Comment on "Adiabatic holonomy and evolution of fermionic coherent state"

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We argue that the way to get the true adiabatic evolution of fermionic coherent states (FCS's) is to use an averaged version of the quantum variational principle. Then Hannay's angle does not appear in the global phase picked by the FCS during its evolution but in the argument of the parameter specifying the FCS. In fact Abe's derivation of the classical holonomy must be understood in terms of canonical transformations.

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In a recent paper [1] concerning the extension of the classical adiabatic holonomy (Hannay's angle) to the Grassmannian case [2], Abe sheds new light on it by calling for the introduction of fermionic coherent states (FCS's). Indeed FCS's, like the usual coherent states (CS's), are known to be a suitable tool for exhibiting classical features at the quantum level. The model under consideration is the Grassmannian spin which, after quantization, becomes the well-known Pauli-spin system with the Hamiltonian $H(t)=\frac{1}{2}\mathbf{B}(t)\cdot\boldsymbol{\sigma}$. This Hamiltonian also reads

$$H(t)=|\mathbf{B}(t)|[b^\dagger(t)b(t)-\tfrac{1}{2}]$$

in terms of fermionic operators which connect the lowest eigenstate $|0,\mathbf{B}(t)\rangle$ of $H(t)$ to the highest one $|1,\mathbf{B}(t)\rangle$. One of the important results of [1] is that Hannay's angle for the Grassmannian system (which is known [2] to be the difference $\gamma_1^B(t)-\gamma_0^B(t)$ of Berry's phases $\gamma_n^B(t)$ ($n=0,1$) relative to the states $|n,\mathbf{B}(t)\rangle$) can be recovered by the introduction of the FCS

$$|\zeta(t),\mathbf{B}(t)\rangle=\exp[b^\dagger(t)\zeta(t)+b(t)\zeta^*(t)]|0,\mathbf{B}(t)\rangle$$

and the evolution of the states $|\zeta(0),\mathbf{B}(0)\rangle$. (Another result in [1], which we shall not comment on, concerns the inverse problem of deriving Berry's phase from the classical Hannay angle through quantization.) Our point concerns the precaution to take when dealing with adiabatic evolution of FCS's (or CS's). Although, following Ref. [1], we consider the simplest two-level system example, our discussion remains relevant for N-level systems and also in field theory (where Berry's phase is closely related to anomalies in some gauge theories with fermions). Our remarks, which contradict several intermediary results of Ref. [1] focus on the necessity, when dealing with FCS's (or CS's) in the adiabatic limit, to consider, not the Schrödinger equation as one does usually to get the Berry's phase of an energy eigenstate, but an averaged variational principle (where the average is over the argument of the parameter of the FCS). As we shall show, this is the only correct way to obtain Hannay's angle from the true adiabatic evolution of FCS's. The result is that, in contradiction with a generally received opinion, Hannay's angle does not appear as an external phase in front of the FCS but inside its parameter's argument. Then we shall prove that Abe's approach does not connect, as claimed, the holonomy with the evolution of FCS's but with canonical transformations. (This latter approach has been, in the classical commutative case, the one used in the pioneering papers [3,4].) Finally we shall comment on a recent assertion according to which, in the Grassmannian case, Berry's phase and the corresponding Hannay angle may be identified [5].

Our first remark concerns the true adiabatic evolution of FCS's. The simplest way to obtain it is to start from the adiabatic evolution

$$|0,\mathbf{B}(0)\rangle\to\exp[i\gamma_0(t)]|0,\mathbf{B}(t)\rangle$$

and

$$|1,\mathbf{B}(0)\rangle\to\exp[i\gamma_1(t)]|1,\mathbf{B}(t)\rangle$$

of the energy eigenstates of $H(0)$. As is well known, the global phases $\gamma_n(t)$ ($n=0,1$) contain a dynamical part $\gamma_n^D(t)=-\int_0^t E_n(s)ds$ and a geometrical one $\gamma_n^B(t)=i\int_0^t\langle n,\mathbf{B}(s)|\partial_s|n,\mathbf{B}(s)\rangle ds$ (Berry's phase). From these evolutions it is a trivial task to derive that of an initial FCS:

$$|\zeta(0),\mathbf{B}(0)\rangle=\exp[-\tfrac{1}{2}\zeta^*(0)\zeta(0)]\times[|0,\mathbf{B}(0)\rangle-\zeta(0)|1,\mathbf{B}(0)\rangle]\,.$$

It simply reads

$$|\zeta(0),\mathbf{B}(0)\rangle\to\exp[i\gamma_0(t)]|\zeta(0)\exp\{i[\gamma_1(t)-\gamma_0(t)]\},\mathbf{B}(t)\rangle.$$

