PHYSICAL REVIEW D

PARTICLES, FIELDS, GRAVITATION, AND COSMOLOGY

THIRD SERIES, VOLUME 46, NUMBER 6 — 15 SEPTEMBER 1992

RAPID COMMUNICATIONS

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Ashtekar's variables for arbitrary gauge group

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(Received 24 April 1992)

A generally covariant gauge theory for an arbitrary gauge group with dimension ≥ 3 that reduces to Ashtekar's canonical formulation of gravity for SO(3,C) is presented. The canonical form of the theory is shown to contain only first-class constraints.

PACS number(s): 04.60.+n, 04.20.Cv, 04.20.Fy

When Ashtekar [1] managed to reformulate Einstein gravity on Yang-Mills phase space it rekindled the old dream of finding a unified theory of gravity and Yang-Mills theory. However, it soon became clear that this Ashtekar formulation relied heavily on the use of the gauge group SO(3) (or a locally isomorphic one), and the simple structure-constant identity that exists for these groups. Without this identity the constraint algebra fails to close, the theory is not diffeomorphism invariant, and it contains second-class constraints. In an attempt to find an Ashtekar formulation for an arbitrary gauge group, there are no problems with the generator of gauge transformations (Gauss's law), or the generator of spatial diffeomorphisms (the vector constraint). They form a system of first-class constraints by themselves, for an arbitrary gauge group. The difficult part is the generator of diffeomorphisms off the spatial hypersurface (the Hamiltonian constraint). This constraint is constructed with the help of the structure constants, and in the Poisson brackets between two Hamiltonian constraints, the identity, mentioned above, is needed to give a weakly vanishing result. So, one strategy to construct a generalized Ashtekar theory is to write down a Hamiltonian constraint without the use of the structure constants, such that, when choosing the gauge group SO(3), the constraint reduces to the ordinary Ashtekar constraint. The hope is then that the construction might work for an arbitrary gauge group, since one does not use any particular feature of a special gauge group any more. To do this in practice, first define a scalar with the help of the four fundamental scalar densities $\epsilon_{abc} f_{ijk} \Pi^a_i \Pi^b_j B^c_k$, $\epsilon_{abc} f_{ijk} \Pi^a_i \Pi^b_j \Pi^c_k$, $\epsilon_{abc} f_{ijk} \Pi^a_i B^b_j B^c_k$, $\epsilon_{abc} f_{ijk} B^a_i B^b_j B^c_k$, where Π^a_i is the momenta and B^a_i is the magnetic field constructed from the SO(3) curvature, in Ashtekar's canonical formulation of gravity. Then, multiply the ordinary Ashtekar Hamiltonian constraint with this scalar, and, finally, use the SO(3) structure-constant identity to eliminate all structure constants. After this elimination, the gauge group can be considered as arbitrary. This new Hamiltonian will then in general give a closed constraint algebra for an arbitrary gauge group.

In this Rapid Communication, I will show how to obtain this Ashtekar theory for an arbitrary gauge group, through a Legendre transform from a pure connection Lagrangian of the form discovered by Capovilla, Jacobson, and Dell [2] (CDJ). The resulting canonical theory will correspond to multiplying the Hamiltonian constraint by the determinant of the "magnetic" field, in the strategy above.

In order to find the Ashtekar theory for a general gauge group, I will start with the generally covariant and gauge-invariant CDJ action [2]

$$S=\tfrac{1}{8}\int d^4x\, \eta[\mathrm{Tr}\Omega^2+a(\mathrm{Tr}\Omega)^2]\ , \tag{1}$$

where $\Omega^{ij}:=\epsilon^{\alpha\beta\gamma\delta}F^i_{\alpha\beta}F^j_{\gamma\delta}$ and $F^i_{\alpha\beta}=\partial_{[\alpha}A^i_{\beta]}+f_{ijk}A^j_\alpha A^k_\beta$

