Chiral perturbation theory for the quenched approximation of QCD

Claude W. Bernard and Maarten F. L. Golterman
Department of Physics, Washington University, St. Louis, Missouri 63130
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We describe a technique for constructing the effective chiral theory for quenched QCD. The effective theory which results is a Lagrangian one, with a graded symmetry group which mixes Goldstone bosons and fermions, and with a definite (though slightly peculiar) set of Feynman rules. The straightforward application of these rules gives automatic cancellation of diagrams which would arise from virtual quark loops. The techniques are used to calculate chiral logarithms in f_K/f_π, m_π, m_K, and the ratio of $\langle \bar{s}s \rangle$ to $\langle \bar{u}u \rangle$. The leading finite-volume corrections to these quantities are also computed. Problems for future study are described.

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I. INTRODUCTION AND MOTIVATION

The quenched approximation [1] to QCD, in which virtual quark loops are neglected, is a necessary evil in lattice QCD simulations and will be with us for the foreseeable future. Even with the proposed QCD teraflop machine [2], the quenched approximation will be needed to approach the crucial corners of the parameter space: large volumes, physical quark masses, and the continuum limit. We therefore need to learn as much as possible analytically about the quenched approximation in order to have good control over the systematics of such calculations.

In the full theory, chiral perturbation theory (ChPT) is a key analytic tool. It gives the following.

1. It gives the detailed form of the approach to the chiral limit. The universal terms ("chiral logarithms") can be calculated order by order in the loop expansion. Comparison with this expected chiral behavior provides, for example, a crucial check of lattice weak matrix element calculations.

2. It gives the leading finite-volume corrections at large volume [3]. As the lightest particles, the pseudoscalar mesons clearly control these corrections; ChPT is simply the effective theory of their interactions.

It is therefore clear why one would like to have a ChPT corresponding to the quenched approximation. In fact, there have been several previous attempts to calculate quenched chiral logarithms. Morel [4] and Sharpe [5] use the strong coupling and $1/d$ expansions; Kilcup *et al.* [6] and Sharpe [7] use the quark-flow approach (see below). The papers by Sharpe in particular emphasize the importance of quenched ChPT and mention several of the key issues (in particular, the problems caused by the η'). A preliminary version of the current work has been presented in Ref. [8].

II. QUARK-FLOW APPROACH

In this approach, one starts with ordinary ChPT for full QCD and writes down all meson diagrams which contribute to the process of interest. To each meson diagram one then associates one or more quark-flow diagrams in QCD. Next, one eliminates all those quark-flow diagrams which have virtual quark loops. Finally, one attempts to reinterpret this elimination as conditions on the meson diagrams. Note that, in the case where more than one quark-flow diagram corresponds to a given meson diagram, it is by no means obvious that the final step can always be performed. We have been able to carry it through, more or less satisfactorily, in simple cases (see below), but have been unable to prove, within the context of this approach, that it can always be done.

In order to go back and forth between quark-flow diagrams and meson diagrams, the natural basis to use is the $q\bar{q}$ basis. In the neutral sector, this means that one works with $u\bar{u}$, $d\bar{d}$, and $s\bar{s}$ states rather than π^0, η, and η'. The latter basis is convenient in the full theory since one can treat the η' mass as "large," decouple it, and work only with π^0 and η. This turns out not to be possible in the quenched theory. In full QCD the η' gets the singlet part of its mass ($\equiv \mu$) through the iteration of quark-loop diagrams joined by gluons (see Fig. 1). In the approximation where the η' mass is much greater than the masses of the octet mesons, the η' decouples and may be neglected. In the quenched approximation, on the other hand, only the first two diagrams in Fig. 1 survive, and only the second diagram (the "two-hairpin" diagram—Fig. 2) depends on μ. The "two-hairpin vertex" in Fig. 2 is $\sim \mu^2$. Since the vertex is not iterated, μ^2 appears in the numerator, not in the denominator, of the η' propagator. Thus the η' cannot be neglected in the quenched approximation.

In many cases, it is immediately clear which full ChPT diagrams should be dropped in the quenched approximation. For example, consider the lowest-order correction to a π^+ propagator: a meson tadpole, shown in Fig. 3.

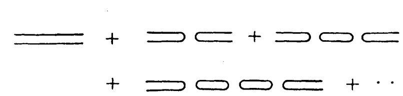

FIG. 1. Quark flow diagrams for the η' propagator in full QCD.