Therefore, up to an external phase factor, the FCS associated with the Hamiltonian at time zero moves into a FCS associated with the Hamiltonian at time t. The main point of this elementary result is that it differs from that of Ref. [1] [formulas (22) and (23)]; in the true evolution, the argument $\gamma_1(t)-\gamma_0(t)$ of the parameter $\zeta(t)$ of the FCS does not only contain a dynamical part $\gamma_1^D(t)-\gamma_0^D(t)$ but also a geometrical one $\gamma_1^B(t)-\gamma_0^B(t)$. This geometrical part is nothing but (minus) Hannay's angle [2] and its

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presence at this place is quite natural. Indeed, as explained in Ref. [6] for ordinary CS's, the argument of the parameter specifying the CS is (minus) the classical angular coordinate of a representative point of the system in phase space, and therefore Hannay's angle which describes a geometrical drift of the canonical angle variable must appear in this argument. What the above calculus shows is that the Grassmannian case looks like the commutative one: Hannay's angle does not appear as the geometrical part of the external phase in front of the FCS [equal to $\gamma_0^B(t)$ and not to $\gamma_1^B(t)-\gamma_0^B(t)$] but as the geometrical part of the argument of $\zeta(t)$.

A second question of interest concerns the method which consists, in order to find the adiabatic evolution of coherent states (and FCS's in particular), of putting into the Schrödinger equation test vectors $|\psi(t)\rangle$ which are the product of an unknown phase factor with a predetermined coherent state: $|\psi(t)\rangle=\exp[i\phi(t)]|\psi_{\rm CS}(t)\rangle$. In Ref. [1] (and in some other papers) the overall phase $\phi(t)$ is actually obtained through the relation $\langle\psi(t)|i\partial_t-H(t)|\psi(t)\rangle=0$ [understanding, of course, that the predetermined state $|\psi_{\rm CS}(t)\rangle$ is the right one]. We claim that this apparently natural procedure is ill founded. The reason is that the so-called "solution" of the Schrödinger equation in the adiabatic limit is not an exact solution but an approximate one which becomes valid only in the limit. Consequently, searching for an *approximate solution,* one must call for some variational principle and not for the Schrödinger equation to get the right answer. As explained in Ref. [6], when dealing with adiabatic approximation, the usual variational principle (which is equivalent to the Schrödinger equation) has to be replaced by an averaged version:

$$\delta\left[\int\overline{\langle\psi(s)|i\partial_s-H(s)|\psi(s)\rangle}ds\right]=0\ .$$

[The overbar means that one first calculates the expression in the angular brackets for the value $\zeta(s)\exp(i\theta)$ of the parameter of the coherent state; then one averages over θ before varying the integral.] One way to understand the necessity of averaging is to remind ourselves that, in the classical case, Hannay's angle is obtained from a one-form averaged over the classical trajectory and that (as previously quoted) there is an equivalence between the argument of $\zeta(t)$ and the angular coordinate on the classical trajectory. Another way to convince oneself is to make an explicit calculation and to recover the result of the previous paragraph. Let $|\psi(t)\rangle=\lambda(t)|\zeta(t),\mathbf{B}(t)\rangle$ be the test vector. The quantity to extremize is then

$$i\lambda^*(s)\dot\lambda(s)+|\lambda(s)|^2\overline{\langle\zeta(s),\mathbf{B}(s)|(i\partial_s-H(s))|\zeta(s),\mathbf{B}(s)\rangle}\ .$$

A variation with respect to $\lambda^*(s)$ shows that the external phase $\phi(t)$ is the extremum value of the *average* quantity

$$\int_0^t\overline{\langle\zeta(s),\mathbf{B}(s)|(i\partial_s-H(s))|\zeta(s),\mathbf{B}(s)\rangle}ds\ .$$

From the definition of $|\zeta(s),\mathbf{B}(s)\rangle$ the calculation of the integrand is easy and one shows that $\phi(t)$ is the extremum of

$$\int_0^t\left\{\frac{i}{2}[\zeta^*(s)\dot\zeta(s)-\dot\zeta^*(s)\zeta(s)]+\dot\gamma_0(s)+\zeta^*(s)\zeta(s)[\dot\gamma_1(s)-\dot\gamma_0(s)]\right\}ds\ .$$

[Notice that the effect of averaging is to eliminate cross terms such as $\langle 0,\mathbf{B}(s)|\partial_s|1,\mathbf{B}(s)\rangle$.] If one now varies this later quantity with respect to $\zeta^*(s)$ one recovers the results of the first paragraph, which are

$$\zeta(t)=\zeta(0)\exp\{i[\gamma_1(t)-\gamma_0(t)]\}$$

and

$$\phi(t)=\gamma_0(t)\ .$$

In conclusion, our second remark is a serious criticism of the method followed in Ref. [1] [formulas (24)–(26)] to get the global phase in front of a coherent state.

