Bifurcating and screening solutions of the topologically massive Yang-Mills gauge field theories with external matter sources

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Using a suitable ansatz for the gauge potentials and the external source currents, we demonstrate that solutions for the topologically massive gauge field theories can be constructed to exhibit branching in the total energy versus total external charge plot. As the gauge field is massive, charge screening of the external matter source also occurs.

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I. INTRODUCTION

Recently there has been some interest in constructing classical solutions of $(2+1)$-dimensional field theories involving the Chern-Simons (CS) term. Since for the CS action alone the classical solution is trivial, the Yang-Mills (YM) action [1] or the charged-scalar field terms [2,3] or both [4] are usually incorporated. Classical solutions are useful as they may provide insight into the full quantified theories. The purpose of this paper is to present some solutions to topological massive gauge field theory in the presence of external (nondynamical) matter sources. These solutions exhibit bifurcation (branching) in the total energy H versus total external charge Q plot when the controlling parameters of the external source are varied; furthermore, complete charge (color) screening of the external source also occurs. Solutions for $(2+1)$-dimensional gauge field equations interacting with external sources have been discussed in Ref. [5], but without the above two features.

For the $(3+1)$-dimensional YM theories, complete charge screening solutions were first found by Sikivie and Weiss [6]. These solutions can have arbitrarily low energies and the external sources are screened by the YM gauge fields, which vanish fast enough at large distances so that the total non-Abelian charge of the system (YM field plus external matter source) is zero. The external source distribution can be regarded as an assembly of many quarks and the gauge-invariant measure of the non-Abelian charge has been discussed in Ref. [7]. Bifurcating solutions were first constructed by Jackiw, Jacobs, and Rebbi [8]. Their solutions require the support of minimal nonzero source strengths and cusplike behavior is seen in the energy H versus total external charge Q diagram. As a working definition, bifurcation is said to occur if there exists at least a common point in the parametric space at which the total energy and the total external source strength have their respective stationary values [9]. Consequently the branching of the total energy occurs as the total external source strength is varied.

In the following section we introduce our notation and definition of the non-Abelian charge. In Sec. III we construct an ansatz for the gauge field by imposing the axial symmetry, the resulting solutions have a finite energy as well as finite total external source strength. The total non-Abelian charge of the whole system vanishes, in contrast with the finite nonzero total charge of the external matter source. This indicates charge screening of the matter source by the gauge fields at the classical level. By manipulating the parameters specifying the external sources, we find that our solutions can lead to two branches emanating from the stationary point in the H versus Q diagram. The locus of bifurcation points is also displayed in Sec. IV. We end with some brief remarks in Sec. V.

II. THE EQUATIONS

For the SU(2) gauge group and in $(2+1)$-dimensional spacetime, the YM equations with the CS term and the Bianchi identity are, respectively,

$$D_\mu F^{\mu\nu}+\xi\tilde{F}^\nu=J^\nu\ , \tag{1a}$$

$$D_\nu\tilde{F}^\nu=0,\quad \tilde{F}^\nu\equiv\tfrac{1}{2}\epsilon^{\nu\alpha\beta}F_{\alpha\beta}\ , \tag{1b}$$

$$D_\nu J^\nu=0\ , \tag{1c}$$

where J^ν is the external source current and the metric is $g_{\mu\nu}=(-++)$. The coefficient ξ of the CS term is replaced by $-i\xi$ in Euclidean spacetime. The total energy H of the system is obtained from the energy-momentum tensor $T_{\mu\nu}$:

$$\begin{aligned}H&=\int d^2x\ T^{00}\\&=\int d^2x\left[\tfrac{1}{2}(E_i^aE_i^a+B^aB^a)+J_i^aA_i^a\right]\ ,\end{aligned} \tag{2}$$

where $E_i^a=F_{i0}^a$ and $B^a=\frac{1}{2}\epsilon^{ij}F_{ij}^a$ are, respectively, the non-Abelian electric and magnetic fields. A gauge-invariant measure of the total non-Abelian charge of the system can be defined as [7]