and η is a scalar density of weight -1. The trace is taken with the invariant bilinear Killing form of the Lie algebra. [For some Lie algebras, such as so(1,3), so(4), and so(2,2), there exist two different "traces" that could be used in the action (1), meaning that for these groups there exists an even more general Lagrangian, quartic in the field strength [3].] A 3+1 canonical decomposition of this action, with $a=-\frac{1}{2}$ and the gauge group $SO(3,C)$, is known to give Ashtekar's Hamiltonian for pure gravity without a cosmological constant. It is important to note that in doing this 3+1 canonical decomposition one has to exclude field configurations with a "degenerate" Weyl tensor [2], and allow the fields to be complex valued, in order to find the Ashtekar Hamiltonian. For other values of a but keeping the gauge group $SO(3,C)$, the action still describes a theory that has an interpretation in terms of Riemannian geometry [5,6]. This pure connection formulation of general relativity has also been studied with a cosmological constant and matter couplings in three and four dimensions [2,4,7].

It is correct to state that, at the configuration space Lagrangian level, the action (1) is already a generalization of Ashtekar's general relativity for an arbitrary gauge group; however, no one has yet given the canonical form of the theory for an arbitrary gauge group. Since this action is invariant under diffeomorphisms and gauge transformations, it should be quite clear that a canonical decomposition of it should give a set of first-class constraints generating these symmetries. And since one knows that with the gauge group $SO(3,C)$ it gives Ashtekar's variables, the general theory must be what one would call "Ashtekar's variables for arbitrary gauge group." However, there are at least two things that could ruin this construction. The first is that it could be "impossible" to perform the Legendre transform for a general gauge group. [This is not so strange since previous work on this action has relied heavily on the fact that the gauge group is three dimensional, and that the structure constant satisfies the simple SO(3) identity.] The other thing that could have ruined the beauty of the Hamiltonian formulation is that additional complicated second-class constraints would have appeared. However, none of these are the case, and as I soon will show, the only thing that happens for an arbitrary gauge group is that the Hamiltonian constraint splits up into three pieces.

First I define the momenta conjugated to A_a^i:

$$\Pi_i^a := \frac{\partial \mathcal{L}}{\partial \dot{A}_a^i} = \eta(\Omega_j^i + a\,\mathrm{Tr}\Omega\delta_j^i)B^{aj}\ , \tag{2}$$

where $B^{aj} := \epsilon^{abc}F_{bc}^j$ is the "magnetic field." a,b,c denote spatial indices and i,j,k denote gauge indices. Now, it is rather straightforward to perform the Legendre transform, provided the "magnetic metric" $b^{ab} := B^{ai}B_i^b$ is invertible. It is here that one must require the dimension of the gauge group to be ≥ 3 in order to have a nondegenerate "magnetic metric." The difference between this Legendre transform and the one performed by Capovilla [5] is that Capovilla chose the gauge group to be $SO(3,C)$ and could therefore use the magnetic field itself as a three-by-three matrix, while keeping the gauge group arbitrary makes it necessary to use the "magnetic metric" instead. Defining the inverse of the "magnetic metric"

$$b_{ab} := \frac{1}{2\det(b^{cd})}\epsilon_{aef}\epsilon_{bgh}b^{eg}b^{fh}$$

and performing the Legendre transform gives

$$\mathcal{H}_{\mathrm{tot}} = N\mathcal{H} + N^a\mathcal{H}_a + \Lambda^i\mathcal{G}_i\ , \tag{3}$$

where

$$\mathcal{H} = \frac{\sqrt{\det(b^{ab})}}{4}\left\{2\Pi^{ai}\Pi_i^b b_{ab} - (\Pi^{ai}B_i^b)b_{bc}(\Pi^{cj}B_j^d)b_{da} - \frac{a}{1+3a}(\Pi^{ai}B_i^b b_{ab})^2\right\} \approx 0\ ,$$

$$\mathcal{H}_a = \tfrac{1}{2}\epsilon_{abc}\Pi^{bi}B_i^c \approx 0\ ,$$

$$\mathcal{G}_i = \mathcal{D}_a\Pi_i^a \approx 0\ ,$$

$$N := \frac{1}{2\eta\sqrt{\det(b^{ab})}}\ ,$$

$$\Lambda^i := -A_0^i\ .$$

$\mathcal{H}$ is usually called the Hamiltonian constraint, $\mathcal{H}_a$ the vector constraint, and $\mathcal{G}_i$ Gauss's law. The value $a=-\frac{1}{3}$ must be handled separately. For that case, the Hamiltonian constraint splits up in two separate pieces which will become second-class constraints. Notice that the form of Gauss's law and the vector constraint are independent of the gauge group while the Hamiltonian constraint looks a bit more complicated for a general gauge group compared with the ordinary Ashtekar form for SO(3):

$$\mathcal{H}_{\mathrm{Ash}} = \frac{i}{4}\epsilon_{abc}f_{ijk}\Pi_i^a\Pi_j^b B_k^c\ .$$

