(r) - \frac{4}{3}\frac{\alpha_s}{r}. \tag{28}$$

Then the mass formula [Eq. (27)] becomes, using Eq. (25),

$$M_{\rm meson} = \sqrt{M^2 + \langle \mathbf{p}^2 \rangle} + \sqrt{m^2 + \langle \mathbf{p}^2 \rangle} + \langle V(r) \rangle + \alpha Q q B + \left(\alpha Q q - \frac{4}{3}\alpha_s\right)\left[\frac{1}{2mM}C - \frac{\pi}{2}\left(\frac{1}{m^2} + \frac{1}{M^2} + \frac{4\mathcal{G}}{mM}\right) D\right]. \quad (29)$$

Now at last we are in a position to write explicit formulas for the mass splittings of interest. Denoting the mass of a meson of spin S and valence quarks $\bar{Q},q$ as $M^S(\bar{Q}q)$, we define

$$\begin{aligned} \Delta_Q^0 &\equiv M^0(\bar{Q}u) - M^0(\bar{Q}d), \\ \Delta_Q^1 &\equiv M^1(\bar{Q}u) - M^1(\bar{Q}d), \\ \Delta_{\bar{Q}u}^* &\equiv M^1(\bar{Q}u) - M^0(\bar{Q}u), \\ \Delta_{\bar{Q}d}^* &\equiv M^1(\bar{Q}d) - M^0(\bar{Q}d), \end{aligned} \quad (30)$$

where u and d, the up and down constituent quarks, are nearly degenerate in mass: Defining $\Delta m \equiv m_u - m_d$ and $m \equiv \frac{m_u + m_d}{2}$, we have $\left|\frac{\Delta m}{m}\right| \ll 1$. Therefore, the differences in Eq. (30) are expanded in Taylor series in $\frac{\Delta m}{m}$ about m. It is also convenient to define

$$\begin{aligned} A &\equiv \langle \mathbf{p}^2 \rangle, \\ \beta &\equiv \frac{1}{1 + m/M}, \\ \mu &\equiv \text{usual reduced mass}, \\ \bar{\mu} &\equiv m\beta, \\ D_{\alpha_s} &\equiv \beta\left(\frac{\mu}{\alpha_s}\frac{\partial \alpha_s}{\partial \mu}\right)\bigg|_{\mu=\bar{\mu}}, \\ D_X &\equiv \beta\left(\mu\frac{\partial X}{\partial \mu}\right)\bigg|_{\mu=\bar{\mu}}, \quad X = A, B, C, D, \langle V(r) \rangle. \end{aligned} \quad (31)$$

Then the expressions for mass splitting are

$$\begin{aligned} \Delta_Q^{0,1} = {} & \left[\frac{2m^2 + D_A}{\sqrt{m^2 + A}} + \frac{D_A}{\sqrt{M^2 + A}}\right]\frac{\Delta m}{2m} + D_{\langle V \rangle}\frac{\Delta m}{m} \\ & -\frac{4}{3}\alpha_s \Delta m \left\{\frac{1}{2m^2 M}\left(D_C - C + C D_{\alpha_s}\right)\right. \\ & \left. -\frac{\pi}{2m^3}\left[\left(1 + 4\mathcal{G}\frac{m}{M} + \frac{m^2}{M^2}\right)\left(D_D + D D_{\alpha_s}\right) - 2\left(1 + 2\mathcal{G}\frac{m}{M}\right) D\right]\right\} \\ & + \alpha Q\left[B + \frac{1}{2mM}C - \frac{\pi}{2m^2}\left(1 + 4\mathcal{G}\frac{m}{M} + \frac{m^2}{M^2}\right) D\right] + o\left[\left(\frac{\Delta m}{m}\right)^3\right] + o\left(\alpha\frac{\Delta m}{m}\right). \end{aligned} \quad (32)$$

Note that no derivatives appear in the $\alpha_{\rm e.m.}$ terms because we take both $\alpha_{\rm e.m.}$ and $\frac{\Delta m}{m}$ (but not α_s) as expansion parameters. Furthermore, the running of $\alpha_s(\mu)$ is explicitly taken into account.