$$Q_T=\int d^2x\ \partial_i(F^{ai0}\eta^a)\ , \tag{3}$$

where $\eta^a(x)$ is a unit vector in the internal space. The total non-Abelian charge of the external source is

$$Q=\int d^2x(J^{a0}\eta^a)\ . \tag{4}$$

Note that as

$$\partial_\mu(F^{a\mu\nu}\eta^a)=(D_\mu F^{\mu\nu})^a\eta^a+F_a^{\mu\nu}(D_\mu\eta)^a \tag{5}$$

the non-Abelian charge carried by the gauge field is [7]

$$Q_F=\int d^2x\, F_a^{i0}(D_i\eta)^a , \tag{6}$$

and that due to the CS term and the external matter source is

$$Q_s=\int d^2x(D_iF^{i0})^a\eta^a=Q+Q_{\rm CS} , \tag{7a}$$

where

$$Q_{\rm CS}=\int d^2x(-\xi\tilde{F}_a^0\eta^a) . \tag{7b}$$

The CS term $(-\xi\tilde{F}^\nu)$ can effectively be regarded as a source for the gauge field. Clearly, $Q_T=(Q+Q_{\rm CS})+Q_F$. Charge screening occurs if $Q\neq 0$ but $Q_T=0$.

III. SOLUTIONS

To construct solutions which exhibit a bifurcation property we begin with a symmetry consideration so as to arrive at a general and suitable ansatz. For the time-independent case the simplest symmetry we can consider on 2-dimensional space is axial symmetry. Using a polar coordinate system the basis vectors are the radial and transverse vectors n^i and ϕ_i $(i=1,2)$, respectively. At each point in two-dimensional space, we characterize the basis vectors in the SU(2) group space by n^a, ϕ^a, and δ^{a3}, $a=1,2,3$ and $n^3=\phi^3=0$. The axial-symmetric ansatz can be constructed by considering terms containing products of the space and group basis vectors; the coefficient of each term is a function of ρ only. The proposed ansatz is

$$\begin{aligned}A_j^a=\delta^{a3}&[n_jg_1(\rho)+\phi_jg_2(\rho)]\\&+\phi_j[n^ag_3(\rho)+\phi^ag_4(\rho)]\\&\quad\times n_j[\phi^ag_5(\rho)+n^ag_6(\rho)] ,\end{aligned} \tag{8a}$$

$$A_0^a=\delta^{a3}f_1(\rho)+n^af_2(\rho)+\phi^af_3(\rho) , \tag{8b}$$

with a similar form for the external source current density $J_\mu^a(\rho)$. This ansatz reduces the field equations (1) to a set of nonlinear and coupled differential equations which are rather complex. Inspection of these reduced equations reveals some redundancy. To resolve this redundancy and for simplicity we set $g_1=g_3=g_5=0=f_1=f_2$ and put $g_2=1/\rho$, $g_4\equiv P$, $g_6\equiv T$, and $f_3=V$. Thus the new ansatz is

$$A_j^a=(\phi^aP+\delta_3^a/\rho)\phi_j+n^an_jT , \tag{9a}$$

$$A_0^a=\phi^aV , \tag{9b}$$

and it leads to the nonlinear reduced equations

$$-V''-\frac{V'}{\rho}+VT^2=\xi(P'+P/\rho)+J^{0a}\phi^a , \tag{10a}$$

$$2V'T+VT'+VT/\rho=-\xi TP+J^{0a}\delta_3^a , \tag{10b}$$

$$-(TP)'-(P'+P/\rho)T=\xi VT+J^{ja}\delta_3^a\phi_j , \tag{10c}$$

$$P''+P'/\rho+P/\rho^2-PT^2=-\xi V'+J^{ja}\phi^a\phi_j , \tag{10d}$$

$$(V^2-P^2)T=J^{ja}n^an_j . \tag{10e}$$

To satisfy the constraint (1c) on the external current and to simplify the above equations further, the source current can now be written as

