Restrictions imposed on relativistic two-body interactions by classical relativistic field theory

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We show that various relativistic potential models (all sharing exact relativistic two-body kinematics and a common nonrelativistic limit) can be distinguished by agreement or disagreement with relativistic corrections produced by classical field theory. We find that the only one of these models whose relativisic corrections duplicate those of classical field theory is the minimal Todorov equation. Conversely, we derive the Todorov equation from the semirelativistic dynamics of classical field theory, thus exposing the classical field-theoretic origins of its characteristic minimal potential structures and dependences on effective one-body variables.

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I. INTRODUCTION

The nonrelativistic Schrödinger equation for two particles permits introduction of interaction through potentials whose forms are restricted only by a proper definition of the quantum-mechanical wave function. Thus, physicists have profitably studied its properties for interactions (such as the harmonic oscillator) selected for pedagogical or phenomenological purposes. However, its first and most important application was to a system, the atom, whose potential (Coulomb) arose as a solution to the relevant classical field equations (Maxwell's equations for the electromagnetic field). In the same spirit, investigators have used field-theoretic potentials obtained from chromodynamics [1,2] in the Schrödinger equation to calculate spectra of bound quark systems composed of heavy quarks. Thus, throughout the history of applications of the Schrödinger equation, physicists have regarded its fruits as fundamental or as merely useful depending on the field-theoretic pedigree of its interactions.

Just like the Schrödinger equation, relativistic quantum wave equations for two interacting particles permit the introduction of interaction through potentials whose forms are restricted only by a proper definition of the wave function. Thus, relativistic treatments have included those of phenomenologically or pedagogically useful interactions such as the many versions of the relativistic harmonic oscillator [3–5]. Recently, however, physicists have begun to use Schrödinger-like two-body relativistic quantum wave equations to treat systems of relativistic particles such as quarks or electrons [6–17] whose interactions arise from a relativistic quantum field theory. Thus, the interactions appearing in such two-body relativistic wave equations are distinguished by origin in relativistic field theory or lack thereof.

When presented with a relativistic wave equation plus potential just how are we to determine whether or not the potential is "field-theoretic"? Two methods immediately suggest themselves. One is to carry out the complete derivation of the dynamical description, relativistic wave equation plus interaction, from relativistic quantum field theory. For a particular form of relativistic dynamics, the relativistic constraint approach [18–24], this has been carried out in part elsewhere [25,14]. The other (whose consequences are the subject of the present paper) is to insist that the correspondence principle be respected for the system of interest. That is, the classical mechanical description complete with interaction must agree with that provided by the relevant classical relativistic field theory. For each form of relativistic dynamics and interaction one tests this agreement by checking the system's expansion in inverse powers of the speed of light around its nonrelativistic limit against that provided by classical field theory. At the very least, these expansions should agree in the first (semirelativistic) order above the nonrelativistic limit.

As the authors of a recent paper have pointed out [13], a number of relativistic wave equations for the two-body system containing a vector interaction that are often treated as equivalently good actually possess inequivalent forms in the semirelativistic (slow-motion, weak-potential) approximation. On the other hand, as we have shown elsewhere [26,27], one of them, the Todorov form [6] that follows from relativistic constraint mechanics for the electromagnetic interaction, is canonically equivalent in the semirelativistic approximation to the time-honored Darwin Hamiltonian of electrodynamics. Do other forms share this agreement with classical electrodynamics or does the correspondence principle single out the Todorov form at the semirelativistic order of approximation?

In fact, as we show in the first section of this paper, classical relativistic field theory selects one of these wave equations, the minimal Todorov equation, as the "correct" one. We do this by first demonstrating that each of the relativistic wave equations discussed in Ref. [13] is actually a special case of the relativistic two-body mass-shell condition with a characteristic relativistic potential evaluated in the c.m. system. Then we derive the semirelativistic corrections implied by each. We show that, of the relativistic wave equations considered in Ref. [13], only the Todorov equation possesses a total c.m. Hamiltonian that is canonically equivalent to the total c.m. Hamiltonian obtained from classical field theory through order $1/c^2$, including relativistic kinematic corrections along with electromagnetic and scalar Darwin interactions [28–29].

