Vacuum structure and chiral-symmetry breaking in (2+1)-dimensional lattice gauge theories with fermions

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We study further the vacuum structure and chiral-symmetry breaking in Hamiltonian lattice gauge theory with fermions. Attention is paid to (2+1)-dimensional systems. For the fermion sector, a unitary transformation and the variational method are employed. The antiferromagnetic nature of the unitarily transformed Hamiltonian and mesons as spin waves are presented at strong coupling, and the relevance for high-T_c superconductivity is discussed. The vacuum state of the full theory is assumed to be the combination of the variational fermion vacuum state and the exact ground state of the modified Hamiltonian for pure gauge theory of Guo and co-workers. The existing problems in the modified theory are pointed out as well. The chiral condensates in QED_3 and QCD_3 are calculated and the scaling behavior is observed in the crossover regime.

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I. INTRODUCTION

Gauge theories in 2+1 dimensions have acquired increasing attention [1–4], because (a) they possess many similarities to (3+1)-dimensional QCD and are easier to study and (b) $SU(2)_3$ is evidently relevant to high-temperature superconductivity [5–7]. In the new high-T_c superconductivity materials such as La-BaCu-O, it is widely accepted that the essential physics can be described by the Hubbard model defined on a two-dimensional lattice (the Cu-O planes). In the strong-coupling regime when the system is half-filled, it reduces to the antiferromagnetic (Néel-ordered) Heisenberg model

$$H=J\sum_{x,j}\mathbf{S}_x\mathbf{S}_{x+j}\ , \qquad (1.1)$$

where $\mathbf{S}_x$ is a spin-$\frac{1}{2}$ operator. The materials become superconducting when they are doped with a certain concentration of holes. Equation (1.1) obviously owns a global SU(2) symmetry. By making the spin-color correspondences, Eq. (1.1) can be rewritten as [6,7]

$$H=\frac{J}{8}\sum_{x,j}[M_xM_{x+j}+2(B_x^\dagger B_{x+j}+B_{x+j}^\dagger B_x)] -\frac{Jd}{4}\sum_x(M_x-\tfrac{1}{2})\ . \qquad (1.2)$$

Here d is the spatial dimensions and ($c=1,2$ are "color" indices and ψ are fermion fields)

$$M_x=\sum_c\psi_{x,c}^\dagger\psi_{x,c}\ ,\qquad B_x=\psi_{x,1}\psi_{x,2}\ . \qquad (1.3)$$

One sees apparently that Eq. (1.2) has a local SU(2) gauge symmetry. As will be shown in Sec. II, the Heisenberg model is equivalent to lattice $SU(2)_3$ in the strong-coupling regime. Chiral-symmetry breaking ($\langle\bar\psi\psi\rangle\neq0$) in (2+1)-dimensional lattice gauge theories (LGT_3) corresponds to the magnetization ($\langle S^3\rangle\neq0$) in the Heisenberg model.

Spontaneous chiral-symmetry breaking, which should be investigated nonperturbatively, is an important character in quantum field theory. It was conjectured that chiral-symmetry breaking, fermion confinement in LGT, and the process of pairing in the Hubbard model may be closely related. LGT presents a well-defined technique in which nonperbative evaluations of physical quantities can be performed from first principles. LGT_3 has been formulated and discussed using numerical [8–15] and analytical [16–20] methods. For a small number of fermion flavors, confident evidence for chiral-symmetry breaking in $U(1)_3$ [11–13,15] and preliminary results for $SU(2)_3$ [14,16,19,20] and $SU(3)_3$ [19,20] were obtained. With the development of dedicated supercomputers and algorithms, numerical simulations have become a powerful tool for this task, giving many encouraging results. In order to understand the physical insights of LGT, such as the local structure of the vacuum and wave functions, several analytical methods [21–26] were also developed. The lattice computations are reliable only when the correct continuum limit is approached and the expected scaling behaviors are observed. The answers to these questions are far from trivial. In the Hamiltonian formalism, the second one may be partly reduced to the di-

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agonalization of the lattice Hamiltonian.

Recently, we developed a different approach to $(d+1)$-dimensional Hamiltonian LGT with fermions [19,20,27–33], which consists of a unitary transformation and the variational method. In [27–33] our approach was successfully applied to the Schwinger model and QCD_2. In Refs. [19,20,27] we extended it to $2+1$ and $3+1$ dimensions by incorporating the variational fermion vacuum state and exact ground state of the modified Hamiltonian of pure gauge theory proposed by Guo and co-workers [34–36]. Preliminary results are encouraging up to the crossover regime. However, as addressed in [38–42] and Sec. III, there may still be some existing problems in the modified Hamiltonian of pure gauge theory.

This paper is organized as follows. In Sec. II a unitary transformation and the variational method [27–33] are used to reproduce the effective Hamiltonian at strong coupling [43,44], its equivalence to the antiferromagnetic Heisenberg model is presented, and the existence of pseudo Goldstone bosons is demonstrated. In Sec. III the vacuum structure of the full theory (the fermionic Hamiltonian plus the modified pure gauge one of Guo and co-workers) is analyzed. The existing problems in the modified Hamiltonian are discussed as well. In Sec. IV the chiral condensates in $U(1)_3$, $SU(2)_3$, and $SU(3)_3$ are systematically calculated. Conclusions are summarized in Sec. V.

