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Four-family lepton mixing

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We introduce a fourth family of heavy leptons and discuss the lepton mixing matrix. In the minimal case of three massless neutrino types, the lepton mixing matrix is parametrized by three real angles and no phases, and leads to electron, μ, and τ number nonconservation which provides experimental limits on the three angles.

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Since we do not understand the origin of family replication, it is possible that there is a fourth family waiting to be discovered at a higher energy. Indeed there is a hint from experimental data that the τ lifetime may be longer than is consistent with the minimal three-family standard model [1],[1] although the neutral-current data are in good agreement with τ universality [3]. If this effect is interpreted in terms of a heavy fourth neutrino [4] one obtains [5] $m_{\nu_4} \geq 45.3$ GeV, $\sin^2\theta_{34} \approx 0.06$. Of course it is quite possible that the τ problem will go away, as new measurements of the τ lifetime, mass and leptonic branching ratio become available. Nevertheless there is some theoretical motivation for contemplating a fourth family, for example, the idea of fourth-family condensates as the origin of dynamical electroweak symmetry breaking.

In this Rapid Communication we discuss the problem of four-family lepton mixing. For simplicity we shall assume to begin with that the first three neutrino types are massless while the fourth neutrino is heavy $m_{\nu_4} > M_Z/2$ as required by results from the CERN e^+e^- collider LEP [6]. Theoretically such a model may result from adding a single right-handed neutrino field ν_R with zero Majorana mass to the theory [7]. We show that in this minimal four-family model (MFFM) the lepton mixing matrix may be parametrized by three real angles and no phases, and leads to electron, μ, and τ number nonconservation. Although L_e, L_μ, L_τ are not separately conserved in the MFFM, total lepton number L is conserved, and enforces the masslessness of the first three neutrino species.

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[1]The results quoted for the τ lifetime from the four LEP experiments are [2]

$$\tau_\tau(\text{OPAL}) = [30.8 \pm 1.3(\text{comb})] \times 10^{-14}\ \text{s}\ ;$$

$$\tau_\tau(\text{DELPHI}) = [31.4 \pm 2.5(\text{comb})] \times 10^{-14}\ \text{s}\ ;$$

$$\tau_\tau(\text{ALEPH}) = [29.1 \pm 1.3(\text{stat}) \pm 0.6(\text{syst})] 10^{-14}\ \text{s}\ ;$$

$$\tau_\tau(L3) = [30.9 \pm 2.3(\text{stat}) \pm 3.0(\text{syst})] 10^{-14}\ \text{s}\ .$$

The standard model (SM) expectation for the τ lifetime is [1]

$$\tau_\tau(\text{SM}) = (28.3 \pm 0.7) \times 10^{-14}\ \text{s}\ .$$

Note that the OPAL experiment has the smallest quoted error.

In the MFFM [7] the charged weak currents in the mass eigenstate basis are of the form

$$J_\mu^{CC} = (\bar{e}, \bar{\mu}, \bar{\tau}, \bar{\sigma})_L \gamma_\mu U_0 \begin{bmatrix} \tilde{\nu}_e \\ \tilde{\nu}_\mu \\ \tilde{\nu}_\tau \\ \nu_\sigma \end{bmatrix}_L \tag{1}$$

where e, μ, τ, σ are the charged lepton mass eigenstates, and $\tilde{\nu}_e, \tilde{\nu}_\mu, \tilde{\nu}_\tau$ are massless orthonormal neutrino eigenstates, while $\nu_{\sigma L}$ is identified as the fourth-family neutrino with a Dirac mass given by

$$\mathcal{L}_{\text{mass}}^{\nu} = -m_{\nu_4} \bar{\nu}_R \nu_{\sigma L} + \text{H.c.} \tag{2}$$