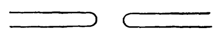

FIG. 2. The "two-hairpin" diagram, the only diagram which distinguishes the singlet-from the octet-meson propagator in the quenched approximation.

When the tadpole is a K^+, then the diagram must be absent in the quenched approximation, since the s quark in the K^+ is not present in the external states and must come from a virtual loop. When the tadpole is itself a π^+, however, the situation is less clear. If the s quark of the previous K^+ tadpole is replaced by a d, then again the diagram is absent in the quenched approximation. But there is now a second possibility: the valence quarks, themselves, could make the tadpole as in Fig. 4. The vertex in such a diagram is a meson-meson scattering vertex with no quark exchange. It turns out that such a vertex vanishes at $O(p^2)$ in ChPT, although we have never been able to prove this to our complete satisfaction within the quark-flow approach.[1] Thus π^+ tadpoles are also absent in the quenched approximation. Indeed, the only correction to the quenched π^+ propagator at this order comes from an η' tadpole with a single two-hairpin vertex, Fig. 5.

The end result of this approach can be described as a "Lagrangian + rules." The Lagrangian is the ordinary chiral Lagrangian corresponding to full QCD. The rules give the weighting of the diagrams and prescribe how to replace η' contributions by two-hairpin diagrams. However, we find this approach unsatisfactory for two reasons. First of all, it is difficult to make the application of the rules routine. As mentioned above, it is not obvious that one can always interpret the elimination of quark-flow diagrams with virtual loops as conditions on the mesons diagrams. One is, at the minimum, forced to prove the vanishing of various vertices (such as the meson-meson no-exchange vertex). Not only are convincing proofs elusive, but new processes may bring up

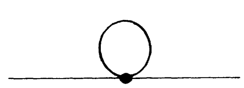

FIG. 3. The one-loop contribution to the pion propagator for full QCD, with π^+, π^0, K^+, K^0, η, and η' on the loop.

[1]For the number of flavors $N_F \geq 4$, it is easy to show that this vertex vanishes. In that case, one can choose all four participating quarks to be different and thereby make a unique correspondence between the quark vertex and a meson vertex. Examination of the trace structure of the $O(p^2)$ chiral Lagrangian then immediately gives the desired result [12]. However, the proof in this context for $N_F = 3$ escapes us, though the vertex certainly does vanish, as can be seen by working backward from the known result derived in the Lagrangian framework of Sec. III.

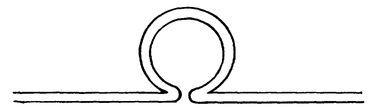

FIG. 4. A possible valence-quark contribution to the pion propagator.

new such vertices, so it is never clear that all the necessary proofs have been produced.

A second, more fundamental, problem with this approach is that the presence of the "rules" implies that one does not have a true Lagrangian theory. This means, for example, that the invariance of the physics under field redefinitions is not guaranteed. Such redefinitions are needed to reduce the number of terms in the Lagrangian involving the η' — see Ref. [9]. (These terms are not constrained much by symmetry because of the anomaly.) Similarly, the cancellation of the quartic divergences in ordinary ChPT is guaranteed by the chiral invariance of the Lagrangian and the measure. It is not clear (at least to us) whether the rules automatically respect this cancellation.

We therefore turn now to an alternative approach to quenched ChPT which gives a true Lagrangian framework and makes the calculation of quenched chiral logarithms routine.

III. A LAGRANGIAN FRAMEWORK

We start with QCD. To make a Lagrangian that describes the quenched approximation, we take the ordinary QCD Lagrangian and add, for each quark q_a ($a = u,d,s$), a scalar (ghost) quark $\tilde{q}_a$ with the same mass [4]. The ghost determinant then cancels the quark determinant. Of course, the resulting theory is not unitary in the quark sector; this is acceptable since the quenched approximation is not unitary.

Assuming that quark confinement still holds, the low-energy effective theory for this quenched QCD Lagrangian may now be constructed. It will describe the interactions of all possible pseudoscalar bound states of quarks or scalar quarks with their antiparticles: ordinary $q\bar{q}$ mesons ($\pi, K, \ldots$) which we denote, generically, by ϕ; ghost $\tilde{q}\bar{\tilde{q}}$ mesons denoted by $\tilde{\phi}$; and fermionic mesons $\tilde{q}\bar{q}$ and $q\bar{\tilde{q}}$ denoted by χ and $\chi^\dagger$, respectively.

