Finite-element quantum electrodynamics: Canonical formulation, unitarity, and the magnetic moment of the electron

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This is the first in a series of papers dealing with four-dimensional quantum electrodynamics on a finite-element lattice. We begin by studying the canonical structure of the theory without interactions. This tells us how to construct momentum expansions for the field operators. Next we examine the interaction term in the Dirac equation. We construct the transfer matrix explicitly in the temporal gauge, and show that it is unitary. Therefore, fermion canonical anticommutation relations hold at each lattice site. Finally, we expand the interaction term to second order in the temporal-lattice spacing and deduce the magnetic moment of the electron in a background field, consistent with the continuum value of $g = 2$.

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I. INTRODUCTION

The finite-element lattice approach to quantum field theory has been under development for several years now [1–9]. Its principal advantage is that, of all possible discretizations of the operator equations of motion, the finite-element prescription alone is unitary, in the sense that if the canonical commutation relations hold at some initial time, they continue to hold exactly at each later time on the lattice. This is in addition to its numerical advantage, well known in classical contexts [10], of providing the most accurate approximation. Linear finite elements have a relative error, after N time steps, of N^{-2}, rather than of N^{-1} for an arbitrary finite-difference scheme. A third remarkable feature was observed early on: the species-doubling problem for fermions does not occur [4].

By the finite-element approach we mean a formulation based on a representation of the field equations as operator difference equations on a Minkowski space-time lattice. In addition to being unitary, this formulation is causal, in the sense that fields at a given time are determined in terms of fields at earlier times only. In principle, then, the difference equations may be solved for fields at an arbitrary time in terms of fields at some initial time, i.e., in terms of appropriate boundary conditions consistent with the canonical commutation relations. Physical quantities are then extracted from relations between matrix elements of suitable operators. Evidently, this general approach is quite different from the line of attack of conventional lattice gauge theory, where Euclidean functional integrals are approximately evaluated by Monte Carlo simulation, a procedure subject to statistical fluctuations.

More than six years ago we showed how to implement Abelian gauge invariance on a finite-element lattice [6]. This led to an interaction term in the lattice Dirac equation which explicitly exhibited the point-splitting implicit in the continuum formulation. We solved the equations for two-dimensional massless electrodynamics (the Schwinger model) and obtained an excellent approximation for the axial-vector anomaly:

$$\langle \text{“}\partial_\mu j_5^\mu\text{”} \rangle = -\frac{e^2}{M\sin(\pi/M)}E, \tag{1.1}$$

where j_5^μ is the axial-vector current, e is the charge of the electron, E is the electric field, and M is the number of spatial-lattice sites. The quotation marks are a reminder that the derivative is taken according to the finite-element prescription: a forward difference in the direction of the derivative, an average over adjacent lattice sites in the other directions. As expected, the relative error is of order M^{-2}. This same result was later found by using a method [2, 5] in which one expands in the temporal-lattice spacing, h, and computes matrix elements by a variational technique, thereby deriving a dispersion relation [7]. This method will be followed in this paper.

More recently, we showed how to extend the gauge-invariant equations of motion to the non-Abelian regime [8]. Now there are nonlocal interaction terms in Yang-Mills equations as well as in the Dirac equation. The field strength is expressed locally in terms of the potential, in terms of nested commutators. The transformations of the gauge potentials are similarly determined to be given by a series of nested commutators.

These formulations, for simplicity, were stated in $1+1$ dimensions. The generalization to four dimensions is trivial for the Abelian case. (There are nontrivial aspects of this generalization for the non-Abelian case, but the generalization is straightforward [11].) It is the purpose of this paper to begin the study of the finite-element formulation of gauge theories in four dimensions. Here we discuss the basic framework in terms of finite-element electrodynamics. Further investigations, specifically involving nonperturbative evaluations of form factors (which should reveal something of their analytic dependence on the fine-structure constant) will provide both a familiar context in which to study technical issues, and a confirmation of the validity of the method. In the future, the finite-element technique will be applied to non-Abelian theories such as QCD, Chern-Simons theories, and models of symmetry breaking.

The plan of this paper is as follows. In Sec. II we study the noninteracting equations of motion and examine the canonical structure in the radiation gauge. We will see that appropriate canonical variables are the fields averaged over adjacent spatial-lattice sites. We derive the momentum expansions for these field operators. In particular we find the lattice form of the electron spinors. In Sec. III we rewrite the interaction term in the Dirac equation as a transfer matrix, which expresses ψ_{n+1} in terms of ψ_n, where n is the lattice time. We prove that the transfer matrix is unitary, and therefore that the Dirac fields are canonical at each lattice time. Finally, in Sec. IV we expand the transfer matrix in terms of a complete set of Dirac matrices, and derive a simple set of equations for these expansion coefficients. By expanding in powers of h, the temporal-lattice unit, we are able to derive the lattice analogue to the continuum of the dispersion relation

$$\omega^2 = \mu^2 + \Pi^2 - e\boldsymbol{\sigma}\cdot\mathbf{H}, \tag{1.2}$$

and thereby derive the $g = 2$ value for the electron. Radiative corrections to the magnetic moment will be given in the second paper in this series.

II. CANONICAL FORMULATION IN THE RADIATION GAUGE

We begin by recalling the canonical formulation of continuum electrodynamics in the radiation gauge. Without interactions, we take the potentials to satisfy

$$A^0 = 0, \quad \boldsymbol{\nabla}\cdot\mathbf{A} = 0. \tag{2.1}$$

We regard the spatial components of the potentials to be the canonical coordinates, and the canonial momenta to be given in terms of the electric field

$$\pi_k = E_k. \tag{2.2}$$

Thus, the canonical equal-time commutation relations are

$$[A_i(\mathbf{x},t), A_j(\mathbf{y},t)] = 0, \quad [E_i(\mathbf{x},t), E_j(\mathbf{y},t)] = 0, \tag{2.3a}$$

and

$$-[E_i(\mathbf{x},t), A_j(\mathbf{y},t)] = i\delta^T_{ij}(\mathbf{x}-\mathbf{y}), \tag{2.3b}$$

where

$$\delta^T_{ij}(\mathbf{x}-\mathbf{y}) = \int \frac{d^3k}{(2\pi)^3} e^{i\mathbf{k}\cdot(\mathbf{x}-\mathbf{y})}\left(\delta_{ij} - \frac{k_i k_j}{\mathbf{k}^2}\right). \tag{2.3c}$$

