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Stability of electroweak strings

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We map the parameter space that leads to stable Z vortices in the electroweak model. For $\sin^2\theta_W=0.23$, we find that the strings are unstable for a Higgs-boson mass larger than 24 GeV. Given the latest constraints on the Higgs-boson mass from the CERN e^+e^- collider LEP, this shows that, if the standard electroweak model is realized in nature, the Z vortex (in the bare model) is unstable.

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The presence of vortices in a physically realized particle physics model is an exciting prospect. The presence of such vortices in the electroweak model is even more exciting since it immediately opens up the possibility of new signatures in accelerator experiments at accessible energies. If these signatures are found, it would be the first time a coherent state would have been detected in particle physics.

In previous papers [1,2] (see also Refs. [3–5]), one of us showed the existence of vortex (and/or string) solutions in the Weinberg-Salam model [6] and also discussed the stability of the solutions. An existence proof of the stability was provided and it was clear that the stability crucially depends on the values of the parameters in the model. In this paper [7], we map the range of parameters for which the electroweak string solution is stable and, in particular, we study the physical case in which $\sin^2\theta_W=0.23$, $m_Z=92$ GeV, and $m_H>57$ GeV (where m_H is the mass of the Higgs particle).

Our approach to the problem is to consider the variation in the energy up to second order in the perturbations of the fields about the vortex solution. In principle, there are four scalar fields and four vector fields each with four components. This makes a total of twenty fields, each of which has to be perturbed. However, remarkably, we are able to reduce the problem down to only *one* field. For this perturbation mode, we construct a Schrödinger equation and numerically find the range of parameters for which there is no bound state. This gives us the parameter values for which the vortex solution is stable to small perturbations.

We will use the standard notation defined in Ref. [8]. In addition, we make the usual definitions $Z^\mu\equiv\cos\theta_W W^{\mu 3}-\sin\theta_W B^\mu$, $A^\mu\equiv\sin\theta_W W^{\mu 3}+\cos\theta_W B^\mu$, where $\tan\theta_W\equiv g'/g$. Also, $\alpha\equiv\sqrt{g^2 g'^2}$ where g and g' are the couplings of the W^a_μ and B_μ gauge fields to the Higgs field.

The energy functional for static field configurations in the Weinberg-Salam model is

$$E=\int d^2x\, dz[\tfrac{1}{4}G^a_{ij}G^a_{ij}+\tfrac{1}{4}F_{\beta ij}F_{\beta ij}+(D_j\phi)^\dagger(D_j\phi)+\lambda(\phi^\dagger\phi-\eta^2/2)^2]\ , \tag{1}$$

where $i,j,a=1,2,3$. The integral over the z coordinate has been shown explicitly since we shall show that it is sufficient to consider field configurations in the xy plane alone.

The vortex solution [1,2] that extremizes the above energy functional is

$$W^{\mu 1}=0=W^{\mu 2}=A^\mu,\quad Z^\mu=[A^\mu]_{\rm NO}=-\frac{v_{\rm NO}(r)}{r}\hat{e}_\theta\ ,$$
$$\phi=f_{\rm NO}(r)e^{im\theta}\Phi,\quad \Phi\equiv\begin{bmatrix}0\\1\end{bmatrix}\ , \tag{2}$$

where the coordinates r and θ are polar coordinates in the xy plane. The integer m is the winding number of the vortex and, here, we shall restrict ourselves to the case $m=1$. The subscript NO on the functions f and A^μ means that they are identical to the corresponding functions found by Nielsen and Olesen [9] for the usual Abelian-Higgs string. (We now drop the subscript NO on the functions f and v for ease of writing.) These functions are given by the equations of motion:

$$f''+\frac{f'}{r}-\left[1-\frac{e}{2}v\right]^2\frac{f}{r^2}-2\lambda\left[f^2-\frac{\eta^2}{2}\right]f=0\ , \tag{3}$$

$$v''-\frac{v'}{r}+e\left[1-\frac{e}{2}v\right]f^2=0\ , \tag{4}$$

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where primes denote differentiation with respect to r. The functions f and v also satisfy the boundary conditions $f(0)=0=v(0)$, $f(\infty)=\eta/\sqrt{2}$, $v(\infty)=2/\alpha$. The string solutions resulting from these equations have been studied previously by several authors in much detail. A sample of these papers may be found in the collection of Ref. [10].