As in ordinary ChPT, the symmetries at the quark level determine the form of the interactions among the mesons. The symmetry is $\mathrm{U}(3|3)_L \times \mathrm{U}(3|3)_R$, where

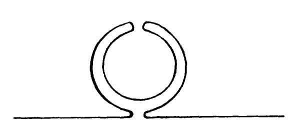

FIG. 5. The quark flow diagram for the one-loop contribution to the pion propagator in quenched QCD.

$U(3|3)$ is "almost" a $U(6)$ among $u,d,s,\tilde{u},\tilde{d},\tilde{s}$, but has a graded structure since it mixes fermions and bosons [10]. If we write a matrix $U \in U(3|3)$ in block form as

$$U = \begin{bmatrix} A & C \\ D & B \end{bmatrix} , \tag{1}$$

then A and B are 3×3 matrices of commuting numbers, C and D, of anticommuting. Unitarity is defined as usual: $U^\dagger U = I$. Hermitian conjugation ($\dagger$) also has the usual definition (complex conjugation of the usual transpose), but complex conjugation is defined to switch the order of anticommuting variables: $(\epsilon_1\epsilon_2)^* = \epsilon_2^*\epsilon_1^*$. There is also a cyclic "supertrace" defined by $\mathrm{str}(U) = \mathrm{tr}(A) - \mathrm{tr}(B)$, and a "superdeterminant," $\mathrm{sdet}(U) = \exp(\mathrm{str} \ln U)$, with the property $\mathrm{sdet}(U_1 U_2) = \mathrm{sdet}(U_1)\mathrm{sdet}(U_2)$. Explicitly,

$$\mathrm{sdet}(U) = \det(A - CB^{-1}D)/\det(B) . \tag{2}$$

Now define the Hermitian field Φ and the mass matrix $\mathcal{M}$ by

$$\Phi \equiv \begin{bmatrix} \phi & \chi^\dagger \\ \chi & \tilde{\phi} \end{bmatrix} , \quad \mathcal{M} \equiv \begin{bmatrix} M & 0 \\ 0 & M \end{bmatrix} , \tag{3}$$

where

$$M = \begin{bmatrix} m_u & 0 & 0 \\ 0 & m_d & 0 \\ 0 & 0 & m_s \end{bmatrix} , \tag{4}$$

is the usual quark-mass matrix. Note that, to lowest order in M, these ChPT quark masses are the same as those of QCD.

The unitary field $\Sigma \equiv \exp(2i\Phi/f)$ transforms as $\Sigma \to U_L \Sigma U_R^\dagger$. The Lagrangian invariant under the full $U(3|3)_L \times U(3|3)_R$ is then

$$\mathcal{L}_{\mathrm{inv}} = \frac{f^2}{8}\mathrm{str}(\partial_\mu\Sigma\partial^\mu\Sigma^\dagger) + v\,\mathrm{str}(\mathcal{M}\Sigma + \mathcal{M}\Sigma^\dagger) , \tag{5}$$

where f and v are as yet undetermined bare coupling constants. This looks very much like ordinary ChPT.

The anomaly breaks the symmetry group down to $SU(3|3)_L \times SU(3|3)_R \times U(1)$. The anomalous field is $\Phi_0 \equiv (\eta' - \tilde{\eta}')/\sqrt{2}$, where the minus sign comes from the relative minus sign between boson and fermion loops. Under the reduced group, $\Phi_0 \propto \mathrm{str} \ln\Sigma = \ln \mathrm{sdet}\Sigma$ is invariant, so arbitrary functions of Φ_0 can be included in the full Lagrangian $\mathcal{L}$. However, in the current framework one can redefine Σ to simplify $\mathcal{L}$, much as in Ref. [9]. The result is

$$\begin{aligned} \mathcal{L} = {} & V_1(\Phi_0)\mathrm{str}(\partial_\mu\Sigma\partial^\mu\Sigma^\dagger) + V_2(\Phi_0)\mathrm{str}(\mathcal{M}\Sigma + \mathcal{M}\Sigma^+) \\ & - V_0(\Phi_0) + V_5(\Phi_0)(\partial_\mu\Phi_0)^2 , \end{aligned} \tag{6}$$