It is easy to show that these equations are consistent with the equations of motion:

$$\mathbf{E} = \frac{\partial\mathbf{A}}{\partial \mathrm{t}}, \quad \mathbf{B} = \boldsymbol{\nabla}\times\mathbf{A}, \tag{2.4a}$$

$$\boldsymbol{\nabla}\cdot\mathbf{E} = 0, \quad \frac{\partial\mathbf{E}}{\partial t} + \boldsymbol{\nabla}\times\mathbf{B} = 0. \tag{2.4b}$$

Similarly, in the fermionic sector, we have canonical equal-time anticommutation relations:

$$\{\psi(\mathbf{x},t), \psi(\mathbf{y},t)\} = 0, \quad \{\psi(\mathbf{x},t), \psi^\dagger(\mathbf{y},t)\} = \mathbb{1}\delta(\mathbf{x}-\mathbf{y}), \tag{2.5}$$

which are consistent with the free Dirac equation

$$i\gamma^\mu\partial_\mu\psi + \mu\psi = 0. \tag{2.6}$$

A. The photon sector

Let us now write down the finite-element equations of motion in the absence of interactions in a gauge where $A^0 = 0$. We use the notation that $\mathbf{m} = (m_1, m_2, m_3)$ denotes the spatial-lattice coordinate, n denotes the temporal-lattice coordinate, Δ is the lattice unit in a space direction, and h is the lattice spacing in the time direction. Then the electric field $\mathbf{E}$ is constructed from the vector potential $\mathbf{A}$ according to

$$\mathbf{E}_{\overline{\mathbf{m}},\overline{n}} = \frac{1}{h}(\mathbf{A}_{\overline{\mathbf{m}},n+1} - \mathbf{A}_{\overline{\mathbf{m}},n}), \tag{2.7}$$

while the magnetic field $\mathbf{B}$ is given by

$$(B_i)_{\overline{\mathbf{m}},\overline{n}} = \frac{1}{\Delta}\epsilon_{ijk}[(A_k)_{m_j+1,\overline{\mathbf{m}}_\perp,\overline{n}} - (A_k)_{m_j,\overline{\mathbf{m}}_\perp,\overline{n}}]. \tag{2.8}$$

Here an overbar represents a forward average over that coordinate:

$$x_{\overline{m}} \equiv \frac{1}{2}(x_{m+1} + x_m). \tag{2.9}$$

In (2.8) the notation $\overline{\mathbf{m}}_\perp$ means that all spatial coordinates but m_j are averaged. Summation is to be understood over repeated indices. The field equations corresponding to (2.4b) are

$$\frac{1}{\Delta}[(E_i)_{m_i+1,\overline{\mathbf{m}}_\perp,\overline{n}} - (E_i)_{m_i,\overline{\mathbf{m}}_\perp,\overline{n}}] = 0 \tag{2.10}$$

and

$$\frac{1}{h}[(E_i)_{\overline{\mathbf{m}},n+1} - (E_i)_{\overline{\mathbf{m}},n}] + \frac{1}{\Delta}\epsilon_{ijk}[(B_k)_{m_j+1,\overline{\mathbf{m}}_\perp,\overline{n}} - (B_k)_{m_j,\overline{\mathbf{m}}_\perp,\overline{n}}] = 0. \tag{2.11}$$

Our goal in this section is to examine the canonical structure of the lattice theory and establish that the spatially averaged fields satisfy canonical commutation relations. To do this, it seems simplest to expand in the time lattice unit h. We do not, however, expand in powers of Δ. That is, we relate variables at time $n+1$ to those at n by expanding in a Taylor series in h:

$$\mathbf{A}_{\mathbf{m},n+1} = \mathbf{A}_{\mathbf{m},n} + h\boldsymbol{\mathcal{A}}_{\mathbf{m},n} + h^2\boldsymbol{\mathcal{B}}_{\mathbf{m},n} + \cdots, \tag{2.12a}$$

$$\mathbf{E}_{\mathbf{m},n+1} = \mathbf{E}_{\mathbf{m},n} + h\boldsymbol{\mathcal{C}}_{\mathbf{m},n} + h^2\boldsymbol{\mathcal{D}}_{\mathbf{m},n} + \cdots, \tag{2.12b}$$

$$\mathbf{B}_{\mathbf{m},n+1} = \mathbf{B}_{\mathbf{m},n} + h\boldsymbol{\mathcal{F}}_{\mathbf{m},n} + h^2\boldsymbol{\mathcal{G}}_{\mathbf{m},n} + \cdots . \quad (2.12c)$$

The operators $\boldsymbol{\mathcal{A}}$, $\boldsymbol{\mathcal{B}}$, $\boldsymbol{\mathcal{C}}$, $\boldsymbol{\mathcal{D}}$, $\boldsymbol{\mathcal{F}}$, and $\boldsymbol{\mathcal{G}}$ are to be determined by the equations of motion (2.7), (2.8), (2.10), and (2.11). Thus, substituting (2.12a) and (2.12b) into (2.7), and equating powers of h, we find

$$\boldsymbol{\mathcal{A}}_{\overline{\mathbf{m}},n} = \mathbf{E}_{\overline{\mathbf{m}},n}, \quad (2.13a)$$

$$\boldsymbol{\mathcal{B}}_{\overline{\mathbf{m}},n} = \frac{1}{2}\boldsymbol{\mathcal{C}}_{\overline{\mathbf{m}},n}. \quad (2.13b)$$