For vector-pseudoscalar splittings, we have

$$\begin{aligned} \Delta_{\bar{Q}q}^* = {} & \frac{8\pi}{3mM}\left[\left(\frac{4}{3}\alpha_s - \alpha Q q\right) D \pm \frac{4}{3}\alpha_s \frac{\Delta m}{2m}\left(D_D + D D_{\alpha_s} - D\right)\right] \\ & + o\left[\left(\frac{\Delta m}{m}\right)^2\right] + o\left(\alpha\frac{\Delta m}{m}\right), \quad \text{with } \pm \text{ for } q = u(d). \end{aligned} \quad (33)$$

Let us remind ourselves of the physical significance of the terms in the previous two equations. Terms containing A signify kinetic energy contributions, including intrinsic quark masses. The potential term is identified, of course, by V; B, C, and D denote static Coulomb, Darwin, and hyperfine terms, respectively.

V. QUANTUM-MECHANICAL THEOREMS

In order to apply the foregoing results, we will need to evaluate the expectation values A, B, C, D, and $\langle V(r) \rangle$ for our potential $V(r)$. There are two quantum-mechanical theorems which make the evaluation of these expectation values and their mass derivatives simpler [11].

Theorem 1 (Feynman-Hellmann theorem). For normalized eigenstates of a Hamiltonian depending on a parameter λ,

$$\frac{\partial E}{\partial \lambda} = \left\langle \frac{\partial H(\lambda)}{\partial \lambda} \right\rangle. \quad (34)$$

In the particular case that $\lambda = \mu$,

$$\frac{\partial E}{\partial \mu} = -\frac{1}{\mu}\left[E - \langle V(r) \rangle\right] + \left\langle \frac{\partial V}{\partial \mu} \right\rangle. \quad (35)$$

The other result may be less familiar. For reasons that will become clear, let us call it the generalized virial theorem (GVT).

Theorem 2 (generalized virial theorem). Consider bound eigenstates $u_\ell(r)$ in a spherically symmetric potential $V(r)$ such that

$$\lim_{r\to 0} r^2 V(r) = 0.$$

Then, writing the Schrödinger equation as

$$u_\ell''(r) + \frac{2\mu}{\hbar^2}\left[E - V(r) - \frac{\hbar^2 \ell(\ell+1)}{2\mu r^2}\right] u_\ell(r) = 0,$$

and defining a_ℓ by

$$\lim_{r\to 0} \frac{u_\ell(r)}{r^{l+1}} \equiv a_\ell,$$

then (i) a_ℓ is a nonzero constant; (ii) for $q \geq -2\ell$,

$$(2\ell+1)^2 a_\ell^2 \delta_{q,-2l} = -\frac{2\mu}{\hbar^2}\left\langle r^{q-1}\left(2q[E-V(r)] - r\frac{dV}{dr}\right)\right\rangle + (q-1)\left[2\ell(\ell+1) - \frac{1}{2}q(q-2)\right] \times \left\langle r^{q-3}\right\rangle . \tag{36}$$

Clearly this theorem will prove most useful for potentials easily expressed as a sum of terms which are powers in r. But in fact there are some interesting general results included. For example, the $q = \ell = 0$ case generates the well-known result for s waves,

$$|\Psi(0)|^2 = \frac{\mu}{2\pi\hbar^2}\left\langle \frac{dV}{dr}\right\rangle , \tag{37}$$

whereas the $q = 1$ case produces

$$E - \langle V(r)\rangle = \frac{1}{2}\left\langle r\frac{dV}{dr}\right\rangle , \tag{38}$$

the quantum-mechanical virial theorem.