In the second section of this paper, we show that the agreement between the semirelativistic dynamics of classical field theory and that of the minimal Todorov equation is no accident. We derive the minimal Todorov constraint equation (in the semirelativistic approximation) from the semirelativistic dynamics of classical field theory assuming only the generic constraint form (exact two-body relativistic kinematics) and that the quasipotential is an analytic function of the total c.m. energy w through the relative internal energy $(w-m_1-m_2)/(m_1+m_2)$. As a by-product of our derivation, we learn the origin of two important properties of the minimal Todorov equation in the semirelativistic dynamics of classical field theory. First, we observe the emergence of the minimal Todorov equation's characteristic minimal coupling forms of interaction for scalar and vector potentials in its semirelativistic form. Second, we find that one additional assumption, that the minimal structures persist to all orders in $1/c^2$, extrapolates the semirelativistic Todorov constraint to the fully relativistic Todorov equation. In particular, we demonstrate the origin in classical field theory of the role played by Todorov's kinematic relativistic reduced m_w mass and energy ϵ_w in the dynamical structure of the minimal Todorov equation.

II. RESTRICTIONS IMPOSED BY CLASSICAL FIELD THEORY ON THE CONSTRUCTION OF RELATIVISTIC POTENTIAL MODELS WITH EXACT RELATIVISTIC KINEMATICS

We begin by establishing our notation in order to review various relativistic potential models in a common format. Let w be the total c.m. energy with square given by

$$w^2=-(p_1+p_2)^2=-P^2 , \tag{1}$$

in which p_i is the four-momentum of particle i. For the nonrelativistic case

$$\mathbf{p}^2+2\mu U=2\mu\epsilon_B , \tag{2}$$

where $\mathbf{p}$ is the relative momentum, μ is the reduced mass, and ϵ_B is related to w through

$$\epsilon_B=w-m_1-m_2 . \tag{3}$$

Here we will consider only those relativistic systems whose nonrelativistic limit is governed by two potential terms, $U=\mathcal{V}+S$, in which $\mathcal{V}$ is generated by the timelike component (parallel to the total four-momentum P^μ) of a Lorentz four-vector potential and S arises from a Lorentz scalar. For a one-body problem the relativistic version of $\mathcal{V}$ enters through minimal subtraction from the particle's energy while S/c^2 appears as an addition to the particle's rest mass. All of the classical forms that we shall examine possess the nonrelativistic limit:

$$w=m_1+m_2+\frac{\mathbf{p}^2}{2\mu}+\mathcal{V}+S \tag{4}$$

when $\mathbf{p}^2/2\mu$, $\mathcal{V}$, and S become small compared to the particle rest masses.

The authors of Ref. [13] consider several relativistic wave equations whose classical versions each take the form (in the c.m. system)

$$\mathbf{p}^2+\Phi(r,w)=b^2(w) , \tag{5}$$

in which $b^2(w)$ is the relativistic kinematic form

$$b^2(w)=\frac{1}{4w^2}[w^4-2(m_1^2+m_2^2)w^2+(m_1^2-m_2^2)^2] , \tag{6}$$

which is equal to the square of the on-shell value of the relative momentum in the c.m. system. We remind the reader that in Eq. (5) the three-dimensional forms $\mathbf{r}^2$ and $\mathbf{p}^2$ are the c.m. restrictions of invariant four-dimensional forms. In particular

$$\mathbf{r}^2=x_\perp^2|_{\text{c.m.}} , \tag{7}$$

where

$$x_\perp^\mu=(g^{\mu\nu}+\hat{P}^\mu\hat{P}^\nu)(x_1-x_2)_\nu , \tag{8}$$

with $\hat{P}^\mu=P^\mu/w$. This vector has a zero time component in the c.m. system. The relative four-momentum conjugate to this position four-vector is

$$p_\perp^\mu=(g^{\mu\nu}+\hat{P}^\mu\hat{P}^\nu)p_\nu , \tag{9}$$

where

$$p=\frac{\epsilon_2 p_1-\epsilon_1 p_2}{w} , \tag{10}$$

with $\epsilon_1+\epsilon_2=w$ and $\epsilon_i=\epsilon_i(w)$ guaranteeing that $\{x_\mu,p_\nu\}=g_{\mu\nu}$. The ϵ_i's are the c.m. values of the time components of the p_i. The form (5) is then the c.m. ($\mathbf{P}=0$) version of the Lorentz-invariant form [30]