$$\mathcal{L}(\Phi_0 = 0) \equiv \mathcal{L}_{\mathrm{inv}} , \tag{7}$$

where the functions V_i can be chosen to be real and even by making use of the freedom allowed by field redefinitions. In Ref. [9] a different choice is made: V_5 is set to 0 but V_2 is kept complex. The potentials V_3 and V_4 from Ref. [9] are not needed for the purposes of this paper and have been dropped. Note that the notation in Eqs. (6) and (7) is slightly different from that used in Ref. [8]. For the purposes of this paper, we need only the quadratic terms in $\mathcal{L}_{\Phi_0}$. We have

$$\begin{aligned} & \mathcal{L} = \mathcal{L}_{\mathrm{inv}} + \alpha(\partial_\mu\Phi_0)^2 - \mu^2\Phi_0^2 + \cdots , \\ & \alpha \equiv V_5(0) , \\ & \mu^2 \equiv \tfrac{1}{2}V_0''(0) . \end{aligned} \tag{8}$$

One can now calculate straightforwardly with $\mathcal{L}$. Note that because of the minus sign in the definition of str, some of the fields will have negative metrics. An unusual feature occurs in the $\eta', \tilde{\eta}'$ sector: terms from V_0 and V_5 have a different matrix structure from those of $\mathcal{L}_{\mathrm{inv}}$, and one cannot diagonalize the quadratic Lagrangian in a momentum-independent way. This leads us to treat the quadratic terms from V_0 and V_5 as vertices. Iterations of these vertices on the same line then automatically vanish due to cancellation between the η' and the negative metric $\tilde{\eta}'$. This is a manifestation of the fact that the iteration of the two-hairpin vertex is forbidden in quenched QCD. When $m_s \neq m$ another peculiarity occurs: the π^0 is the only well-behaved neutral particle. The propagators of the orthogonal states do not have simple-pole structures. When also $m_u \neq m_d$, even the π^0 propagator becomes ill behaved.

Because of the unusual structure of the neutral sector, it is convenient, both in the formalism and in actual computations, to write the neutral meson propagators in the basis of the states corresponding to $u\bar{u}$, $d\bar{d}$, $s\bar{s}$ and their ghost counterparts. As mentioned above, this is unlike the case of full QCD, where, due to the singlet part of the η' mass, the propagators in this sector are diagonal in the π^0, η, η' basis.

Since we have a true Lagrangian theory, the symmetry should guarantee that any quartic divergences in the diagrammatic expansion will be canceled by contributions from the measure, just as in the full theory. We have explicitly checked this in the $SU(1|1)_L \times SU(1|1)_R$ case. The $SU(3|3)_L \times SU(3|3)_R$ case is considerably more complicated; however, it turns out that no quartic divergences appear in any of the calculations presented below.

The vertices from V_0 and V_5 can in principle appear more than once in a diagram if they occur on different lines. However, it is our philosophy to treat the parameters μ^2 and α as small. This is certainly true in the $1/N_c$ expansion. (Recall that in full QCD the η' gets its singlet mass at order $1/N_c$ [11]). Moreover, it appears that the real expansion parameters are $\alpha/3$ and $\mu^2/3$ (see below). To estimate the size of the one-loop corrections, one may take $\alpha \equiv 0$, neglect η-η' mixing, and use the physical η' mass. One gets $\mu^2/3 \simeq (500\ \mathrm{MeV})^2 \simeq m_K^2$, which leads one to expect that quenched ChPT should be roughly as good as full ChPT for the koan.

IV. RESULTS AND CONCLUSIONS

We have calculated, at one loop, m_π, m_K, f_π, f_K, $\langle \bar{u}u \rangle$, and $\langle \bar{s}s \rangle$. In the isospin limit ($m_u = m_d \equiv m$) and at infinite volume, we get

