Maxwell-Chern-Simons Casimir effect. II. Circular boundary conditions

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In odd-dimensional spaces, gauge invariance permits a Chern-Simons mass term for the gauge fields in addition to the usual Maxwell-Yang-Mills kinetic energy term. We study the Casimir effect in such a $(2+1)$-dimensional Abelian theory. The case of parallel conducting lines was considered by us in a previous paper. Here we discuss the Casimir effect for a circle and examine the effect of finite temperature. The Casimir stress is found to be attractive at both low and high temperatures.

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I. INTRODUCTION

By now it is well known that, for theories in odd-dimensional spaces, one can add a gauge-invariant Chern-Simons mass term for the gauge field in addition to the usual Maxwell-Yang-Mills term [1, 2]. Recently there has been considerable interest in such a $(2+1)$-dimensional Abelian theory in connection with the studies of the fractional quantum Hall effect [3] in semiconductors and of high-T_c superconductivity [4] in copper-oxide crystals. The Lagrangian for the Maxwell-Chern-Simons theory written in curvilinear coordinates is

$$\mathcal{L} = -\sqrt{-g}\tfrac{1}{4}F^{\mu\nu}F_{\mu\nu} + \tfrac{1}{4}\mu\epsilon^{\mu\alpha\beta}F_{\alpha\beta}A_\mu, \tag{1.1}$$

where g is the determinant of the metric $g_{\mu\nu}$ and

$$\epsilon^{\mu\alpha\beta} = \sqrt{-g}e^{\mu\alpha\beta} \tag{1.2}$$

is a tensor density, with $\epsilon^{012} = 1$. In terms of the dual tensor

$$\begin{aligned} F^\lambda &= \tfrac{1}{2}e^{\lambda\alpha\beta}F_{\alpha\beta} \\ &= \frac{1}{2\sqrt{-g}}\epsilon^{\lambda\alpha\beta}(\partial_\alpha A_\beta - \partial_\beta A_\alpha), \end{aligned} \tag{1.3}$$

we can rewrite (1.1) as

$$\mathcal{L} = \tfrac{1}{2}\sqrt{-g}(F^\lambda F_\lambda + \mu F^\lambda A_\lambda). \tag{1.1$'$}$$

Varying $\mathcal{L}$ with respect to A_μ we find the equations of motion

$$\epsilon^{\mu\alpha\beta}\partial_\alpha F_\beta + \mu\sqrt{-g}F^\mu = 0, \tag{1.4}$$

which satisfy the Bianchi identity

$$\partial_\mu(\sqrt{-g}F^\mu) = 0, \tag{1.5}$$

consistent with (1.3). We can identify μ as the mass of the gauge field by using (1.4) to show that in Cartesian coordinates

$$(-\partial^\lambda\partial_\lambda + \mu^2)F^\nu = 0. \tag{1.6}$$

The equations of motion (1.4) are obviously invariant under a gauge transformation, while the Lagrangian changes only by an irrelevant total derivative.

In a previous paper [5] we considered the Casimir effect between parallel conducting lines in two spatial dimensions for the Maxwell-Chern-Simons theory defined by (1.4). At zero temperature we found an attractive force per unit length given by (47) of Ref. [5]:

$$f = -\frac{1}{16\pi a^3}\int_{2\mu a}^\infty dy\frac{y^2}{e^y - 1}, \tag{1.7}$$

where a is the separation of the two parallel lines. At high temperature, $\beta = (kT)^{-1} \ll 4\pi a$, we found

$$f^{T\to\infty} \approx -\frac{1}{4\pi\beta a^2}\int_{2\mu a}^\infty dy\frac{y^2}{e^y - 1}\frac{1}{\sqrt{y^2 - 4\mu^2a^2}}. \tag{1.8}$$

In this paper we turn to the case of a circular boundary. In Sec. II we compute the Casimir self-stress starting with the reduced Green's functions, both inside and outside the conducting circle. The representation of the product of two fields at a point is obtained by allowing the two field points in the Green's function to become infinitesimally close. A formula for the force on the circle is obtained in terms of Bessel functions. In Sec. III uniform asymptotic expansions for the Bessel functions are used to obtain an approximate numerical result for various values of the Chern-Simons mass μ. In Sec. IV we examine this effect in the limit of high temperatures. Physically, to obtain a (effectively) two-dimensional system, one needs to freeze the degrees of freedom in the third direction perpendicular to the two-dimensional plane. According to quantum mechanics, it takes a finite amount of energy to excite the motion in the third direction. So, if the temperature is low enough, all the particles will remain in the ground state for those degrees of freedom and the system behaves as if there were only two spatial directions. Implicit in our discussion of the high-temperature limit in Sec. IV is the assumption that the temperature is sufficiently low so that the degrees of freedom in the perpendicular directions are not excited. However, the high-temperature result for the $2+1$ theory is of field-theoretic interest in its own right. In Sec. V we give a brief discussion, and a comparison with the massless $(3+1)$-dimensional Casimir effect for a cylinder [6]. For that comparison we also need the result of the (massless) scalar field case for the circle, the derivation of which is relegated to the Appendix.

II. CASIMIR SELF-STRESS ON A CIRCLE

We can rewrite the Lagrangian in (1.1′) in terms of the fundamental variable A_μ as

$$\mathcal{L} = \frac{1}{2}\frac{1}{\sqrt{-g}} g_{\lambda\mu}\epsilon^{\lambda\alpha\beta}\epsilon^{\mu\sigma\tau}\partial_\alpha A_\beta \partial_\sigma A_\tau + \tfrac{1}{2}\mu\epsilon^{\lambda\alpha\beta}\partial_\alpha A_\beta A_\lambda. \tag{1.1''}$$

Note that the last term is independent of $g_{\mu\nu}$. Varying the Lagrangian (1.1″) with respect to $g_{\mu\nu}$,

$$\delta\mathcal{L} = \tfrac{1}{2}\delta g_{\alpha\beta} t^{\alpha\beta}, \tag{2.1}$$

we find

$$t^{\alpha\beta} = \sqrt{-g}(F^\alpha F^\beta - \tfrac{1}{2}g^{\alpha\beta}F_\lambda F^\lambda) \tag{2.2}$$

for the stress tensor density for the photon, where we have used $\delta\sqrt{-g} = \sqrt{-g}\frac{1}{2}g^{\mu\nu}\delta g_{\mu\nu}$.

Next, we introduce the propagator $D_{\mu\nu}$ for the A_μ field according to

$$A_\mu(x) = \int dx' \sqrt{-g(x')} D_\mu{}^\nu(x,x') J_\nu(x'), \tag{2.3}$$

where J_ν is the source for the A_μ field. Equivalently, $D_{\mu\nu}$ is given by the time-ordered vacuum expectation values

$$D_{\mu\nu}(x,x') = i\langle A_\mu(x)A_\nu(x')\rangle. \tag{2.4}$$

Similarly, we introduce the Green's function $G_{\mu\nu}$ according to

$$F_\mu(x) = \int dx' \sqrt{-g(x')} G_\mu{}^\nu(x,x') J_\nu(x'). \tag{2.5}$$

The equations of motion (1.4) imply that $G_{\mu\nu}$ satisfies the equation

$$e_\mu{}^{\nu\lambda}\partial_\nu G_\lambda{}^\alpha + \mu G_\mu{}^\alpha = -\frac{1}{\sqrt{-g}} g_\mu{}^\alpha \delta(x-x'), \tag{2.6}$$

where $e_\mu{}^{\nu\lambda} = g_{\mu\beta}\epsilon^{\beta\nu\lambda}/\sqrt{-g}$. Equations (1.3), (2.3), and (2.5) can be used to show that

$$G_{\mu\nu}(x,x') = \frac{1}{\sqrt{-g}}\epsilon_\mu{}^{\alpha\beta}\partial_\alpha D_{\beta\nu}(x,x') \tag{2.7}$$