$$p_\perp^2+\Phi(\sqrt{x_\perp^2},w)=b^2(w) . \tag{11}$$

Such a generalized Schrödinger-like form can be derived from more general principles using constraint mechanics [18–23,31] (see also the Appendix). The constraint form provides a covariant Hamiltonian formalism (with one parametric time) that can be easily quantized and adapted to phenomenological studies. The effective generalized mass-shell condition displays exact relativistic two-body kinematics with an effective potential that is covariant but otherwise arbitrary.

For convenience we will work in the c.m. frame. In this frame the solution of (5) for the total energy leads to

$$w=(\mathbf{p}^2+m_1^2+\Phi)^{1/2}+(\mathbf{p}^2+m_2^2+\Phi)^{1/2} . \tag{12}$$

This equation is in general an implicit equation since Φ may also depend on w. The simplest equation treated by the authors of Ref. [13] is the relativistic Schrödinger equation obtained by reinterpreting the potentials of the nonrelativistic limit as relativistic potentials,

$$\Phi=2\mu(\mathcal{V}+S) , \tag{13}$$

in Eq. (5) with the unapproximated relativistic $b^2(w)$ of (6) ensuring that the resulting equation possesses correct two-body relativistic kinematics. Notice that this form for Φ does not distinguish between scalar and vector potentials and is thus destined to produce relativistic corrections to the dynamics that disagree with classical field-theoretic corrections which do. The authors of Ref. [13] next treat the minimal Todorov equation [6,14,32,33]. This equation takes the effective one-body form

$$\mathbf{p}^2+(m_w+S)^2-(\epsilon_w-\mathcal{V})^2=0 , \tag{14}$$

in which $\epsilon_w=(w^2-m_1^2-m_2^2)/2w$, $m_w=m_1m_2/w$ are the relativistic energy and reduced mass of a particle of relative motion [6,27] whose energy and momentum satisfy the Einstein mass-shell condition. In terms of these effective particle variables, $b^2(w)=\epsilon_w^2-m_w^2$. Then (14) implies that, for the Todorov equation,

$$\Phi=2m_wS+2\epsilon_w\mathcal{V}+S^2-\mathcal{V}^2 . \tag{15}$$

In a wave equation recently proposed by Lichtenberg [34],

$$\Phi=2\mu S+2\epsilon\mathcal{V}+\rho^2(S^2-\mathcal{V}^2) , \tag{16}$$

in which $\epsilon=\epsilon_1\epsilon_2/w$, $\epsilon_i=[m_i^2+b^2(w)]^{1/2}$, $\epsilon_1+\epsilon_2=w$, $\rho=(m_1^2+m_2^2)/M^2$, and $M=m_1+m_2$. The Todorov equation and that of Lichtenberg share the same heavy-particle limit ($m_2\to\infty$) in which each reduces to the single-particle Klein-Gordon form

$$\mathbf{p}^2+(m_1+S)^2-(\epsilon_1-\mathcal{V})^2=0 . \tag{17}$$

The authors of Ref. [13] go on to consider two other wave equations. One, the spinless Salpeter equation, takes the classical form

$$w=(\mathbf{p}^2+m_1^2)^{1/2}+(\mathbf{p}^2+m_2^2)^{1/2}+\mathcal{V}+S . \tag{18}$$

[Note that the form for Φ that would be obtained from this equation would not distinguish between scalar and vector potentials and thus, just like Eq. (13), is destined to produce relativistic corrections to the dynamics that disagree with classical field-theoretic corrections which do distinguish between those potentials.] The other takes the classical form

$$w=[\mathbf{p}^2+(m_1+\tfrac{1}{2}S)^2]^{1/2}+[\mathbf{p}^2+(m_2+\tfrac{1}{2}S)^2]^{1/2}+\mathcal{V} . \tag{19}$$

This equation differs from the spinless Salpeter equation in that it distinguishes between S and $\mathcal{V}$. As the authors of Ref. [13] point out, neither of these equations reduces to the Klein-Gordon equation when one of the particles becomes very heavy.

