PHYSICAL REVIEW D VOLUME 46, NUMBER 9 1 NOVEMBER 1992

Remark on the effect of renormalization scheme dependence on the determination of $\Lambda_{\rm QCD}$ from τ-lepton decay

Piotr A. Rączka
Warsaw University, Department of Physics, Institute of Theoretical Physics, ul. Hoża 69, 00-681 Warsaw, Poland
(Received 17 August 1992)

The perturbative QCD corrections to the semileptonic decay width of the τ lepton are analyzed in various renormalization schemes. Significant differences are found between *a priori* admissible schemes, which indicates that it is impossible to use these corrections to obtain a precise determination of the QCD scale parameter Λ.

PACS number(s): 13.35.+s, 12.38.Aw, 12.38.Qk

Recently there has been some interest in the quantum chromodynamic effects in τ-lepton decay [1–17], which are represented by the quantity called R_τ:

$$\mathcal{R}_\tau=\frac{\Gamma(\tau^-\to\nu_\tau+\text{hadrons})}{\Gamma(\tau^-\to\nu_\tau e^-\bar{\nu}_e)}=3(1+R_\tau)[1+O(\alpha_{\rm em})]\ . \tag{1}$$

The QCD prediction for R_τ is dominated by the perturbative contribution, which is now known up to next-next-to-leading (NNL) order [1–4,8,10,11,13]. One of the reasons for the interest in R_τ is that it appears to be quite sensitive to the QCD scale parameter Λ, allowing in principle for a surprisingly accurate determination of this parameter from the available data on τ-lepton decay [4,7,9,13,14]. However, the running coupling constant at the energy scale of m_τ is relatively large, so that the perturbation series for R_τ is less reliable than in most of the high-energy QCD predictions. Therefore in the case of R_τ one may expect a relatively stronger renormalization scheme (RS) dependence, which should be properly taken into account in the fits to the experimental data. In this article the problem of the theoretical uncertainty of the predictions for R_τ due to RS dependence is considered in some detail. A general picture of the RS dependence of R_τ is obtained as the parameters determining the scheme are varied over a reasonable range. It is shown that there are significant differences between various *a priori* admissible schemes. This implies that it is impossible to use R_τ to obtain a precise determination of the QCD scale parameter Λ, despite the fact that in many schemes the prediction for R_τ is very sensitive to this parameter.

Let us begin by recalling some facts about the RS dependence of the perturbative expressions. Neglecting the effects of nonzero masses of the three "active" quarks we may write the renormalization group improved QCD perturbation expansion for R_τ in the form

$$R_\tau=a(km_\tau)[1+r_1(k)a(km_\tau)+r_2(k)a^2(km_\tau)+\cdots]\ , \tag{2}$$

$$r_1(k)=r_1^{(0)}+\beta_0\ln k\ , \tag{3}$$

$$r_2(k)=r_2^{(0)}+(c_1+2r_1^{(0)})\beta_0\ln k+(\beta_0\ln k)^2\ , \tag{4}$$

where $k=\mu/m_\tau$ is a constant scale parameter, μ is the renormalization point, and $a(\mu)=g^2(\mu)/4\pi^2$ is the running coupling constant. The numerical value of $a(\mu)$ is determined by the implicit equation

$$\beta_0\ln\frac{\mu}{\Lambda}=\Phi^{(1)}(a)+\int_0^a dz\left[\frac{1}{z^2(1+c_1z)}-\frac{1}{z^2(1+c_1z+c_2z^2+\cdots)}\right], \tag{5}$$

$$\Phi^{(1)}(a)=c_1\ln\left[\frac{\beta_0}{2c_1}\right]+\frac{1}{a}+c_1\ln\left[\frac{c_1a}{1+c_1a}\right], \tag{6}$$

which is obtained by integrating the renormalization group equation

$$\mu\frac{da}{d\mu}=-\beta_0a^2(1+c_1a+c_2a^2+\cdots)\ , \tag{7}$$

with an appropriate boundary condition. The results of perturbative calculations are usually expressed in the modified minimal subtraction ($\overline{\rm MS}$) scheme [18]. In this scheme we have for R_τ [13]