Initially one might expect the unitary matrix U_0 in Eq. (1) to be parametrized by six real angles plus three irremovable phases, as in the quark sector [8]. However there is a threefold degeneracy in the massless neutrino subspace so that we can rotate the massless basis without changing the physics:

$$\begin{bmatrix} \tilde{\nu}_e \\ \tilde{\nu}_\mu \\ \tilde{\nu}_\tau \end{bmatrix}_L \to V \begin{bmatrix} \tilde{\nu}_e \\ \tilde{\nu}_\mu \\ \tilde{\nu}_\tau \end{bmatrix}_L \tag{3}$$

where V is any three-dimensional unitary matrix. By performing such rotations it is always possible to write U_0 in the form

$$U_0 = \begin{bmatrix} U_{11} & 0 & 0 & U_{14} \\ U_{21} & U_{22} & 0 & U_{24} \\ U_{31} & U_{32} & U_{33} & U_{34} \\ U_{41} & U_{42} & U_{43} & U_{44} \end{bmatrix} . \tag{4}$$

Given such a choice of basis, it is possible to write the mixing matrix in Eq. (4) in terms of three real angles and no phases. We are free to perform phase transformations on each of the eight mass eigenstate fields $e_L, \mu_L, \tau_L, \sigma_L, \tilde{\nu}_{eL}, \tilde{\nu}_{\mu L}, \tilde{\nu}_{\tau L}, \nu_{\sigma L}$, but only seven such transformations are useful in removing phases from U_0 since the overall $\text{U}(1)_L$ transformation leaves U_0 invariant. By performing such phase transformations we can remove the phase from the elements

$U_{11}, U_{14}, U_{21}, U_{22}, U_{31}, U_{33}, U_{44}$. It is then easy to see that unitarity requires the remaining elements $U_{24}, U_{32}, U_{34}, U_{41}, U_{42}, U_{43}$ to be phaseless. We conclude that the unitary matrix U_0 in Eq. (4) is purely real.

Given that the mixing matrix U_0 is real, we can define some weak eigenstates in the usual way,

$$\begin{bmatrix} \nu'_e \\ \nu'_\mu \\ \nu'_\tau \\ \nu'_\sigma \end{bmatrix}_L = U_0 \begin{bmatrix} \tilde{\nu}_e \\ \tilde{\nu}_\mu \\ \tilde{\nu}_\tau \\ \nu_\sigma \end{bmatrix}_L \tag{5}$$

then write the weak eigenstates in terms of some massless states $\nu_{eL}, \nu_{\mu L}, \nu_{\tau L}, \nu_{0L}$ as

$$\begin{aligned} \nu'_{eL} &= c_{14}\nu_{eL} + s_{14}\nu_{\sigma L} \ , \quad \nu'_{\mu L} = c_{24}\nu_{\mu L} + s_{24}\nu_{\sigma L} \ , \\ \nu'_{\tau L} &= c_{34}\nu_{\tau L} + s_{34}\nu_{\sigma L} \ , \quad \nu'_{\sigma L} = c_{44}\nu_{\sigma L} - s_{44}\nu_{0L} \ , \end{aligned} \tag{6}$$

where $c_{ij} = \cos\theta_{ij}$, $s_{ij} = \sin\theta_{ij}$. The introduction of three different massless neutrino bases may require some words of explanation. The tilde basis in Eq. (1) is some orthonormal massless neutrino basis, defined up to the transformations in Eq. (3). The primed basis in Eq. (5) is obviously the weak eigenstate basis, as is clear by substituting Eq. (5) into Eq. (1). The third basis $(\nu_e, \nu_\mu, \nu_\tau)$ defined by Eq. (6) consists of linear combinations of the tilde neutrino states, as may be seen explicitly from Eqs. (4) and (5). It is precisely these linear combinations that will be produced in weak decays, which is why the third neutrino basis is useful.

The four angles introduced in Eq. (6) are not all independent, since we can use unitarity to express θ_{44} in terms of $\theta_{14}, \theta_{24}, \theta_{34}$ as

$$s_{44}^2 = s_{14}^2 + s_{24}^2 + s_{34}^2 \ . \tag{7}$$

The massless neutrinos $\nu_{eL}, \nu_{\mu L}, \nu_{\tau L}, \nu_{0L}$ are the neutrino states that are produced in weak decays associated with e, μ, τ, σ, respectively. These four massless neutrinos are not mutually orthogonal, although they are all correctly normalized. We find, from Eq. (6),

$$\begin{aligned} &\langle \nu_{eL} | \nu_{\mu L} \rangle = -t_{14}t_{24} \ , \quad \langle \nu_{eL} | \nu_{\tau L} \rangle = -t_{14}t_{34} \ , \\ &\langle \nu_{\mu L} | \nu_{\tau L} \rangle = -t_{24}t_{34} \ , \quad \langle \nu_{eL} | \nu_{0L} \rangle = t_{14}/t_{44} \ , \\ &\langle \nu_{\mu L} | \nu_{0L} \rangle = t_{24}/t_{44} \ , \quad \langle \nu_{\tau L} | \nu_{0L} \rangle = t_{34}/t_{44} \ , \end{aligned} \tag{8}$$

where $t_{ij} = \tan\theta_{ij}$.