We shall now study the stability of the vortex solution given in Eq. (2) by considering infinitesimal perturbations around it and finding if the variation in the energy is positive or negative. The perturbations are time independent and therefore can also set the zero components of the gauge field to be zero.

Let us write

$$\phi=\begin{bmatrix}\phi_1\\ \phi_{\rm NO}+\phi_2\end{bmatrix}, \tag{5}$$

$$Z^\mu=Z^\mu_{\rm NO}+\delta Z^\mu , \tag{6}$$

$$T^1\equiv{\rm diag}(-\cos 2\theta_W,1) , \tag{7}$$

and

$$\mathbf{d}_j\equiv(\partial_j\mathbb{1}+i\tfrac{1}{2}\alpha T^1 Z_j) , \tag{8}$$

where $\mathbb{1}$ is the 2×2 matrix. Now, since we are considering perturbations on top of the vortex solution, the fields ϕ_1, ϕ_2, δZ^μ, $W^{\mu\bar{a}}$ ($\bar{a}=1,2$), and A^μ are infinitesimal.

The perturbations can depend on the z coordinate and the z components of the vector fields can also be nonzero. From (1) the relevant z-dependent terms in the integrand are

$$\tfrac{1}{2}G^a_{i3}G^a_{i3}+\tfrac{1}{2}F_{Bi3}F_{Bi3}+(D_3\phi)^\dagger(D_3\phi) . \tag{9}$$

This contribution to the energy is strictly non-negative and is minimized (that is, made to vanish) by setting the z components of the gauge fields to zero and also considering the perturbations to be independent of the z coordinate. For this reason, we shall drop all reference to the z coordinate in the calculations below and it will be understood that the energy is actually the energy *per unit length* of the string.

Now we write (1) after discarding terms of cubic and higher order in the infinitesimal perturbations. We find

$$E=E_{\rm NO}[f,v]+\delta E_{\rm NO}[f,v;\phi_2,\delta Z]+E_1[f,v;\phi_1]$$
$$+E_c[f,v;\phi_1,W^{\bar a}]+E_W[f,v;W^{\bar a},A] , \tag{10}$$

where $\bar a=1,2$ $E_{\rm NO}$ is the energy of the Nielsen-Olesen string, and $\delta E_{\rm NO}$ is the energy variation due to the perturbations ϕ_2 and δZ^μ. The variation E_1 is due to the perturbation ϕ_1 in the upper component of the Higgs field:

$$E_1=\int d^2x\,[\,|\bar d_j\phi_1|^2+2\lambda(f^2-\eta^2/2)|\phi_1|^2] , \tag{11}$$

where $\bar d_j\equiv\partial_j-i(\alpha/2)\cos(2\theta_W)Z_j$. The contribution from the ϕ and $\mathbf{W}^{\bar a}$ interaction is

$$E_c=\cos\theta_W\int d^2x\; J^{\bar a}_j W^{\bar a}_j , \tag{12}$$

$$J^{\bar a}_j\equiv\tfrac{1}{2}i\alpha[\phi^\dagger\tau^{\bar a}\mathbf{d}_j\phi-(\mathbf{d}_j\phi)^\dagger\tau^{\bar a}\phi] , \tag{13}$$

and the energy in the $\mathbf{W}^{\bar a}$ and $\mathbf{A}$ bosons is [11]

$$E_W\equiv\int d^2x\,[\gamma\mathbf{W}^1\times\mathbf{W}^2\cdot\nabla\times\mathbf{Z}+\tfrac{1}{2}|\nabla\times\mathbf{W}^1+\gamma\mathbf{W}^2\times\mathbf{Z}|^2+\tfrac{1}{2}|\nabla\times\mathbf{W}^2+\gamma\mathbf{Z}\times\mathbf{W}^1|^2+\tfrac{1}{4}g^2f^2(\mathbf{W}^{\bar a})^2+\tfrac{1}{2}(\nabla\times\mathbf{A})^2] , \tag{14}$$

where $\gamma\equiv g\cos\theta_W$. It may be noted that the f and $\mathbf{Z}$ fields in Eqs. (11)–(14) are the unperturbed fields of the string since we are only keeping up to quadratic terms in the infinitesimal quantities. Also, note that the current $J^{\bar a}_j$ is first order in the perturbation ϕ_1 because the $\tau^{\bar a}$ matrices are off diagonal and mix the upper and lower components of the Higgs doublet. That is, $(0,1)\tau^{\bar a}(0,1)^T=0$.