$$r_1^{(0)\overline{\rm MS}}=5.2023,\quad r_2^{(0)\overline{\rm MS}}=26.366\ ,$$

and $\beta_0=\frac{9}{2}$, $c_1=\frac{16}{9}$, $c_2^{\overline{\rm MS}}=3863/864\cong4.47$ [19]. Other schemes are related to the $\overline{\rm MS}$ scheme by a finite renormalization, which in our approximation amounts to the redefinition of the coupling constant:

$$a_{\overline{\rm MS}}(\mu)=a(\mu)[1+A_1a(\mu)+A_2a^2(\mu)+\cdots]\ , \tag{8}$$

where the constants A_i are specific to the considered scheme. Using this relation we find the following formulas for the expansion coefficients r_i and c_i in an arbitrary scheme:

$$r_1=r_1^{\overline{\rm MS}}+A_1\ , \tag{9}$$

$$r_2=r_2^{\overline{\rm MS}}+2A_1r_1^{\overline{\rm MS}}+A_2\ , \tag{10}$$

$$c_2=c_2^{\overline{\rm MS}}+A_1c_1+A_1^2-A_2\ . \tag{11}$$

Also the scale parameter Λ depends on the choice of the scheme [20]:

 ©1992 The American Physical Society

$$\Lambda=\Lambda_{\overline{\rm MS}}\exp(-A_1/\beta_0)\ . \tag{12}$$

It should be emphasized that this relation is exact to all orders of the perturbation expansion. The coefficients A_i are simply related to the finite parts of the renormalization constants, and in principle they may be arbitrary. Consequently, the coefficients c_i (for $i\geq 2$) and r_i may vary over a broad range of values. For example, one may adjust the constants A_i so that $r_i=0$ at each order. This defines the so-called fastest apparent convergence (FAC) scheme [21,22]. There exist, however, RS independent combinations of the expansion coefficients [22–25]. At NNL order we have

$$\rho_2=c_2+r_2-c_1r_1-r_1^2\ . \tag{13}$$

In the case of R_τ $\rho_2\cong -5.475$. The numerical value of the predictions obtained with the truncated perturbative expression does depend on the choice of the RS. In the Nth order of the perturbation expansion the differences between the predictions in various renormalization schemes are formally always of the order $N+1$, but numerically they may become significant if the expansion coefficients are large or the coupling constant is not very small. A proper estimate of the uncertainty in the predictions due to the RS dependence is crucial for a meaningful comparison of the theory with the experimental data. The effect of the RS dependence of R_τ was discussed to a certain extent in [6,7] (these, however, used an incorrect value for r_2) and in [9,14,15]. The analysis performed in these papers concentrates on various "optimal" schemes, which are distinguished by some additional requirements such as the principle of minimal sensitivity (PMS) [22], the condition of fastest apparent convergence, or others [14]. It should be stressed, however, that all the optimization conditions are in fact heuristic rules, which pick up one scheme among a continuum of possibilities. Therefore, restricting our attention only to the "optimized" predictions we do not obtain a proper picture of the RS dependence. Instead, for a full estimate of the RS dependence ambiguity one should compare predictions in all schemes which *a priori* seem to be admissible, without the requirement that they be "optimal" in any sense. Such an approach is adopted in the following.

It is instructive to discuss first the RS dependence of the predictions for R_τ in the next-to-leading (NL) order, even though in the fits to the experimental data the NNL-order expression is usually used. The NL-order expression for R_τ in a general scheme may be written in the form

$$R_\tau^{(1)}=a(1+r_1a)\ , \tag{14}$$

$$\beta_0\ln\left[\frac{m_\tau}{\Lambda_{\overline{\rm MS}}}\right]=r_1^{(0)\overline{\rm MS}}-r_1+\Phi^{(1)}(a)\ , \tag{15}$$