Similarly, (2.12a) and (2.12c) when substituted into (2.8) give

$$(B_i)_{\overline{\mathbf{m}},n} = \frac{1}{\Delta}\epsilon_{ijk}[(A_k)_{m_j+1,\overline{\mathbf{m}}_\perp,n} - (A_k)_{m_j,\overline{\mathbf{m}}_\perp,n}], \quad (2.14a)$$

$$(\mathcal{F}_i)_{\overline{\mathbf{m}},n} = \frac{1}{\Delta}\epsilon_{ijk}[(\mathcal{A}_k)_{m_j+1,\overline{\mathbf{m}}_\perp,n} - (\mathcal{A}_k)_{m_j,\overline{\mathbf{m}}_\perp,n}], \quad (2.14b)$$

$$(\mathcal{G}_i)_{\overline{\mathbf{m}},n} = \frac{1}{\Delta}\epsilon_{ijk}[(\mathcal{B}_k)_{m_j+1,\overline{\mathbf{m}}_\perp,n} - (\mathcal{B}_k)_{m_j,\overline{\mathbf{m}}_\perp,n}]. \quad (2.14c)$$

When (2.12b) is substituted into (2.10) we find

$$\frac{1}{\Delta}[(E_i)_{m_i+1,\overline{\mathbf{m}}_\perp,n} - (E_i)_{m_i,\overline{\mathbf{m}}_\perp,n}] = 0, \quad (2.15a)$$

$$\frac{1}{\Delta}[(\mathcal{C}_i)_{m_i+1,\overline{\mathbf{m}}_\perp,n} - (\mathcal{C}_i)_{m_i,\overline{\mathbf{m}}_\perp,n}] = 0, \quad (2.15b)$$

$$\frac{1}{\Delta}[(\mathcal{D}_i)_{m_i+1,\overline{\mathbf{m}}_\perp,n} - (\mathcal{D}_i)_{m_i,\overline{\mathbf{m}}_\perp,n}] = 0. \quad (2.15c)$$

Finally, from (2.11) we find

$$(\mathcal{C}_i)_{\overline{\mathbf{m}},n} = -\frac{1}{\Delta}\epsilon_{ijk}[(B_k)_{m_j+1,\overline{\mathbf{m}}_\perp,n} - (B_k)_{m_j,\overline{\mathbf{m}}_\perp,n}], \quad (2.16a)$$

$$(\mathcal{D}_i)_{\overline{\mathbf{m}},n} = -\frac{1}{2\Delta}\epsilon_{ijk}[(\mathcal{F}_k)_{m_j+1,\overline{\mathbf{m}}_\perp,n} - (\mathcal{F}_k)_{m_j,\overline{\mathbf{m}}_\perp,n}]. \quad (2.16b)$$

Evidently, given the canonical variables at the initial time, $\mathbf{E}_{\overline{\mathbf{m}},n}$ and $\mathbf{A}_{\overline{\mathbf{m}},n}$, we can find the other quantities in (2.12) as follows: (2.13a) gives $\boldsymbol{\mathcal{A}}$, (2.14a) gives $\mathbf{B}$, (2.16a) gives $\boldsymbol{\mathcal{C}}$, (2.13b) gives $\boldsymbol{\mathcal{B}}$, (2.14b) gives $\boldsymbol{\mathcal{F}}$, (2.16b) gives $\boldsymbol{\mathcal{D}}$, and (2.14c) gives $\boldsymbol{\mathcal{G}}$. The remaining equations (2.15) are constraints.

We solve these equations by adopting a Fourier decomposition for the fields. We take, as the canonical variables,

$$\mathbf{A}_{\overline{\mathbf{m}},n} = \sum_{k_i=1}^{M} \gamma_{\mathbf{k}} \left(\mathbf{a}_{\mathbf{k}} e^{i\mathbf{k}\cdot\mathbf{m}2\pi/M} + \mathbf{a}_{\mathbf{k}}^\dagger e^{-i\mathbf{k}\cdot\mathbf{m}2\pi/M}\right), \quad (2.17a)$$

$$\mathbf{E}_{\overline{\mathbf{m}},n} = \sum_{k_i=1}^{M} \frac{i}{2\gamma_{\mathbf{k}} L^3} \left(-\mathbf{a}_{\mathbf{k}} e^{i\mathbf{k}\cdot\mathbf{m}2\pi/M} + \mathbf{a}_{\mathbf{k}}^\dagger e^{-i\mathbf{k}\cdot\mathbf{m}2\pi/M}\right), \quad (2.17b)$$

where we have introduced a variational parameter $\gamma_{\mathbf{k}}$. Here $L = M\Delta$ is the length of the (cubic) spatial lattice. We must now impose the radiation-gauge condition, the lattice analogue of (2.1). We solve for the unaveraged fields using the identity

$$x_m = \sum_{l=m}^{M+m-1} (-1)^{l+m} x_{\bar{l}}, \quad (2.18)$$

valid for periodic fields, $x_{m+M} = x_m$, with M odd, a condition that we will henceforward assume. Thus the condition

$$(A_i)_{m_i+1,\overline{\mathbf{m}}_\perp,n} - (A_i)_{m_i,\overline{\mathbf{m}}_\perp,n} = 0 \quad (2.19)$$

implies the constraint

$$\mathbf{a}_{\mathbf{k}} \cdot \mathbf{t}_{\mathbf{k}} = 0, \quad (2.20)$$

where $(\mathbf{t}_{\mathbf{k}})_i = \tan k_i\pi/M$. The commutation relations then satisfied by the creation and annihilation operators $\mathbf{a}_{\mathbf{k}}^\dagger$ and $\mathbf{a}_{\mathbf{k}}$ are

$$[a_{\mathbf{k}}^i, a_{\mathbf{k}'}^j] = 0, \quad [a_{\mathbf{k}}^i, a_{\mathbf{k}'}^{j\dagger}] = \delta_{\mathbf{k},\mathbf{k}'} f^{ij}(\mathbf{k}), \quad (2.21)$$

where

$$f^{ij}(\mathbf{k}) = \delta_{ij} - \frac{(\mathbf{t}_{\mathbf{k}})_i(\mathbf{t}_{\mathbf{k}})_j}{(\mathbf{t}_{\mathbf{k}})^2}. \quad (2.22)$$

Then it follows that the canonical commutation relation between $\mathbf{A}$ and $\mathbf{E}$ is

$$-[E^i_{\overline{\mathbf{m}},n}, A^j_{\overline{\mathbf{m}}',n}] = \frac{i}{\Delta^3}\delta^{Tij}_{\mathbf{m},\mathbf{m}'}, \quad (2.23)$$

where

$$\delta^{Tij}_{\mathbf{m},\mathbf{m}'} = \frac{1}{M^3}\sum_{\mathbf{k}} e^{i\mathbf{k}\cdot(\mathbf{m}-\mathbf{m}')2\pi/M} f^{ij}(\mathbf{k}). \quad (2.24)$$

The potentials, at equal times, commute with each other,

$$[A^i_{\overline{\mathbf{m}},n}, A^j_{\overline{\mathbf{m}}',n}] = 0, \quad (2.25)$$

provided that

$$\gamma_{\mathbf{k}} = \gamma_{\mathbf{M}-\mathbf{k}}, \quad \mathbf{M} = (M, M, M). \quad (2.26)$$

Note that the constraint (2.20) follows also from (2.15a). This is a small consistency check.