The three angles $\theta_{14}, \theta_{24}, \theta_{34}$, are sufficient to completely parametrize lepton mixing in the minimal model in which $\nu_{eL}, \nu_{\mu L}, \nu_{\tau L}$ are massless. In the limit that $\theta_{14}, \theta_{24}, \theta_{34} \to 0$, Eq. (6) reduces to the minimal standard model in which the individual lepton numbers L_e, L_μ, L_τ are separately conserved. For example, the absence of $\mu \to e\gamma$ puts a tight constraint on the product $s_{14}s_{24}$ [7]. The relevant diagram involves a virtual $\nu_{\sigma L}$ and assuming $m_{\nu_4} = M_W$ the branching ratio is given by [9]

$$B(\mu \to e\gamma) \approx \frac{3\alpha}{32\pi} s_{14}^2 s_{24}^2 \ . \tag{9}$$

The experimental limit $B(\mu \to e\gamma) < 5 \times 10^{-11}$ [1] then implies

$$s_{14}s_{24} < 5 \times 10^{-4} \ . \tag{10}$$

The constraint from $\mu N \to eN$ conversion is expected to give a more restrictive constraint $s_{14}s_{24} < 10^{-4}$ [5]. Interestingly s_{14} and s_{24} taken separately are not so tightly constrained. Neutrino oscillation experiments which average over the oscillations give typical limits of [1]

$$\text{Prob}(\bar{\nu}_e \not\to \bar{\nu}_e) < 0.07 \ , \tag{11a}$$

$$\text{Prob}(\nu_\mu \to \nu_e) < 0.0017 \ , \tag{11b}$$

$$\text{Prob}(\nu_e \to \nu_\tau) < 0.06 \ . \tag{11c}$$

In the MFFM a ν_e propagating in vacuo will remain a ν_e, and not oscillate into a different species of neutrino, so limit (11a) is irrelevant. However, from Eq. (8) a ν_μ has a probability amplitude $-t_{14}t_{24}$ to interact with an e^-, so we deduce from (11b)

$$t_{14}t_{24} < 0.04 \tag{12}$$

which is a much weaker limit than Eq. (10). Similarly Eq. (11c) implies

$$t_{14}t_{34} < 0.25 \ . \tag{13}$$

As already noted, the τ-lifetime experiments favor $s_{34}^2 \approx 0.06$ [5] or $s_{34} \approx 0.25, \theta_{34} \approx 14°$, so that Eq. (13) gives no limit on θ_{14}. However assuming typical limits of Cabibbo-type universality of up to about 1% in the muon constant we find [9]

$$s_{14} < 0.1 \ , \quad s_{24} < 0.1 \ . \tag{14}$$

Although Eq. (6) is perfectly adequate to describe neutrino mixing in the MFFM, we may wish to express this mixing in the orthonormal basis of Eq. (4). Expanding to quadratic order in θ_{14} and θ_{24}, but allowing for large θ_{34} we find

$$U_0 \approx \begin{bmatrix} 1-\frac{1}{2}\theta_{14}^2 & 0 & 0 & \theta_{14} \\ -\theta_{14}\theta_{24} & 1-\frac{1}{2}\theta_{24}^2 & 0 & \theta_{24} \\ -\theta_{14}s_{34} & -\theta_{24}s_{34} & c_{34}[1-\frac{1}{2}t_{34}^2(\theta_{14}^2+\theta_{24}^2)] & s_{34} \\ -\theta_{14}c_{34} & -\theta_{24}c_{34} & -s_{34}[1+\frac{1}{2}(\theta_{14}^2+\theta_{24}^2)] & c_{34}\left[1-\dfrac{1}{2}\dfrac{\theta_{14}^2+\theta_{24}^2}{c_{34}^2}\right] \end{bmatrix} . \tag{15}$$