The common form, Eq. (12), leads to a generic $O(1/c^2)$ expansion for w, the total c.m. energy. When we evaluate this for each of the relativistic interactions discussed, we find in each case that the c-independent term reproduces the common nonrelativistic limit, Eq. (4). In addition, by construction, these equations share the same kinematics to all orders in $1/c^2$. However, each differs from the others in its dynamical $O(1/c^2)$ parts. One finds (after restoring units) that each takes the form [13]

$$w(i)=m_1c^2+m_2c^2+\frac{\mathbf{p}^2}{2\mu}+S+\mathcal{V}+\frac{1}{c^2}\left[-\frac{\mathbf{p}^4}{8}\left[\frac{1}{m_1^3}+\frac{1}{m_2^3}\right]\right]+w^{(2)}(i) . \tag{20}$$

The simple Schrödinger model yields

$$w^{(2)}(S)=\frac{1}{c^2}\left[-\frac{\mu\mathbf{p}^2}{2}(\mathcal{V}+S)\left[\frac{1}{m_1^3}+\frac{1}{m_2^3}\right]-\frac{\mu^2(\mathcal{V}+S)^2}{2}\left[\frac{1}{m_1^3}+\frac{1}{m_2^3}\right]\right] . \tag{21}$$

Note that, as anticipated, $\mathcal{V}$ and S are not distinguished. The Todorov equation gives

$$w^{(2)}(T)=\frac{1}{c^2}\left[\frac{\mathbf{p}^2\mathcal{V}}{m_1m_2}-\frac{\mathbf{p}^2S}{2}\left[\frac{1}{m_1^2}+\frac{1}{m_2^2}\right]+\frac{(\mathcal{V}+S)^2}{2M}\right] , \tag{22}$$

so that $\mathcal{V}$ and S are distinguished. The new phenomenological equation of Ref. [34] yields

$$w^{(2)}(L)=\frac{1}{c^2}\left[-\frac{\mu\mathbf{p}^2S}{2}\left[\frac{1}{m_1^3}+\frac{1}{m_2^3}\right]+\frac{(\mathcal{V}^2-S^2)(m_1-m_2)^2}{2M^3}\right] , \tag{23}$$

so that (just as in the Todorov equation) $\mathcal{V}$ and S appear differently. The spinless Salpeter equation gives

$$w^{(2)}(\mathrm{SS})=0 . \tag{24}$$

Finally, the modified spinless Salpeter equation (called the "two-body Klein-Gordon equation" [35] in Refs. [13,15,34]) gives

$$w^{(2)}(K)=\frac{1}{c^2}\left[-\frac{\mathbf{p}^2s}{4}\left[\frac{1}{m_1^2}+\frac{1}{m_2^2}\right]\right] . \tag{25}$$

Classical relativistic field theories have exact covariant dynamics built in through the field equations and the particle equations of motion and, like constraint dynamics, display exact two-body kinematics. However, unlike constraint dynamics, these theories do not have an associated covariant Hamiltonian structure. Nevertheless any classical field theory determines its own classical expression for the semirelativistic (slow-motion, weak-potential) one-time Hamiltonian that governs the corresponding two-particle system. If we compare that Hamiltonian for the relevant field theory to the dynamical descriptions provided by the $1/c^2$ expansions [Eqs. (21)–(25)] of the relativistic constraint descriptions we have just listed, we can determine which of these equations are actually capable of treating perturbative field-theoretic dynamics (i.e., yield the appropriate spin-independent terms beyond the nonrelativistic potential). This procedure can be carried out for any (weak) field-theoretic interaction for which the potentials are known. Here we restrict our treatment to massless vector and scalar fields coupled to two particles. By matching the well-known nonrelativistic Hamiltonians generated by field theory to the common form, Eq. (20), we first fix $\mathcal{V}$ and S. We then compare the higher-order terms generated by classical field theory to those appearing in Eqs. (20)–(25).