Now from (2.14a) we find that the magnetic field has the form

$$\mathbf{B}_{\overline{\mathbf{m}},n} = \frac{2i}{\Delta}\sum_{\mathbf{k}} \gamma_{\mathbf{k}}(\mathbf{a}_{\mathbf{k}} \times \mathbf{t}_{\mathbf{k}} e^{i\mathbf{k}\cdot\mathbf{m}2\pi/M} - \mathbf{a}_{\mathbf{k}}^\dagger \times \mathbf{t}_{\mathbf{k}} e^{-i\mathbf{k}\cdot\mathbf{m}2\pi/M}). \quad (2.27)$$

From (2.16a) we find

$$\boldsymbol{\mathcal{C}}_{\overline{\mathbf{m}},n} = -\frac{4}{\Delta^2}\sum_{\mathbf{k}} \gamma_{\mathbf{k}}(\mathbf{a}_{\mathbf{k}} \mathbf{t}_{\mathbf{k}}^2 e^{i\mathbf{k}\cdot\mathbf{m}2\pi/M} + \mathbf{a}_{\mathbf{k}}^\dagger \mathbf{t}_{\mathbf{k}}^2 e^{-i\mathbf{k}\cdot\mathbf{m}2\pi/M}). \quad (2.28)$$

Our first test of unitarity is satisfied, in that

$$[A^i_{\overline{\mathbf{m}},n}, \mathcal{C}^j_{\overline{\mathbf{m}}',n}] = 0, \quad (2.29)$$

this being the entire $O(h)$ contribution to $[A^i_{\overline{\mathbf{m}},n+1}, E^j_{\overline{\mathbf{m}}',n+1}]$ because $\boldsymbol{\mathcal{A}}$ is identical to $\mathbf{E}$.

Similarly, from (2.14b) we find

$$\mathcal{F}_{\overline{\mathbf{m}},n} = \frac{1}{\Delta L^3}\sum_{\mathbf{k}}\frac{1}{\gamma_{\mathbf{k}}}(\,\mathbf{a_k}\times\mathbf{t_k}e^{i\mathbf{k}\cdot\mathbf{m}2\pi/M} +\mathbf{a}^\dagger_{\mathbf{k}}\times\mathbf{t_k}e^{-i\mathbf{k}\cdot\mathbf{m}2\pi/M}). \quad (2.30)$$

Equation (2.14c) gives

$$\mathcal{G}_{\overline{\mathbf{m}},n} = -\frac{4i}{\Delta^3}\sum_{\mathbf{k}}\gamma_{\mathbf{k}}(\,\mathbf{a_k}\times\mathbf{t_k}\,\mathrm{t}^2_{\mathbf{k}}e^{i\mathbf{k}\cdot\mathbf{m}2\pi/M} -\mathbf{a}^\dagger_{\mathbf{k}}\times\mathbf{t_k}\,\mathrm{t}^2_{\mathbf{k}}e^{-i\mathbf{k}\cdot\mathbf{m}2\pi/M}), \quad (2.31)$$

while (2.16b) gives

$$\mathcal{D}_{\overline{\mathbf{m}},n} = \frac{i}{\Delta^2 L^3}\sum_{\mathbf{k}}\frac{1}{\gamma_{\mathbf{k}}}(\,\mathbf{a_k}\,\mathrm{t}^2_{\mathbf{k}}e^{i\mathbf{k}\cdot\mathbf{m}2\pi/M} -\mathbf{a}^\dagger_{\mathbf{k}}\,\mathrm{t}^2_{\mathbf{k}}e^{-i\mathbf{k}\cdot\mathbf{m}2\pi/M}). \quad (2.32)$$

Note that constraints (2.15b) and (2.15c) are satisfied by $\mathcal{C}$ and $\mathcal{D}$ given in (2.28) and (2.32), by virtue of (2.20).

Before we check all the commutators to $O(h^2)$, we determine the variational parameter $\gamma_{\mathbf{k}}$. We consider a matrix element between the vacuum and a one photon state of momentum $\boldsymbol{l}$. Because such a state is an energy eigenstate, we have, approximately, on the one hand

$$\begin{aligned}\langle \boldsymbol{l}|\mathbf{E}_{\overline{\mathbf{m}},n+1}|0\rangle &\approx \exp(i\omega_l h)\langle \boldsymbol{l}|\mathbf{E}_{\overline{\mathbf{m}},n}|0\rangle \\ &\approx \left(1+i\omega_l h-\frac{\omega_l^2h^2}{2}\right)\langle \boldsymbol{l}|\mathbf{E}_{\overline{\mathbf{m}},n}|0\rangle. \end{aligned} \quad (2.33)$$

On the other hand, we have, from (2.12b),

$$\begin{aligned}\langle \boldsymbol{l}|\mathbf{E}_{\overline{\mathbf{m}},n+1}|0\rangle &= \langle \boldsymbol{l}|\mathbf{E}_{\overline{\mathbf{m}},n}|0\rangle + h\langle \boldsymbol{l}|\mathcal{C}_{\overline{\mathbf{m}},n}|0\rangle \\ &\quad +h^2\langle \boldsymbol{l}|\mathcal{D}_{\overline{\mathbf{m}},n}|0\rangle+\cdots. \end{aligned} \quad (2.34)$$