The perturbations of the fields that make up the string do not couple to the other available perturbations; i.e., the perturbations in the fields f and v only occur inside the variation $\delta E_{\rm NO}$. However, we know that the Nielsen-Olesen string with unit winding number is stable to perturbations for any values of the parameters. Therefore, necessarily, $\delta E_{\rm NO}\geq 0$ and the perturbations ϕ_2 and δZ^μ cannot destabilize the vortex. Then we are justified in ignoring these perturbations and setting $\delta E_{\rm NO}=0$.

Also note that the variation in the energy vanishes to linear order in the perturbations. Therefore the vortex solution given in (2) extremizes the energy and is a solution of the Weinberg-Salam model regardless of the values of the parameters in the model.

We now consider the expansion of the remaining perturbations in Fourier modes. This gives

$$\phi_1=\chi^m(r)e^{im\theta} \tag{15}$$

for the mth mode where m is any integer. For the gauge fields we have

$$\mathbf{W}^1=\Big[\{\bar f^n_1(r)\cos(n\theta)+f^n_1\sin(n\theta)\}\hat e_r+\frac{1}{r}\{-\bar h^n_1\sin(n\theta)+h^n_1\cos(n\theta)\}\hat e_\theta\Big] , \tag{16}$$

$$\mathbf{W}^2=\Big[\{-\bar f^n_2(r)\sin(n\theta)+f^n_2\cos(n\theta)\}\hat e_r+\frac{1}{r}\{\bar h^n_2\cos(n\theta)+h^n_2\sin(n\theta)\}\hat e_\theta\Big] , \tag{17}$$

for the nth mode where n is a non-negative integer. The functionals E_1, E_c, and E_W may now be expressed in terms of the modes χ^μ, f^n_i, and h^n_i. Our goal is to focus on $\delta E=E_1+E_c+E_W$ and obtain the parameter space ($\beta\equiv 8\lambda/\alpha^2,\theta_W$) for which there are no modes with $\delta E<0$. Here we will only sketch the basic steps of the calculation. The full calculation will be presented elsewhere [12].

It may be shown, by examining the form of δE, that we only need to look at the $m=0$ and $n=1$ modes to consider the stability of the electroweak string. This is because these are the modes with the most negative contribution to the energy variation at the center of the string. Inserting (15)–(17) in δE, we find that the stability problem in the barred functions completely separates from the stability problem in the unbarred variables. In addition, it may be shown that if the string is stable to perturbations in the unbarred variables it will also be stable to perturbations in the barred variables. Therefore, we are left with five perturbations: χ^0, f_1^1, f_2^1, h_1^1, and h_2^1. [In what follows we will drop the mode (upper) indices for simplicity.]

We now define

$$F_{\pm}=\frac{f_2 \pm f_1}{2}\ , \tag{18}$$

$$\xi_{\pm}=\frac{h_2 \pm h_1}{2}\ , \tag{19}$$

$$\zeta=(1-\gamma v)\chi+\tfrac{1}{2}gf\xi_+\ . \tag{20}$$

After a lot of algebra, we find that the energy variation is

$$\delta E=2\pi\int dr\, r\left[\left\{\frac{\zeta'^2}{P_+}+U(r)\zeta^2\right\} + \text{sum of whole squares}\right]\ , \tag{21}$$

where primes denote differentiation with respect to r,

$$P_+=(1-\gamma v)^2+\tfrac{1}{2}g^2r^2f^2\ , \tag{22}$$

$$U(r)=\frac{f'^2}{P_+f^2}+\frac{2S_+}{g^2r^2f^2}+\frac{1}{r}\frac{d}{dr}\left[\frac{rf'}{P_+f}\right]\ , \tag{23}$$

and

$$S_+=\frac{g^2f^2}{2}-\frac{\gamma^2v'^2}{P_+}+\gamma r\frac{d}{dr}\left[\frac{v'}{r}\frac{1-\gamma v}{P_+}\right]\ . \tag{24}$$

The sum of whole squares in (21) can be made to vanish by suitably choosing $F_\pm$ and by setting $\xi_-=0$. This simply leaves us with a problem in ζ.