or, with the help of (2.4),

$$G_{\mu\nu}(x,x') = i\langle F_\mu(x)A_\nu(x')\rangle. \tag{2.8}$$

The vacuum expectation value of the stress tensor can now be put in terms of the Green's function by using

$$i\langle F^\mu(x)F^\nu(x')\rangle = \frac{i}{\sqrt{-g'}}\epsilon^{\nu\alpha\beta}\partial'_\alpha\langle F^\mu(x)A_\beta(x')\rangle = \frac{1}{\sqrt{-g'}}\epsilon^{\nu\alpha\beta}\partial'_\alpha G^\mu{}_\beta(x,x'), \tag{2.9}$$

where $g' = g(x')$. We then have

$$\langle t^{\alpha\beta}\rangle = \lim_{x'\to x}\frac{1}{i}(\epsilon^{\beta\gamma\sigma}\partial'_\gamma G^\alpha{}_\sigma - \tfrac{1}{2}g^{\alpha\beta}g_{\mu\nu}\epsilon^{\nu\gamma\sigma}\partial'_\gamma G^\mu{}_\sigma), \tag{2.10}$$

where the limit $x' \to x$ is to be taken symmetrically. For the problem at hand we use polar coordinates

$$x^\mu = (t, r, \theta), \tag{2.11a}$$

so that the metric is given by

$$g_{\mu\nu} = (-1, 1, r^2), \quad \sqrt{-g} = r. \tag{2.11b}$$

Then the $\langle t^{11}\rangle$ component is given by

$$\begin{aligned}\langle t^{11}\rangle &= \langle t^{rr}\rangle \\ &= \frac{1}{2i}\left(\frac{\partial}{\partial\theta'}G^1{}_0 - \frac{\partial}{m\partial t'}G^1{}_2\right) \\ &\quad + \frac{1}{2i}\left(\frac{\partial}{\partial r'}G^0{}_2 - \frac{\partial}{\partial\theta'}G^0{}_1\right) \\ &\quad - \frac{r^2}{2i}\left(\frac{\partial}{\partial t'}G^2{}_1 - \frac{\partial}{\partial r'}G^2{}_0\right).\end{aligned} \tag{2.12}$$

We further introduce the Fourier transform appropriate to the polar coordinates:

$$G_\mu{}^\nu(x,x') = \int\frac{d\omega}{2\pi}e^{-i\omega(t-t')}\sum_{m=-\infty}^{\infty} e^{im(\theta-\theta')}\mathcal{G}_\mu{}^\nu(r,r'), \tag{2.13}$$

where we have suppressed the dependence of the reduced Green's function $\mathcal{G}$ on m and ω. In terms of the Fourier transform

$$\begin{aligned}\langle t^{11}\rangle &= -\frac{m}{2}(\mathcal{G}^1{}_0 - \mathcal{G}^0{}_1) - \frac{\omega}{2}(\mathcal{G}^1{}_2 + r^2\mathcal{G}^2{}_1) \\ &\quad + \frac{1}{2i}\left(\frac{\partial}{\partial r'}\mathcal{G}^0{}_2 + r^2\frac{\partial}{\partial r'}\mathcal{G}^2{}_0\right) \\ &= \frac{m}{2}(\mathcal{G}_1{}^0 - \mathcal{G}_0{}^1) - \frac{\omega}{2}(r^2\mathcal{G}_1{}^2 + \mathcal{G}_2{}^1) \\ &\quad - \frac{1}{2i}\left(\frac{\partial}{\partial r'}r'^2\mathcal{G}_0{}^2 + \frac{\partial}{\partial r'}\mathcal{G}_2{}^0\right),\end{aligned} \tag{2.14}$$

where the limit $r' \to r$ is understood, and we have suppressed all the (obvious) arguments. Henceforth, unless stated otherwise, by $\mathcal{G}$ we mean $\mathcal{G}(r,r')$.

We now must solve the Green's function equation (2.6) for the various components which appear in (2.14). The corresponding equations for the reduced Green's functions fall into three groups. The first involves the ${}_0{}^0$, ${}_1{}^0$, and ${}_2{}^0$ components:

$$-\frac{1}{r}\frac{\partial}{\partial r}\mathcal{G}_2{}^0 + \frac{im}{r}\mathcal{G}_1{}^0 + \mu\mathcal{G}_0{}^0 = -\frac{1}{2\pi r}\delta(r-r'), \quad (2.15a)$$

$$\frac{im}{r}\mathcal{G}_0{}^0 + \frac{i\omega}{r}\mathcal{G}_2{}^0 + \mu\mathcal{G}_1{}^0 = 0, \quad (2.15b)$$

$$-i\omega r\mathcal{G}_1{}^0 - r\frac{\partial}{\partial r}\mathcal{G}_0{}^0 + \mu\mathcal{G}_2{}^0 = 0. \quad (2.15c)$$

We combine these equations to find the second-order equation satisfied by $\mathcal{G}_0{}^0$:

$$\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} - \frac{m^2}{r^2} + \omega^2 - \mu^2\right)\mathcal{G}_0{}^0 = -\frac{\omega^2-\mu^2}{2\pi\mu r}\delta(r-r'). \quad (2.16a)$$

From $\mathcal{G}_0{}^0$ we can determine the two other Green's functions according to

$$\mathcal{G}_2{}^0 = -\frac{m\omega + \mu r\frac{\partial}{\partial r}}{\omega^2-\mu^2}\mathcal{G}_0{}^0, \quad (2.16b)$$

$$\mathcal{G}_1{}^0 = -\frac{im}{\mu r}\mathcal{G}_0{}^0 - \frac{i\omega}{\mu r}\mathcal{G}_2{}^0. \quad (2.16c)$$

Similarly, the ${}_0{}^1$, ${}_1{}^1$, and ${}_2{}^1$ components of (2.6) are

$$-\frac{1}{r}\frac{\partial}{\partial r}\mathcal{G}_2{}^1 + \frac{im}{r}\mathcal{G}_1{}^1 + \mu\mathcal{G}_0{}^1 = 0, \quad (2.17a)$$

$$\frac{im}{r}\mathcal{G}_0{}^1 + \frac{i\omega}{r}\mathcal{G}_2{}^1 + \mu\mathcal{G}_1{}^1 = -\frac{1}{2\pi r}\delta(r-r'), \quad (2.17b)$$

$$-i\omega r\mathcal{G}_1{}^1 - r\frac{\partial}{\partial r}\mathcal{G}_0{}^1 + \mu\mathcal{G}_2{}^1 = 0, \quad (2.17c)$$

which can be combined to yield

$$\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} - \frac{m^2}{r^2} + \omega^2 - \mu^2\right)\mathcal{G}_0{}^1 = -\frac{i}{2\pi\mu r}\left(\frac{m\mu}{r} - \omega\frac{\partial}{\partial r}\right)\delta(r-r'), \quad (2.18a)$$

$$\mathcal{G}_1{}^1 = \frac{1}{\omega^2-\mu^2}\left(\frac{im\mu}{r} + i\omega\frac{\partial}{\partial r}\right)\mathcal{G}_0{}^1 - \frac{\mu}{2\pi r}\frac{\delta(r-r')}{\mu^2-\omega^2}, \quad (2.18b)$$

$$\mathcal{G}_2{}^1 = -\frac{m}{\omega}\mathcal{G}_0{}^1 + \frac{i\mu r}{\omega}\mathcal{G}_1{}^1 + \frac{i}{2\pi\omega}\delta(r-r'). \quad (2.18c)$$

Henceforth, we will ignore the δ functions in (2.18b) and (2.18c) because we are interested in the *limit* $r \to r'$.