By comparing Eqs. (6) and (15) it is clear that ν_{eL} is aligned along the $\tilde{\nu}_{eL}$ axis, while $\nu_{\mu L}$ is in the $\tilde{\nu}_{eL}\tilde{\nu}_{\mu L}$ plane, at a small angle $\beta \approx \theta_{14}\theta_{24}$ to the $\tilde{\nu}_{\mu L}$ axis. The state $\nu_{\tau L}$ may be described by polar angles (θ,ϕ) where $\theta \approx (\theta_{14}^2+\theta_{24}^2)^{1/2}\theta_{34}$ is the small angle that $\nu_{\tau L}$ makes to the $\tilde{\nu}_{\tau L}$ axis. Similarly the state ν_{0L} may be described by polar angles (θ',ϕ') where $\theta' \approx (\theta_{14}^2+\theta_{24}^2)^{1/2}/\theta_{34}$ so that ν_{0L} is also quite accurately aligned along the $\tilde{\nu}_{\tau L}$ axis. The azimuthal angles of $\nu_{\tau L}$ and ν_{0L} are $\phi \approx \pi + \arctan(\theta_{24}/\theta_{14})$ and $\phi' \approx \arctan(\theta_{24}/\theta_{14})$. In the limit that $\theta_{14},\theta_{24} \to 0$ we see that $\nu_{\mu L} \to \tilde{\nu}_{\mu L}, \nu_{\tau L} \to \tilde{\nu}_{\tau L}, \nu_{0L} \to \tilde{\nu}_{\tau L}$. This implies that in this limit $\nu_{eL},\nu_{\mu L},\nu_{\tau L}$ becomes a massless orthonormal basis as in the minimal standard model, regardless of θ_{34}. L_τ is only conserved in the limit $\theta_{34} \to 0$, since in the $\theta_{14},\theta_{24} \to 0$ limit, Eq. (15) describes two-state mixing between the third and fourth family.

The neutral weak current has the form [7]

$$J_\mu^{\mathrm{NC}} = (\bar{\tilde{\nu}}_{eL},\bar{\tilde{\nu}}_{\mu L},\bar{\tilde{\nu}}_{\tau L})\gamma^\mu \begin{pmatrix} \tilde{\nu}_{eL} \\ \tilde{\nu}_{\mu L} \\ \tilde{\nu}_{\tau L} \end{pmatrix} + \bar{\nu}_{\sigma L}\gamma^\mu \nu_{\sigma L} \ . \tag{16}$$

As pointed out previously [7] there is a Glashow-Iliopoulos-Maiani (GIM) mechanism, and the Z^0 will decay into precisely three massless neutrino channels $\bar{\tilde{\nu}}_{eL}\tilde{\nu}_{eL}, \bar{\tilde{\nu}}_{\mu L}\tilde{\nu}_{\mu L}, \bar{\tilde{\nu}}_{\tau L}\tilde{\nu}_{\tau L}$, assuming that $m_{\nu_4} > M_Z/2$. In the nonorthonormal basis $(\nu_{eL},\nu_{\mu L},\nu_{\tau L})$ there will be off-diagonal Z^0 couplings since the matrix M which describes the change of basis,

$$\begin{pmatrix} \nu_{eL} \\ \nu_{\mu L} \\ \nu_{\tau L} \end{pmatrix} = M \begin{pmatrix} \tilde{\nu}_{eL} \\ \tilde{\nu}_{\mu L} \\ \tilde{\nu}_{\tau L} \end{pmatrix} , \tag{17}$$

is not unitary, $M^\dagger M \neq 1$. However, such off-diagonal couplings vanish in the limit $\theta_{14},\theta_{24} \to 0$ that $(\nu_{eL},\nu_{\mu L},\nu_{\tau L})$ become an orthonormal basis. The small off-diagonal couplings are not particularly significant since in any case L_e, L_μ, L_τ are violated according to Eq. (8), regardless of Z^0 interactions.