$$J_0^a=(I-VT^2)\phi^a+M\delta_3^a , \tag{11a}$$

$$J_i^a=T(V^2-P^2)n^an_i-[T^2P\phi^a+(TP)'\delta_3^a]\phi_i , \tag{11b}$$

where I and M are functions of ρ only. This choice reduces Eqs. (10) tremendously:

$$-V''-V'/\rho=\xi(P'+P/\rho)-I , \tag{12a}$$

$$-(P'+P/\rho)=\xi V , \tag{12b}$$

$$2V'T+VT'+VT/\rho=-\xi TP-M . \tag{12c}$$

Evidently many solutions can be found for the above reduced equations. Since we demand our solution to exhibit bifurcation properties, the total energy and the total external charge must have finite nonzero values. We find after many trials that a viable solution can be constructed by first solving Eqs. (12a) and (12b):

$$V=K(\alpha+2-sz),\quad K\equiv z^\alpha e^{-sz} , \tag{13a}$$

$$P=zK,\quad z\equiv\xi\rho , \tag{13b}$$

$$\begin{aligned}I=\xi^2K[&s^2(3\alpha+5)-(\alpha+2)-s(s^2-1)z\\&-s(3\alpha^2+7\alpha+3)/z+\alpha^2(\alpha+2)/z^2] ,\end{aligned} \tag{13c}$$

where α and s are positive parameters for the charge distribution. To solve the remaining nonlinear Eq. (12c), we expand the functions T and M by power series and after some manipulation we obtain

$$T=\xi L,\quad L\equiv z^\beta e^{-tz} , \tag{14a}$$

$$\begin{aligned}M-\xi^2zKL\{&2s^2+st-1-[s(4\alpha+\beta+7)+t(\alpha+2)]z^{-1}\\&+(\alpha+2)(2\alpha+\beta+1)z^{-2}\} ,\end{aligned} \tag{14b}$$

where β and t, like α and s, are positive parameters for the charge distribution.

The non-Abelian electric and magnetic fields are given by

$$E_i^a=(\phi^aV'-\delta_3^aVT)n_i , \tag{15a}$$

$$B^a=\delta_3^aPT+\phi^a\xi V . \tag{15b}$$

At large distances the fields and the external source all vanish exponentially fast:

$$E_i^a\approx sz^{\alpha+1}e^{-sz}(\phi^as+\delta_3^a\xi z^\beta e^{-tz})n_i , \tag{16a}$$

$$B^a\approx-\xi z^{\alpha+1}e^{-sz}(\phi^as+\delta_3^az^\beta e^{-tz}) , \tag{16b}$$

$$J_0^a\approx\xi^2z^{\alpha+1}e^{-sz}[\phi^as(1-s^2)+\delta_3^az^\beta e^{-tz}(2s^2+st-1)] , \tag{16c}$$

$$J_i^a \approx \xi z^{\alpha+\beta+1} e^{-(t+s)z} \{ n^a n_i (s^2-1) z^{\alpha+1} e^{-sz} + \phi^i [\phi^a \xi z^\beta e^{-tz} - \delta_3^a (s+t)\xi] \} , \tag{16d}$$

while near the origin we have

$$E_i^a \approx n_i(\alpha+2) z^\alpha (\phi^a \alpha/z - \delta_3^a \xi z^\beta) , \tag{17a}$$

$$B^a \approx \xi z^\alpha [\phi^a(\alpha+2) - \delta_3^a z^{\beta+1}] , \tag{17b}$$

$$J_0^a \approx \xi^2(\alpha+2) z^\alpha \{ \phi^a [(\alpha/z)^2 - z^\beta/\xi] + \delta_3^a z^{\beta-1}(2\alpha+\beta+1) \} , \tag{17c}$$

$$J_i^a \approx \xi z^{\alpha+\beta} \{ n^a n_i (\alpha+2)^2 z^\alpha + \phi_i [\phi^a \xi z^{\beta+1} + \delta_3^a(\alpha+\beta+1)] \} . \tag{17d}$$