II. ANTIFERROMAGNETISM AND MESON SPECTRUM IN THE STRONG-COUPLING REGIME

By discretizing the spatial dimensions and choosing the $A_t^\alpha=0$ gauge, the Hamiltonian of $(d+1)$-dimensional LGT is

$$\begin{aligned}
H&=H_m+H_k+H_r+H_g ,\\
H_m&=m\sum_x\bar\psi(x)\psi(x) ,\\
H_k&=\frac{1}{2a}\sum_{x,k}\bar\psi(x)\gamma_k U(x,k)\psi(x+k) ,\\
H_r&=\frac{r}{2a}\sum_{x,k}\bar\psi(x)[\psi(x)-U(x,k)\psi(x+k)] ,\\
H_g&=\frac{g^2}{2a}\sum_{y,j}E_j^\alpha(y)E_j^\alpha(y)-\frac{f}{ag^2}\sum_p\mathrm{Tr}(U_p+U_p^\dagger-2)+\cdots ,
\end{aligned}\tag{2.1}$$

where m, a, r, g, and $U(x,k)$ are the fermion mass, lattice spacing, Wilson parameter, bare coupling constant, and gauge link variables. In $2+1$ dimensions, ψ are still taken to be four-component spinors, because γ_3 and γ_5, which anticommute with γ_1, γ_2, and γ_4, can be employed to define global chiral symmetries. H_g is a mixture of gauge electric and magnetic energies [$f=\frac{1}{2}$ for U(1) and $f=1$ for $SU(N_c)$], and the dots represent some possible terms which will be discussed in next section. The strong coupling or bare vacuum state $|0\rangle$ is defined by

$$\xi(x)|0\rangle=\eta(x)|0\rangle=E_j^\alpha(y)|0\rangle=0 .\tag{2.2}$$

We note that H_k, which is the only nondiagonal term in the fermionic Hamiltonian, consists of fermion creation and annihilation operators. In the strong-coupling limit $1/g^2=0$, $|0\rangle$ is the zeroth-order approximation to the vacuum state without color-electric flux. When g decreases, fermion-antifermion pairs connected by gauge fields are created in the vacuum, which results in the diagonalization of the Hamiltonian. The fermion vacuum can be described by [27–33]

$$\begin{aligned}
|\Omega_f\rangle&=\exp(iS_f)|0\rangle ,\\
S_f&=i\sum_{x,k_i}{}'\sum_{n=0}^{\infty}[\theta_n\psi^\dagger(x)\gamma_{k_1}\cdots\gamma_{k_{2n+1}}U(x,k_1,\ldots,k_{2n+1})\psi(x+k_1+\cdots+k_{2n+1})\\
&\qquad+\theta_n'\psi^\dagger(x)\gamma_{k_1}\cdots\gamma_{k_{2n+1}}U(x,k_1,\ldots,k_{2n+2})\psi(x+k_1+\cdots+k_{2n+2})] ,
\end{aligned}\tag{2.3}$$

where summations over k_i satisfy $k_i+k_{i+1}\neq0$, and θ_n and θ_n' are variational parameters determined by minimizing the vacuum energy

$$E_{\Omega_f}=\frac{\langle\Omega_f|H_f|\Omega_f\rangle}{\langle\Omega_f|\Omega_f\rangle}=\langle 0|H_f'|0\rangle ,\tag{2.4}$$

while

$$H_f'=\exp(-iS_f)H_f\exp(iS_f)\tag{2.5}$$

is just the unitarily transformed fermionic Hamiltonian.

For free fermions [$U(x,k)=1$ and $H_g=0$], the unitary transformation and variational method [30] not only diagonalize exactly the Hamiltonian, but also give the correct vacuum energy (dispersion law). For the Schwinger model and QCD_2 (there is no plaquette energy in $1+1$ dimensions), this approach results in satisfactory agreement [27–33] with the continuum predictions.

Now let us look how the fermion ansatz works in higher dimensions. For simplicity, the Hamiltonian with naive fermions ($r=0$) is considered here so that θ_n' vanish automatically [27–33]. In the strong-coupling region $1/g^2\ll1$, θ_n are so small that the one-link term ($n=0$) in Eq. (2.3),

$$S_f \approx i\theta_0 \sum_{x,k} \psi^\dagger(x)\gamma_k U(x,k)\psi(r+k) , \tag{2.6}$$

is dominant. By expanding the transformed Hamiltonian up to two-link terms, we obtain

$$H' = H'_m + H'_k + H'_g ,$$

$$H'_m = m\left[\sum_x \bar\psi(x)\psi(x) + (-2\theta_0)\sum_{x,k}\bar\psi(x)\gamma_k U(x,k)\psi(x+k) + \frac{(-2\theta_0)^2}{2!}\sum_{x,k_1,k_2}\bar\psi(x)\gamma_{k_1}\gamma_{k_2}U(x,k_1,k_2)\psi(x+k_1+k_2)\right] ,$$

$$H'_k = \frac{1}{2a}\left[\sum_{x,k}\bar\psi(x)\gamma_k U(x,k)\psi(x+k) + (-2\theta_0)\sum_{x,k_1,k_2}\bar\psi(x)\gamma_{k_1}\gamma_{k_2}U(x,k_1,k_2)\psi(x+k_1+k_2)\right] , \tag{2.7}$$

$$H'_g = \frac{g^2}{2a}\sum_{y,j}\Big\{E_j^\alpha(y)E_j^\alpha(y) + \theta_0 E_j^\alpha(y)\sum_{x,k}\psi(x)^\dagger\gamma_k[U(x,k),E_j^\alpha(y)]\psi(x+k) + \theta_0\sum_{x,k}\psi^\dagger(x)\gamma_k[U(x,k),E_j^\alpha(y)]\psi(x+k)E_j^\alpha(y)$$

$$+\theta_0^2\sum_{x_1,k_1}\psi^\dagger(x_1)\gamma_{k_1}[U(x_1,k_1),E_j^\alpha(y)]\psi(x_1+k_1)\sum_{x_2,k_2}\psi^\dagger(x_2)\gamma_{k_2}[U(x_2,k_2),E_j^\alpha(y)]\psi(x_2+k_2)\Big\} ,$$

where the magnetic fluctuations have been neglected. The vacuum energy is

$$E_{\Omega_f} = (2N_c N_f V)\varepsilon_{\Omega_f}$$

$$= (2N_c N_f V)\left\{-m[1-(2\theta_0)^2 d] - \frac{(2\theta_0)d}{a} + \frac{g^2 C}{2a}\frac{(2\theta_0)^2 d}{2}\right\} , \tag{2.8}$$