Finally, the ${}_0{}^2$, ${}_1{}^2$, and ${}_2{}^2$ components satisfy

$$-\frac{1}{r}\frac{\partial}{\partial r}\mathcal{G}_2{}^2 + \frac{im}{r}\mathcal{G}_1{}^2 + \mu\mathcal{G}_0{}^2 = 0, \quad (2.19a)$$

$$\frac{im}{r}\mathcal{G}_0{}^2 + \frac{i\omega}{r}\mathcal{G}_2{}^2 + \mu\mathcal{G}_1{}^2 = 0, \quad (2.19b)$$

$$-i\omega r\mathcal{G}_1{}^2 - r\frac{\partial}{\partial r}\mathcal{G}_0{}^2 + \mu\mathcal{G}_2{}^2 = -\frac{1}{2\pi r}\delta(r-r'), \quad (2.19c)$$

which can be combined to yield

$$\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} - \frac{m^2}{r^2} + \omega^2 - \mu^2\right)\mathcal{G}_0{}^2 = -\left(\frac{m\omega}{\mu r^2} - \frac{1}{r}\frac{\partial}{\partial r}\right)\frac{1}{2\pi r}\delta(r-r'), \quad (2.20a)$$

$$\mathcal{G}_2{}^2 = -\frac{1}{\omega^2-\mu^2}\left(m\omega + \mu r\frac{\partial}{\partial r}\right)\mathcal{G}_0{}^2 + \frac{\mu\delta(r-r')}{2\pi r(\omega^2-\mu^2)}, \quad (2.20b)$$

$$\mathcal{G}_1{}^2 = -\frac{im}{\mu r}\mathcal{G}_0{}^2 - \frac{i\omega}{\mu r}\mathcal{G}_2{}^2. \quad (2.20c)$$

Again we ignore the δ function in (2.20b) in what follows.

We solve these equations for the reduced Green's functions subject to perfect conductor boundary conditions at $r = a$. That is, the tangential electric field must vanish on the circle, or in terms of the dual field,

$$F_1 = F_r = 0 \quad \text{at} \quad r = a. \quad (2.21)$$

It is interesting to note that this is precisely the condition necessary to ensure the gauge invariance of the Lagrangian (1.1′). That is, the mass term

$$\tfrac{1}{2}\mu\int dx\,\sqrt{-g}F^\lambda A_\lambda \quad (2.22)$$

is gauge invariant only if we neglect the surface term [see (1.5)]

$$\tfrac{1}{2}\mu\int dx\,\sqrt{-g}F^\lambda\partial_\lambda\Lambda = \tfrac{1}{2}\mu\int dS_\lambda\sqrt{-g}F^\lambda\Lambda = 0, \quad (2.23)$$

which is true if the normal component of F^λ vanishes on the bounding surfaces.

We begin by solving the system of Eqs. (2.16). The solution to (2.16a) is

$$\mathcal{G}_0{}^0 = -\frac{\lambda^2}{4i\mu}J_m(\lambda r_<)H_m(\lambda r_>) + AJ_m(\lambda r) + BH_m(\lambda r), \quad (2.24)$$

where $\lambda^2 = \omega^2 - \mu^2$, J_m is the Bessel function of the first kind, $H_m = H_m^{(1)}$ is the Hankel function of the first kind, and $r_<$ ($r_>$) is the lesser (greater) of r and r'. The constants A and B are to be determined by the boundary condition (2.21). When we insert (2.24) into (2.16b) we find

$$\mathcal{G}_2{}^0 = \frac{m\omega}{4i\mu}(\widetilde{HJ})(r,r') - \frac{m\omega}{\lambda^2}[A\tilde{\mathcal{J}}_m(\lambda r) + B\tilde{\mathcal{H}}_m(\lambda r)], \quad (2.25)$$

where we have introduced the abbreviations (the prime stands for the derivative with respect to the argument)

$$\tilde{\mathcal{J}}_m(x) = J_m(x) + \frac{\mu x}{m\omega}J_m'(x), \quad (2.26a)$$

$$\tilde{\mathcal{H}}_m(x) = H_m(x) + \frac{\mu x}{m\omega}H_m'(x), \quad (2.26b)$$

and

$$(\widetilde{HJ})(r,r') = \begin{cases} H_m(\lambda r')\tilde{\mathcal{J}}_m(\lambda r), & r < r', \\ J_m(\lambda r')\tilde{\mathcal{H}}_m(\lambda r), & r > r'. \end{cases} \quad (2.27)$$

When (2.24) and (2.25) are inserted into (2.16c) we obtain

$$\begin{aligned} \mathcal{G}_1{}^0 &= \frac{m\lambda^2}{4\mu^2 r}[J_m(\lambda r_<)H_m(\lambda r_>) + \tilde{A}J_m(\lambda r) + \tilde{B}H_m(\lambda r)] \\ &\quad + \frac{\omega^2 m}{4\mu^2 r}[-(\widetilde{HJ})(r,r') - \tilde{A}\tilde{\mathcal{J}}_m(\lambda r) - \tilde{B}\tilde{\mathcal{H}}_m(\lambda r)] \\ &= -\frac{m}{4r}[(HJ)(r,r') + \tilde{A}\mathcal{J}_m(\lambda r) + \tilde{B}\mathcal{H}_m(\lambda r)], \end{aligned} \quad (2.28)$$

where we have rescaled the constants

$$\tilde{A} = -\frac{4i\mu}{\lambda^2}A, \quad \tilde{B} = -\frac{4i\mu}{\lambda^2}B, \quad (2.29)$$

and have defined

$$\mathcal{J}_m(x) = J_m(x) + \frac{\omega x}{m\mu}J_m'(x) \quad (2.30a)$$

$$= \frac{\omega^2}{\mu^2}\tilde{\mathcal{J}}_m(x) - \frac{\lambda^2}{\mu^2}J_m(x), \quad (2.30b)$$

$$\mathcal{H}_m(x) = H_m(x) + \frac{\omega x}{m\mu}H_m'(x) \quad (2.30c)$$

$$= \frac{\omega^2}{\mu^2}\tilde{\mathcal{H}}_m(x) - \frac{\lambda^2}{\mu^2}H_m(x), \quad (2.30d)$$

$$(HJ)(r,r') = \begin{cases} H_m(\lambda r')\mathcal{J}_m(\lambda r), & r < r', \\ J_m(\lambda r')\mathcal{H}_m(\lambda r), & r > r'. \end{cases} \quad (2.31)$$

Now we are in a position to impose the boundary condition (2.21) on $\mathcal{G}_1{}^0$. First, we consider points inside the circle, $r, r' < a$. From (2.28) we see that $B = 0$ in order that the solution be finite at the origin. Then the boundary condition $\mathcal{G}_1{}^0 = 0$ at $r = a$ implies from (2.28) that

$$\tilde{A} = -\frac{\mathcal{H}_m(\lambda a)}{\mathcal{J}_m(\lambda a)}J_m(\lambda r'), \quad (2.32)$$

from which we deduce the explicit form for these components, for $r, r' < a$:

$$\mathcal{G}_0{}^0 = -\frac{\lambda^2}{4i\mu}\left(J_m(\lambda r_<)H_m(\lambda r_>) - \frac{\mathcal{H}_m(\lambda a)}{\mathcal{J}_m(\lambda a)}J_m(\lambda r)J_m(\lambda r')\right), \quad (2.33a)$$

$$\mathcal{G}_2{}^0 = -\frac{m\omega}{4i\mu}\left(-(\widetilde{HJ})(r,r') + \frac{\mathcal{H}_m(\lambda a)}{\mathcal{J}_m(\lambda a)}\tilde{\mathcal{J}}_m(\lambda r)J_m(\lambda r')\right), \quad (2.33b)$$

$$\mathcal{G}_1{}^0 = -\frac{m}{4r}\left((HJ)(r,r') - \frac{\mathcal{H}_m(\lambda a)}{\mathcal{J}_m(\lambda a)}\mathcal{J}_m(\lambda r)J_m(\lambda r')\right). \quad (2.33c)$$