where

$$r_1=r_1^{(0)\overline{\rm MS}}+\beta_0\ln k+A_1\ . \tag{16}$$

The presence of $\Lambda_{\overline{\rm MS}}$ in this general expression is a consequence of the relation (12), which has been explicitly taken into account in order to simplify the comparison of the predictions in different schemes. (This is $\Lambda^{(3)}_{\overline{\rm MS}}$, as is appropriate for three flavors—it is related to $\Lambda^{(4)}_{\overline{\rm MS}}$ and $\Lambda^{(5)}_{\overline{\rm MS}}$ via the matching relation [26].) A conventional way of estimating the renormalization scheme dependence in the NL order in QCD is to use the $\overline{\rm MS}$ scheme and vary the scale coefficient k over some "reasonable" range of values, usually close to unity. This is justified by the fact that in the NL order a change of the renormalization scale coefficient from k to k' has formally the same effect on the predictions as a change of the RS via (9) with $A_1=\beta_0\ln(k'/k)$. Unfortunately, such a procedure does not give a full picture of the RS dependence in the NL order. To show this explicitly, let us note that any choice of the RS or of the scale coefficient k in the expressions (14)–(16) amounts simply to some choice of the expansion coefficient r_1. Therefore this coefficient may be used to distinguish the approximants which are available in the NL order. Varying the scale parameter k in the $\overline{\rm MS}$ scheme in the range $\frac{1}{3}\leq k\leq 3$ we obtain variation of r_1 in the range $0.26\leq r_1\leq 10.15$. However, the same variation of the scale parameter in some other scheme may correspond to an essentially different range of values for r_1. For example, in the so-called momentum subtraction scheme, in which the coupling constant is defined via the three-gluon vertex at the symmetric configuration of Euclidean momenta [20] [$A_1=-4.09$ in the formula (9), for three flavors in the Landau gauge], the variation of the scale coefficient in the same range gives $-3.83\leq r_1\leq 6.06$. It is thus evident that in the conventional analysis of the RS dependence in the NL order some of the approximants are not taken into account. The NL-order predictions for R_τ in various renormalization schemes are shown in Fig. 1. The experimental value $R_\tau^{\rm exp}=0.20\mp 0.02$ [27] has been indicated to mark the range of values of R_τ which are of phenomenological

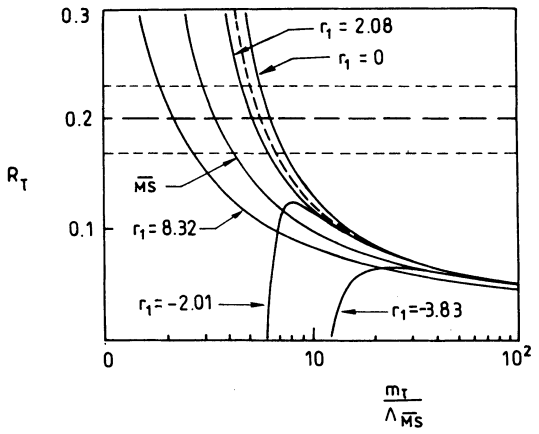

FIG. 1. The NL-order predictions for R_τ as a function of $m_\tau/\Lambda_{\overline{\rm MS}}$, as obtained in the $\overline{\rm MS}$ scheme with the scale coefficient $k=2$, 1, $\frac{1}{2}$ ($r_1=8.32$, 5.20, 2.08, respectively), in the FAC scheme ($r_1=0$), and in the symmetric momentum subtraction scheme with $k=1$, $\frac{1}{2}$, $\frac{1}{3}$ [$r_1=1.11$ (dashed line), $r_1=-2.01$, -3.83, respectively]. The dashed horizontal lines represent the experimental value of $R_\tau^{\rm exp}=0.20\pm 0.02$ [27].

interest. The predictions for R_τ in each scheme are shown as a function of $m_\tau/\Lambda_{\overline{\rm MS}}$. This has the advantage that the effect of the scheme dependence on the fit of $\Lambda_{\overline{\rm MS}}$ to the experimental data may be clearly seen. The indicated curves correspond to the values of r_1 from $r_1=8.32$ ($k=2$ in the $\overline{\rm MS}$ scheme) to $r_1=-3.83$ ($k=\frac{1}{3}$ in the symmetric momentum subtraction scheme). We see that the difference between the schemes are large in the NL order, and that they are significant compared to the accuracy of the experimental data. By changing the scheme a qualitatively different dependence of the predictions on $\Lambda_{\overline{\rm MS}}$ may be obtained. In particular, in some schemes there is no $\Lambda_{\overline{\rm MS}}$ that would fit the central value of $R_\tau^{\rm exp}$.

