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Causally spinning anyonic cosmic string

Y. M. Cho and D. H. Park
Department of Physics, Seoul National University, Seoul 151-742, Korea
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We present new gravitational anyonic solutions of (2+1)-dimensional Einstein-Chern-Simons electrodynamics which can describe spinning cosmic strings made of fermions, and discuss the physical implications of the solutions. Remarkably we show that a spinning string which does not allow any closed timelike curves, and thus does not violate causality, is possible.

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Physics in 2+1 dimensions has been known to have many interesting features. For instance, in a (2+1)-dimensional space-time an anyonic elementary particle is possible which carries a fractional angular momentum [1,2]. This is because the rotation group in the two-dimensional space is U(1), whose representation is characterized by an arbitrary real number. A well-known example of the anyon is a charge-flux composite state in (2+1)-dimensional electrodynamics [2,3]. The existence of anyons which obey fractional statistics implies the existence of new physics in (2+1)-dimensional space-time. Indeed, the anyons are expected to play a crucial role in our understanding of the fractional quantum Hall effect [4] and possibly high-T_c superconductivity [5].

Recently it has been pointed out that (2+1)-dimensional gravitation allows gravitational anyons [6,7]. In a (2+1)-dimensional space-time a point particle bound to a spinning gravitational point source forms a gravitational anyon. This is because the spin-energy composite state in gravitation behaves very much like the charge-flux composite state in electrodynamics. Indeed, one can easily show that the angular momentum of a point particle with energy E (in units of the gravitational constant) moving around a massless spinning gravitational point source is given by [7]

$$l=n+\frac{\sigma E}{2\pi}\ , \tag{1}$$

where n is an integer and σ is the spin of the gravitational source. The result should be compared with the well-known angular momentum of a charge-flux composite state in electrodynamics [2]:

$$l=n-\frac{q\Phi}{2\pi}\ , \tag{2}$$

where q is the electric charge and Φ is the magnetic flux. This shows that energy and spin in gravitation play the role of electric charge and magnetic flux in electrodynamics. Exactly the same correspondence can be established when one compares the gravitational scattering of a neutral test particle around a spinning string [8] with the Aharonov-Bohm scattering of an electron around a magnetic vortex [9]. This correspondence is not accidental, and has a deep physical implication. In particular, the correspondence implies the existence of gravitational monopoles which can radically generalize Einstein's theory of gravitation [10].

So far, however, a self-consistent exact solution of Einstein's theory which can describe a regular gravitational anyon has been missing. The difficulty of finding such a solution is twofold. First, in (2+1)-dimensional space-time the gravitational attraction between two point particles is in general too weak to combine them into a bound state. This can be understood from the fact that a gravitational point source in 2+1 dimensions does not admit a closed geodesic in general. Second, any regular spin-energy composite solution is most likely to violate causality, because the spinning gravitational point source necessarily has closed timelike curves around the source [11,12]. So it is a nontrivial task to find a gravitationally coupled spin-energy composite state which does not violate causality. The purpose of this Rapid Communication is to present such a solution. We consider the (2+1)-dimensional Chern-Simons electrodynamics coupled to Einstein's gravitation, and obtain a set of causally spinning gravitational anyons which also carries nonvanishing electromagnetic charge and flux. Our result clearly demonstrates the fact that *the existence of a closed timelike curve is not a necessity for a spinning cosmic string*. This tells us that any suggestion to dismiss the spinning cosmic strings as unphysical is ill conceived and unfounded.