where V, N_c, N_f, ε_{Ω_f}, and C are, respectively, the total number of lattice sites, number of colors, number of flavors, vacuum energy density, and Casmir invariant [$C=1$ for U(1) and $C=(N_c^2-1)/(2N_c)$ for SU(N_c)]. By solving the equation $\partial\varepsilon_{\Omega_f}/(\partial\theta_0)=0$, θ_0 as a function of the fermion mass and coupling constant is obtained:

$$2\theta_0 = \frac{2}{4ma+g^2C} . \tag{2.9}$$

In the zero- and one-link subspace, this condition also eliminates the one-link terms in H' so that the matrix elements of one-link terms in the substates vanish. This is the first step to the diagonalization of H. In the fluxless state $|0\rangle$, H becomes the effective Hamiltonian in the strong-coupling region:

$$H_{\text{eff}} = \left\{m[1-(2\theta_0)^2 d] + \frac{(2\theta_0)d}{a}\right\}\sum_x \bar\psi(x)\psi(x) + \frac{g^2 C}{2a}\frac{(2\theta_0)^2}{4N_c}\sum_{x,k}\psi^\dagger_{c_1,f_1}(x)\gamma_k\psi_{c_2,f_1}(x+k)\psi^\dagger_{c_2,f_2}(x)\gamma_k\psi_{c_1,f_2}(x+k) . \tag{2.10}$$

Here the color and flavor indices of the four-fermion term are specified. This Hamiltonian represents the nearest-neighbor interactions. Smit [43] and Greensite and Primack [44] derived very similar results in 3+1 dimensions by using a strong-coupling expansion method. As seen in Refs. [19,20,27–33] and the following sections, our approach to the fermion sector is self-consistent and can at least be extended to the crossover regime. For $SU(2)_3$, Eq. (2.10) is equivalent to the antiferromagnetic Heisenberg Hamiltonian [Eq. (1.2)]. To understand this further, we make a Fierz rearrangement so that Eq. (2.10) becomes

$$H_{\text{eff}} = (2N_c N_f)\left\{m[1-(2\theta_0)^2 d] + \frac{(2\theta_0)d}{a}\right\}\sum_x \bar\psi(x)\psi(x) + \frac{g^2 C d(2\theta_0)^2}{4a}\sum_x \psi^\dagger(x)\psi(x)$$

$$-\frac{g^2 C(2\theta_0)^2}{32aN_c}\sum_{x,k,f_1,f_2}\psi^\dagger_{f_1}(x)\Gamma^A\psi_{f_2}(x+k)\psi^\dagger_{f_2}(x)\Gamma^A\psi_{f_1}(x+k)L^A , \tag{2.11}$$

where ($\Sigma_q = i\varepsilon_{jkq}\gamma_j\gamma_k$)

$$\Gamma^A: 1,\gamma_4,\gamma_3,\gamma_5,i\gamma_4\gamma_3,i\gamma_4\gamma_5,\gamma_q,i\gamma_4\gamma_q,\Sigma_3,\Sigma_q,i\gamma_4\Sigma_3,i\gamma_4\Sigma_q ,$$

$$L^A: 1,-1,-1,-1,1,1,-1+2\delta_{k,q},1-2\delta_{k,q},-1,-1+2\delta_{k,q},-1,-1+2\delta_{k,q} . \tag{2.12}$$

By making a linear approximation [45], the effective Hamiltonian can be represented by the operators (related to pseudoscalars and vectors)

$$P_3(x)_{f_1f_2}=\frac{1}{2(2N_c)^{1/2}}\psi^\dagger_{f_1}(x)(1-\gamma_4)\gamma_3\psi_{f_2}(x)\ ,$$
$$P_5(x)_{f_1f_2}=\frac{1}{2(2N_c)^{1/2}}\psi^\dagger_{f_1}(x)(1-\gamma_4)\gamma_5\psi_{f_2}(x)\ , \qquad (2.13)$$
$$V_q(x)_{f_1f_2}=\frac{1}{2(2N_c)^{1/2}}\psi^\dagger_{f_1}(x)(1-\gamma_4)\gamma_q\psi_{f_2}(x)\ ,$$

as

$$H_{\rm eff}=2N_cN_fV\varepsilon_{\Omega_f}+H_3+H_5+\sum_q H_q\ ,$$
$$H_3=\sum_{p,f_1,f_2}\left\{G_1A_3^\dagger(p)_{f_1,f_2}A_3(p)_{f_2,f_1}+G_2[A_3^\dagger(p)_{f_1,f_2}A_3(-p)_{f_2,f_1}+\text{H.c.}]\sum_j \cos pja\right\}\ ,$$
$$H_5=\sum_{p,f_1,f_2}\left\{G_1A_5^\dagger(p)_{f_1,f_2}A_5(p)_{f_2,f_1}+G_2[A_5^\dagger(p)_{f_1,f_2}A_5(-p)_{f_2,f_1}+\text{H.c.}]\sum_j \cos pja\right\}\ , \qquad (2.14)$$
$$H_q=\sum_{p,f_1,f_2}\left\{G_1B_q^\dagger(p)_{f_1,f_2}B_q(p)_{f_2,f_1}+G_2[B_q^\dagger(p)_{f_1,f_2}B_q(-p)_{f_2,f_1}+\text{H.c.}]\left[\sum_j \cos pja-2\cos pqa\right]\right\}\ ,$$

where $G_1=-2\varepsilon_{\Omega_f}$, $G_2=g^2Cd(2\theta_0)^2/(4a)$, and $A_3(p)$, $A_5(p)$, and $B_q(p)$ are the Fourier-transformed operators of those in Eq. (2.13). The antiferromagnetic nature of this Hamiltonian is obvious, which also shows that mesons behave as spin waves. It can be exactly diagonalized by the Bogoliubov transformation:

$$A_3(p)=\cosh u_3(p)a_3(p)+\sinh u_3(p)a_3^\dagger(-p)\ ,$$
$$A_5(p)=\cosh u_5(p)a_5(p)+\sinh u_5(p)a_5^\dagger(-p)\ , \qquad (2.15)$$
$$B_q(p)=\cosh v_q(p)b_q(p)+\sinh v_q(p)b_q^\dagger(-p)\ ,$$