By comparing (2.17b) and (2.28) we have the relation coming from the $O(h)$ term

$$-\frac{\omega_{\mathbf{k}}}{2\gamma_{\mathbf{k}}L^3} = -\frac{4}{\Delta^2}\gamma_{\mathbf{k}}\mathrm{t}^2_{\mathbf{k}}. \quad (2.35)$$

Using (2.32) we deduce, from the order h^2 terms,

$$\omega^2_{\mathbf{k}} = \frac{4}{\Delta^2}\mathrm{t}^2_{\mathbf{k}} = \frac{4}{\Delta^2}\sum_{i=1}^{3}\tan^2\left(\frac{k_i\pi}{M}\right), \quad (2.36)$$

which is the familiar massless finite-element dispersion relation. [We see that this has the correct continuum limit because the lattice momentum is $(\mathbf{p_k})_i = 2\pi k_i/L$.] Combining (2.36) and (2.35) we find

$$\gamma^2_{\mathbf{k}} = \frac{1}{2L^3\omega_{\mathbf{k}}}, \quad (2.37)$$

which is the obvious four-dimensional generalization of the two-dimensional result given in Ref. [5].

Now it is easy to check that the canonical commutation relations hold at time $n+1$. We first examine

$$\begin{aligned}[A^i_{\overline{\mathbf{m}},n+1},A^j_{\overline{\mathbf{m}}',n+1}] &= [A^i_{\overline{\mathbf{m}},n},A^j_{\overline{\mathbf{m}}',n}] + h\left([A^i_{\overline{\mathbf{m}},n},\mathcal{A}^j_{\overline{\mathbf{m}}',n}]+[\mathcal{A}^i_{\overline{\mathbf{m}},n},A^j_{\overline{\mathbf{m}}',n}]\right) \\ &\quad +h^2\left([A^i_{\overline{\mathbf{m}},n},\mathcal{B}^j_{\overline{\mathbf{m}}',n}]+[\mathcal{B}^i_{\overline{\mathbf{m}},n},A^j_{\overline{\mathbf{m}}',n}]+[\mathcal{A}^i_{\overline{\mathbf{m}},n},\mathcal{A}^j_{\overline{\mathbf{m}}',n}]\right)+\cdots. \end{aligned} \quad (2.38)$$

The order h terms vanish because the two (nonzero) commutators cancel due to the symmetry of δ^T in its indices. The order h^2 terms vanish individually because each commutator is zero in view of (2.26). In just the same way the $[E^i_{\overline{\mathbf{m}},n+1},E^j_{\overline{\mathbf{m}}',n+1}]$ commutator can be shown to be zero. We have already seen in (2.29) that the order h term in

$$\begin{aligned}[A^i_{\overline{\mathbf{m}},n+1},E^j_{\overline{\mathbf{m}}',n+1}] &= [A^i_{\overline{\mathbf{m}},n},E^j_{\overline{\mathbf{m}}',n}] + h\left([\mathcal{A}^i_{\overline{\mathbf{m}},n},E^j_{\overline{\mathbf{m}}',n}]+[A^i_{\overline{\mathbf{m}},n},\mathcal{C}^j_{\overline{\mathbf{m}}',n}]\right) \\ &\quad +h^2\left([A^i_{\overline{\mathbf{m}},n},\mathcal{D}^j_{\overline{\mathbf{m}}',n}]+[\mathcal{B}^i_{\overline{\mathbf{m}},n},E^j_{\overline{\mathbf{m}}',n}]+[\mathcal{A}^i_{\overline{\mathbf{m}},n},\mathcal{C}^j_{\overline{\mathbf{m}}',n}]\right)+\cdots \end{aligned} \quad (2.39)$$

vanishes. And it is easy to see, from (2.13a), (2.13b), (2.28), and (2.32), that the three nonzero commutators in the order h^2 term combine to give zero. We expect that this proof of unitarity of the lattice Maxwell equations can be carried through to all orders in h.

Of course, this consistency can be verified without using the momentum expansion (2.17a), (2.17b), but the demonstration is somewhat more elaborate.

B. The electron sector

The finite-element lattice Dirac equation is

$$\frac{i\gamma^0}{h}(\psi_{\overline{\mathbf{m}},n+1}-\psi_{\overline{\mathbf{m}},n}) + \frac{i\gamma^j}{\Delta}(\psi_{m_j+1,\overline{\mathbf{m}}_\perp,\overline{n}}-\psi_{m_j,\overline{\mathbf{m}}_\perp,\overline{n}}) + \mu\psi_{\overline{\mathbf{m}},\overline{n}} = 0. \quad (2.40)$$

Let us begin by finding the momentum-space spinors, the eigenvectors of the transfer matrix. That is, write, for a plane wave at time n,

$$\psi_{\overline{\mathbf{m}},n} = u_n e^{-i\mathbf{p}\cdot\mathbf{m}2\pi/M}, \tag{2.41a}$$

and, at time $n+1$,

$$\psi_{\overline{\mathbf{m}},n+1} = u_{n+1} e^{-i\mathbf{p}\cdot\mathbf{m}2\pi/M}. \tag{2.41b}$$

The transfer matrix is defined by

$$u_{n+1} = T u_n. \tag{2.42}$$

By substituting (2.41a) and (2.41b) into the Dirac equation (2.40) we easily find that

$$\begin{aligned} T &= \left(\frac{i\gamma^0}{h} + \frac{\gamma\cdot\mathbf{t}}{\Delta} + \frac{\mu}{2}\right)^{-1}\left(\frac{i\gamma^0}{h} - \frac{\gamma\cdot\mathbf{t}}{\Delta} - \frac{\mu}{2}\right) \\ &= \left(1 + \frac{\mu^2h^2}{4} + \frac{h^2}{\Delta^2}t^2\right)^{-1}\left(1 - \frac{\mu^2h^2}{4} - \frac{h^2}{\Delta^2}t^2 + \frac{2h}{\Delta}i\gamma^0\gamma\cdot\mathbf{t} + \mu h i\gamma^0\right), \end{aligned} \tag{2.43}$$

where

$$\mathbf{t} = \mathbf{t_p}, \quad (\mathbf{t_p})_i = \tan p_i\pi/M. \tag{2.44}$$