Outside the circle, $r, r' > a$, we must have *outgoing* cylindrical waves, so $A = 0$, and the boundary condition $\mathcal{G}_1{}^0 = 0$ at $r = a$ implies from (2.28) that

$$\tilde{B} = -\frac{\mathcal{J}_m(\lambda a)}{\mathcal{H}_m(\lambda a)}H_m(\lambda r'), \quad (2.34)$$

from which we deduce the explicit form for these components, for $r, r' > a$:

$$\mathcal{G}_0{}^0 = -\frac{\lambda^2}{4i\mu}\left(J_m(\lambda r_<)H_m(\lambda r_>) - \frac{\mathcal{J}_m(\lambda a)}{\mathcal{H}_m(\lambda a)}H_m(\lambda r)H_m(\lambda r')\right), \quad (2.35a)$$

$$\mathcal{G}_2{}^0 = -\frac{m\omega}{4i\mu}\left(-(\widetilde{HJ})(r,r') + \frac{\mathcal{J}_m(\lambda a)}{\mathcal{H}_m(\lambda a)}\tilde{\mathcal{H}}_m(\lambda r)H_m(\lambda r')\right), \quad (2.35b)$$

$$\mathcal{G}_1{}^0 = -\frac{m}{4r}\left((HJ)(r,r') - \frac{\mathcal{J}_m(\lambda a)}{\mathcal{H}_m(\lambda a)}\mathcal{H}_m(\lambda r)H_m(\lambda r')\right). \quad (2.35c)$$

Next, we solve the system (2.18). It is slightly harder to solve (2.18a). We write

$$\mathcal{G}_0{}^1 = A_\pm J_m(\lambda r) + B_\pm H_m(\lambda r), \quad (2.36)$$

where the upper (lower) sign holds if $r > r'$ ($r < r'$). Equation (2.18a) implies that the derivative of $\mathcal{G}_0{}^1$ is discontinuous at $r = r'$:

$$-\frac{im}{2\pi r'} = (A_+ - A_-)\lambda r' J_m'(\lambda r') + (B_+ - B_-)\lambda r' H_m'(\lambda r'), \quad (2.37a)$$

while the function itself is discontinuous:

$$\frac{i\omega}{2\pi\mu r'} = (A_+ - A_-)J_m(\lambda r') + (B_+ - B_-)H_m(\lambda r'). \quad (2.37b)$$

These equations are solved by

$$A_+ - A_- = \frac{m}{4r'}\mathcal{H}_m(\lambda r'), \quad (2.38a)$$

$$B_+ - B_- = -\frac{m}{4r'}\mathcal{J}_m(\lambda r'). \quad (2.38b)$$

Inside the circle, $r, r' < a$, we have $B_- = 0$ and the boundary condition given through (2.18b) implies

$$B_+\mathcal{H}_m(\lambda a) + A_+\mathcal{J}_m(\lambda a) = 0. \quad (2.39)$$

Solving these equations gives, for $r, r' < a$,

$$\mathcal{G}_0{}^1 = \frac{m}{4r'}\left(-(JH)(r,r') + \frac{\mathcal{H}_m(\lambda a)}{\mathcal{J}_m(\lambda a)}J_m(\lambda r)\mathcal{J}_m(\lambda r')\right), \quad (2.40a)$$

$$\mathcal{G}_1{}^1 = \frac{im^2\mu}{4\lambda^2 rr'}\left(-\mathcal{J}_m(\lambda r_<)\mathcal{H}_m(\lambda r_>) + \frac{\mathcal{H}_m(\lambda a)}{\mathcal{J}_m(\lambda a)}\mathcal{J}_m(\lambda r)\mathcal{J}_m(\lambda r')\right), \tag{2.40b}$$

$$\mathcal{G}_2{}^1 = -\frac{m^2\omega}{4\lambda^2 r'}\left(-[\widetilde{HJ}](r,r') + \frac{\mathcal{H}_m(\lambda a)}{\mathcal{J}_m(\lambda a)}\tilde{\mathcal{J}}_m(\lambda r)\mathcal{J}_m(\lambda r')\right). \tag{2.40c}$$

Here we have introduced the abbreviations

$$(JH)(r,r') = \begin{cases} J_m(\lambda r)\mathcal{H}_m(\lambda r'), & r < r' \\ H_m(\lambda r)\mathcal{J}_m(\lambda r'), & r > r' \end{cases} = (HJ)(r',r) \tag{2.41}$$

and

$$[\widetilde{HJ}](r,r') = \begin{cases} \mathcal{H}_m(\lambda r')\tilde{\mathcal{J}}_m(\lambda r), & r < r', \\ \mathcal{J}_m(\lambda r')\tilde{\mathcal{H}}_m(\lambda r), & r > r'. \end{cases} \tag{2.42}$$

Outside the circle, $r, r' > a$, $A_+ = 0$ and (2.18b) implies

$$B_-\mathcal{H}_m(\lambda a) + A_-\mathcal{J}_m(\lambda a) = 0. \tag{2.43}$$

So now the solution to the system (2.18) is, for $r, r' > a$

$$\mathcal{G}_0{}^1 = \frac{m}{4r'}\left(-(JH)(r,r') + \frac{\mathcal{J}_m(\lambda a)}{\mathcal{H}_m(\lambda a)}H_m(\lambda r)\mathcal{H}_m(\lambda r')\right), \tag{2.44a}$$

$$\mathcal{G}_1{}^1 = \frac{im^2\mu}{4\lambda^2 rr'}\left(-\mathcal{J}_m(\lambda r_<)\mathcal{H}_m(\lambda r_>) + \frac{\mathcal{J}_m(\lambda a)}{\mathcal{H}_m(\lambda a)}\mathcal{H}_m(\lambda r)\mathcal{H}_m(\lambda r')\right), \tag{2.44b}$$

$$\mathcal{G}_2{}^1 = -\frac{m^2\omega}{4\lambda^2 r'}\left(-[\widetilde{HJ}](r,r') + \frac{\mathcal{J}_m(\lambda a)}{\mathcal{H}_m(\lambda a)}\tilde{\mathcal{H}}_m(\lambda r)\mathcal{H}_m(\lambda r')\right). \tag{2.44c}$$

The system (2.20) is solved in just the same way. The result is, inside the circle $(r, r' < a)$,

$$\mathcal{G}_0{}^2 = -\frac{im\omega}{4\mu r'^2}\left(-(\widetilde{JH})(r,r') + \frac{\mathcal{H}_m(\lambda a)}{\mathcal{J}_m(\lambda a)}J_m(\lambda r)\tilde{\mathcal{J}}_m(\lambda r')\right), \tag{2.45a}$$

$$\mathcal{G}_2{}^2 = \frac{im^2\omega^2}{4\mu\lambda^2 r'^2}\left(-\tilde{\mathcal{J}}_m(\lambda r_<)\tilde{\mathcal{H}}_m(\lambda r_>) + \frac{\mathcal{H}_m(\lambda a)}{\mathcal{J}_m(\lambda a)}\tilde{\mathcal{J}}_m(\lambda r)\tilde{\mathcal{J}}_m(\lambda r')\right), \tag{2.45b}$$

$$\mathcal{G}_1{}^2 = \frac{m^2\omega}{4\lambda^2 rr'^2}\left(-[\widetilde{JH}](r,r') + \frac{\mathcal{H}_m(\lambda a)}{\mathcal{J}_m(\lambda a)}\mathcal{J}_m(\lambda r)\tilde{\mathcal{J}}_m(\lambda r')\right), \tag{2.45c}$$

and outside the circle $(r, r' > a)$,

$$\mathcal{G}_0{}^2 = -\frac{im\omega}{4\mu r'^2}\left(-(\widetilde{JH})(r,r') + \frac{\mathcal{J}_m(\lambda a)}{\mathcal{H}_m(\lambda a)}H_m(\lambda r)\tilde{\mathcal{H}}_m(\lambda r')\right), \tag{2.46a}$$

$$\mathcal{G}_2{}^2 = \frac{im^2\omega^2}{4\mu\lambda^2 r'^2}\left(-\tilde{\mathcal{J}}_m(\lambda r_<)\tilde{\mathcal{H}}_m(\lambda r_>) + \frac{\mathcal{J}_m(\lambda a)}{\mathcal{H}_m(\lambda a)}\tilde{\mathcal{H}}_m(\lambda r)\tilde{\mathcal{H}}_m(\lambda r')\right), \tag{2.46b}$$