$$
\begin{aligned}
(m_\pi^{1\,\text{loop}})^2 &= m_\pi^2\left\{1+\frac{1}{8\pi^2 f^2}\left[\frac{\alpha}{3}\Lambda^2-\frac{\mu^2}{3}+\frac{\alpha}{3}m_\pi^2+\left[\frac{\mu^2}{3}-\frac{2\alpha}{3}m_\pi^2\right]\ln(\Lambda^2/m_\pi^2)\right]\right\},\\
(m_K^{1\,\text{loop}})^2 &= m_K^2\left\{1+\frac{1}{8\pi^2 f^2}\left[\frac{\alpha}{3}\Lambda^2+\left[\frac{\mu^2}{3}-\frac{2\alpha}{3}m_K^2\right]\ln(\Lambda^2/m_\pi^2)\right.\right.\\
&\qquad\left.\left.-\left[\frac{\mu^2}{3}-\frac{\alpha}{3}(2m_K^2-m_\pi^2)\right]\frac{2m_K^2-m_\pi^2}{2(m_K^2-m_\pi^2)}\ln\left[\frac{2m_K^2}{m_\pi^2}-1\right]\right]\right\},\\
f_\pi^{1\,\text{loop}} &= f\,,\\
\left[\frac{f_K}{f_\pi}\right]^{1\,\text{loop}} &= 1+\frac{1}{16\pi^2 f^2}\left[\frac{\alpha}{3}m_K^2-\frac{\mu^2}{3}+\frac{(\mu^2/3)m_K^2-(\alpha/3)m_\pi^2(2m_K^2-m_\pi^2)}{2(m_K^2-m_\pi^2)}\ln\left[\frac{2m_K^2}{m_\pi^2}-1\right]\right],\\
m\langle\bar{u}u\rangle^{1\,\text{loop}} &= -\frac{1}{4}(m_\pi^{1\,\text{loop}})^2 f^2\\
m_s\langle\bar{s}s\rangle^{1\,\text{loop}} &= -\frac{1}{4}(2m_K^2-m_\pi^2)f^2\left\{1+\frac{1}{8\pi^2 f^2}\left[\frac{\alpha}{3}\Lambda^2+\left[\frac{\mu^2}{3}-\frac{2\alpha}{3}(2m_K^2-m_\pi^2)\right]\ln[\Lambda^2/(2m_K^2-m_\pi^2)]\right.\right.\\
&\qquad\left.\left.+\frac{\alpha}{3}(2m_K^2-m_\pi^2)-\frac{\mu^2}{3}\right]\right\},
\end{aligned}
\tag{9}
$$

where Λ is the cutoff, and m_K, m_π, and f are the bare parameters:

$$m_\pi^2=\frac{8vm}{f^2},\quad m_K^2=\frac{4v(m_s+m)}{f^2}\,. \tag{10}$$

It should be noticed that, except for the Λ^2 terms, α terms are actually higher order in a combined expansion in $1/N_c$ and M. This implies that, apart from quadratically divergent terms, we may set $\alpha=0$ in Eq. (9) systematically.

Note that in the quenched approximation the ratio f_K/f_π is finite at one loop, unlike the full theory where this quantity contains a logarithmic divergence. In fact, if we consider ratios in which the quadratic divergence cancels, and then set $\alpha=0$ as argued above, the ratios $(m_K^{1\,\text{loop}}/m_\pi^{1\,\text{loop}})^2$ and $(\langle\bar{s}s\rangle/\langle\bar{u}u\rangle)^{1\,\text{loop}}$ are also finite. Expressed in terms of the bare quark masses we have

$$
\begin{aligned}
\left[\frac{m_K^{1\,\text{loop}}}{m_\pi^{1\,\text{loop}}}\right]^2 &= \frac{m+m_s}{2m}\left[1+\frac{\mu^2/3}{8\pi^2 f^2}\left[1-\frac{m_s}{m_s-m}\ln(m_s/m)\right]\right],\\
\frac{\langle\bar{s}s\rangle^{1\,\text{loop}}}{\langle\bar{u}u\rangle^{1\,\text{loop}}} &= 1-\frac{\mu^2/3}{8\pi^2 f^2}\ln(m_s/m)\,.
\end{aligned}
\tag{11}
$$

Despite the fact that many of our one-loop results are finite, they are not quantitative predictions because the terms in the $O(p^4)$ Lagrangian (the L_i's of Gasser and Leutwyler [9]) may also contribute. In other words, we have computed the "chiral logarithms" only, not what are usually called the "finite terms" (which are always uncomputable in ChPT). One would need in general to take further ratios of physical quantities to eliminate such uncertainties.