Using the definition (2), the total energy H for the above solution can be straightforwardly evaluated. We find, after some calculation,

$$H=(as^2+b)s^{-2(\alpha+1)} + (cs^2+ds+e)(s+t)^{-2(\alpha+\beta+2)} \tag{18a}$$

where

$$a=\Gamma_1[2(6\alpha^2+16\alpha+9)(2\alpha+1) - 3(2\alpha+3)(2\alpha+1)(\alpha+1) - 8\alpha(2\alpha+3)(\alpha+2) + 4\alpha(\alpha+2)^2] , \tag{18b}$$

$$b=\Gamma_1[(2\alpha+3)(2\alpha+1)(\alpha+1) - 4(\alpha+1)(\alpha+2)(2\alpha+1) + 2(2\alpha+1)(\alpha+2)^2] , \tag{18c}$$

$$c=\Gamma_2[(2\alpha+2\beta+3)(\alpha+\beta+1) - 4(\alpha+\beta+1)(\alpha+2) + 2(\alpha+2)^2] , \tag{18d}$$

$$d=-4\Gamma_2(\alpha+2)(\beta-1)t , \tag{18e}$$

$$e=\Gamma_2[2t^2(\alpha+2)^2 - (2\alpha+2\beta+3)(\alpha+\beta+1)] , \tag{18f}$$

$$\Gamma_1 \equiv (\pi\alpha) 4^{-(\alpha+1)} \Gamma(2\alpha) ,$$

$$\Gamma_2 \equiv (6\pi) 4^{-(\alpha+\beta+2)} \Gamma(2\alpha+2\beta+2) ,$$

and $\alpha > 0$, $\beta > -(\alpha+1)$. As for the total external charge strength, we use definition (4) and project along the direction ϕ^a to obtain

$$Q=\int d^2x (VT^2 + \xi^2 V) = 4\pi(s+2t)^{-(\alpha+2\beta+3)} \Gamma(\alpha+2\beta+2)[t(\alpha+2) - \beta s] , \tag{19}$$

where the integral over $(\xi^2 V)$ is zero by using solution (13a), and we require $\alpha > 0$ and $\beta > -\frac{1}{2}(\alpha+2)$. Again using the definition (3), the total non-Abelian charge of the total system projecting along the direction $\eta^a = \phi^a$ is

$$Q_T = \int d^2x \, \partial_i (F^{ai0}\phi^a) = -\int d^2x \left[V'' + \frac{V'}{\rho} \right] = 0 . \tag{20}$$

The total charge carried by the gauge field as computed from Eq. (6) with $\eta^a = \phi^a$ is

$$Q_F = -\int d^2x \, VT^2 \tag{21}$$

whereas that due to the CS term $(-\xi \tilde{F}^\nu)$ is

$$Q_{\rm CS} = -\xi^2 \int d^2x \, V = 0 . \tag{22}$$

Clearly, $(Q+Q_{\rm CS}+Q_F)=0$ as it should be since the sum is equal to Q_T. As Q_T vanishes but Q is finite, charge (color) screening is said to occur. In passing we note that if we project along the direction $\eta^a = \delta_3^a$, the total charge of the system Q_T again vanishes while the total charge of the external matter source as given by

$$Q = \int d^2x \, J^{a0}\delta_3^a = \int d^2x (V'T + \xi TP) \tag{23a}$$

is a finite quantity, indicating color screening. However in contrast with Eq. (22), here the charge due to the CS term,

$$Q_{\rm CS} = -\int d^2x (\xi \tilde{F}_a^0 \delta_3^a) = -\xi \int d^2x \, PT, \tag{23b}$$

remains finite. The charge carried by the gauge field projected along $\eta^a = \delta_3^a$ is

$$Q_F = -\int d^2x \, TV' . \tag{23c}$$

Again we have $Q+Q_{\rm CS}+Q_F=0$.