Let us adopt a representation of the Dirac matrices in which

$$\gamma^0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad i\gamma^0\gamma^j = \sigma^{0j} = i\begin{pmatrix} 0 & \sigma^j \\ \sigma^j & 0 \end{pmatrix}. \tag{2.45}$$

Then the eigenvalues of T are easily found:

$$\lambda = \left(1 + \frac{\mu^2h^2}{4} + \frac{h^2}{\Delta^2}t^2\right)^{-1}\left(1 - \frac{\mu^2h^2}{4} - \frac{h^2}{\Delta^2}t^2 \pm 2i\sqrt{\frac{\mu^2h^2}{4} + \frac{h^2}{\Delta^2}t^2}\right). \tag{2.46}$$

It is easily checked that λ has modulus unity, so it can be written in the form $\lambda = \exp(-i\omega h)$, where ω is, of course, a function of h. The corresponding eigenvectors may also be found straightforwardly. They are to be normalized according to

$$u_\pm^\dagger\gamma^0 u_\pm = \pm 1. \tag{2.47}$$

They are

$$u_\pm = \begin{pmatrix} \pm[(\tilde\omega \pm \mu)/2\mu]^{1/2}\boldsymbol{\sigma}\cdot\mathbf{t}/t \\ [(\tilde\omega \mp \mu)/2\mu]^{1/2} \end{pmatrix}\chi, \tag{2.48}$$

where χ is a two-component, rest-frame spinor, normalized by $\chi^\dagger\chi = 1$. Here $\tilde\omega$ is an abbreviation for

$$\tilde\omega = \sqrt{\frac{4t^2}{\Delta^2} + \mu^2}, \tag{2.49}$$

which is the massive analogue of the dispersion relation (2.36). Thus, with

$$i\gamma_5 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \tag{2.50}$$

we have

$$u_\pm = \left[\left(\frac{\tilde\omega + \mu}{2\mu}\right)^{1/2} \pm i\gamma_5\frac{\boldsymbol{\sigma}\cdot\mathbf{t}}{t}\left(\frac{\tilde\omega - \mu}{2\mu}\right)^{1/2}\right]u_\pm^{(0)}, \tag{2.51}$$

where $u_\pm^{(0)}$ is a four-component rest-frame spinor with γ^0 eigenvalue of ± 1. Therefore, in terms of the spinors $\tilde u_+(\mathbf{p}) = u_+(\mathbf{p})$, $\tilde u_-(\mathbf{p}) = u_-(-\mathbf{p})$, we have the completeness relations

$$\sum_{\text{spins}} \tilde u_\pm \tilde u_\pm^\dagger \gamma^0 = \pm\frac{1}{2\mu}\left(\mu \pm \gamma^0\tilde\omega \mp \frac{2\gamma\cdot\mathbf{t}}{\Delta}\right), \tag{2.52}$$

which in the continuum limit reduces to $\pm(\mu \mp \gamma\cdot p)/2\mu$. We have the same result on the lattice, provided we define

$$\tilde p^0 = \tilde\omega, \quad \tilde{\mathbf{p}} = \frac{2\mathbf{t}}{\Delta}. \tag{2.53}$$

All of this tells us that the momentum expansion of the Dirac field has the form

$$\begin{aligned} \psi_{\overline{\mathbf{m}},n} = \sum_{s,\mathbf{p}} \sqrt{\frac{\mu}{\tilde\omega}}(\, & b_{\mathbf{p},s}u_{\mathbf{p},s}e^{i\mathbf{p}\cdot\mathbf{m}2\pi/M} \\ & + d_{\mathbf{p},s}^\dagger v_{\mathbf{p},s}e^{-i\mathbf{p}\cdot\mathbf{m}2\pi/M}), \end{aligned} \tag{2.54}$$

where we now use the standard notation $u = i\gamma_5\tilde u_-$, $v = i\gamma_5\tilde u_+$, with the usual interpretation that $d^\dagger$ creates a positive-energy positron, while b annihilates a positive-energy electron. The canonical lattice anticommutation relations

$$\{\psi_{\overline{\mathbf{m}},n}, \psi_{\overline{\mathbf{m}}',n}^\dagger\} = \frac{1}{\Delta^3}\delta_{\mathbf{m},\mathbf{m}'} \tag{2.55}$$

will now be satisfied if

$$\{b_{\mathbf{p},s}, b^\dagger_{\mathbf{p}',s'}\} = \frac{1}{L^3}\delta_{\mathbf{p},\mathbf{p}'}\delta_{s,s'},$$
$$\{d_{\mathbf{p},s}, d^\dagger_{\mathbf{p}',s'}\} = \frac{1}{L^3}\delta_{\mathbf{p},\mathbf{p}'}\delta_{s,s'}, \tag{2.56}$$

and all other anticommutors of these operators vanish.

The unitarity of T is sufficient to establish the unitarity of the fermion sector in the noninteracting theory. We now turn to the effect of interactions, at least with a background field.

III. UNITARITY OF THE DIRAC EQUATION IN THE TEMPORAL GAUGE

The incorporation of interactions into Abelian electrodynamics in such a way as to preserve lattice gauge invariance was carried out in Ref. [6]. We will merely quote the lattice Dirac equation with interactions here, expressed in terms of the spatially averaged electron field:

$$\frac{i}{h}(\psi_{\overline{\mathbf{m}},n+1} - \psi_{\overline{\mathbf{m}},n}) - \frac{2i\gamma^0\gamma^j}{\Delta}\left[\sum_{m_j'=m_j+1}^{M}(-1)^{m_j+m_j'}\psi_{\overline{m}_j',\overline{\mathbf{m}}_\perp,\overline{n}} - \sum_{m_j'=1}^{m_j-1}(-1)^{m_j+m_j'}\psi_{\overline{m}_j',\overline{\mathbf{m}}_\perp,\overline{n}}\right]$$
$$+\frac{\mu\gamma^0}{2}(\psi_{\overline{\mathbf{m}},n+1} + \psi_{\overline{\mathbf{m}},n}) + 2\frac{\gamma^0\gamma^j}{\Delta}\sum_{m_j'}\alpha^{(j)}_{\mathbf{m}_\perp;m_j,m_j'}\psi_{\overline{m}_j',\overline{\mathbf{m}}_\perp,\overline{n}} = 0, \tag{3.1}$$

where a sum over the repeated index j is understood. (We recall [6] that with M odd, ψ is periodic on the spatial lattice.) Here, we have adopted a temporal gauge, $A^0 = 0$, and expressed the interaction in terms of (only the jth index is explicit and the spatial coordinates refer to the jth direction)

$$\alpha^{(j)}_{m,m'} = \frac{i}{2}(-1)^{m+m'}\sec\zeta\sum_{m''=1}^{M}\text{sgn}(m''-m)\text{sgn}(m''-m')\left(e^{2i\zeta_{m''}}-1\right)$$
$$\times\exp\left[i\sum_{m'''=1}^{M}\text{sgn}(m'''-m)\text{sgn}(m'''-m'')\text{sgn}(m''-m)\zeta_{m'''}\right]. \tag{3.2}$$