$$\mathcal{G}_1{}^2 = \frac{m^2\omega}{4\lambda^2 rr'^2}\left(-[\widetilde{JH}](r,r') + \frac{\mathcal{J}_m(\lambda a)}{\mathcal{H}_m(\lambda a)}\mathcal{H}_m(\lambda r)\tilde{\mathcal{H}}_m(\lambda r')\right), \tag{2.46c}$$

where

$$(\widetilde{JH})(r,r') = \begin{cases} \tilde{\mathcal{H}}_m(\lambda r')J_m(\lambda r), & r < r' \\ \tilde{\mathcal{J}}_m(\lambda r')H_m(\lambda r), & r > r' \end{cases} = (\widetilde{HJ})(r',r) \tag{2.47}$$

and

$$[\widetilde{JH}](r,r') = \begin{cases} \tilde{\mathcal{H}}_m(\lambda r')\mathcal{J}_m(\lambda r), & r < r' \\ \tilde{\mathcal{J}}_m(\lambda r')\mathcal{H}_m(\lambda r), & r > r' \end{cases} = [\widetilde{HJ}](r',r). \tag{2.48}$$

Note that there are only six independent Green's functions because of the following symmetry relations between them:

$$r'^2\mathcal{G}_1{}^2(r,r') = -\mathcal{G}_2{}^1(r',r), \tag{2.49a}$$

$$r'^2\mathcal{G}_0{}^2(r,r') = -\mathcal{G}_2{}^0(r',r), \tag{2.49b}$$

$$\mathcal{G}_0{}^1(r,r') = \mathcal{G}_1{}^0(r',r), \tag{2.49c}$$

$$\mathcal{G}_0{}^0(r,r') = \mathcal{G}_0{}^0(r',r), \tag{2.49d}$$

$$\mathcal{G}_1{}^1(r,r') = \mathcal{G}_1{}^1(r',r), \tag{2.49e}$$

$$r'^2\mathcal{G}_2{}^2(r,r') = r^2\mathcal{G}_2{}^2(r',r). \tag{2.49f}$$

Using the above symmetry relations we can write the expression for the vacuum expectation value of the rr component of the stress tensor (2.14) as (recall that the limit $r' \to r$ is understood)

$$\langle t^{11}\rangle = -\frac{1}{2i}\left(\frac{\partial}{\partial r'}r'^2\mathcal{G}_0{}^2 + \frac{\partial}{\partial r'}\mathcal{G}_2{}^0\right). \qquad (2.50)$$

What we require, in fact, is the discontinuity across the surface of the sphere:

$$\Delta\langle t^{11}\rangle = \langle t^{11}\rangle\big|_{r=r'=a-} - \langle t^{11}\rangle\big|_{r=r'=a+}. \qquad (2.51)$$

From (2.45a), (2.46a), (2.33b), and (2.35b) we find, for this discontinuity,

$$\Delta\langle t^{11}\rangle = -\frac{\lambda^2 a}{8}\left\{\frac{\mathcal{H}_m(\lambda a)}{\mathcal{J}_m(\lambda a)}\left[\left(1-\frac{m^2}{(\lambda a)^2}\right)[J_m(\lambda a)]^2 + [J_m'(\lambda a)]^2\right] - \frac{\mathcal{J}_m(\lambda a)}{\mathcal{H}_m(\lambda a)}\left[\left(1-\frac{m^2}{(\lambda a)^2}\right)[H_m(\lambda a)]^2 + [H_m'(\lambda a)]^2\right]\right\}. \qquad (2.52)$$

Here, we have used (2.26a) and (2.26b) as well as the Bessel equation

$$(zJ_m'(z))' = -z\left(1-\frac{m^2}{z^2}\right)J_m(z). \qquad (2.53)$$

Equation (2.52), when integrated over the frequency ω and summed over m, is our general analytic expression for the Casimir stress on a conducting circle:

$$F = 2\pi\sum_{m=-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{d\omega}{2\pi}\Delta\langle t^{11}\rangle \qquad (2.54)$$

is the total force on the circle.[1] (Recall t^{11} is the stress tensor density, so a factor of $a=\sqrt{-g}$ on the surface is already absorbed.)

III. NUMERICAL RESULTS AT ZERO TEMPERATURE

We now turn to the task of extracting a numerical result from expressions (2.54) and (2.52). To do so, it is convenient to rotate the contour of frequency integration, $\omega \to i\zeta$, define the dimensionless real variable x by $x^2 = \zeta^2 a^2 + \mu^2 a^2$, and introduce the modified Bessel functions

$$I_\nu(x) = e^{-\frac{1}{2}\nu\pi i}J_\nu(xe^{\frac{1}{2}\pi i}), \qquad (3.1a)$$

$$K_\nu(x) = \frac{\pi}{2}ie^{\frac{1}{2}\nu\pi i}H_\nu(xe^{\frac{1}{2}\pi i}). \qquad (3.1b)$$

(For details of the contour rotation see, for example, Ref. [7].) When we explicitly symmetrize between positive and negative values of x (or m), we find the result

$$F = -\frac{1}{2\pi a^2}\frac{1}{\mu^2 a^2}\sum_{m=-\infty}^{\infty}\frac{1}{m^2}\int_{\mu a}^{\infty}\frac{dx\,x^4}{\sqrt{x^2-\mu^2a^2}}(x^2-\mu^2a^2+m^2)\frac{[I_m(x)K_m'(x)+K_m(x)I_m'(x)]}{I_m^2(x)+x^2(x^2-\mu^2a^2)I_m'^2(x)/m^2\mu^2a^2} \times\frac{K_m(x)I_m(x)+x^2(x^2-\mu^2a^2)K_m'(x)I_m'(x)/m^2\mu^2a^2}{K_m^2(x)+x^2(x^2-\mu^2a^2)K_m'^2(x)/m^2\mu^2a^2}. \qquad (3.2)$$

In the massless limit, $\mu a \to 0$, (3.2) simplifies dramatically:

$$F = -\frac{1}{2\pi a^2}\sum_{m=-\infty}^{\infty}\int_0^\infty dx\,x\frac{d}{dx}\ln[x^2 I_m'(x)K_m'(x)]. \qquad (3.3)$$

We consider the $m=0$ and the $m\neq 0$ terms in (3.3) separately. As usual in Casimir calculations [8], we ignore terms in the integrand of (3.3) which are powers of x (contact terms). (Note that a power of x corresponds to a polynomial in derivatives of δ functions in time.) In particular, for the $m=0$ term, we add a contact term to the integrand to make it converge, so that the corresponding contribution to the force becomes

$$F_0 = -\frac{1}{2\pi a^2}\int_0^\infty dx\,x\frac{d}{dx}\ln[2xI_1(x)K_1(x)]. \qquad (3.4)$$

Straightforward numerical integration gives the attractive result

$$F_0 = -\frac{1}{2\pi a^2}1.59 = -\frac{0.254}{a^2}. \qquad (3.5)$$

For $m\neq 0$ we make use of the uniform asymptotic expansions for the modified Bessel functions [9]. The leading term gives

$$I_m'(x)K_m'(x) \sim -\frac{1}{2m}\frac{1}{z^2}(1+z^2)^{1/2}, \qquad (3.6)$$

where $z=x/m$. The corresponding contribution to the force is

$$F_{\rm LT} \sim -\frac{1}{2\pi a^2}\int_0^\infty dx\,x^2\,2\sum_{m=1}^{\infty}\frac{1}{m^2+x^2}, \qquad (3.7)$$

where the sum is performed according to

[1]We should also be able to derive the result (2.54), (2.52) from the vacuum energy. It is easy to do so for $\mu=0$, but much more elaborate for $\mu\neq 0$, so we forego further discussion of this point.

$$2\sum_{m=1}^{\infty}\frac{1}{m^2+x^2}=\frac{\pi}{x}\coth\pi x-\frac{1}{x^2}. \tag{3.8}$$

Again, we supply appropriate contact terms, so that this leading term $m\neq 0$ contribution is

$$F_{\rm LT}=-\frac{1}{4\pi^2a^2}\int_0^\infty\frac{dy\,y}{e^y-1}=-\frac{1}{24a^2}, \tag{3.9}$$

only 16% of (3.5). We should now correct (3.9) by including the next-to-leading corrections. However, it is not hard to see that these possess an infrared divergence, a phenomenon which is associated with the low dimensionality of the problem. This divergence is probably spurious: each integrand in (3.3) is quite accurately represented by the leading term given in (3.6). {Even at $m = 1$, the maximum value of $\ln[-2x^2I'_m(x)K'_m(x)(m^2+x^2)^{-1/2}]$ is less than 7% of the value of $\ln[-x^2I'_m(x)K'_m(x)]$, and globally the fit is excellent.} This divergence, of course, is regulated by the mass μ, so we will discuss this point further below.