IV. BIFURCATION

As mentioned earlier, bifurcation means the branching of the total energy $H(\lambda)$ of the gauge field and external source system when the total external charge $Q(\lambda)$ is varied [8], where λ is a set of parameters. The existence of local extrema of $H(\lambda)$ and $Q(\lambda)$ at common parametric values, say $\lambda=\lambda_c$, will imply the bifurcation of the $H(Q)$ curve [9]. We have four parameters here, $\lambda=(\alpha,\beta,s,t)$, and if we vary all of them at the same time then it is not easy to find the common parametric values of (α,β,s,t) at which $H(\lambda)$ and $Q(\lambda)$ assume their respective extrema. However by trial and error we find after much effort that H and Q do posses their respective local minimum when $\beta=1.000\,000$, $t=0.400\,000$, $s=2.224\,259$, and $\alpha=1.829\,673$. In our search for the bifurcation point, each time we fix the values of β, s, and t when the parameter α is continuously varied. In this way, H and Q essentially depend on only one parameter, namely α. In Fig. 1, we present the bifurcating curve with the characteristic cusp in the H versus Q plot; the values of the parameters β, t, and s are fixed as above while the parameter α is varied. Note that once a bifurcation point has been ascertained, many other bifurcation

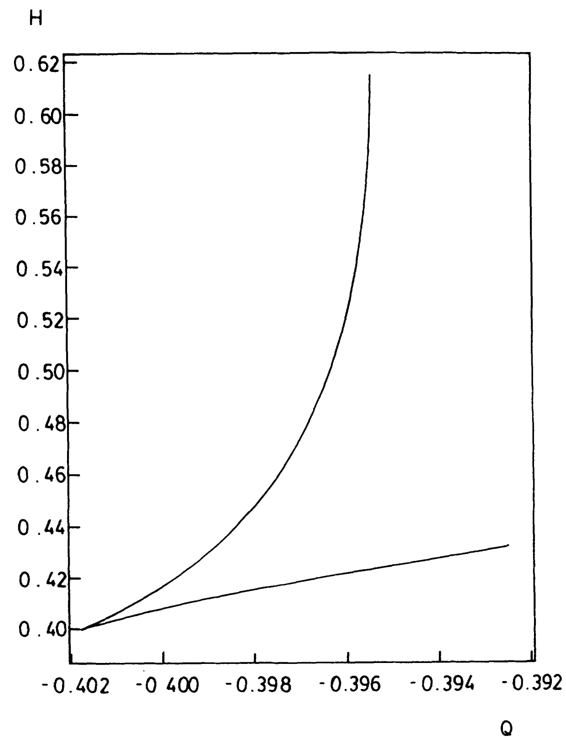

FIG. 1. Plot of H vs Q for the solution of the reduced Eqs. (12) when the parameter α is varied while other parameters are kept fixed: $\beta=1.000000$, $s=2.224259$, and $t=0.400000$.

points can be traced out by varying, say, the parameter t. The locus of the bifurcation point is shown in Fig. 2.

One can adjust the parameters so that the maximum of the energy coincides with the minimum of the total external charge. For instance, the parameter s is varied while parameters α, β, and t are kept fixed. We find that when $t=1.535850$, $\alpha=0.500000$, and $\beta=4.000000$, H has its maximum and Q has its minimum at $s=1.343869$ simultaneously. The branching of H when Q is varied is displayed in Fig. 3 and, as before many other branching points, can be traced out as shown by the dashed curve in the same figure.