We have used the abbreviations

$$\zeta_{m_j} = \frac{e\Delta}{2}A^j_{\overline{m_j-1}},\quad \zeta = \sum_{m_j=1}^{M}\zeta_{m_j}, \tag{3.3}$$

and

$$\text{sgn}(x) = \begin{cases} +1, & x > 0, \\ -1, & x \le 0. \end{cases} \tag{3.4}$$

We can now carry out the sum over m'' in (3.2):

$$\alpha^{(j)}_{m,m'} = i\epsilon_{m',m}(-1)^{m+m'}\left[-1+\cos\left(\sum_{m''=1}^{M}\text{sgn}(m''-m)\text{sgn}(m''-m')\zeta_{m''}\right)\sec\zeta\right]$$
$$-(-1)^{m+m'}\sin\left(\sum_{m''=1}^{M}\text{sgn}(m''-m)\text{sgn}(m''-m')\zeta_{m''}\right)\sec\zeta, \tag{3.5}$$

where

$$\epsilon_{m',m} = \begin{cases} 1, & m' > m, \\ 0, & m' = m, \\ -1, & m' < m. \end{cases} \tag{3.6}$$

It is obvious that $\alpha^{(j)}$ is Hermitian.

Let us write the Dirac equation (3.1) in the form

$$U\psi_{n+1} = V\psi_n. \tag{3.7}$$

It is apparent that $V = 2 - U$, so the transfer matrix is

$$T = 2U^{-1} - 1. \tag{3.8}$$

The condition that T is unitary translates into the following condition on U:

$$U + U^\dagger = 2. \tag{3.9}$$

From (3.1) the matrix U is explicitly

$$U_{\mathbf{m},\mathbf{m}'} = \delta_{\mathbf{m},\mathbf{m}'} + \frac{h}{\Delta}\gamma^0\gamma^j(-1)^{m_j+m_j'}\epsilon_{m_j,m_j'}\delta_{\mathbf{m}_\perp,\mathbf{m}_\perp'}$$
$$-\frac{ih\mu\gamma^0}{2}\delta_{\mathbf{m},\mathbf{m}'} - \frac{ih\gamma^0\gamma^j}{\Delta}\alpha^{(j)}_{\mathbf{m}_\perp;m_j,m_j'}\delta_{\mathbf{m}_\perp,\mathbf{m}_\perp'}. \tag{3.10}$$

Therefore, the unitarity condition (3.9) is equivalent to the condition that $\alpha^{(j)}$ be Hermitian:

$$\alpha^{(j)\dagger} = \alpha^{(j)}, \tag{3.11}$$

which is satisfied as noted above.

IV. EXTRACTION OF THE MAGNETIC MOMENT OF THE ELECTRON

We now want to write the transfer matrix (3.8) in terms of the complete set of gamma matrices:

$$T = 2U^{-1} - 1 = \alpha\gamma^0 + \beta_\mu\gamma^0\gamma^\mu + \delta\gamma^0\gamma_5 + \epsilon_\mu\gamma^0\gamma_5\gamma^\mu + \tfrac{1}{2}\rho_{\mu\nu}\gamma^0\sigma^{\mu\nu}. \tag{4.1}$$

Let us write (3.10) in matrix form as

$$U = 1 - \frac{ih\mu\gamma^0}{2}1 + \frac{h}{\Delta}\gamma^0\boldsymbol{\gamma}\cdot\boldsymbol{\mathcal{D}}, \tag{4.2}$$

where $\boldsymbol{\mathcal{D}} = \boldsymbol{\xi} + i\boldsymbol{\eta}$ is given by

$$\xi^j_{\mathbf{m},\mathbf{m}'} = (-1)^{m_j+m'_j}\epsilon_{m_j,m'_j}\delta_{\mathbf{m}_\perp,\mathbf{m}'_\perp}, \tag{4.3a}$$

$$\eta^j_{\mathbf{m},\mathbf{m}'} = -\alpha^{(j)}_{\mathbf{m}_\perp,m_j,m'_j}\delta_{\mathbf{m}_\perp,\mathbf{m}'_\perp}. \tag{4.3b}$$

We note that $2\mathcal{D}/\Delta$ is the lattice version of the covariant derivative. It is then straightforward to work out equations satisfied by the parameters in (4.1). We define

$$\rho_i = \rho_{0i}, \quad \text{and} \quad \tau_i = 2\epsilon_{ijk}\rho_{jk}. \tag{4.4}$$