Let us start from the following Lagrangian of Einstein-Chern-Simons electrodynamics:

$$\mathcal{L}=-\sqrt{-g}\left[\frac{1}{2\kappa}R-\frac{\mu}{4}\epsilon^{\mu\nu\rho}A_\mu F_{\nu\rho}-i\bar{\Psi}\gamma^\mu\nabla_\mu\Psi+m\bar{\Psi}\Psi-\frac{\lambda}{2}(\bar{\Psi}\Psi)^2\right]\ , \tag{3}$$

where κ, μ, and λ are the gravitational constant, the Chern-Simons coupling constant, and the fermionic quartic coupling constant, and ∇_μ is the generally and gauge-covariant derivative. Notice that when the Chern-Simons interaction is induced by the quantum fluctuation, one must have $\mu=\pm e^2/2\pi$ [13]. But in the following we will keep μ arbitrary to be general. From (3) one has the equations of motion

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$$R_{\mu\nu}-\tfrac{1}{2}g_{\mu\nu}R=-\kappa T_{\mu\nu}\ ,$$
$$\frac{\mu}{2}\epsilon^{\mu\nu\rho}F_{\nu\rho}=e\overline{\Psi}\gamma^{\mu}\Psi\ , \tag{4}$$
$$(i\gamma^{\mu}\nabla_{\mu}-m)\Psi=-\lambda(\overline{\Psi}\Psi)\Psi\ ,$$

where

$$T_{\mu\nu}=-\frac{1}{2}(i\overline{\Psi}\gamma_{\mu}\nabla_{\nu}\Psi+i\overline{\Psi}\gamma_{\nu}\nabla_{\mu}\Psi)+g_{\mu\nu}\left[i\overline{\Psi}\gamma^{\alpha}\nabla_{\alpha}\Psi-m\overline{\Psi}\Psi+\frac{\lambda}{2}(\overline{\Psi}\Psi)^2\right]$$
$$=-\frac{1}{2}[i\overline{\Psi}\gamma_{\mu}\nabla_{\nu}\Psi+i\overline{\Psi}\gamma_{\nu}\nabla_{\mu}\Psi+g_{\mu\nu}\lambda(\overline{\Psi}\Psi)^2]\ .$$

Notice that in a coordinate basis $(\partial_t\oplus\partial_i)$ the most general stationary metric can be written as

$$g_{\mu\nu}=\begin{bmatrix}-N^2 & -\frac{\kappa}{2\pi}N^2\sigma_i\\ -\frac{\kappa}{2\pi}N^2\sigma_i & \gamma_{ij}-\frac{\kappa^2}{4\pi^2}N^2\sigma_i\sigma_j\end{bmatrix}\ , \tag{5}$$

where N, σ_i, and γ_{ij} $(i,j=1,2)$ are functions of the space coordinates only. But in the block-diagonal basis $(\hat{\partial}_t\oplus\hat{\partial}_i)$ [14]

$$\hat{\partial}_t=\partial_t,\quad \hat{\partial}_i=\partial_i-\frac{\kappa}{2\pi}\sigma_i\partial_t\ , \tag{6}$$

where

$$[\hat{\partial}_t,\hat{\partial}_i]=0,\quad [\hat{\partial}_i,\hat{\partial}_j]=-\Pi_{ij}\hat{\partial}_t\ ,$$
$$\Pi_{ij}=\frac{\kappa}{2\pi}(\partial_i\sigma_j-\partial_j\sigma_i)\ ,$$

the metric (5) can be written as

$$g_{\mu\nu}=\begin{bmatrix}-N^2 & 0\\ 0 & \gamma_{ij}\end{bmatrix}\ .$$

To obtain the desired solutions we assume that the two-dimensional space is conformally flat, and put

$$\gamma_{ij}=\rho^2\delta_{ij}\ , \tag{7}$$

where δ_{ij} is the flat two-dimensional metric. Now, with

$$N=1,\quad A_{\mu}=(0,A_i(\mathbf{x}))$$
$$\Psi=e^{-iEt}\psi_{\pm}(\mathbf{x}),\quad \psi_+=\begin{bmatrix}\phi_+\\ 0\end{bmatrix},\quad \psi_-=\begin{bmatrix}0\\ \phi_-\end{bmatrix}\ , \tag{8}$$

the equation of motion (4) is reduced to [15]