Let us now consider the NNL-order predictions for R_τ. The renormalization group improved expression for R_τ in the NNL order has the form

$$R_\tau^{(2)}=a(1+r_1a+r_2a^2)\ , \tag{17}$$

where the coupling constant is determined by the equation

$$\beta_0\ln\left[\frac{m_\tau}{\Lambda_{\overline{\rm MS}}}\right]=r_1^{(0)\overline{\rm MS}}-r_1+\Phi^{(2)}(a)\ . \tag{18}$$

An explicit form of $\Phi^{(2)}(a,c_2)$ is given for example in [28]. The expression for r_2 in an arbitrary scheme and with an arbitrary choice of the renormalization scale may be easily obtained from (4) and (10). In order to fully characterize the RS dependence of the NNL-order approximants we have to use two independent parameters. The arbitrariness in the predictions, which is related to the freedom of choice of the renormalization scale, is most conveniently parametrized by the coefficient r_1, similarly as in the NL order. The second degree of freedom, which is characteristic of the NNL order, is related to the scheme transformations which change the β function. To parametrize this arbitrariness we use the coefficient c_2 [22]. To obtain an estimate of the RS dependence one should now find the differences in the predictions when the parameters r_1 and c_2 are varied over some reasonable range. The choice of a proper range of variation for these parameters is a delicate matter if we want to argue for a strong RS dependence, since for artificially large parameters we may always obtain significant differences in the predictions. It seems natural to relate the condition on r_1, r_2, and c_2 to the RS invariant ρ_2—the values of r_1, r_2, and c_2 may be considered to be "reasonable" or "natural" when their contributions to ρ_2 do not involve extensive cancellations. Below we show two figures which illustrate some characteristic features of the RS dependence in the NNL order. In Fig. 2 it is shown how the NNL-order predictions for R_τ depend on the value of r_1 with fixed value of $c_2=c_2^{\overline{\rm MS}}$. The indicated curves correspond to the same values of r_1 as in the case of the NL-order predictions shown in Fig. 1. We see that although the curves corresponding to larger positive r_1 lie closer to each other than in the NL case, the differences between the schemes are large for phenomenologically relevant values of R_τ. Again, we find schemes with qualitatively different dependence on

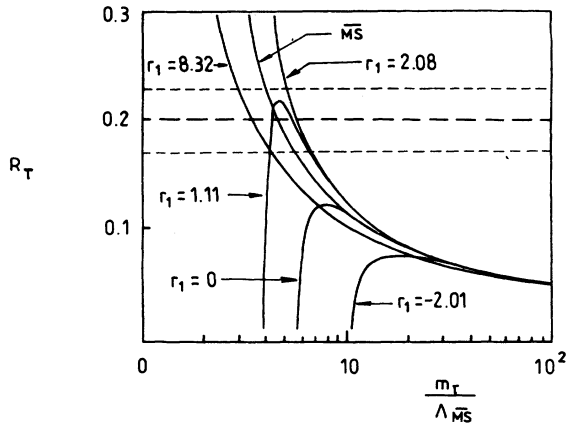

FIG. 2. The NNL-order predictions for R_τ as a function of $m_\tau/\Lambda_{\overline{\rm MS}}$, as obtained for $c_2=4.47=c_2^{\overline{\rm MS}}$ and $r_1=8.32$, 5.20, 2.08, 1.11, 0, -2.01. The dashed horizontal lines represent the experimental value of $R_\tau^{\rm exp}=0.20\pm0.02$ [27].