Using partial integration, the Schrödinger equation, and the GVT, it is possible to show the following ($\hbar = 1$):

$$A = 2\mu\left[E - \langle V(r)\rangle\right],$$

$$C = 4\mu\left[E\left\langle\frac{1}{r}\right\rangle - \left\langle\frac{V(r)}{r}\right\rangle - \frac{1}{4}\left\langle\frac{dV}{dr}\right\rangle(1+\delta_{\ell,0})\right], \tag{39}$$

$$D = \frac{\mu}{2\pi}\left\langle\frac{dV}{dr}\right\rangle\delta_{\ell,0},$$

$$\int_0^\infty \left(\frac{du_\ell(r)}{dr}\right)^2 dr = A - \ell(\ell+1)\left\langle\frac{1}{r^2}\right\rangle .$$

In addition, we must also uncover what we can about the μ dependence of expectation values. For a general potential this is actually an unsolved problem. However, unless the potential has very special μ dependence, it can be shown that only in the case $V(r) = V_0 r^\nu$ is it possible to scale away the dimensionful parameters V_0 and μ in the Schrödinger equation. In that case, the μ dependence will be entirely contained in the scaling factors, and computing D_X will be trivial. Unfortunately, in the potential we consider in the next section, we will see that this is not the case, and we must resort to subterfuge to obtain the required information.

VI. EXAMPLE: $V(r) = \frac{r}{a^2} - \frac{\kappa}{r}$

The potential $V(r) = \frac{r}{a^2} - \frac{\kappa}{r}$, where $\kappa = \frac{4}{3}\alpha_s$, is interesting because it phenomenologically includes quark confinement via the linear term. This potential was considered in greatest detail by Eichten *et al.* [12] to describe the mass splitting structure of the charmonium system (and was later applied to bottomonium). The Schrödinger equation was solved numerically; currently, no analytic solution is known. However, it is possible to extract a great deal of information from their tabulated results, as we shall see below.

This is possible because of the GVT. If we rescale the Schrödinger equation with the linear-plus-Coulomb potential to

$$\left(\frac{d^2}{d\rho^2} - \frac{\ell(\ell+1)}{\rho^2} + \frac{\lambda}{\rho} + \zeta - \rho\right) w_\ell(\rho) = 0, \tag{40}$$

where

$$\rho \equiv \left(\frac{2\mu}{a^2}\right)^{1/3} r, \qquad \lambda \equiv \kappa(2\mu a)^{2/3},$$
$$\zeta \equiv (2\mu a^4)^{1/3} E, \qquad w_\ell(\rho) \equiv u_\ell(r)\left(\frac{a^2}{2\mu}\right)^{1/6}, \tag{41}$$

then the GVT gives

$$(q=0) \quad a_0^2\,\delta_{0,\ell} = \left(\frac{2\mu}{a^2}\right)\left[1 + \lambda\left\langle\frac{1}{\rho^2}\right\rangle - 2\ell(\ell+1)\left\langle\frac{1}{\rho^3}\right\rangle\right],$$
$$(q=1) \qquad 0 = 3\langle\rho\rangle - 2\zeta - \lambda\left\langle\frac{1}{\rho}\right\rangle . \tag{42}$$

Also, defining

$$\left\langle v^2\right\rangle \equiv \int_0^\infty \left(\frac{dw_\ell(\rho)}{d\rho}\right)^2 d\rho, \tag{43}$$

we find

$$\left\langle v^2\right\rangle = -\langle\rho\rangle + \zeta + \lambda\left\langle\frac{1}{\rho}\right\rangle - \ell(\ell+1)\left\langle\frac{1}{\rho^2}\right\rangle . \tag{44}$$

It is a happy accident of this potential that all of the quantities in the expectation values we need, for any ℓ, may be expressed in terms of the three quantities ζ, $\left\langle\frac{1}{\rho^2}\right\rangle$, and $\left\langle v^2\right\rangle$, which are exactly those values tabulated for the $1s$ state, as functions of λ, in Eichten *et al.* (Table I). Defining $\sigma \equiv \left(\frac{2\mu}{a^2}\right)^{1/3}$ and taking $\ell = 0$ (as per our mesonic model), we find

$$\begin{aligned} A &= \sigma^2 \langle v^2 \rangle , \\ B &= \frac{\sigma}{2\lambda} \left[3 \langle v^2 \rangle - \zeta \right] , \\ C &= \sigma^2 \left[2B\zeta + \sigma \left(-3 + \lambda \left\langle \frac{1}{\rho^2} \right\rangle \right) \right] , \\ D &= \frac{\sigma^3}{4\pi} \left[\lambda \left\langle \frac{1}{\rho^2} \right\rangle + 1 \right] . \end{aligned} \tag{45}$$

So now we can compute all of the necessary expectation values numerically. The superficial singularity in $B(\lambda = 0)$ is false; $B(0)$ is computed by extrapolation of the computed values of B for nonzero λ and is found to be finite.