$$E=\pm m,\quad \lambda=\frac{\kappa}{8}$$

and

$$\partial_i\alpha_j-\partial_j\alpha_i=\pm 2\pi\kappa m\epsilon_{ij}|\phi_{\pm}|^2\ ,$$
$$\partial_i\sigma_j-\partial_j\sigma_i=\mp\pi\epsilon_{ij}|\phi_{\pm}|^2\ , \tag{9}$$
$$\partial_i A_j-\partial_j A_i=-\frac{e}{\mu}\epsilon_{ij}|\phi_{\pm}|^2\ ,$$
$$(\nabla_1\pm i\nabla_2)\phi_{\pm}=0\ ,$$

where

$$\alpha_i=2\pi\epsilon_i{}^j\partial_j\ln\rho$$
$$\nabla_i=\partial_i+i\left[\pm\frac{\kappa m\sigma_i}{2\pi}\pm\frac{\alpha_i}{4\pi}+eA_i\right]\ ,$$

and ϵ_{ij} is the two-dimensional totally antisymmetric tensor field normalized by $\epsilon_{12}=\sqrt{\det\gamma}$. Note that $F_{\pm}=\rho^2|\phi_{\pm}|^2$ satisfies the Liouville equation

$$\nabla^2\ln F_{\pm}=\mp 2(\kappa m+e^2/\mu)F_{\pm}\ , \tag{10}$$

where ∇^2 is the Laplacian of flat two-dimensional space. The equation is known to admit a nonsingular nontrivial solution when $\pm(\kappa m+e^2/\mu)>0$.

To obtain explicit solutions we further assume rotational symmetry, and let

$$\rho=\rho(r),\quad \alpha_i=\alpha(r)\partial_i\varphi,\quad \sigma_i=\sigma(r)\partial_i\varphi\ ,$$
$$\phi_{\pm}=e^{\pm in\varphi}f_{\pm}(r),\quad A_i=A(r)\partial_i\varphi\ , \tag{11}$$

where r and φ are the polar coordinates. Then with the boundary condition

$$\alpha(0)=\sigma(0)=A(0)=0\ , \tag{12}$$

we obtain the following solutions when $\pm\mu/(\mu\kappa m+e^2)>0$:

$$\alpha=\frac{4\pi\mu\kappa m}{\mu\kappa m+e^2}(n+1)\frac{(r/a)^{2n+2}}{(r/a)^{2n+2}+1}\ ,$$
$$\sigma=-\frac{2\pi\mu}{\mu\kappa m+e^2}(n+1)\frac{(r/a)^{2n+2}}{(r/a)^{2n+2}+1}\ ,$$
$$eA=\mp\frac{2e^2}{\mu\kappa m+e^2}(n+1)\frac{(r/a)^{2n+2}}{(r/a)^{2n+2}+1}\ , \tag{13}$$
$$f_{\pm}=\frac{2}{b}\left[\pm\frac{\mu}{\mu\kappa m+e^2}\right]^{1/2}(n+1)\frac{(r/a)^n}{[(r/a)^{2n+2}+1]^{1-c}}\ ,$$

where a and b are positive integration constants,

$$c=\frac{\mu\kappa m}{\mu\kappa m+e^2}\ ,$$

and n is a non-negative integer. Notice that since the regularity at the origin requires $f_{\pm}(0)$ to be finite, n must be non-negative.

In addition to the regular solution (13) we can obtain a more general solution if we allow a singularity at the origin. With a singular point source at the origin we can choose the boundary condition [15]

$$\alpha(0)=\alpha_0,\quad \sigma(0)=\sigma_0,\quad A(0)=A_0\ . \tag{14}$$

With this we obtain the solutions

$$
\begin{aligned}
\alpha-\alpha_0 &= \frac{4\pi\mu\kappa m}{\mu\kappa m+e^2}\gamma_\pm \frac{(r/a)^{2\gamma_\pm}}{(r/a)^{2\gamma_\pm}+1} ,\\
\sigma-\sigma_0 &= -\frac{2\pi\mu}{\mu\kappa m+e^2}\gamma_\pm \frac{(r/a)^{2\gamma_\pm}}{(r/a)^{2\gamma_\pm}+1} ,\\
e(A-A_0) &= \mp\frac{2e^2}{\mu\kappa m+e^2}\gamma_\pm \frac{(r/a)^{2\gamma_\pm}}{(r/a)^{2\gamma_\pm}+1} ,\\
f_\pm &= \frac{2}{b}\left[\pm\frac{\mu}{\mu\kappa m+e^2}\right]^{1/2}\gamma_\pm \frac{(r/a)^{\gamma_\pm+\alpha_0/2\pi-1}}{[(r/a)^{2\gamma_\pm}+1]^{1-c}} ,
\end{aligned} \tag{15}
$$

where

$$\gamma_\pm = n\pm eA_0+1+\frac{\kappa m\sigma_0}{2\pi}-\frac{\alpha_0}{4\pi} .$$