When $\mu\neq 0$, the calculation proceeds similarly. We first treat the $m = 0$ term, which is easily seen from (3.2) to be the obvious generalization of (3.4):

$$F_0=-\frac{1}{2\pi a^2}\int_{\mu a}^\infty dx\,\frac{x^2}{\sqrt{x^2-\mu^2a^2}}\frac{d}{dx}\ln[2xI_1(x)K_1(x)]. \tag{3.10}$$

The results of numerical integration of (3.10) are shown in Fig. 1. This contribution to the force decreases rapidly from the massless value (3.5) to zero as $\mu a\to\infty$. For $m\neq 0$ we use the uniform asymptotic expansion for the modified Bessel functions. Doing so with the general expression (3.2) requires only the leading terms for three of the factors there:

$$I_m(x)K_m(x)+\frac{x^2(x^2-\mu^2a^2)}{m^2\mu^2a^2}I'_m(x)K'_m(x)\sim-\frac{mz^2}{2\mu^2a^2t}, \tag{3.11a}$$

$$I_m^2(x)+\frac{x^2(x^2-\mu^2a^2)}{m^2\mu^2a^2}I'^2_m(x)\sim\frac{m^2z^2}{\mu^2a^2t}\frac{e^{2m\eta}}{2\pi m}, \tag{3.11b}$$

$$K_m^2(x)+\frac{x^2(x^2-\mu^2a^2)}{m^2\mu^2a^2}K'^2_m(x)\sim\frac{m^2z^2}{\mu^2a^2t}\frac{\pi}{2m}e^{-2m\eta}. \tag{3.11c}$$

(The value of η is, evidently, irrelevant here.) The fourth factor requires that we go out to the next-to-leading order:

$$I_m(x)K'_m(x)+K_m(x)I'_m(x)\sim-\frac{z}{2m^2}t^3. \tag{3.12}$$

Here, adopting the notation of [9], we have the abbreviations

$$t=(1+z^2)^{-1/2},\quad z=\frac{x}{m}. \tag{3.13}$$

We substitute these asymptotic expressions into (3.2) and carry out the sum on m using (3.8), again omitting contact terms. The result is

$$F_{\rm LT}=-\frac{1}{2a^2}\frac{1}{(2\pi)^2}\int_{2\pi\mu a}^\infty dy\,\frac{y^2}{\sqrt{y^2-(2\pi\mu a)^2}}\frac{1}{e^y-1}\left\{2-\frac{(2\pi\mu a)^2}{y}\left[\left(\frac{1}{y}+1\right)+\frac{1}{(e^y-1)}\right]\right\}. \tag{3.14}$$

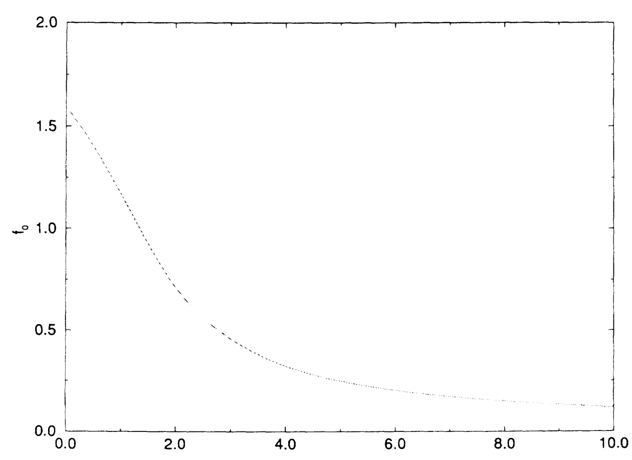

FIG. 1. The contribution of $m = 0$ to the Casimir force at zero temperature, F_0, given in (3.10). Plotted is $f_0 = -2\pi a^2F_0$ as a function of μa.

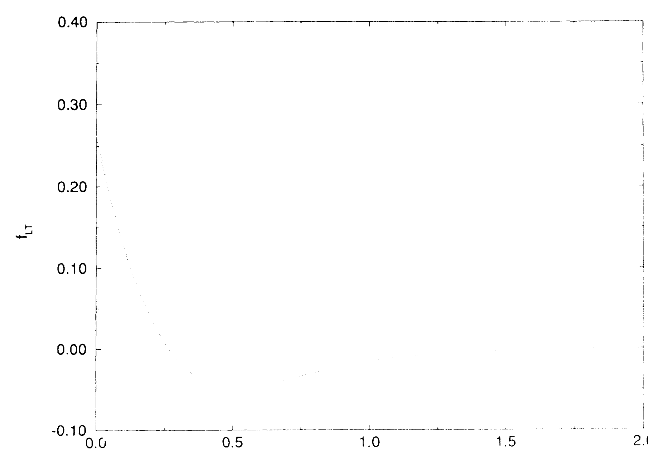

FIG. 2. The leading uniform asymptotic approximation to the $m\neq 0$ contributions to the Casimir force at zero temperature, $F_{\rm LT}$, given by (3.14). Plotted is $f_{\rm LT} = -2\pi a^2F_{\rm LT}$ as a function of μa.

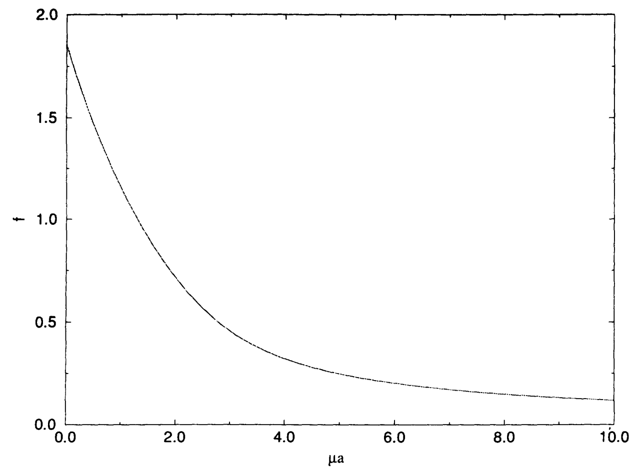

FIG. 3. The sum of the contributions shown in Figs. 1 and 2. Plotted is $f = -2\pi a^2(F_0 + F_{\rm LT})$.

This result, of course, generalizes the formula in (3.9). Numerical integration of (3.14) yields the contribution to the force shown in Fig. 2. This partial result is extremely interesting, because of the sign change, from attractive to repulsive at about $\mu a = 0.27$. However, the $m = 0$ term given by (3.10) and Fig. 1 is much larger, so that these terms together always give an attractive force. The sum of these two terms is plotted in Fig. 3.

We have, of course, worked out the next-to-leading contributions to the force. As noted above, these are finite when $\mu \neq 0$, but are larger than the leading term given by (3.14) and Fig. 2 even for large μa. Based on our previous experience [10, 6] we expect that the leading terms should in fact be quite accurate, and that this breakdown of the asymptotic expansion, which must always occur at some order, occurs early here because of the lowness of the dimensionality of space. We therefore believe that the $m = 0$ and leading-order contributions here constitutes a reliable estimate of the Casimir self-stress in this case. Clearly, there are issues here requiring further investigation.