From the explicit expressions of $H(\lambda)$ and $Q(\lambda)$, Eqs. (18) and (19), it is easy to locate the bifurcation points analytically if we regard H and Q as dependent solely on just one parameter, say, t or s. For example, taking α, β, and s as constants we have $H=H(t)$ and $Q=Q(t)$; then the extremum conditions $dH/dt=0=dQ/dt$ lead to an equation involving s, α, and β which can be solved for s:

$$s=[(2\alpha+2\beta+3)(\alpha+\beta+2)/(\alpha+\beta+\tfrac{7}{2})]^{1/2} . \tag{24}$$

The critical t value is then given by

$$t_c=(2\beta+1)s/(2\alpha+4) \tag{25}$$

and at this t_c, H and Q assume their respective local maximum values. Assigning $\alpha=\beta=0.01$, we find $s=1.3208$, $t_c=0.3351$, and

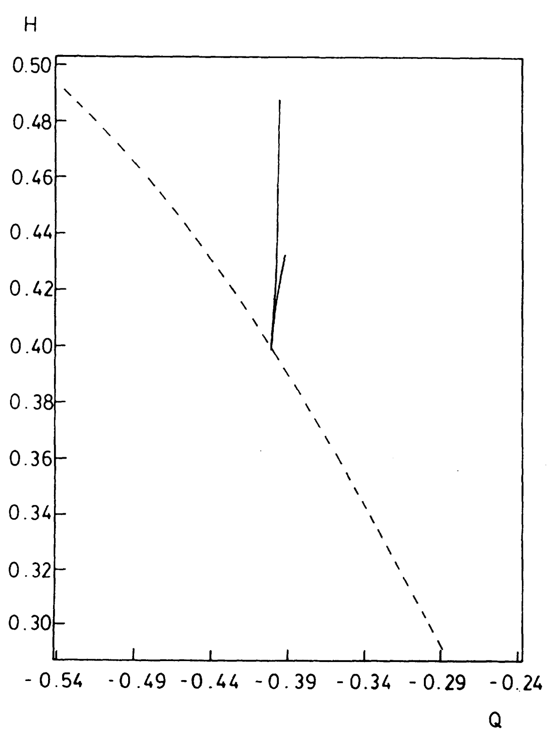

FIG. 2. The dashed line represents the locus of bifurcation points. The bifurcating curve is that of Fig. 1.

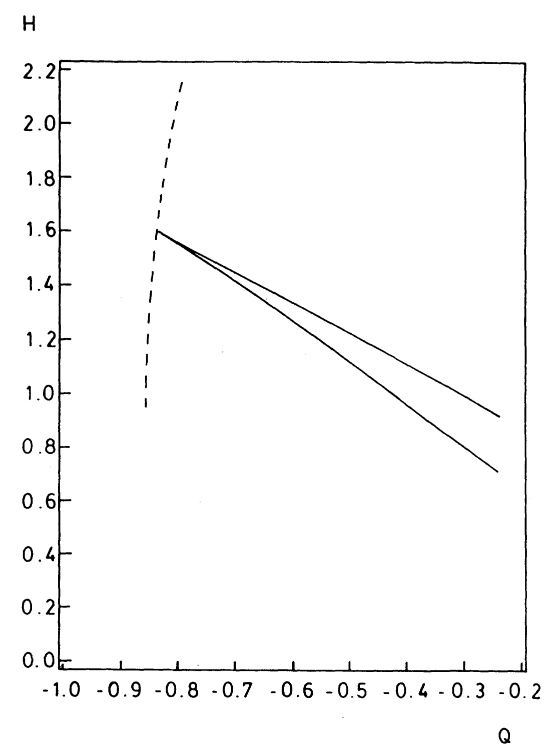

FIG. 3. Plot of H vs Q for the solution of the reduced Eqs. (12) when the parameter s is varied while the rest of parameters are kept fixed: $t=1.535850$, $\alpha=0.500000$, and $\beta=4.000000$. The dashed curve represents the locus of bifurcation points.