Our last remark concerns the reason why, nevertheless, Abe obtains the exact Hannay angle from its formula (26). We now show that the right-hand side (RHS) $[(\zeta(t)|i\partial_t|\zeta(t))-(\zeta(t)|H(t)|\zeta(t))]$ of this formula (26) must not be interpreted as the time derivative of the global phase of a FCS but as the new Hamiltonian obtained after a time-dependent canonical transformation. In order to understand this point it is useful to recall two results concerning the connection between the quantum problem and its Grassmannian counterpart. (These remarks lie at the heart of Abe's approach and are quite explicit in his paper.) The first one is that the mean value of the quantum Hamiltonian in a FCS is the classical Hamiltonian. (For ordinary CS's this is true only in the classical limit.) The second one is that the mean value of the operator $i\partial_t$ in a FCS is the canonical one-form [i.e., the Grassmann analogue of $p(t)\dot q(t)$ of the commutative case]. Therefore the quantum variational principle with fermionic coherent states

$$\delta\left[\int_0^t\langle\psi_{\rm FCS}(s)|[i\partial_s-H(s)]|\psi_{\rm FCS}(s)\rangle ds\right]=0$$

is identical to the classical Hamilton's equations for the variables $\zeta(t)$. The question now is what should we take for FCS's? One may choose the FCS 's $|\zeta_0(t),\mathbf{B}(0)\rangle$ associated with the initial Hamiltonian $H(0)$. In that case the canonical variables ζ_0 are chosen once for all. One can also, at each time, choose the FCS $|\zeta(t),\mathbf{B}(t)\rangle$. This change of reference FCS, at each time, is equivalent to a time-dependent canonical transformation [7] $\zeta_0\to\zeta(\zeta_0,t)$. Such transformations have been the starting point for the derivation of Hannay's angle in the cummutative case [4]: the Hamiltonian which describes the evolution of the new canonical coordinates is equal to the old one (written in terms of this new coordinates) plus an extra term (the time derivative of the generating functional of the transformation) and Hannay's angle is the derivative of this averaged extra term with respect to the action. Let us now sketch what this procedure becomes in the Grassmannian case. When one inserts the FCS $|\zeta(t),\mathbf{B}(t)\rangle$ [instead of $|\zeta_0(t),\mathbf{B}(0)\rangle$] into the quantum variational principle, the mean value of $i\partial_t-H(t)$ gains the additional contribution

$$-i\langle \zeta(t),\mathbf{B}(t)|\partial_{\mathbf{B}}|\zeta(t),\mathbf{B}(t)\rangle\dot{\mathbf{B}}(t)\ .$$

This term, which originates from the mean of the time derivative operator, may be seen as a modification of the Hamiltonian [to compare with the first term of the RHS of formula (26)]. Then taking the derivative of the *average* $\zeta^*(t)\zeta(t)[\gamma_1^B(t)-\gamma_0^B(t)]$ of this quantity with respect to the Grassmannian action $\zeta^*(t)\zeta(t)$ [1] one recovers Hannay's angle $\theta^H(t)=\gamma_1^B(t)-\gamma_0^B(t)$.

In conlusion, we have clarified three important points: the correct expression of the adiabatic evolution of FCS's, the inadequacy of the Schrödinger equation to derive this evolution, and the distinction which must be done between evolution of FCS's and time-dependent canonical transformations. In so doing we have provided three different derivations of Hannay's angle (for the Grassmannian spin system) from FCS's. Abe's derivation seems closer to the last one although he never speaks of canonical transformations. We think that this ambiguity originates in the contradiction existing between its definition (15) of the FCS [which coincides with our states $|\zeta(t),\mathbf{B}(t)\rangle$] and relations (16)–(18) where he unduely identifies the *different* "vacua" $|0,\mathbf{B}(t)\rangle$ to a *unique* invariant one $|0\rangle$. [In practice (18) helps Abe to eliminate the above-mentioned cross terms which, in fact, can disappear only if one calls for the averaging procedure.] Nevertheless, the "second quantization" idea suggested by Abe's relation (18), although not valid in his approach, is interesting. The right way to introduce it is to define a true invariant vacuum $|0\rangle$ and creation operators $b_n^\dagger(t)$ ($n=0,1$), which bring it to the "one-particle" states $|n,\mathbf{B}(t)\rangle$. In such an approach, which has been generalized to an N-level system in Ref. [8], the FCS depends on two (N) Grassmannian parameters ζ_0 and ζ_1. The evolution of these FCS's is such that $\zeta_n(0)\rightarrow\zeta_n(0)\exp[i\gamma_n(t)]$; i.e., Hannay's angles coincide with the corresponding Berry phases. This identification has already been noticed in another context in Ref. [5]. It is, in our opinion, the most natural relation between the classical and quantum holonomies in the Grassmannian case because, contrary to the commutative case, the classical situation is obtained from the quantum one without any limit procedure ($\hbar\rightarrow 0$ or the action $I\rightarrow\infty$).

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