IV. HIGH-TEMPERATURE LIMIT

It is easy, in principle, to see how to extract the finite-temperature Casimir effect. We take the general zero-temperature result (3.2) and replace the continuous frequency variable ζ by the discrete variable $2\pi n/\beta$, where $\beta = 1/kT$ and n is an integer. That is,

$$\int_{\mu a}^{\infty} dx\, x f(x) \to \frac{4\pi^2 a^2}{\beta^2} \sum_{n=0}^{\infty}{}' n f(x_n), \tag{4.1}$$

$$x_n = \left[\left(\frac{2\pi a n}{\beta}\right)^2 + (\mu a)^2\right]^{1/2}, \tag{4.2}$$

where the prime on the summation sign means that $n = 0$ is counted with half weight.

Given the complicated form of the integrand in (3.2), it is very hard to work out the general temperature dependence in this case. The high-temperature limit, however, would seem to be tractable. That is because we anticipate that only the $n = 0$ term contributes when $\beta \to 0$. Indeed, if $n \neq 0$, $x_n \to 2\pi a n/\beta \to \infty$, and the corresponding contribution to F coming from (3.2) is

$$F_{n\neq 0}^{T\to\infty} = -\frac{1}{a\beta} \sum_{m=-\infty}^{\infty} \sum_{n=1}^{\infty} x_n \left(\frac{I_m(x_n)}{I_m'(x_n)} + \frac{K_m(x_n)}{K_m'(x_n)}\right)$$
$$= -\frac{kT}{a} \sum_{m=-\infty}^{\infty} \sum_{n=1}^{\infty}, \tag{4.3}$$

where we have used the asymptotic behavior of the Bessel functions. This divergent contribution, in fact, should be subtracted off, for the constant summand may again be identified with a contact term.

The high-temperature limit thus arises from the $n = 0$ term. We will use the uniform asymptotic approximation employed in Sec. III, and hence we will make the replacement (4.1) in (3.10) and (3.14). For the former, $m = 0$, term we have

$$F_0^{T\to\infty} = -\frac{1}{2a\beta} x \frac{d}{dx} \ln[2x I_1(x) K_1(x)], \quad x = \mu a, \tag{4.4}$$

which is plotted in Fig. 4. This attractive contribution equals $-1/2a\beta$ at $\mu a = 0$, and vanishes as $\mu a \to \infty$. For $m \neq 0$ we have, from (3.14),

$$F_{\rm LT}^{T\to\infty} \sim -\frac{\pi x}{a\beta} \frac{1}{e^{2\pi x} - 1} \left\{\frac{1}{2} - \pi x - \frac{\pi x}{e^{2\pi x} - 1}\right\},$$
$$x = \mu a. \tag{4.5}$$

This repulsive term vanishes both at $\mu a = 0$ and as $\mu a \to \infty$, and, like (3.14) is rather small compared to $F_0^{T\to\infty}$. The combined high-temperature Casimir force $F_0^{T\to\infty} + F_{\rm LT}^{T\to\infty}$ is plotted in Fig. 5.

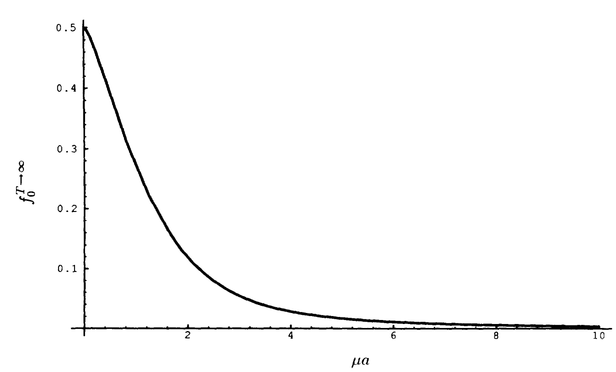

FIG. 4. The contribution of $m = 0$ to the Casimir force at high temperature, $F_0^{T\to\infty}$, given in (4.4). Plotted is $f_0^{T\to\infty} = -a\beta F_0^{T\to\infty}$ as a function of μa.

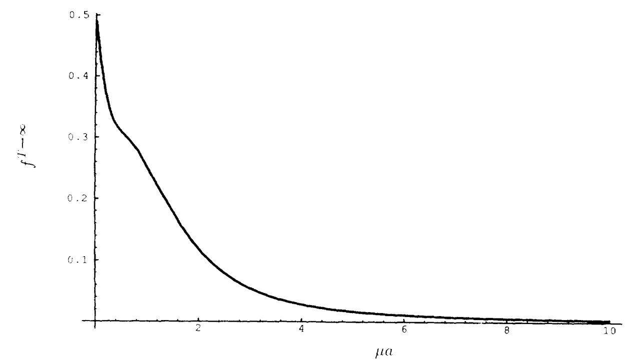

FIG. 5. The total Casimir force at high temperature, including the leading uniform asymptotic approximation for $m \neq 0$. Plotted is $f^{T\to\infty} = -a\beta F^{T\to\infty}$.

V. DISCUSSION

The process of quantization automatically leads to unavoidable vacuum fluctuations. Usually, the vacuum energy of a medium is irrelevant. But the physics changes drastically (i) when a phase transition between two states of the medium can occur, (ii) when the medium is the whole Universe and one couples gravity to the vacuum energy (leading to the vexing cosmological constant problem [11]), or (iii) when geometric boundary effects are taken into account. In this paper we have studied the effect of vacuum fluctuations in the last case, i.e., the Casimir effect. The example of the Casimir effect that we have considered is particularly interesting since it is associated with topology, the non-Abelian generalization of the "photon" mass term being the Chern-Simons secondary characteristic.

In a previous paper [5] we examined the Casimir effect between parallel lines due to the topologically massive photon in the $(2+1)$-dimensional theory of quantum electrodynamics. We found that the Casimir force is attractive and the result is the same as for a massive spin-zero field [12]. The agreement of the respective Casimir forces is not surprising since, as the scalar field, the topologically massive spin-1 field in $2+1$ dimensions has one (polarization) degree of freedom. We anticipated that this agreement may not persist for other geometries; for example, in $(3+1)$-dimensional electrodynamics, the Casimir force between perfectly conducting parallel plates is twice that for a scalar field [13], but such is not the case for a spherical shell [10, 14].

In this paper we have calculated the Casimir self-stress for a circle. *A priori*, it is hard to guess the sign of the self-stress in this case, since the $(3+1)$-dimensional analogue of a circle can be a spherical shell (for which the stress is repulsive [10, 14]) or a cylindrical shell (for which the stress is attractive [6]). We have found, in fact, the Casimir stress, at zero and at high temperatures, to be attractive. We have also found that the respective Casimir forces are not the same for the spin-1 field (3.2) and for the spin-0 field (A11) discussed in the Appendix, in accordance with our expectation.

Actually, there is a way to read off the Casimir stress for the *massless* spin-1 field for the circle from that for the cylindrical shell and the result for the massless scalar field for the circle. We see this by returning to (2.52) and taking the $\mu \to 0$ limit:

$$\Delta\langle t^{11}\rangle = \frac{i\lambda}{4\pi}\left(\frac{J''_m(z)}{J'_m(z)} + \frac{H''_m(z)}{H'_m(z)} + \frac{2}{z}\right), \tag{5.1}$$

where $z = \lambda a \to \omega a$ and use has been made of the equation of motion and the Wronskian. We can, in fact, read this off directly from Ref. [6], if, there, we make appropriate $(2+1)$-dimensional restrictions. That is, we set the momentum h along the cylinder axis equal to zero, and include only B_z, E_r, and E_θ:

$$T_{rr} = \tfrac{1}{2}(B_z^2 - E_r^2), \tag{5.2}$$

where the discontinuities are

$$\Delta B_z^2 = \frac{\lambda^2}{iz}\left(\frac{H_m(z)}{H'_m(z)} + \frac{J_m(z)}{J'_m(z)}\right), \tag{5.3a}$$

$$\Delta E_r^2 = \frac{1}{iz}\frac{m^2\omega^2}{z^2}\left(\frac{J_m(z)}{J'_m(z)} + \frac{H_m(z)}{H'_m(z)}\right). \tag{5.3b}$$

These follow from (3.5) and (3.3) of Ref. [6]. This agrees precisely with (5.1) when we recognize the connection $\Delta T_{rr} = 2\pi\Delta\langle t_{rr}\rangle/a$.