$m_\tau/\Lambda_{\overline{\rm MS}}$ and schemes in which there is no $\Lambda_{\overline{\rm MS}}$ that would fit the experimental data. In Fig. 3 it is shown how the NNL predictions for R_τ depend on the parameter c_2 for fixed values of $r_1=r_1^{(0)\overline{\rm MS}}$ and $r_1=1.11$. The predictions for R_τ in the FAC scheme are also indicated on this figure. Also in this case we find significant differences between the schemes. Let us remark that the fact that the $\overline{\rm MS}$ predictions lie close to the FAC predictions is of little relevance for the overall picture of the RS dependence.

It should be pointed out that when the results of the fits to the experimental data are expressed in terms of the value of the running coupling constant in the $\overline{\rm MS}$ scheme, as is common in the contemporary QCD literature, it is very difficult to extract correctly the error in the fit due to the change of RS involving a change of the β function. This is to be contrasted with the transparent way of es-

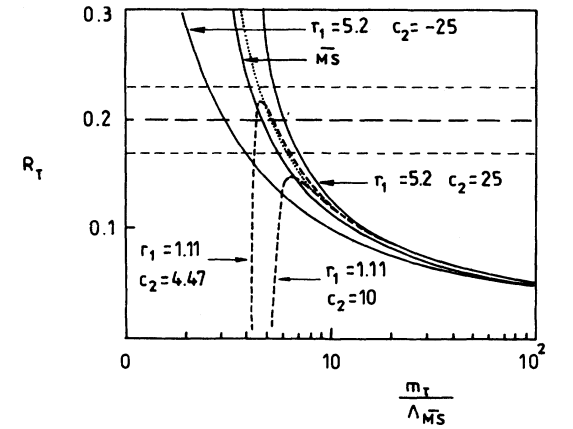

FIG. 3. The NNL-order predictions for R_τ as a function of $m_\tau/\Lambda_{\overline{\rm MS}}$, as obtained for $r_1=5.20=r_1^{(0)\overline{\rm MS}}$ with $c_2=-25$, 4.47, 25 and for $r_1=1.11$ with $c_2=4.47$, 10 (dashed curves). The dotted line indicates predictions in the FAC scheme ($r_1=0=r_2$).

timating the effects of the RS dependence on the fits of $\Lambda_{\overline{\rm MS}}$. The use of the parameter Λ is preferable in the fits because the one-loop relation (12), translating it from one scheme to another, is exact to all orders.

Summarizing, we may say that we have found strong RS dependence of the perturbative predictions for R_τ both in the NL and the NNL order. This strong RS dependence is caused by the fact that the coupling constant is not small, so that the perturbation expansion is poorly convergent. For example in the $\overline{\rm MS}$ scheme the phenomenologically relevant value is $a \cong 0.1$, in which case the NL-order and the NNL-order corrections are of a comparable magnitude. Therefore even small variations of the scheme may have significant effect on the prediction. A strong RS dependence of the predictions for R_τ seems to indicate that the truncated perturbation expansion is not adequate in this case, even if one uses some optimization methods. Presumably to obtain reliable predictions one has to go beyond the truncated perturbation series, perhaps including the information on the high-order behavior of the perturbation series and constructing nonpolynomial approximants, as has been discussed in [29].

From our analysis above, it follows that it is impossible to obtain a precise value of $\Lambda_{\overline{\rm MS}}$ from the fit to the experimental data on R_τ. It is true of course, that in some schemes—including the $\overline{\rm MS}$ scheme—the QCD predictions for R_τ are very sensitive to the value of $\Lambda_{\overline{\rm MS}}$, as may be clearly seen in Figs. 1–3, and that the fits to experimental data in these schemes yield tightly constrained values of $\Lambda_{\overline{\rm MS}}$. This sensitivity is a consequence of a relatively large value of the coupling constant, which in turn implies an increased rate of running of the coupling. However, when the coupling constant is large, the RS dependence is also large. This introduces an additional uncertainty in the fits, which in the case of R_τ completely compensates for the increase in the accuracy of the fits due to the strong sensitivity to $\Lambda_{\overline{\rm MS}}$. This resolves an apparent paradox, that using a poorly convergent perturbation series for R_τ in, say, the $\overline{\rm MS}$ scheme, we obtain a more precise value of $\Lambda_{\overline{\rm MS}}$ than from the fits of many of the high-energy QCD predictions, for which the perturbation expansion is much better behaved.