Using $\alpha=0$ and μ as estimated above and neglecting the "finite terms," $(f_K/f_\pi)^{1\,\text{loop}}\cong 1.07$, indicating that quenched ChPT is working well. Note however that $(m_\pi^{1\,\text{loop}}/m_\pi)^2\cong 1.5$ for $\Lambda\cong 1$ GeV and $(\langle\bar{s}s\rangle/\langle\bar{u}u\rangle)^{1\,\text{loop}}\cong 0.4$, although these ratios are not directly physical, and the large corrections should perhaps not be worrisome.

We have also computed the leading finite-volume corrections to the above results. The calculation is straightforward: we simply replace the infinite-volume meson propagators by their finite-volume counterparts. We find

$$
\begin{aligned}
\Delta((m_\pi^{1\,\text{loop}})^2) &= \frac{m_\pi^2}{4\pi^2 f^2}(\mu^2-\alpha m_\pi^2)\left[\frac{2\pi}{m_\pi L}\right]^{1/2} e^{-m_\pi L},\\
\Delta((m_K^{1\,\text{loop}})^2) &= 0\,,\\
\Delta(f_\pi^{1\,\text{loop}}) &= 0\,,\\
\Delta((f_K/f_\pi)^{1\,\text{loop}}) &\\
&= \frac{1}{16\pi^2 f^2}(\mu^2-\alpha m_\pi^2)\left[\frac{2\pi}{m_\pi L}\right]^{1/2} e^{-m_\pi L},\\
\Delta(m\langle\bar{u}u\rangle^{1\,\text{loop}}) &= -\frac{m_\pi^2}{16\pi^2}(\mu^2-\alpha m_\pi^2)\left[\frac{2\pi}{m_\pi L}\right]^{1/2} e^{-m_\pi L},\\
\Delta(m_s\langle\bar{s}s\rangle^{1\,\text{loop}}) &= 0\,,
\end{aligned}
\tag{12}
$$

where Δ (quantity) denotes leading order finite volume corrections to be added to the infinite volume one-loop expressions for the quantities given in Eq. (9). L is the spatial size of the box. We have assumed periodic boundary conditions[2] and $T \gg L$, where T is the temporal size

[2]In order to preserve the graded symmetry, the ghost quarks must have the same boundary conditions (antiperiodic or periodic) as are chosen for the quarks. All mesons, including the fermionic ones, will thus have periodic boundary conditions.

of the box. In addition we have neglected terms of order $e^{-m_\pi(\sqrt{2}L)}$ and of order $e^{-m_K L}$. Thus the leading corrections come only from pions propagating from the closest periodic images of the original box. Finally, we have taken only the leading contribution to the pion propagator and neglected terms of order $(m_\pi L)^{-3/2}e^{-m_\pi L}$. Note that the terms we have neglected may very well not be small in many current lattice simulations. In such cases, the exact one-loop finite volume corrections (computed by using the exact finite-volume propagators) should be used.

Some comments on our results and directions for future work follow.

1. The absence of chiral logarithms in m_π seen in Ref. [5] is presumably a feature of the leading term in the $1/d$ expansion. Indeed, the η' diagrams which give such logarithms are mentioned in Ref. [7].

2. Many of these results [e.g., $(f_K/f_\pi)^{1\text{ loop}}$, $(\langle\bar{s}s\rangle/\langle\bar{u}u\rangle)^{1\text{ loop}}$, or $m_\pi^{1\text{ loop}}/m_\pi$] blow up as $m\to 0$. This is an IR effect coming from the double pole of the two-hairpin diagram and is absent in the full theory where the vertex is iterated. It is not clear at this point whether this is a sickness of the quenched approximation or only of the current quenched chiral expansion.

3. A Gasser-Leutwyler [9] program for quenched ChPT at one loop is possible: we expect there to be interesting numerical relations involving only computable (on the lattice) quantities. Such relations could give quantitative insight into the effects of quenching.

4. The techniques described here can be easily used to generate the effective theory for a QCD in which the quark loops are not neglected, but the masses of valence and virtual quarks are not identical. Such an effective theory is relevant to many "full QCD" simulations. Similarly, one can generate effective theories which correspond to the quenching of some, but not all the light quarks.

5. It seems to be straightforward to extend these ideas to the calculation of the chiral logarithms in weak matrix elements.

ACKNOWLEDGMENTS

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