The field-theoretic semirelativistic correction terms are those first derived by Darwin [28]. For particles of charges e_1 and e_2 interacting through half-advanced, half-retarded fields the total c.m. energy through terms of order $1/c^2$ is [28,29]

$$w=m_1c^2+m_2c^2+\frac{\mathbf{p}^2}{2\mu}+\frac{e_2e_2}{r}+\frac{1}{c^2}\left[-\frac{(\mathbf{p}^2)^2}{8}\left[\frac{1}{m_1^3}+\frac{1}{m_2^3}\right]+\frac{\mathbf{p}^2e_1e_2}{2m_1m_2r}+\frac{(\mathbf{p}\cdot\mathbf{r})^2e_1e_2}{2m_1m_2r^3}\right]+O\left[\frac{1}{c^4}\right]. \tag{26}$$

Comparison of the common nonrelativistic limit of the relativistic wave equations for the case $S=0$ yields $\mathcal{V}=e_1e_2/r$. The corresponding classical field-theoretic expression for total c.m. energy with scalar interactions alone is [29]

$$w=m_1c^2+m_2c^2+\frac{\mathbf{p}^2}{2\mu}-\frac{g_1g_2}{r}+\frac{1}{c^2}\left[-\frac{(\mathbf{p}^2)^2}{8}\left[\frac{1}{m_1^3}+\frac{1}{m_2^3}\right]+\frac{\mathbf{p}^2g_1g_2}{2r}\left[\frac{1}{m_1^2}+\frac{1}{m_2^2}\right]+\frac{g_1g_2}{2rm_1m_2}\mathbf{p}^2-\frac{g_1g_2}{2m_1m_2}\frac{(\mathbf{p}\cdot\mathbf{r})^2}{r^3}\right]+O\left[\frac{1}{c^4}\right]. \tag{27}$$

Comparison with the nonrelativistic limit of Eqs. (15), (16), (18), and (19) for $\mathcal{V}=0$ yields $S=-g_1g_2/r$. For combined scalar and vector interactions, the interaction terms simply add (at this order) yielding

$$w=m_1c^2+m_2c^2+\frac{\mathbf{p}^2}{2\mu}+\frac{e_1e_2}{r}-\frac{g_1g_2}{r}+\frac{1}{c^2}\left[-\frac{(\mathbf{p}^2)^2}{8}\left[\frac{1}{m_1^3}+\frac{1}{m_2^3}\right]+\frac{\mathbf{p}^2e_1e_2}{2m_1m_2r}+\frac{(\mathbf{p}\cdot\mathbf{r})^2e_1e_2}{2m_1m_2r^3}+\frac{\mathbf{p}^2g_1g_2}{2r}\left[\frac{1}{m_1^2}+\frac{1}{m_2^2}\right]+\frac{g_1g_2}{2rm_1m_2}\mathbf{p}^2-\frac{g_1g_2}{2m_1m_2}\frac{(\mathbf{p}\cdot\mathbf{r})^2}{r^3}\right]+O\left[\frac{1}{c^4}\right]. \tag{28}$$

Note that at the semirelativistic $O(1/c^2)$ level no cross terms between scalar and vector interactions appear. [Such cross terms appear in $O(1/c^4)$ and higher [36].] At first sight, when we insist on the nonrelativistic identifications $\mathcal{V}=e_1e_2/r$ and $S=-g_1g_2/r$, none of the c.m. energy expressions in Eqs. (20)–(25) agrees with the result of the field-theoretic expression at the level of the dynamical $O(1/c^2)$ terms. This is the sort of disagreement that one would expect if the only criteria for introducing relativity to the two-body problem were correct two-body kinematics and correct heavy-particle limits. Note that the only parts that do agree naively with the field-theoretic results are the scalar $O(1/c^2)$ pieces of Eq. (22) and Eq. (23) that do not vanish in the $m_2\rightarrow\infty$ limit. The parts that do vanish in the $m_2\rightarrow\infty$ limit do not appear to match the field-theoretic results. Note particularly that the field-theoretic forms depend not only on the momentum [as do the terms of Eqs. (20)–(25)] but also on the angular momentum since $(\mathbf{p}\cdot\mathbf{r})^2=\mathbf{r}^2\mathbf{p}^2-\mathbf{L}^2$. As was originally shown by Schwinger [37], and subsequently used by us [27] to show the connection between Todorov and Fermi-Breit equations for electromagnetic interactions, such terms can be eliminated at this order by a canonical transformation. Thus, we perform the transformation