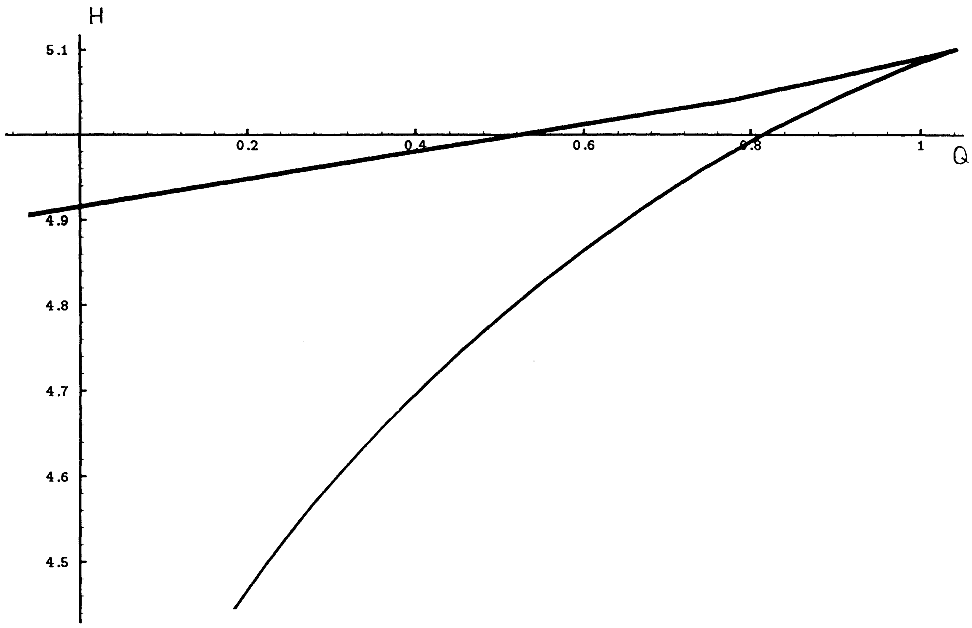

FIG. 4. Plot of H vs Q for the expressions (26) and (27). Here $\alpha=\beta=0.01$, and $s=1.3208$.

$$Q(t)=(25.5883t-0.1682)(2t+1.3208)^{-3.03}\,, \tag{26}$$

$$H(t)=4.1093+(9.4217t^2+12.2585t+2.4466)\times(t+1.3208)^{-4.04}\,. \tag{27}$$

The maximum values of the energy and the total external charge are respectively given by $H_{\max}=5.101\,56$ and $Q_{\max}=1.043\,36$. The bifurcation curve is different from Figs. 1 and 3 and is shown in Fig. 4.

V. REMARKS

We end with some remarks.

(i) It is not difficult to construct solutions for Eqs. (1) once an appropriate ansatz has been written down. However, if we require the solution to have a finite total energy H and a finite total charge Q so that branching occurs in the H versus Q plot, then much effort is demanded. In Sec. III we have demonstrated that such a solution can indeed be found. Of course our solution is not unique and it is possible that other solutions with the above properties can also be found. Incidentally, because the gauge field is topologically massive due to the Chern-Simons term, charge screening is anticipated.

(ii) The bifurcation picture presented in Sec. IV corresponds to the weak bifurcation as suggested in Ref. [9]. This is because the two branches correspond to different external charge density distributions although the total external charges are identical. The weak bifurcation [9] can always be engineered to occur by manipulating the parameters specifying the external charge distribution.

(iii) The energy expression (2) is gauge dependent owing to the fact that A_i^a transforms gauge noncovariantly. Consequently, for our solution in Sec. III, the energy H given by Eqs. (18) is not gauge invariant. If we set $J_i^a=0$ so as to make the energy gauge independent, then the construction of a solution exhibiting charge screening and bifurcation is much harder. Note that J_i^a can be related to the color magnetization density if J_μ is regarded as the quark current [10].

(iv) The charge carried by the CS term as defined in Eq. (7) is in fact the magnetic flux since $\tilde{F}_a^0=B_a$. The CS term may play some dynamic role as suggested in Ref. [3].

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