The mass derivatives must be handled in a different fashion. We begin by defining

$$\begin{aligned} &\tilde{D}_\zeta \equiv \mu \frac{\partial \zeta}{\partial \mu}, \quad \tilde{D}_v \equiv \mu \frac{\partial \langle v^2 \rangle}{\partial \mu}, \quad \tilde{D}_\rho \equiv \mu \frac{\partial \left\langle \frac{1}{\rho^2} \right\rangle}{\partial \mu}, \\ &\tilde{D}_{\alpha_s} \equiv \frac{\mu}{\alpha_s} \frac{\partial \alpha_s}{\partial \mu}. \end{aligned} \tag{46}$$

From the Feynman-Hellmann theorem [Eq. (35)] we may show

$$\tilde{D}_\zeta = \left(\frac{\zeta}{3} - \langle v^2 \rangle \right) \left(1 + \frac{3}{2} \tilde{D}_{\alpha_s} \right) . \tag{47}$$

As mentioned in the previous section, scaling of the Schrödinger equation can be accomplished for μ-independent potentials that are monomials. In the case $\lambda = 0$ (a purely linear potential), the scaling would be perfect, and ζ, $\left\langle \frac{1}{\rho^2} \right\rangle$, and $\langle v^2 \rangle$ would be μ independent. In the $\lambda \neq 0$ case, the derivatives must be found numerically. Again, fortunately, we have a table of numerical values of the desired expectation values, as functions of $\lambda(\mu)$. We fit the expectation values Y ($= \left\langle \frac{1}{\rho^2} \right\rangle$, $\langle v^2 \rangle$) to the functional form

$$Y(\lambda) = Y_0 + K \lambda^{n_Y} . \tag{48}$$

Then, using Eq. (41), we find

$$\tilde{D}_Y = \left(\tfrac{2}{3} + \tilde{D}_{\alpha_s} \right) n_Y \left(Y - Y_0 \right) . \tag{49}$$

Finally, define

$$\tilde{D}_X \equiv \mu \frac{\partial X}{\partial \mu} \quad \text{for } X = A, B, C, D, \tag{50}$$

so that

$$D_X = \beta \, \tilde{D}_X \Big|_{\mu = \bar{\mu}} . \tag{51}$$

Then we find

$$\begin{aligned} \tilde{D}_A &= \tfrac{2}{3} A + \sigma^2 \tilde{D}_v , \\ \tilde{D}_B &= \frac{3\sigma}{2\lambda} \tilde{D}_v - \frac{1}{2} B \tilde{D}_{\alpha_s} \quad (\lambda \neq 0), \\ \tilde{D}_C &= \frac{5}{3} C + 2\sigma^2 \Bigg\{ \left(-\zeta + \tilde{D}_\zeta \right) B + \zeta \tilde{D}_B \\ &\quad + \sigma \left[\frac{\lambda}{2} \left(\tilde{D}_\rho + \tilde{D}_{\alpha_s} \left\langle \frac{1}{\rho^2} \right\rangle \right) + 1 \right] \Bigg\} , \\ \tilde{D}_D &= \frac{\sigma^3}{4\pi} \left\{ \lambda \left[\left(\frac{5}{3} + \tilde{D}_{\alpha_s} \right) \left\langle \frac{1}{\rho^2} \right\rangle + \tilde{D}_\rho \right] + 1 \right\} . \end{aligned} \tag{52}$$

In the exceptional case of $\tilde{D}_B$, we simply note that, for $\lambda = 0$, we have perfect scaling of the wave equation, and we can quickly show that $\tilde{D}_B \Big|_{\lambda=0} = \frac{1}{3} B \big|_{\lambda=0}$. This provides us with everything we need to produce numerical results.

Before leaving the topic, let us mention that many complications of μ derivatives of expectation values vanish if the potential itself has the appropriate μ dependence, for then scaling of the wave equation is possible. For example, one can scale the Schrödinger equation for the potential

$$V(r) = c\mu^2 r - \frac{\kappa}{r} , \tag{53}$$

where c is a pure number.