It is however easy to check that with $a=-\frac{1}{2}$ and the gauge group SO(3), using the identity

$$f^{ijk}f_{lmn} = \delta_l^{[i}\delta_m^{j]}\delta_n^k + \delta_n^{[i}\delta_l^{j]}\delta_m^k + \delta_m^{[i}\delta_n^{j]}\delta_l^k\ ,$$

valid for SO(3), $\mathcal{H}$ can be rewritten as $\mathcal{H} = i\mathcal{H}_{\mathrm{Ash}}$. [The structure constant identity follows from the fact that for SO(3), $f_{ijk} = \epsilon_{ijk}$.]

Now, for an arbitrary gauge group, one must still check that the constraints form a first-class set. And, as mentioned earlier, there are no problems with Gauss's law and the vector constraint. We know that they generate gauge transformations and spatial diffeomorphisms, and all constraints are gauge covariant and diffeomorphism covariant, which means that all Poisson brackets including these constraints are weakly vanishing. So, the only nontrivial calculation is the Poisson brackets between two Hamiltonian constraints. A straightforward calculation gives

$$\{\mathcal{H}[N],\mathcal{H}[M]\} = \mathcal{H}_a[q^{ab}(N\partial_b M - M\partial_b N)]\ , \tag{4}$$

where

$$\mathcal{H}[N] = \int d^3x\, \mathcal{H}N$$

and

$$q^{ab}=2\Pi^{ai}\Pi_i^b-3(\Pi^{ai}B_i^c)b_{cd}(B_j^d\Pi^{bj})+\frac{1+2a}{1+3a}(b_{cd}\Pi^{ci}B_i^d)\tfrac{1}{2}(\Pi^{(aj}B_j^{b)})-\frac{(1+a)(\frac{1}{2}+a)}{(1+3a)^2}(b_{cd}\Pi^{ci}B_i^d)^2b^{ab}+3[b_{ef}(\Pi^{ei}B_i^c)b_{cd}(B_j^d\Pi^{fj})-b_{ef}\Pi^{ei}\Pi_i^f]b^{ab}$$

on the constraint surface. And, according to Hojman, Kuchař, and Teitelboim [9] the object q^{ab} in the Poisson brackets above is to be interpreted as the spatial metric on the hypersurface.

From now on I will put $a=-\frac{1}{2}$ which is the value that for SO(3,C) gives ordinary gravity. This means that the spatial metric is

$$q^{ab}=2\Pi^{ai}\Pi_i^b-3(\Pi^{ai}B_i^c)b_{cd}(B_j^d\Pi^{bj})+3[b_{ef}(\Pi^{ei}B_i^c)b_{cd}(B_j^d\Pi^{fj})-b_{ef}\Pi^{ei}\Pi_i^f]b^{ab} \quad (5)$$

a form that makes it very hard to ensure positive definiteness of the metric, for a general gauge group.

During the calculation of the Poisson brackets (4), it becomes clear that the only parts that could give a nonclosure of the constraint algebra come from the first term in the Hamiltonian constraint. That means that there exists another Hamiltonian constraint, quadratic in momenta, that gives a closed constraint algebra: namely,

$$\mathcal{H}^{\rm Alt}=\frac{\sqrt{\det(b^{ab})}}{4}[(\Pi^{ai}B_i^b)b_{bc}(\Pi^{cj}B_j^d)b_{da}-(\Pi^{ai}B_i^bb_{ab})^2]=-\frac{1}{4\sqrt{\det(b^{ab})}}\epsilon_{abc}\epsilon_{def}(\Pi^{ai}B_i^d)(\Pi^{bj}B_j^e)b^{cf}\ . \quad (6)$$