$$\mathbf{r}\rightarrow\mathbf{r}'=\left[1-\frac{e_1e_2}{2Mrc^2}+\frac{g_1g_2}{2Mrc^2}\right]\mathbf{r}, \tag{29}$$

$$\mathbf{p}\rightarrow\mathbf{p}'=\mathbf{p}-\left[\frac{e_1e_2}{2Mc^2}-\frac{g_1g_2}{2Mc^2}\right]\frac{1}{r^3}(\mathbf{r}\times\mathbf{L}) \tag{30}$$

(where $M=m_1+m_2$) in order to transform the field-theoretic result to the angular-momentum-independent forms possessed by Eqs. (21)–(25). Since $\{r_i,p_i\}=\delta_{ij}$, we find

$$\{r_i',p_j'\}=\delta_{ij}+O\left[\frac{1}{c^4}\right], \tag{31}$$

where we have used $\{r_i,L_l\}=\epsilon_{ilm}r_m$ and $\{p_i,1/r\}=-r_i/r^3$ to show that the $O(1/c^2)$ terms cancel. Thus, through order $1/c^2$, these new variables are also canonical. Since these two sets of canonical variables differ from one another by terms of order $1/c^2$, the transformation will only change the forms of those terms [through order $1/c^2$ in Eq. (28)] that survive in the nonrelativistic limit. Under this canonical transformation

$$\frac{1}{r}\rightarrow\frac{1}{r'}=\frac{1}{r}+\frac{e_1e_2-g_1g_2}{2Mc^2r^2}+O\left[\frac{1}{c^4}\right], \tag{32}$$

$$\mathbf{p}^2\rightarrow\mathbf{p}'^2=\mathbf{p}^2+\frac{e_1e_2-g_1g_2}{2Mc^2r^3}\mathbf{L}^2+O\left[\frac{1}{c^4}\right]. \tag{33}$$

Since $\mathbf{L}^2=\mathbf{r}^2\mathbf{p}^2-(\mathbf{r}\cdot\mathbf{p})^2$, the field-theoretic c.m. energy expression becomes

$$w=(m_1+m_2)c^2+\frac{\mathbf{p}^2}{2\mu}+\frac{e_1e_2}{r}-\frac{g_1g_2}{r}+\frac{1}{c^2}\left[-\frac{(\mathbf{p}^2)^2}{8}\left[\frac{1}{m_3^2}+\frac{1}{m_2^3}\right]+\frac{\mathbf{p}^2e_1e_2}{m_1m_2r}+\frac{g_1g_2\mathbf{p}^2}{2r}\left[\frac{1}{m_1^2}+\frac{1}{m_2^2}\right]+\left[\frac{e_1e_2}{r}-\frac{g_1g_2}{r}\right]^2\frac{1}{2M}\right]+O\left[\frac{1}{c^4}\right]. \tag{34}$$

Comparing this result with our list of candidates (21)–(25), we see that the canonical transformation produces the Todorov form, and only that form.

However, the fact that we have found a canonical transformation (the Schwinger transformation) that demonstrates the classical equivalence of field-theoretic dynamics and Todorov dynamics through $O(1/c^2)$ does not rule out the possibility that each of the other relativistic equations might possess its own (heretofore unknown) canonical transformation producing it from the field-theoretical form. In that case some or all of the descriptions provided by Eqs. (21)–(25) would be canonically equivalent through $O(1/c^2)$. To simplify our investigation of this issue, we assume the absence of scalar interaction. Then

$$w_{\rm FT}=m_1c^2+m_2c^2+\frac{\mathbf{p}^2}{2\mu}+\frac{e_1e_2}{r}+\frac{1}{c^2}\left[-\frac{(\mathbf{p}^2)^2}{8}\left[\frac{1}{m_1^3}+\frac{1}{m_2^3}\right]+\frac{\mathbf{p}^2e_1e_2}{2m_1m_2r}+\frac{(\mathbf{p}\cdot\mathbf{r})^2e_1e_2}{2m_1m_2r^3}\right], \tag{35}$$