VII. NUMERICAL RESULTS

The method of obtaining results from the theory requires us to choose several numerical inputs, most of which are believed known to within a few percent. Let us choose the following inputs to the model:

$$\begin{aligned} &m = 340 \text{ MeV}, \quad M_s = 540 \text{ MeV}, \\ &M_c = 1850 \text{ MeV}, \quad M_b = 5200 \text{ MeV}, \\ &a = 1.95 \text{ GeV}^{-1}. \end{aligned} \tag{54}$$

The light-quark constituent mass is arrived at by assuming that nucleons consist of quarks with negligible anomalous magnetic moments, which can be added nonrelativistically to provide the full nucleonic magnetic moment. Likewise, the strange-quark mass issues from the same considerations applied to strange baryons [5]. The c- and b-quark masses are simply found by dividing the threshold energy value for charm and bottom mesons by 2 (however, smaller masses have been predicted using semileptonic decay results in addition to meson masses [13]). The confinement constant is inferred from charmonium levels [12].

One important input not yet mentioned is Δm, the up-down quark mass difference. Traditionally, this assumes a value of ≈ -3 to -8 MeV, in order to account for the electromagnetic mass splittings of the lighter hadrons. In this model, with the inputs listed in Eq. (54), we find that the experimental splittings for the D and B mesons (both vector and pseudoscalar) can be satisfied within one standard deviation of experimental error for values of Δm in the narrow range of -4.05 to -4.10 MeV. In contrast, it is found that for no choice of Δm can one simultaneously fit D- and K-meson data simultaneously, as was done in the earlier models.

Before exhibiting the quantitative results, let us describe the method by which they are obtained. Once particular inputs for the above variables are chosen, one can compute the various mass splittings for the values of $\lambda \propto \alpha_s$ that occur in Table I of Ref. [12], and in-between values may be interpolated. We then fit vector-pseudoscalar splittings [computed via Eq. (33)] to the corresponding experimental data (since these numbers have the smallest relative errors of the splittings we consider) and thus obtain a value of α_s. For the three systems K, D, and B, we use the three values of α_s to estimate graphically (and admittedly rather crudely) its mass derivative. Applying the values of the strong-coupling constant and its derivative to the splittings in Eq. (32), we generate all of the other values. If the resultant numbers do not fall within the experimental error bars for such splittings, we vary the input parameters (most importantly, Δm) until a simultaneous fit is achieved.

Tables I and II display the various contributions to mass splittings derived in this fashion for B and D mesons. Although the kinetic term (which includes the explicit difference Δm) and the static Coulomb term are unsurprisingly large, a significant contribution to the mass splitting arises in the strong hyperfine term. That strong contributions to the so-called electromagnetic mass splittings could be important was observed by Chan [2], and was exploited in the subsequent literature. It is exactly this term which is most significant in driving the B splittings toward zero. Note also the decrease in the derived value of α_s as the reduced mass of the system increases when we move from the D system to the B system, consistent with asymptotic freedom in QCD. It was this running which motivated the inclusion of mass derivatives of the strong-coupling constant in this model. If they are not included, one actually obtains a value of $\Delta m > 0$, in contrast with all estimates from both nonrelativistic and chiral models.

TABLE I. Contributions to mass splittings of heavy mesons: isospin pairs.

	D mesons	B mesons
α_s	0.363	0.312
Source	(MeV)	(MeV)
Kinetic energy	−4.109	−3.523
Potential energy	1.057	−1.645
Strong Darwin	−0.834	−0.635
e.m. Darwin	−0.769	0.147
Static Coulomb	−2.442	1.252
Δ_Q^0		
Strong hyperfine	2.148	4.075
e.m. hyperfine	0.424	−0.561
Total	−4.525	−0.889
Δ_Q^1		
Strong hyperfine	3.683	5.244
e.m. hyperfine	1.817	−0.825
Total	−1.596	0.017

TABLE II. Contributions to mass splittings of heavy mesons: 1^-- 0^- pairs.