if $u_3(p)$, $u_5(p)$, and $v_3(q)$ obey

$$\tanh 2u_3(p)=-\frac{2G_2}{G_1}\sum_j\cos pja\ ,$$
$$u_5(p)=u_3(p)\ , \qquad (2.16)$$
$$\tanh 2v_q(p)=-\frac{2G_2}{G_1}\left[\sum_j\cos pja-2\cos pqa\right]\ ,$$

which is the condition of minimizing the vacuum energy as well. The Bogoliubov transformed Hamiltonian is

$$H'_{\rm eff}=2N_cN_fV\varepsilon_{\Omega'_f}+G_1\sum_{p,f_1,f_2}\left\{[1-\tanh^2 2u_3(p)]^{1/2}a_3^\dagger(p)_{f_1,f_2}a_3(p)_{f_2,f_1}+[1-\tanh^2 2u_5(p)]^{1/2}a_5^\dagger(p)_{f_1,f_2}a_5(p)_{f_2,f_1}\right.$$
$$\left.+\sum_q[1-\tanh^2 2v_q(p)]^{1/2}b_q^\dagger(p)_{f_1,f_2}b_q(p)_{f_2,f_1}\right\}\ . \qquad (2.17)$$

Here

$$\varepsilon_{\Omega'_f}=\varepsilon_{\Omega_f}-\frac{G_1N_f}{2N_cV}\sum_p\{1-[1-\tanh^2 2u_3(p)]^{1/2}\}-\frac{G_1N_f}{4N_cV}\sum_{p,q}\{1-[1-\tanh^2 2v_q(p)]^{1/2}\} \qquad (2.18)$$

is the energy density of the exact ground state of H in the strong-coupling region:

$$|\Omega'_f\rangle=\exp(i\theta_0S_{f_0})R|0\rangle\ , \qquad (2.19)$$

with

$$R=\exp\left\{\frac{1}{2}\sum_{p,f_1,f_2}\left[u_3(p)A_3(-p)_{f_1,f_2}A_3(p)_{f_2,f_1}+u_5(p)A_5(-p)_{f_1,f_2}A_5(p)_{f_2,f_1}\right.\right.$$
$$\left.\left.+\sum_q v_q(p)B_q(-p)_{f_1,f_2}B_q(p)_{f_2,f_1}-\text{H.c.}\right]\right\}. \tag{2.20}$$

The vacuum state describes that when the bare coupling is reduced below the infinitive, fermions can move and exchange between neighboring mesons, which leads to meson pair condensation and generates these nearest-neighbor interactions.

Using the Feynman-Hellmann theorem, we obtain the chiral condensate

$$\frac{\langle \Omega'_f|\sum_x\bar{\psi}(x)\psi(x)|\Omega'_f\rangle}{2N_cN_fV}=\left.\frac{\partial\varepsilon_{\Omega'_f}}{\partial m}\right|_{m=0}=-\left[1-\frac{4d}{g^4C^2}\right]\left[1-\frac{2N_f}{N_c}I_1-\frac{N_f}{N_c}I_2\right], \tag{2.21}$$

where, for $d=2$,

$$I_1=\frac{a^2}{2(2\pi)^2}\int_{-\pi/a}^{\pi/a}d^2p\left[\frac{1}{\left[1-\left[\frac{1}{2}\sum_j\cos p_ja\right]^2\right]^{1/2}}-1\right],$$
$$I_2=\frac{a^2}{2(2\pi)^2}\sum_q\int_{-\pi/a}^{\pi a}d^2p\left\{\frac{1}{\left\{1-\left[\frac{1}{2}\left[\sum_j\cos p_ja-2\cos p_qa\right]\right]^2\right\}^{1/2}}-1\right\}. \tag{2.22}$$

Note that H in Eq. (2.1) is chirally invariant. The nonvanishing value of $\langle\bar{\psi}\psi\rangle$ indicates spontaneous chiral-symmetry breaking in the vacuum, which results in two kinds of massless pseudoscalars. These Goldstone bosons are generated by applying, respectively, $a_3^\dagger(p)$ and $a_5^\dagger(p)$ to the vacuum. Their masses are

$$M_{P_3}^2=M_{P_5}^2=G_1^2[1-\tanh^2 2u_3(0)]\xrightarrow{1/g^2\ll 1,\,m\to 0} m\left[\frac{8d}{ag^2C}\right], \tag{2.23}$$

where the current-algebra relation is well reproduced. Similarly, the vector mass at momentum $p=0$ is

$$M_q=G_1[1-\tanh^2 2v_3(0)]^{1/2}=G_1\xrightarrow{1/g^2\ll 1,\,m\to 0}\frac{2d}{ag^2C}. \tag{2.24}$$

In the strong-coupling regime $1/g^2\ll 1$, the creation operators for the pseudoscalars and vector at $p=0$ are

$$a_3^\dagger(x)_{f_1f_2}=\frac{1}{2(2N_c)^{1/2}}\left[\left[\frac{2d}{g^2Cma}\right]^{1/4}\psi_{f_1}^\dagger(x)\gamma_3\psi_{f_2}(x)+\left[\frac{g^2Cma}{2d}\right]^{1/4}\psi_{f_1}^\dagger(x)\gamma_4\gamma_3\psi_{f_2}(x)\right],$$
$$a_5^\dagger(x)_{f_1f_2}=\frac{1}{2(2N_c)^{1/2}}\left[\left[\frac{2d}{g^2Cma}\right]^{1/4}\psi_{f_1}^\dagger(x)\gamma_5\psi_{f_2}(x)+\left[\frac{g^2Cma}{2d}\right]^{1/4}\psi_{f_1}^\dagger(x)\gamma_4\gamma_5\psi_{f_2}(x)\right], \tag{2.25}$$
$$b_q^\dagger(x)_{f_1f_2}=\frac{1}{2(2N_c)^{1/2}}\psi_{f_1}^\dagger(x)(1+\gamma_4)\gamma_q\psi_{f_2}(x).$$

In $a_3^\dagger(x)$ and $a_5^\dagger(x)$ only the first terms do not vanish in the chiral limit $m\to 0$.