while

$$w^{(2)}(S)=\frac{1}{c^2}\left[-\frac{\mu\mathbf{p}^2}{2}\frac{e_1e_2}{r}\left[\frac{1}{m_1^3}+\frac{1}{m_2^3}\right]-\frac{\mu^2e_1^2e_2^2}{2r^2}\left[\frac{1}{m_1^3}+\frac{1}{m_2^3}\right]\right], \tag{36}$$

$$w^{(2)}(L)=\frac{1}{c^2}\left[\frac{e_1^2e_2^2}{2M^3r^2}(m_1-m_2)^2\right], \tag{37}$$

$$w^{(2)}(\mathrm{SS})=0, \tag{38}$$

$$w^{(2)}(T)=\frac{1}{c^2}\left[\frac{\mathbf{p}^2e_1e_2}{m_1m_2r}+\frac{e_1^2e_2^2}{2Mr^2}\right]. \tag{39}$$

Consider the most general canonical transformation through $O(1/c^2)$, which we parametrize as

$$\mathbf{r}\rightarrow\mathbf{r}'=\mathbf{r}\left[1+\frac{a}{Mc^2}\right]+\hat{\mathbf{p}}\mathbf{r}\cdot\hat{\mathbf{p}}\frac{j}{Mc^2}+O\left[\frac{1}{c^4}\right], \tag{40}$$

$$\mathbf{p}\rightarrow\mathbf{p}'=\mathbf{p}\left[1+\frac{f}{Mc^2}\right]+\hat{\mathbf{r}}\hat{\mathbf{r}}\cdot\mathbf{p}\frac{h}{Mc^2}+O\left[\frac{1}{c^4}\right]. \tag{41}$$

The terms a,f,j, and h are functions of the three independent variables $x=\mathbf{r}^2/2$, $y=\mathbf{r}\cdot\mathbf{p}$, and $z=\mathbf{p}^2/2$. The requirements that $\{r_i',r_j'\}=0=\{p_i',p_j'\}$ and that $\{r_i',p_j'\}=\delta_{ij}$ through order $O(1/c^2)$ yield the equations

$$a=-f, \tag{42}$$

$$\left[1+y\frac{\partial}{\partial y}\right]j=-2z\frac{\partial f}{\partial z}, \tag{43}$$

$$\left[1+y\frac{\partial}{\partial y}\right]h=2x\frac{\partial f}{\partial x}. \tag{44}$$

Now a,j,f, and h must have units of energy. They must also vanish as $e_1e_2\rightarrow0$ if they are to yield models that have the same (correct) two-body relativistic kinematics as $w_{\rm FT}$. Thus,

$$f=-a=\frac{e_1e_2}{r}\eta(x,y,z), \tag{45}$$

$$j=\frac{e_1e_2}{r}\zeta(x,y,z), \tag{46}$$

$$h=\frac{e_1e_2}{r}\xi(x,y,z), \tag{47}$$

where η, ζ, and ξ are dimensionless functions independent of e_1e_2 to this order. But the only dimensionless combination is xz/y^2 or $y^2/xz\equiv\rho$ (so that $\rho^{1/2}=2\hat{\mathbf{r}}\cdot\hat{\mathbf{p}}$). Hence, the partial differential equations become

$$\zeta+2\rho\zeta'=2\rho\eta', \tag{48}$$

$$\xi+2\rho\xi'=-\eta-2\rho\eta', \tag{49}$$

two equations in three unknowns.

Now, we perform the canonical transformation (40) and (41) on the (classical) field-theoretic expression (35). This yields

$$w=m_1c^2+m_2c^2+\frac{\mathbf{p}^2}{2\mu}+\frac{e_1e_2}{r}+\frac{1}{c^2}\left[-\frac{\mathbf{p}^4}{8}\left[\frac{1}{m_1^3}+\frac{1}{m_2^3}\right]\right]+w_{\rm FT}^{(2)}, \tag{50}$$

where

$$w_{\rm FT}=\frac{e_1e_2}{r}\left[\frac{\mathbf{p}^2}{2M\mu c^2}\left[1+2\eta+\frac{\rho}{4}+\frac{\xi}{2}\rho\right]+\frac{e_1e_2}{rM}\left[\eta-\zeta\frac{\rho}{4}\right]\right] . \tag{51}$$

The additional relation among these three functions is obtained by equating $w_{\rm FT}^{(2)}$ to each $w^{(2)}(i)$, $i=S,T,L,{\rm SS}$ in turn.