This Hamiltonian constraint seems more tractable than the original one in (3), at first sight. However, it has two remarkable features. First, doing the Legendre transform backwards for this Hamiltonian does not give a manifestly covariant pure connection action, despite the fact that one has a closed constraint algebra. (The same situation appears for gravity coupled to a massive spinor in the ordinary Ashtekar formulation [7].) Second, the theory only cares about the part of Π^{ai} that is nonorthogonal to B^{ai} in its internal gauge indices. The orthogonal part of Π^{ai} do not have any effect on A_a^i at all. Which Hamiltonian constraint is then the best choice for a generalization of Ashtekar's variables? Both reduce to the Ashtekar Hamiltonian constraint for SO(3); $\mathcal{H}^{\rm Alt}$ has a simpler form, but $\mathcal{H}$ corresponds to a manifestly covariant Lagrangian. My opinion is that the pure connection Lagrangian (1) really has some fundamental role, and therefore one should choose $\mathcal{H}$. And, in addition, there is something strange with a theory which has a lot of "extra" fields (the "orthogonal" part of Π^{ai}) which have no effect on the equations of motion.

If one is only looking for a theory that for the gauge group SO(3) reduces to the Ashtekar formulation, and does not mind whether or not, for instance, the Hamiltonian is quadratic in the momenta, then there exist several different Hamiltonian constraints. They can be found in three different ways. (1) Use the strategy outlined in the beginning of this Rapid Communication. (2) Write down the general CDJ-type Lagrangian with arbitrarily high orders in the field strength, and perform the Legendre transform. (3) Define the Hamiltonian constraint by contracting the following scalar densities with epsilon-tensor densities: $\Pi^{ai}B_i^b$, $B^{ai}B_i^b$, and $\Pi^{ai}\Pi_i^b$. Here is an example of the third way:

$$\mathcal{H}=\epsilon_{abc}\epsilon_{def}(\Pi^{ai}\Pi_i^d)(\Pi^{bj}\Pi_j^e)(\Pi^{ck}B_k^f)\ .$$

This is an interesting Hamiltonian constraint, which has the feature that the spatial metric will be of the familiar form $q^{ab}\sim\Pi^{ai}\Pi_i^b$. Using the third way, one must carefully check the constraint algebra; it will not always be closed.

Now, the existence of a theory for a general gauge group, which reduces to the theory of pure Einstein gravity for a specific choice of the gauge group, makes it tempting to speculate about a unified description of gravity and Yang-Mills theory. The question is then: What kind of gauge group should be used? Could the naive guess of SO(3)$\times G$ give gravity coupled to a Yang-Mills field with gauge group G, or is there need for a more sophisticated construction that in some way could be reduced to the desired result (spontaneous symmetry breakdown?)? The first thing one may notice is that with gauge group SO(3)$\times G$ the Hamiltonian in (3) can never give the "ordinary" gravity-Yang-Mills coupling, given by Ashtekar, Romano, and Tate [8]. [That is because the ordinary coupling is nonhomogeneous in the momenta, while $\mathcal{H}$ in (3) is just quadratic.] Perhaps, some of the other generalized Hamiltonians, mentioned above, have a better chance. However, what is required of the coupling is really just that it reduces to the ordinary Yang-Mills equations for flat space-times in the weak limit. But, even that seems to fail. Trying the gauge group SO(3)$\times$U(1), it is easy to verify that Maxwell's equations do not appear in the weak field limit. So, if there will be no miraculous improvements for some special Yang-Mills gauge group, this naive construction will fail, and one must really think of something more clever.

Also the reality condition of the Ashtekar formulation seems tough to handle in this direct-product approach. In general, the reality condition will have to be matter field dependent, in order to get a real metric.

However, an optimistic speculation regarding the reality conditions is that it could be possible to find a gauge group in which the "gravity part" would give a positive-definite spatial metric without any need of introducing complex fields.

Looking ahead a bit, one could start thinking of using the Rovelli-Smolin loop-representation quantization scheme [10] for this generalized theory. At first sight, there are two obvious things that change: The "spinor identity" will become more complicated, and the definition of the Hamiltonian constraint in terms of the T variables and T operators changes. But otherwise it does not seem to be an impossible task to redo everything for a general gauge group.

I would like to thank Ingemar Bengtsson for numerous discussions.

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