In Eq. (2.21) one sees that $|\langle\bar{\psi}\psi\rangle|$ decreases with the increase of N_f/N_c, which can be interpreted as the screening of the forces that produces chiral-symmetry breaking due to the effects of dynamical fermions or meson pair condensation. Dagotto, Kocic, and Kogut [12] conjectured that there may be critical value for N_f beyond which chiral symmetries would be restored. At a small number of flavors, the effect of the Bogoliubov transformation R is small. In the quenched limit $N_f\ll 1$, Eq. (2.3) is a good approximation.

III. VACUUM STRUCTURE OF THE FULL THEORY

Beyond the strong-coupling region, however, because of the presence of the plaquette energy, the investigations become very difficult. To diagonalize the Hamiltonian in the weak-coupling regime, one has to use arbitrarily large

Wilson loops. The complications involved have hindered most authors in this field from further studying. In [34–36] Guo and co-workers proposed a series modified pure gauge Hamiltonians with exact ground states by adding to the usual one

$$H_g^{\mathrm{KS}}=\frac{g^2}{2a}\sum_{y,j}E_j^\alpha(y)E_j^\alpha(y)-\frac{f}{ag^2}\sum_p\mathrm{Tr}(U_p+U_p^\dagger-2) \quad (3.1)$$

terms which vanish in the classical continuum limit:

$$\Delta H=-\frac{g^2}{2a}\sum_{y,j}[E_j^\alpha(y),S_g][E_j^\alpha(y),S_g] , \quad (3.2)$$

where S_g is some function of plaquette variables such as

$$S_g=\frac{f}{2Cg^4}\sum_p\mathrm{Tr}(U_p+U_p^\dagger) \quad (3.3)$$

or other variants [34–36]. Here we will use this simplest form for S_g.

At first glance their method worked well because the modified Hamiltonians become the usual Hamiltonian for pure gauge theory and very nice scaling behaviors [35–37] for the glueball masses and string tension were obtained, being consistent with the conventional methods. However, Roskies and others [38–40] found that the vacuum expectation values for ΔH may diverge because of high-energy fluctuations. Although the divergence may be subtracted out by some renormalization scheme [41], the resulting operator ΔH may generally be relevant at the nonperturbative level. (A similar problem appears in LGT with Wilson fermions.) A study of the analogous nonlinear σ model [42] indicated that the modified Hamiltonian was not Lorentz invariant. Therefore the modified pure gauge theory may not be in the same universality class as the conventional one in the continuum limit. However, supported by the spectrum calculations, the modified theory may still be a good approximation at least up to the crossover regime. Even though the modified pure gauge theory has several "fundamental drawbacks," it might be a better method than the strong-coupling expansion and deserves further study: for instance, understanding why eventually different theories seem to give approximately the same results or how well the modified theory works.

In this paper the modified pure gauge Hamiltonian [34–36] of Guo and co-workers is adopted to include the magnetic fluctuations, which is rewritten as

$$H_g=\frac{g^2}{2a}\sum_{y,j}\exp(-S_g)E_j^\alpha(y)\exp(2S_g)E_j^\alpha(y)\exp(-S_g) , \quad (3.4)$$

where the constant term has been neglected. It is obvious that H_g possesses an exact ground state

$$|\Omega_g\rangle=\exp(S_g)|0\rangle , \quad (3.5)$$

describing magnetic fluctuations in the vacuum. We propose the physical vacuum of the full theory to be the combination of the exact ground state of pure gauge theory and the variational fermion vacuum state:

$$|\Omega\rangle=\exp(S_g)\exp(iS_f)|0\rangle . \quad (3.6)$$

One may also discuss the operator mixing and renormalization problem [41]. But it is too complicated for a practical calculation and will not be discussed here. Since LGT in 2+1 dimensions is superrenormalizable, the renormalization effects would be very small in the currently investigated coupling regions. In fact, this conjecture was supported by previous results for glueball masses and string tension [35–37].

IV. FERMION CONDENSATES IN QED$_3$ AND QCD$_3$

The energy of the proposed vacuum state Eq. (3.6) is

$$E_\Omega=\frac{\langle\Omega|H|\Omega\rangle}{\langle\Omega|\Omega\rangle}=\frac{\langle\Omega_g|H_f'|\Omega_g\rangle}{\langle\Omega_g|\Omega_g\rangle} . \quad (4.1)$$

Here one notes that the pure gauge vacuum energy vanishes [34–37] so that the variational parameters can be calculated in the same way as in Sec. II. In Refs. [19,20] the one-link approximation (only the first term in the fermion ansatz S_f)

$$|\Omega_{1la}\rangle=\exp(S_g)\exp(i\theta_0S_{f_0})|0\rangle \quad (4.2)$$

was considered. The vacuum energy was

$$E_{\Omega_{1La}}=m\sum_{n=0}^\infty\frac{(2\theta_0)^{2n}}{(2n)!}\sum_{x,k_i}W_{2n+2}(x,k_i)-\frac{1}{2a}\sum_{n=0}^\infty\frac{(2\theta_0)^{2n+1}}{(2n+1)!}\sum_{x,k_i}W_{2n+2}(x,k_i)+\frac{g^2C}{2a}\sum_{n=0}^\infty\frac{(2\theta_0)^{2n+2}}{(2n+2)!}\sum_{y,j,x,k_i}V_{2n+2}(y,j,x,k_i) , \quad (4.3)$$

where the symbols W and V were given in Refs. [19,20,27]. For example,

$$\begin{aligned}\sum_{x,k_i}W_{2n}(x,k_i)&=(-2N_fN_cV)W_{2n}\\&=\sum_{x,k_i}\frac{\langle 0|\exp(S_g)\bar\psi(x)\gamma_{k_1}\cdots\gamma_{k_{2n}}U(x,k_1,\ldots,k_{2n})\psi(x+k_1+\cdots+k_{2n})\exp(S_g)|0\rangle}{\langle 0|\exp(2S_g)|0\rangle}\\&=\left[-\frac{N_fV}{2}\right]\sum_{k_i}\mathrm{Tr}(\gamma_{k_1}\cdots\gamma_{k_{2n}})\frac{\int[dU]\exp(2S_g)\mathrm{Tr}[U(x,k_1,\ldots,k_{2n})]}{\int[dU]\exp(2S_g)} ,\end{aligned} \quad (4.4)$$