In dyadic form, $\boldsymbol{\beta}$, $\boldsymbol{\rho}$, $\boldsymbol{\epsilon}$, and $\boldsymbol{\tau}$ satisfy a remarkable series of equations:

$$\left[\left(\frac{h}{\Delta}\right)^2\left(1+\frac{h^2\mu^2}{4}\right)^{-1}\boldsymbol{\mathcal{D}}\boldsymbol{\mathcal{D}} - \mathbb{1}\right]\cdot\left(\boldsymbol{\beta} + \frac{h\mu}{2}\boldsymbol{\rho}\right) = \frac{h}{\Delta}\boldsymbol{\mathcal{D}}\times\boldsymbol{\epsilon} + \frac{h}{\Delta}2\left(1+\frac{h^2\mu^2}{4}\right)^{-1}\boldsymbol{\mathcal{D}}, \tag{4.5a}$$

$$\left[\left(\frac{h}{\Delta}\right)^2\left(1+\frac{h^2\mu^2}{4}\right)^{-1}\boldsymbol{\mathcal{D}}\boldsymbol{\mathcal{D}} - \mathbb{1}\right]\cdot\left(\boldsymbol{\epsilon} - \frac{h\mu}{2}\boldsymbol{\tau}\right) = \frac{h}{\Delta}\boldsymbol{\mathcal{D}}\times\boldsymbol{\beta}, \tag{4.5b}$$

$$\left[\left(\frac{h}{\Delta}\right)^2\left(1+\frac{h^2\mu^2}{4}\right)^{-1}\boldsymbol{\mathcal{D}}\boldsymbol{\mathcal{D}} - \mathbb{1}\right]\cdot\left(\boldsymbol{\rho} - \frac{h\mu}{2}\boldsymbol{\beta}\right) = -\frac{h}{\Delta}\boldsymbol{\mathcal{D}}\times\boldsymbol{\tau} - \frac{h}{\Delta}h\mu\left(1+\frac{h^2\mu^2}{4}\right)^{-1}\boldsymbol{\mathcal{D}}, \tag{4.5c}$$

and

$$\left[\left(\frac{h}{\Delta}\right)^2\left(1+\frac{h^2\mu^2}{4}\right)^{-1}\boldsymbol{\mathcal{D}}\boldsymbol{\mathcal{D}} - \mathbb{1}\right]\cdot\left(\boldsymbol{\tau} + \frac{h\mu}{2}\boldsymbol{\epsilon}\right) = \frac{h}{\Delta}\boldsymbol{\mathcal{D}}\times\boldsymbol{\rho}. \tag{4.5d}$$

The remaining parameters are given in terms of these by

$$\alpha = \frac{ih\mu}{1+h^2\mu^2/4}\left[1 - \frac{h}{2\Delta}\boldsymbol{\mathcal{D}}\cdot\boldsymbol{\beta} + \frac{1}{\Delta\mu}\boldsymbol{\mathcal{D}}\cdot\boldsymbol{\rho}\right], \tag{4.6a}$$

$$\beta_0 = \frac{1}{1+h^2\mu^2/4}\left[1 - \frac{h^2\mu^2}{4} - \frac{h}{\Delta}\boldsymbol{\mathcal{D}}\cdot\boldsymbol{\beta} - \frac{h^2\mu^2}{2\Delta\mu}\boldsymbol{\mathcal{D}}\cdot\boldsymbol{\rho}\right], \tag{4.6b}$$

$$\delta = \frac{ih\mu/2}{1+h^2\mu^2/4}\frac{h}{\Delta}\left[\boldsymbol{\mathcal{D}}\cdot\boldsymbol{\epsilon} + \frac{2}{h\mu}\boldsymbol{\mathcal{D}}\cdot\boldsymbol{\tau}\right], \tag{4.6c}$$

$$\epsilon_0 = -\frac{1}{1+h^2\mu^2/4}\frac{h}{\Delta}\left[\boldsymbol{\mathcal{D}}\cdot\boldsymbol{\epsilon} - \frac{h\mu}{2}\boldsymbol{\mathcal{D}}\cdot\boldsymbol{\tau}\right]. \tag{4.6d}$$

It is particularly easy to solve these equations if we regard h as small. To order h^2 we have

$$\boldsymbol{\beta} = -\frac{2h}{\Delta}\boldsymbol{\mathcal{D}}, \tag{4.7a}$$

$$\boldsymbol{\epsilon} = 2\left(\frac{h}{\Delta}\right)^2\boldsymbol{\mathcal{D}}\times\boldsymbol{\mathcal{D}}, \tag{4.7b}$$

$$\boldsymbol{\rho} = 0, \tag{4.7c}$$

$$\boldsymbol{\tau} = 0, \tag{4.7d}$$

and

$$\alpha = ih\mu, \tag{4.8a}$$

$$\beta_0 = 1 - \frac{h^2\mu^2}{2} + 2\left(\frac{h}{\Delta}\right)^2\mathcal{D}^2, \tag{4.8b}$$

$$\delta = 0, \tag{4.8c}$$

$$\epsilon_0 = 0. \tag{4.8d}$$

This leads to a very simple form for the transfer matrix (4.1):

$$T = 1 + h\left[i\mu\gamma^0 - \frac{2}{\Delta}\gamma^0\boldsymbol{\gamma}\cdot\boldsymbol{\mathcal{D}}\right] + h^2\left[-\frac{\mu^2}{2} + \frac{2}{\Delta^2}\mathcal{D}^2 + \frac{2i}{\Delta^2}\boldsymbol{\sigma}\cdot(\boldsymbol{\mathcal{D}}\times\boldsymbol{\mathcal{D}})\right] + O(h^3). \tag{4.9}$$

Here we have used the identity

$$\gamma_5\,\boldsymbol{\gamma} = i\gamma^0\boldsymbol{\sigma}. \tag{4.10}$$

We compare (4.9) with the leading terms in the expansion of $\exp(-i\omega h)$. Equating powers of h, we have

$$-\omega = \gamma^0(\mu - \boldsymbol{\gamma} \cdot \mathbf{\Pi}), \tag{4.11}$$

and

$$\omega^2 = \mu^2 + \Pi^2 + i\boldsymbol{\sigma} \cdot (\mathbf{\Pi} \times \mathbf{\Pi}). \tag{4.12}$$

Here we have written the lattice covariant momentum

$$\mathbf{\Pi} = -\frac{2i}{\Delta}\boldsymbol{\mathcal{D}}. \tag{4.13}$$

If we identify, for a classical background field,

$$\mathbf{\Pi} \times \mathbf{\Pi} = ie\mathbf{H} \tag{4.14}$$

as the effective lattice magnetic field strength, we see that we recover the form of the continuum dispersion relation,

$$\omega^2 = \mu^2 + \Pi^2 - e\boldsymbol{\sigma} \cdot \mathbf{H}, \tag{4.15}$$

which tells us that the g factor of the electron is 2. The formalism developed in this section, particularly the result (4.9), will provide the starting point for the calculation of the radiative corrections to the magnetic moment of the electron.

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