	D mesons	B mesons
α_s	0.363	0.312
Source	(MeV)	(MeV)
Strong hyperfine		
(leading)	141.30	46.04
(subleading)	± 0.77	± 0.58
Δ_{Qu}^*		
e.m. hyperfine	0.93	−0.18
Total	143.00	46.45
Δ_{Qd}^*		
e.m. hyperfine	−0.46	0.09
Total	140.07	45.54

The net result is that one can satisfactorily fit the data for the D and B systems simultaneously in the most natural nonrelativistic model with a physically reasonable potential. The comparison of the results of this calculation for $\Delta m = -4.10$ MeV to experimental data is presented in Table III.

However, the table also exhibits very bad agreement for the K system (despite the fact that the fit to vector-pseudoscalar splittings yields the value $\alpha_s = 0.424$, which runs in the correct direction). One may view this as a failure of the nonrelativistic assumptions of the model in a variety of ways. Most obvious are the *Ansatz* Eq. (26), which is certainly not an airtight assumption in even the best of circumstances, and the crudeness of the estimate of $\frac{\partial \alpha_s}{\partial \mu}$. Other possible problems include the assumption that the quarks occur only in a relative $\ell = 0$ state (relevant for K^* mesons), and the assumption that the strong effects are dominated by a confining potential and one-gluon exchange, since at the lower energies associated

TABLE III. Meson mass splittings compared to experiment.

Mass splitting	Notation	Predicted (MeV)	Expt. (MeV)
$K^+ - K^0$	Δ_s^0	−0.98	−4.024 ± 0.032
$K^{*+} - K^{*0}$	Δ_s^1	−0.15	−4.51 ± 0.37 [a]
$K^{*+} - K^+$	Δ_{su}^*	398.6	397.94 ± 0.24 [a]
$K^{*0} - K^0$	Δ_{sd}^*	397.8	398.43 ± 0.28 [a]
$D^0 - D^+$	Δ_c^0	−4.53	−4.77 ± 0.27
$D^{*0} - D^{*+}$	Δ_c^1	−1.60	−2.9 ± 1.3
$D^{*0} - D^0$	Δ_{cu}^*	143.0	142.5 ± 1.3
$D^{*+} - D^+$	Δ_{cd}^*	140.1	140.6 ± 1.9 [a]
$B^+ - B^0$	Δ_b^0	−0.89	−0.1 ± 0.8
$B^{*+} - B^{*0}$	Δ_b^1	0.02	NA
$B^{*+} - B^+$	Δ_{bu}^*	46.5	46.0 ± 0.6 [b]
$B^{*0} - B^0$	Δ_{bd}^*	45.5	46.0 ± 0.6 [b]

[a] Obtained as a difference of world averages.

[b] Average of charged and neutral states.

with the K system, $o(\alpha_s^2)$ terms and more complex models of confinement may be required. The failure of these assumptions can drastically alter the strong hyperfine interaction, which determines the size of α_s, and hence the other mass splittings.

Some may find the small size of α_s somewhat puzzling. This is primarily the result of the confining term of the model potential: It causes the wave function to be large at the origin, and thus a small α_s is required to give the same experimentally measured vector-pseudoscalar splitting [see Eq. (33)]. Such small values for the strong-coupling constant might lead to excessively small values of $\Lambda_{\rm QCD}$ and large values for mesonic decay constants $f_{\bar{Q}q}$. Indeed, given the naive expressions for these quantities,

$$\alpha_s(\mu) = \frac{12\pi}{(33-2n_f)\ln\left(\frac{\mu^2}{\Lambda^2}\right)}, \tag{55}$$

and assuming the relative momenta of the quarks is small,

$$f_{\bar{Q}q}^2 = \frac{12}{M_{\bar{Q}} + m_q}\left|\Psi(0)\right|^2, \tag{56}$$