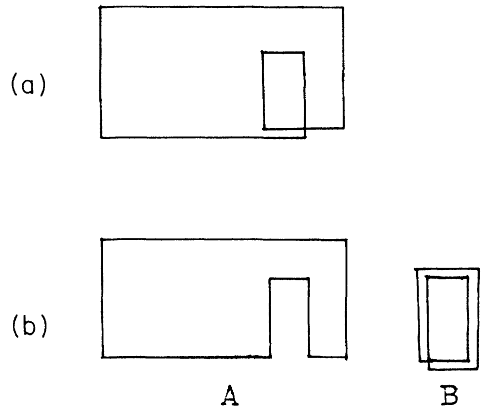

FIG. 1. Factorization of a Wilson loop.

from which one sees that after contractions of fermion fields, various shapes of Wilson loops with $2n$ links satisfying $k_1+\cdots+k_{2n}=0$ appear. The number of these graphs, which grows rapidly, can be counted on a computer [19,20]. [Actually, Eq. (4.3) is calculated until the series converges well.] Once W_{2n} are known, the third term in Eq. (4.3) is obtained. In $2+1$ dimensions the expectation values of Wilson loops can be evaluated exactly. It can be easily proven that, for a Wilson loop such as Fig. 1(a), its expectation value factorizes in a product of those for loops A and B in Fig. 1(b):

$$\langle \mathrm{Tr} W\rangle_g=\frac{1}{N_c}\langle \mathrm{Tr} W_A\rangle_g\langle \mathrm{Tr} W_B\rangle_g \,, \tag{4.5}$$

where $\langle \mathrm{Tr} W_B\rangle_g$ has been calculated in Refs. [35,36] using recurrence relations and

$$\langle \mathrm{Tr} W_A\rangle_g=N_c Y^{n_A}=N_c\left[\frac{\int[dU]\exp(2S_g)\mathrm{Tr}(U_p)}{\int[dU]\exp(2S_g)}\right]^{n_A} , \tag{4.6}$$

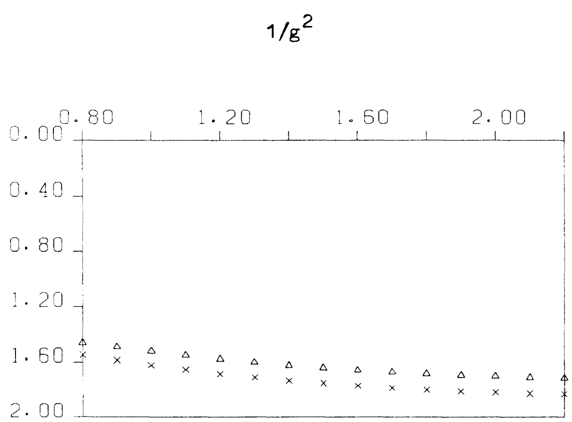

FIG. 2. Vacuum energy density $-\alpha E_\Omega/(N_f N_c V)$ as a function of $1/g^2$ in $\mathrm{U(1)}_3$. The triangles stand for the data of the one-link approximation to S_f, while the crosses represent those of the three-link approximation.

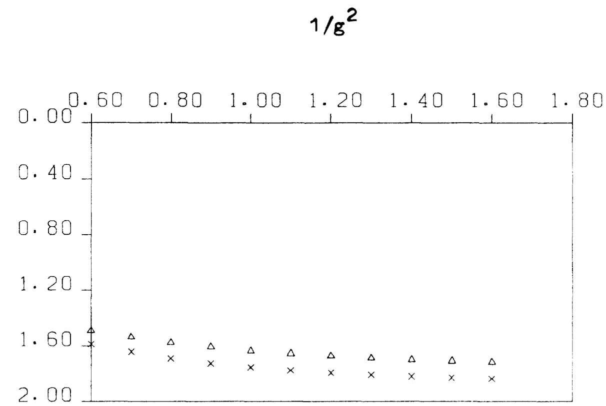

FIG. 3. Vacuum energy density $-aE_\Omega/(N_f N_c V)$ as a function of $1/g^2$ in $\mathrm{SU(2)}_3$. The meanings of the triangles and crosses are the same as those in Fig. 2.

with n_A the number of elementary plaquettes surrounded by this loop and

$$Y=\frac{I_1(1/g^4)}{I_0(1/g^4)} \quad \text{for U(1)} ,$$
$$Y=\frac{I_2(16/3g^4)}{I_1(16/3g^4)} \quad \text{for SU(2)} , \tag{4.7}$$
$$Y=\frac{\sum_{l=0}^{\infty} l z_l t^{l-1}}{6\sum_{l=0}^{\infty} z_l t^l} \quad \text{for SU(3)} ,$$

where I_j are the jth-order modified Bessel functions and $t=3/(4g^4)$. z_l are given in Ref. [20]. In the chiral limit, the vacuum energy densities $-aE_{\Omega_{1la}}/(N_f N_c V)$ of the state [Eq. (4.2)] in $\mathrm{U(1)}_3$, $\mathrm{SU(2)}_3$, and $\mathrm{SU(3)}_3$ versus $1/g^2$ are represented by the triangles in Figs. 2, 3, and 4, respectively. One sees that the vacuum energy is lowered by the inclusion of fermions. According to the Feynman-Hellmann theorem, the corresponding chiral condensate is given by