Furthermore, in the Appendix we calculate the Casimir self-stress for a scalar field, (A9),

$$\Delta\langle t_{rr}\rangle|_{\text{scalar}} = \frac{i\lambda}{4\pi}\left(\frac{J'_m(z)}{J_m(z)} + \frac{H'_m(z)}{H_m(z)}\right), \tag{5.4}$$

which when combined with (5.1) yields the form of the $(3+1)$-dimensional result of Ref. [6]:

$$\Delta\langle t_{rr}\rangle|_{\text{vector}} + \Delta\langle t_{rr}\rangle|_{\text{scalar}} = \frac{i\lambda}{4\pi}\left(\frac{J''_m(z)}{J'_m(z)} + \frac{J'_m(z)}{J_m(z)} + \frac{H''_m(z)}{H'_m(z)} + \frac{H'_m(z)}{H_m(z)} + \frac{2}{z}\right), \tag{5.5}$$

which corresponds to (3.11) in Ref. [6].

Schematically, we write the $h = 0$, $\mu = 0$ correspondence found above as

$$(3+1)_v = (2+1)_v + (2+1)_s. \tag{5.6}$$

We can understand this directly from the $(3+1)_v$ equation of motion

$$\partial_\mu\sqrt{-g}F^{\mu\nu} = 0. \tag{5.7}$$

When there is no z dependence, the $\nu = 0, 1, 2$ components coincide with the $(2+1)_v$ equations of motion, while the $\nu = 3$ component can be written as

$$\left(\frac{1}{r}\frac{\partial}{\partial r}r\frac{\partial}{\partial r} + \omega^2 - \frac{m^2}{r^2}\right)A^3 = 0, \tag{5.8}$$

subject to the boundary condition $A_3 = 0$, so this is the massless scalar problem solved in the Appendix. And explicitly, the $(2+1)_s$ contribution to the stress tensor is

$$T_{rr} = \tfrac{1}{2}\langle H_\theta^2\rangle, \quad H_\theta = -\partial_r A_3, \tag{5.9}$$

where

$$\langle A_3(x)A_3(x')\rangle = \frac{1}{i}G(x,x'), \tag{5.10}$$

in terms of the scalar Green's function, or, in terms of the reduced scalar Green's function (A1),

$$T_{rr} = \frac{2\pi}{2i}\frac{\partial}{\partial r}\frac{\partial}{\partial r'}g(r,r')\bigg|_{r=r'=a}. \tag{5.11}$$

This is just the scalar result (A7).

No such decomposition occurs when $h \neq 0$ or when $\mu \neq 0$.

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APPENDIX: SCALAR CASIMIR EFFECT

Consider a scalar field in 2+1 dimensions with a circular boundary of radius a on which the field vanishes. We write the Green's function in Fourier-transformed form as

$$G(x,x') = \int \frac{d\omega}{2\pi}e^{-i\omega(t-t')}\sum_{m=-\infty}^{\infty} e^{im(\theta-\theta')}g(r,r'), \tag{A1}$$

where we have suppressed the dependence of the reduced Green's function g on m and ω. The reduced Green's function satisfies the differential equation

$$\left(\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}+\omega^2-\mu^2-\frac{m^2}{r^2}\right)g(r,r') = -\frac{1}{2\pi r}\delta(r-r'). \tag{A2}$$

We solve this equation subject to the boundary condition

$$g(a,r') = 0. \tag{A3}$$

The solution is

$$r,r' < a: \quad g(r,r') = \frac{1}{4i}\left(\frac{H_m(\lambda a)}{J_m(\lambda a)}J_m(\lambda r)J_m(\lambda r') - J_m(\lambda r_<)H_m(\lambda r_>)\right), \tag{A4a}$$

$$r,r' > a: \quad g(r,r') = \frac{1}{4i}\left(\frac{J_m(\lambda a)}{H_m(\lambda a)}H_m(\lambda r)H_m(\lambda r') - J_m(\lambda r_<)H_m(\lambda r_>)\right), \tag{A4b}$$

where $\lambda^2 = \omega^2 - \mu^2$. Then we calculate t_{rr} from

$$t^{\mu\nu} = \sqrt{-g}[\partial^\mu\phi\partial^\nu\phi - g^{\mu\nu}\tfrac{1}{2}(\partial^\lambda\phi\partial_\lambda\phi+\mu^2\phi^2)]. \tag{A5}$$

The vacuum expectation value of the product of fields is taken according to

$$\langle\phi(x)\phi(x')\rangle = \frac{1}{i}G(x,x'). \tag{A6}$$

Employing the boundary condition (A3) we find, for the Fourier transform for the stress tensor on the circle,

$$\langle t_{rr}\rangle = \frac{a}{2i}\frac{\partial}{\partial r}\frac{\partial}{\partial r'}g(r,r')\bigg|_{r=r'=a}. \tag{A7}$$

Using the solutions (A4a), (A4b), and the Wronskian

$$J_m(x)H_m'(x) - J_m'(x)H_m(x) = \frac{2i}{\pi x}, \tag{A8}$$

we find

$$\Delta\langle t_{rr}\rangle = -\frac{\lambda}{4\pi i}\left(\frac{J_m'(\lambda a)}{J_m(\lambda a)}+\frac{H_m'(\lambda a)}{H_m(\lambda a)}\right), \tag{A9}$$

so the force on the circle is

$$F = -\frac{1}{4\pi a^2}\frac{1}{i}\int_{-\infty}^{\infty} dz\,\frac{z^2}{\sqrt{z^2+\mu^2a^2}} \times \sum_{m=-\infty}^{\infty}\frac{d}{dz}\ln J_m(z)H_m(z). \tag{A10}$$

To integrate this we perform an imaginary frequency rotation and introduce the modified Bessel functions:

$$F = -\frac{1}{2\pi a^2}\int_{\mu a}^{\infty} dx\frac{x^2}{\sqrt{x^2-\mu^2a^2}} \times \sum_{m=-\infty}^{\infty}\frac{d}{dx}\ln I_m(x)K_m(x). \tag{A11}$$

For $m = 0$ we can easily evaluate the integral numerically, after we insert the appropriate contact term. For example, for $\mu = 0$ we find upon integrating by parts that

$$-\int_0^\infty dx\,\ln 2xI_0(x)K_0(x) = 0.088\,08, \tag{A12}$$

corresponding to the attractive force

$$F_0 = -\frac{0.014}{a^2}. \tag{A13}$$

For $m \neq 0$ we content ourselves with the leading uniform asymptotic expansion

$$\frac{d}{dx}\ln I_m(x)K_m(x) \sim -\frac{x}{m^2+x^2}. \tag{A14}$$

We carry out the sum using (3.8) and find

$$F_{\rm LT} \sim \frac{1}{a^2}\frac{1}{(2\pi)^2}\int_{2\pi\mu a}^{\infty}\frac{dy\,y^2}{\sqrt{y^2-(2\pi\mu a)^2}}\frac{1}{e^y-1}, \tag{A15}$$

which is a repulsive force, precisely the negative of the first term in (3.14). In particular, at $\mu = 0$, we have the negative of (3.9),

$$F_{\rm LT} = \frac{1}{24a^2}, \tag{A16}$$

which overwhelms F_0 above.

This numerical equivalence at $\mu = 0$ is no coincidence. It is a consequence of the theorem (5.6), because the $(3+1)_v$ result described in Ref. [6] has only higher-order contributions in the uniform asymptotic expansion.

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