let us consider, for example, the D system. Then $\alpha_s = 0.363$ and $\mu = 287$ MeV, and with three flavors of quark, we calculate $\Lambda_{\rm QCD} = 42$ MeV and $f_B = 342$ MeV. However, one may state the following objections: First, $\Lambda_{\rm QCD}$ is computed from the full theory of QCD, but the nonrelativistic potential approach includes the confinement in an *ad hoc* fashion, by including a confinement constant a, which is independent of α_s. Furthermore, choosing $\Lambda_{\rm QCD}$ as the renormalization point forces an artificial singularity at $\mu = \Lambda_{\rm QCD}$; the problem is that little is known about the low-energy behavior of strong interactions. At low energies the computation and interpretation of $\Lambda_{\rm QCD}$ requires a more careful consideration of confinement. With respect to the decay constant, the assumption that the quarks are relatively at rest leads to the evaluation of the wave function at zero separation. Inclusion of nonzero relative momentum will presumably result in the necessity of considering separations of up to a Compton wavelength $r \approx \frac{1}{\mu}$, for which the wave function is smaller in the $1s$ state. Thus decay constants may be smaller than computed in the naive model.

There is one further qualitative success of this model, a partial explanation of the experimental facts that $D_s^* - D_s = 141.5\pm1.9$ MeV $\approx D^* - D$, and $B_s^* - B_s = 47.0\pm2.6$ MeV $\approx B^* - B$, namely, the approximate independence of vector-pseudoscalar splitting on the light-quark mass. In our model, the leading term of the splitting is, using Eqs. (33) and (45),

$$\Delta_{\bar{Q}q}^* \approx \frac{16}{9Ma^2}\alpha_s\beta\left[\lambda\left\langle\frac{1}{\rho^2}\right\rangle + 1\right]. \tag{57}$$

Inasmuch as β, $\lambda\left\langle\frac{1}{\rho^2}\right\rangle$, and α_s are slowly varying in the light-quark mass m, the full expression reflects this insensitivity, in accord with experiment. In fact, we may fit the experimental values above to obtain more running values of α_s:

$$\Delta_{\bar{c}s}^* = 141.5 \text{ MeV} \quad \text{for} \quad \alpha_s = 0.351,$$
$$\Delta_{\bar{b}s}^* = 47.0 \text{ MeV} \quad \text{for} \quad \alpha_s = 0.295, \tag{58}$$

and again these decrease as the mass scale increases. Note, however, one kink in this interpretation: The heavy strange mesons all have larger reduced masses than their unflavored counterparts, yet the corresponding values of α_s are nearly the same.

VIII. CONCLUSIONS

In this paper, we have seen how mass contributions to a bound system of particles are derived from an interaction Hamiltonian in field theory, and how this calculation is then reduced to a problem in nonrelativistic quantum mechanics. For the system of a quark and antiquark bound in a meson, the exchange of one mediating vector boson reduces to the Breit-Fermi interaction in the nonrelativistic limit. It is also important to consider contributions to the total energy from the kinetic energy and the long-range potential of the system; in fact, the higher-order momentum expectation values can be so large that it is necessary to impose an *Ansatz* [Eq. (26)] in order to estimate their combined effect. Future work may suggest better estimates.

It is found in the case of a linear-plus-Coulomb potential that the largest contributions to electromagnetic mass splittings originate in the kinetic energy, static Coulomb, and strong hyperfine terms. However, it is likely that similar results hold for other *Ansätze* and potentials. As in other models, vector-pseudoscalar mass differences are determined by strong hyperfine terms.

With typical values for quark masses, the confinement constant, and the up-down quark mass difference, we can obtain agreement for the mass splittings of the D and B mesons. The failure of the model for K mass splittings is attributed to the collapse of the nonrelativistic assumptions in that case. The model also qualitatively explains the similarity of heavy-plus-strange to heavy-plus-unflavored vector-pseudoscalar splittings, although additional work is needed to explain why these numbers are nearly equal, despite the expected inequality of α_s at the two different energy scales.

Another interesting problem is the running of α_s itself at low energies. As mentioned in Sec. VII, this running cannot be neglected if we are to obtain sensible results, and yet our approximation of this running is based on crude assumptions. The size of α_s also enters into another possible development, namely, whether terms of $o(\alpha_s^2)$ are important, particularly for the K system. More reliable estimates are required.

In addition to the explicit formulas derived in this paper, the techniques employed here may be applied to later efforts: in particular, the explicit consideration of the mass dependence of expectation values and the use of quantum-mechanical theorems to relate various expectation values for certain potentials. The methods and formulas in this work may prove to be a starting point for subsequent research.

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