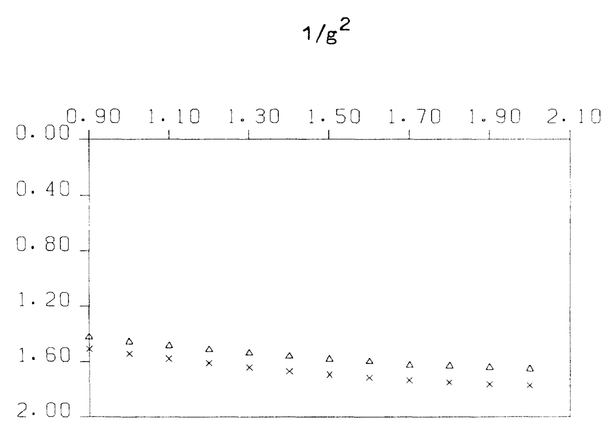

FIG. 4. Vacuum energy density $-aE_\Omega/(N_f N_c V)$ as a function of $1/g^2$ in $\mathrm{SU(3)}_3$. The meanings of the triangles and crosses are the same as those in Fig. 2.

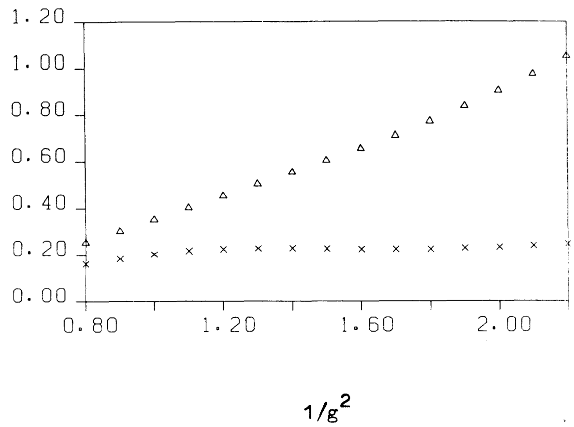

FIG. 5. $-\langle \bar{\psi}\psi \rangle_{\text{lattice}}/(2N_f N_c V g^4)$ as a function of $1/g^2$ in $U(1)_3$. The meanings of the triangles and crosses are the same as those in Fig. 2.

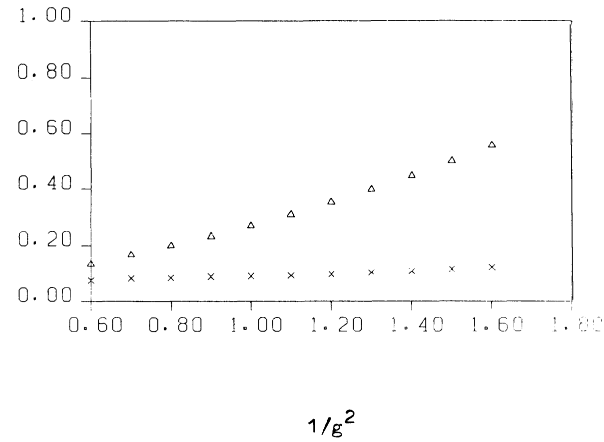

FIG. 6. $-\langle \bar{\psi}\psi \rangle_{\text{lattice}}/(2N_f N_c V g^4)$ as a function of $1/g^2$ in $SU(2)_3$. The meanings of the triangles and crosses are the same as those in Fig. 2.

$$\frac{\langle \Omega_{1la} | \sum_x \bar{\psi}(x)\psi(x) | \Omega_{1la} \rangle}{-2N_c N_f V \langle \Omega_{1la} | \Omega_{1la} \rangle} = \sum_{n=0}^{\infty} \frac{(2\theta_0)^{2n}}{(2n)!} W_{2n} \ . \tag{4.8}$$

Its relation to $1/g^2$ can be obtained by substituting the determined $\theta_0(1/g^2)$ into Eq. (4.8). To extract useful continuum information from LGT, one has to compare it with the scaling behavior. By using the relation between the bare coupling constant and the continuum charge

$$g^2 = e^2 a \tag{4.9}$$

and by dimensional analysis, the lattice chiral condensate $\langle \bar{\psi}\psi \rangle_{\text{lattice}}$ should scale as

$$\frac{\langle \bar{\psi}\psi \rangle_{\text{lattice}}}{g^4} = 2^2 \frac{\langle \bar{\psi}\psi \rangle_{\text{continuum}}}{e^4} \ , \tag{4.10}$$

where the factor 2^2 results from the species doubling of naive fermions. Because LGT_3 is superrenormalizable, the right-hand side (RHS) is a constant. The LHS of Eq. (4.10) in this state in $U(1)_3$, $SU(2)_3$, and $SU(3)_3$ as a function of $1/g^2$ are plotted in Figs. 5, 6, and 7, respectively. It is noted that there are obvious deviations from the expected scaling behavior. These indicate that in the crossover region S_f stretching over only one lattice spacing is not enough. In order to extend the results to weaker coupling, operators with wider separation of fermions in S_f have to be used. These operators correspond to fermion-antifermion interactions at longer distances, which become important at weaker coupling. Let us consider further the three-link approximation to S_f [the first two terms in Eq. (2.3)]:

$$\begin{aligned} S_f &\approx \theta_0 S_{f_0} + \theta_1 S_{f_1} \ , \\ S_{f_1} &= i \sum_{x,k_i}{}' \psi^\dagger(x) \gamma_{k_1} \gamma_{k_2} \gamma_{k_3} U(x,k_1,k_2,k_3) \\ &\quad \times \psi(x+k_1+k_2+k_3) \ . \end{aligned} \tag{4.11}$$