Notice that for the singular solutions the proper boundary condition at the origin must satisfy the condition $f_\pm(0)=0$. Furthermore, consistency requires that $\rho f_\pm(0)$ must be finite. So we must have the following constraint for $\gamma_\pm$ (when $\alpha_0\neq 0$):

$$\gamma_\pm \geq 1, \quad \gamma_\pm > 1-\frac{\alpha_0}{2\pi} .$$

Obviously the solution reduces to the regular solution (13) when $\alpha_0=\sigma_0=A_0=0$. The solution (15) is summarized in Fig. 1.

Now we discuss the physical meaning of the parameters of solution (15). First notice that the parameter a determines the size of the solution, and b determines the amplitude of the fermionic wave function. Next, the parameter A_0 describes a point magnetic flux $2\pi A_0$ at the origin. This is because the magnetic flux $\Phi(r)$ passing through the area encircled by the radius r is given by

$$
\begin{aligned}
\Phi(r) &= \int^r d^2x\sqrt{\gamma}\,\epsilon^{ij}\partial_i A_j \\
&= 2\pi\int^r dr' \frac{dA(r')}{dr'} = 2\pi A(r) .
\end{aligned} \tag{16}
$$

The parameters α_0 and σ_0 describe the point mass α_0/κ and the point spin σ_0 at the origin. This is so because, without the fermion field, Eq. (9) describes nothing but a spinning cone with a singular magnetic flux at the origin.

To discuss the physical content of solution (15) we calculate various quantities carried by the solution. First the total magnetic flux is given by

$$\Phi = 2\pi A_\infty = \mp\frac{4\pi e}{\mu\kappa m+e^2}\gamma_\pm + 2\pi A_0 . \tag{17}$$

This means that the unit of the magnetic flux quanta is given by $(1-c)4\pi/e$. But remember that in flat space-time the unit of the magnetic flux quanta of the anyonic solution of Chern-Simons electrodynamics is given by $4\pi/e$ [3]. Clearly the extra factor $(1-c)$ is due to the gravitational effect. The electric charge q is given by the well-known Chern-Simons relation

$$q = -\mu\Phi . \tag{18}$$

Now the total energy $\mathcal{E}$ and the total angular momentum J of the solution (15) are determined by the asymptotic form of the metric:

$$
\begin{aligned}
\mathcal{E} &= \frac{1}{\kappa}\oint_{r=\infty} dx^i \frac{\alpha_i}{2\pi} = \frac{1}{\kappa}\alpha_\infty = \frac{4\pi\mu m}{\mu\kappa m+e^2}\gamma_\pm + \frac{\alpha_0}{\kappa} ,\\
J &= \frac{1}{2\pi}\oint_{r=\infty} dx^i\sigma_i = \sigma_\infty = -\frac{2\pi\mu}{\mu\kappa m+e^2}\gamma_\pm + \alpha_0 .
\end{aligned} \tag{19}
$$

Notice that α_∞, which determines the total energy, is nothing but the deficit angle at infinity. So when solution (15) has zero total energy, which is possible if $\alpha_0 = -4\pi c\gamma_\pm$, the solution can describe an asymptotically flat but spinning Minkowski space-time.