The method of analysis of the RS dependence formulated in this article applies without modification to any QCD prediction that depends on a single energy variable. For QCD observables at higher energies the RS dependence is less pronounced than in the case of R_τ, and concrete numerical estimates may be obtained for the theoretical uncertainty in the predictions arising from the freedom of choice of the scheme. The results of an extensive study of the RS dependence of various quantities [30] will be published elsewhere.

This work was supported in part by Grant No. KBN 202739101.

[1] C. S. Lam and T. M. Yan, Phys. Rev. D **16**, 703 (1977); S. Narison, Z. Phys. C **2**, 11 (1979); K. Schilcher and M. D. Tran, Phys. Rev. D **29**, 570 (1984).
[2] E. Braaten, Phys. Rev. Lett. **60**, 1606 (1988); **63**, 577 (1989).
[3] E. Braaten, Phys. Rev. D **39**, 1458 (1989).
[4] S. Narison and A. Pich, Phys. Lett. B **211**, 183 (1988).
[5] J. Pumplin, Phys. Rev. Lett. **63**, 576 (1989); Phys. Rev. D **41**, 900 (1990).
[6] C. J. Maxwell and J. A. Nicholls, Phys. Lett. B **236**, 63 (1990).
[7] M. Haruyama, Prog. Theor. Phys. **83**, 841 (1990).
[8] S. G. Gorishny, A. L. Kataev, and S. A. Larin, Phys. Lett. B **259**, 144 (1991).
[9] J. Chyla, A. Kataev, and S. A. Larin, Phys. Lett. B **267**, 269 (1991).
[10] M. A. Samuel and L. R. Surguladze, Phys. Rev. D **44**, 1602 (1991).
[11] A. Pich, in *Proceedings of the Workshop on Tau Lepton Physics,* Orsay, France, 1990, edited by M. Davier and B. Jean-Marie (Editions Frontieres, Gif-sur-Yvette, 1991), p. 321; also CERN Report No. CERN-Th.6237/91 (unpublished).
[12] A. A. Pivovarov, Z. Phys. C **53**, 461 (1992).
[13] E. Braaten, S. Narison, and A. Pich, Nucl. Phys. **B373**, 581 (1992).
[14] M. Haruyama, Phys. Rev. D **45**, 930 (1992).
[15] F. Le Diberder and A. Pich, Phys. Lett. B **286**, 147 (1992).
[16] A. Pich, CERN Report No. CERN-TH.6490/92 (unpublished).
[17] F. Le Diberder and A. Pich, Phys. Lett. B **289**, 165 (1992).
[18] W. A. Bardeen, A. J. Buras, D. W. Duke, and T. Muta, Phys. Rev. D **18**, 3998 (1978).
[19] O. V. Tarasov, A. A. Vladimirov, and A. Yu. Zharkov, Phys. Lett. **93B**, 429 (1980).
[20] W. Celmaster and R. J. Gonsalves, Phys. Rev. D **20**, 1426 (1979).
[21] G. Grunberg, Phys. Lett. **95B**, 70 (1980).
[22] P. M. Stevenson, Phys. Rev. D **23**, 2916 (1981).
[23] A. Dhar, Phys. Lett. **128B**, 407 (1983).
[24] A. Dhar and V. Gupta, Pramāna **21**, 207 (1983); Phys. Rev. D **29**, 2822 (1984).
[25] G. Grunberg, Phys. Rev. D **29**, 2315 (1984).
[26] W. J. Marciano, Phys. Rev. D **29**, 580 (1984).
[27] This is obtained from $\mathcal{R}_\tau = 3.61 \pm 0.05$ (see Ref. [11]) using formula 1 and neglecting electroweak corrections.
[28] C. J. Maxwell, Phys. Rev. D **28**, 2037 (1983).
[29] J. Chyla and C. Burdik, Czech. J. Phys. **40**, 367 (1990); P. A. Rączka, Phys. Rev. D **43**, R9 (1991).
[30] P. A. Rączka (unpublished).