The chiral condensate is

$$\frac{\langle \Omega_{3la} | \sum_x \bar{\psi}(x)\psi(x) | \Omega_{3la} \rangle}{-2N_f N_c V \langle \Omega_{3la} | \Omega_{3la} \rangle} = \frac{1}{2N_f N_c V} \sum_{n=0}^{\infty} \frac{1}{(2n)!} \sum_{n=0}^{n} \frac{(2n)!}{l!(2n-l)!} (2\theta_0)^{2n-l} (2\theta_1)^l \sum_{x,k_{i_0}} \sum_{q_{i_1}}{}' W_{2n-l,l}(x,k_{i_0},q_{i_1}) \ , \tag{4.12}$$

where

$$\begin{aligned} \sum_{x,k_{i_0}} \sum_{q_{i_1}}{}' W_{2n-l,l}(x,k_{i_0},q_{i_1}) &= (-2N_c N_f V) W_{2n-l,l} \\ &= \sum_{x,k_{i_0}} \sum_{q_{i_1}}{}' \langle 0 | \exp(S_g) \bar{\psi}(x) \gamma_{k_1} \cdots \gamma_{k_{2n-l}} \gamma_{q_1} \cdots \gamma_{q_{3l}} U(x,k_1,\ldots,k_{2n-l},q_1,\ldots,q_{3l}) \\ &\quad \times \psi(x+k_1+\cdots+k_{2n-l}+q_1+\cdots+q_{3l}) \exp(S_g) | 0 \rangle / \langle 0 | \exp(2S_g) | 0 \rangle \ , \end{aligned} \tag{4.13}$$

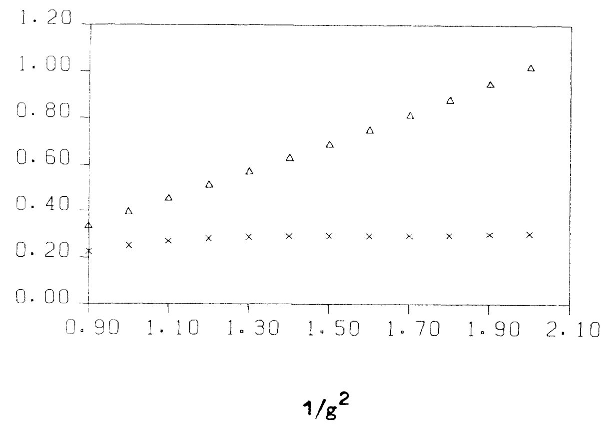

FIG. 7. $-\langle\bar{\psi}\psi\rangle_{\text{lattice}}/(2N_f N_c V g^4)$ as a function of $1/g^2$ in $SU(3)_3$. The meanings of the triangles and crosses are the same as those in Fig. 2.

with

$$\begin{aligned} W_{2n,0} &= W_{2n} , \\ W_{1,1} &= -8Y , \\ W_{3,1} &= 4(-9+22Y-6Y^2) , \\ W_{0,2} &= 4(-9+8Y-6Y^2) , \\ W_{2,2} &= 4(70-98Y+70Y^2-24Y^3+4Y^4) , \\ &\ldots \end{aligned} \tag{4.14}$$

The results for the vacuum energy density and chiral order parameter in the three-link approximations [Eq. (4.11)] are shown by crosses in Figs. 2–7, respectively. One sees that the vacuum energy is further lowered and a nice scaling behavior for the fermion condensate is observed in $U(1)_3$, $SU(2)_3$, and $SU(3)_3$. The plateaus extend to the crossover and weak-coupling regions. The nonvanishing values for the fermion condensates suggest that chiral symmetries (related to γ_3 and γ_5) be spontaneously broken. It is predicted that

$$\begin{aligned} \left.\frac{\langle\bar{\psi}\psi\rangle_{\text{continuum}}}{e^4}\right|_{U(1)_3} &= 2\times(-0.05) , \\ \left.\frac{\langle\bar{\psi}\psi\rangle_{\text{continuum}}}{e^4}\right|_{SU(2)_3} &= 2\times(-2\times 0.0275) , \\ \left.\frac{\langle\bar{\psi}\psi\rangle_{\text{continuum}}}{e^4}\right|_{SU(3)_3} &= 2\times(-3\times 0.072) , \end{aligned} \tag{4.15}$$

up to the three-link approximation to S_f. In the presented regions $0.8 < 1/g^2 \le 2.2$ for $U(1)_3$, $0.6 < 1/g^2 \le 1.6$ for $SU(2)_3$, and $0.9 < 1/g^2 \le 2.0$ for $SU(3)_3$, the first two terms in S_f are enough for obtaining the scaling behavior. As the continuum limit $1/g^2 \to \infty$ is approached, fermion-antifermion pairs of all lengths should be included because they are related to long-range correlations.

V. CONCLUDING REMARKS

In the preceding sections, the vacuum structure and chiral-symmetry breaking in (2+1)-dimensional lattice gauge theories with fermions have been investigated. Some major results may be recapitulated here.

(1) As a first attempt, the effective Hamiltonian in the strong-coupling regime has been obtained and diagonalized exactly. It is shown that this Hamiltonian in strongly coupled $SU(2)_3$ is equivalent to the antiferromagnetic Heisenberg model, which is relevant to superconductivity. It has also been illustrated that spontaneously broken chiral symmetries lead to two Goldstone bosons and one vector particle. Their masses have been calculated as well. The fermion condensate, which corresponds to spontaneous magnetization in the Heisenberg model, seems to decrease with the increase of fermion flavors.

(2) The ground state in the crossover regime is assumed to be the combination of the variational fermion vacuum state and the exact solution to pure gauge fields of Guo and co-workers. Several limitations in the modified theory of Guo and co-workers have been addressed. Because ΔH is relevant and breaks Lorentz invariance, the exact ground state would differ essentially from that of the conventional one in the continuum limit. For long-wavelength configurations [41,46] and superrenormalizable theories such as LGT_3, the modified Hamiltonian seems to work well in the presented regime.

(3) The chiral condensates in $U(1)_3$, $SU(2)_3$, and $SU(3)_3$ have been systematically evaluated. Up to the crossover regime, good scaling behavior has been observed after the inclusion of fermion pairs of longer distances. It is fair to say that some further work is required on the effects of multilink terms of $n \ge 2$, renormalization, and dynamical fermions.

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