Recently the possibility of the violation of causality around cosmic strings has been discussed by many authors [11,12]. It is generally believed that the existence of closed timelike curves and thus a possible violation of causality is a generic feature of (2+1)-dimensional spinning space-time and spinning cosmic strings. However, our result shows that this is not necessarily the case. To see this notice that solution (15) does not allow any closed timelike curves if

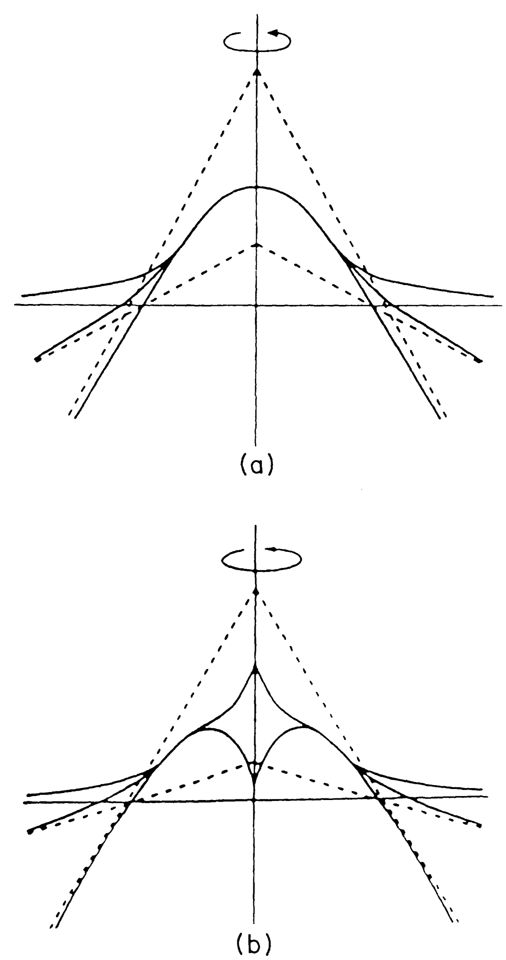

FIG. 1. The causally spinning cosmic strings: (a) shows the regular solutions and (b) shows the singular solutions. The dotted lines describe the asymptotic structure of (2+1)-dimensional space-time.

$$\rho r - \left| \frac{\kappa}{2\pi}\sigma \right| = b\frac{(r/a)^{1-\alpha_0/2\pi}}{[(r/a)^{2\gamma_\pm}+1]^c} - \left| \frac{\kappa\sigma_0}{2\pi} + \frac{c}{m}\gamma_\pm \frac{(r/a)^{2\gamma_\pm}}{(r/a)^{2\gamma_\pm}+1} \right| > 0 \ . \quad (20)$$

The condition at $r=0$ requires $\alpha_0 \geq 2\pi$ when $\sigma_0 \neq 0$. On the other hand the condition at $r=\infty$ requires $\alpha_\infty \leq 2\pi$. So causality requires $\sigma_0 = 0$ when $\alpha_0 < 2\pi$. But aside from this restriction the condition (20) can easily be satisfied with $\sigma_\infty \neq 0$, if the parameter b is sufficiently large. This means that *a cosmic spinning string which does not violate causality is possible.* Indeed, the result shows that one can always restore the causality of the spinning string by smearing out and regularizing the singular spinning source at the origin.

We conclude with the following remarks.

(1) The analytic solution (15) is possible only when the fermion field has a quartic interaction with a particular coupling strength $\lambda = \kappa/8$. When $\lambda \neq \kappa/8$ there may still be a solution, but the analytic solution (15) requires $\lambda = \kappa/8$.

(2) In the limit as κ goes to zero, the above solution reduces to a solution of Chern-Simons electrodynamics on a spinning cone. This is so because even when $\kappa = 0$, α_0 and σ_0 could still become arbitrary constants. If we further restrict the solution with $\alpha_0 = \sigma_0 = 0$, the solution reduces to the known solution [3] of Chern-Simons electrodynamics of a flat $(2+1)$-dimensional space-time.

(3) In this paper we have discussed for simplicity only the rotationally symmetric solutions which could describe an anyonic particle located at the origin. However, one could construct a more general solution which describes n anyonic particles located at different points. Indeed, the Liouville equation (10) guarantees the existence of such a solution.

A more detailed discussion on the subject will be published elsewhere [15].

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