The functional δE is in a form ready to be treated as an eigenvalue problem. That is, upon performing an integration by parts, we can write

$$\delta E|\zeta|=2\pi\int dr\, r\zeta\hat{O}\zeta\ , \tag{25}$$

where $\hat{O}$ is the differential operator:

$$\hat{O}=-\frac{1}{4}\frac{d}{dr}\left[\frac{r}{P_+}\frac{d}{dr}\right]+U(r)\ . \tag{26}$$

The question of stability now reduces to asking if the operator $\hat{O}$ has negative eigenvalues in its spectrum. Therefore we have to determine if the eigenvalue ω of the Schrödinger equation,

$$\hat{O}\zeta=\omega\zeta\ , \tag{27}$$

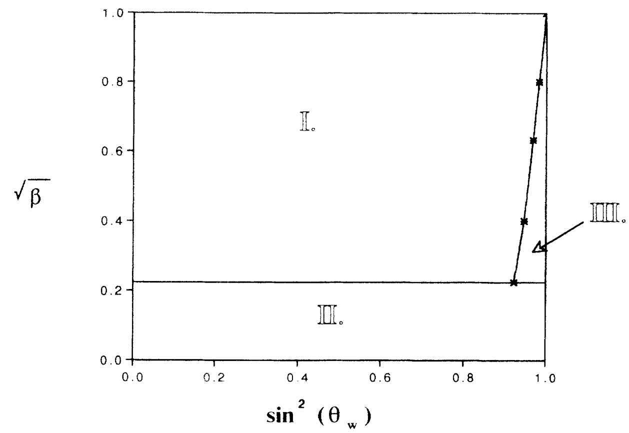

FIG. 1. A map of parameter space showing the region of stability (III) of the vortex solution. The solutions in sector I are unstable and we have not explored the solutions in sector II ($\sqrt{\beta}<0.26$).

can be negative. The eigenfunction ζ must also satisfy the boundary conditions $\zeta(r=0)=1$ and $\zeta\to c$ (c is some constant) as $r\to\infty$.

In this way we have reduced the stability problem to a single eigenvalue problem given by the differential equation in (27) and the corresponding boundary condition. This problem can be solved numerically. But before putting the problem on the computer, we rescale the variables and the coordinates so that the problem only has two free parameters: β and θ_W. These rescalings are standard in the literature and may be found in Ref. [2].

The eigenvalue problem in Eq. (27) was solved by using a fifth-order Runge-Kutta algorithm. We kept β fixed and found θ_W for which the lowest eigenvalue changes sign. We repeated this procedure for several value of β and found the corresponding values of critical parameters ($\sqrt{\beta}$, $\sin^2\theta_W$). The above method was used to scan the range $0.07\le\beta\le 1.0$. Lower values of β make the numerical analysis fairly intensive since then there are two widely different scales in the problem corresponding to the two widely different masses. Our results are shown in Fig. 1 where we plot the critical values of $\sqrt{\beta}$ (the ratio of the Higgs-boson mass to the Z mass) versus the corresponding values of $\sin^2\theta_W$. In sector III, on the right-hand side of the data line, Eq. (27) had no negative eigenvalues implying string stability. Thus we may distinguish three sectors in Fig. 1: sector I where the electroweak strings are unstable, sector III where strings are stable, and, the presently unexplored region shown as sector II ($\beta<0.07$ or $m_H<24$ GeV). It is evident that the physically realized values $\sin^2\theta_W=0.23$ and $\sqrt{\beta}=m_H/m_Z>0.62$ (see Ref. [13]) lie entirely inside sector I. This brings us to the main result of this paper: if the standard electroweak model is the physically realized model, then the existing vortex solutions in the bare model are unstable.

Before closing, we would like to point out that even if the vortex solutions are unstable, their presence may still be felt in various scattering experiments since closed loops and finite string segments could show up as intermediate states. However, it is more exciting to consider the possibility that nature may have chosen an extension of the standard electroweak model in which stable vortices are present.

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[7] In an earlier version of this paper, we found several errors that we have now corrected. The resulting map of parameter space, however, does not change significantly.

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