Although $\nu_{eL},\nu_{\mu L},\nu_{\tau L}$ are massless, since they are not orthonormal they may undergo oscillations in matter as first discussed by Wolfenstein [10]. For example, a beam of $\nu_{\mu L}$ neutrinos contain a small admixture $-\theta_{14}\theta_{24}$ of $\tilde{\nu}_{eL} = \nu_{eL}$, according to Eq. (15). The $\tilde{\nu}_{eL}$ component effectively feels a small "mass" when traversing matter due to its scattering off electrons, whereas the orthogonal $\tilde{\nu}_{\mu L}$ component does not couple to electrons according to Eq. (15), and so remains effectively "massless" (of course, both $\tilde{\nu}_{eL}$ and $\tilde{\nu}_{\mu L}$ gain an equal effective mass due to their equal couplings to the neutral current, which may therefore be ignored). Thus a beam of $\nu_{\mu L}$ neutrinos may undergo small oscillations in matter [10]. However, a beam of ν_{eL} neutrinos does not undergo any oscillations in matter, since $\nu_{eL} = \tilde{\nu}_{eL}$ is effectively a "massive" eigenstate, and contains no admixture of the "massless" component $\tilde{\nu}_{\mu L}$, according to Eq. (15). This means that the MFFM cannot provide a Mikheyev-Smirnov-Wolfenstein- (MSW)-type solution to the solar neutrino problem [10,11]. In order to achieve this the beam of $\nu_{eL} = \tilde{\nu}_{eL}$ neutrinos produced at the core of the Sun would have to transform into either $\tilde{\nu}_{\mu L}$ or $\tilde{\nu}_{\tau L}$. This would require a violation of the GIM mechanism in Eq. (16). Such violations do occur at the one-loop level due for example to the electromagnetic penguin diagram in Fig. 1. The forward scattering amplitude due to processes such as Fig. 1 is finite and proportional to αG_F times small mixing angles. However, it is suppressed by $\alpha \approx 1/137$ times small mixing angles relative to the usual Z^0-induced forward scattering amplitudes. We conclude that the MFFM cannot provide a resolution of the solar neutrino problem.

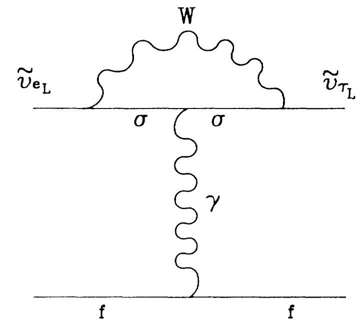

FIG. 1. An electromagnetic penguin diagram giving GIM violation at the one-loop level.

The present generation of solar neutrino experiments such as GALLEX have not confirmed the solar neutrino problem [12]. On the other hand, the GALLEX results serve to constrain the MSW solution to the solar neutrino problem around two very confined regions [12]. The MSW solution to the solar neutrino problem requires neutrino mass differences $\Delta m^2 = (6-8)\times 10^{-6}$ eV^2, while LEP requires $m_{\nu_4} > M_Z/2$ [6,12]. Such a bizarre neutrino spectrum is not possible to arrange by conventional means [13]. However there is a simple modification of the MFFM which can incorporate the MSW effect. Some time ago a model was suggested with a single right-handed neutrino plus four families [14]. Unlike the MFFM, lepton number L is not conserved and so Majorana neutrino masses are allowed [14]. In particular it was shown that if the right-handed neutrino ν_R was given a Majorana mass M_R then three light physical neutrino masses may arise as a result of a two-W-exchange mechanism [14]. MSW-type neutrino mass differences may result [14].

In conclusion, we have shown that in the MFFM [7] the lepton mixing matrix in Eq. (15) [or equivalently Eq. (6)] may be parametrized by three real angles and no phases. In the MFFM lepton number L is conserved even though the separate lepton numbers L_e, L_μ, L_τ are violated. L conservation forbids all Majorana neutrino masses and enforces the masslessness of the first three neutrino species. If L is not imposed then neutrino mass splittings in the MSW range may result [14]. However, it is no longer clear than an MSW solution to the solar neutrino problem is required [12]. The first three neutrino species may in fact be precisely massless as in the MFFM [7]. In this case we have shown that four-family lepton mixing turns out to be very simple.

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