First consider the Todorov model for which

$$w^{(2)}(T)=\frac{e_1e_2}{r}\left[\frac{\mathbf{p}^2}{M\mu c^2}+\frac{e_1e_2}{2Mrc^2}\right] . \tag{52}$$

The additional relation yields

$$\frac{\mathbf{p}^2}{2M\mu c^2}\left[-1+2\eta+\frac{\rho}{4}+\frac{\xi\rho}{2}\right]+\frac{e_1e_2}{Mc^2r}\left[\eta-\frac{1}{2}-\zeta\frac{\rho}{4}\right]=0 . \tag{53}$$

Hence, we have two relations that must be simultaneously satisfied

$$2\eta-1=-\frac{\rho}{4}(1+2\xi)=\rho\frac{\zeta}{2} , \tag{54}$$

whose derivatives are

$$2\eta'=-\frac{1}{4}-\frac{\xi}{2}-\frac{\rho\xi'}{2}=\frac{\zeta}{2}+\frac{\rho\zeta'}{2} . \tag{55}$$

Substitution of these relations into (48) and (49) yields the uncoupled differential equations

$$\zeta'(2\rho-\tfrac{1}{2}\rho^2)=-\zeta(1-\tfrac{1}{2}\rho) , \tag{56}$$

$$\xi'(2\rho-\tfrac{1}{2}\rho^2)+\xi(1-\tfrac{3}{4}\rho)=-\tfrac{1}{2}(1-\tfrac{3}{4}\rho) . \tag{57}$$

These possess the solutions

$$\zeta=k_1(\rho-\tfrac{1}{4}\rho^2)^{-1/2},\quad \xi=-\tfrac{1}{2}+\frac{k_2\rho^{-1/2}}{1-\frac{1}{4}\rho} . \tag{58}$$

But these solutions are not compatible with the right-hand side of Eq. (54) unless $k_1=k_2=0$. Hence,

$$\zeta=0,\quad \eta=\tfrac{1}{2},\quad \xi=-\tfrac{1}{2} , \tag{59}$$

which determines the canonical transformation

$$\mathbf{r}'=\mathbf{r}\left[1-\frac{e_1e_2}{2Mrc^2}\right] , \tag{60}$$

$$\mathbf{p}'=\mathbf{p}\left[1+\frac{e_1e_2}{2Mrc^2}\right]-\hat{\mathbf{r}}\hat{\mathbf{r}}\cdot\mathbf{p}\frac{e_1e_2}{2Mc^2r} , \tag{61}$$

which reproduces (29) and (30) without the scalar interaction.

Now we attempt the same procedure for the spinless Salpeter model. In that case we find the relations

$$2\eta+1=-\tfrac{1}{4}(\rho+2\xi\rho)=\tfrac{1}{2}\zeta\rho+1 , \tag{62}$$

whose derivatives are

$$2\eta'=-\tfrac{1}{4}-\tfrac{1}{2}\xi-\tfrac{1}{2}\rho\xi'=\tfrac{1}{2}\zeta+\tfrac{1}{2}\rho\zeta' . \tag{63}$$

Note that the relations for η' have not changed. Thus, the ζ differential equation arising from Eq. (48) remains unchanged. Hence,

$$\zeta=k_1(\rho-\tfrac{1}{4}\rho^2)^{-1/2} . \tag{64}$$

The other differential equation arising from (49) differs from (57) due to the presence of η in (49):

$$\xi(1-\tfrac{3}{4}\rho)+\xi'(2\rho-\tfrac{1}{2}\rho^2)=\tfrac{1}{2}(1+\tfrac{3}{4}\rho) . \tag{65}$$

The solution to this is

$$\xi=\frac{\rho^{-1/2}}{1-\frac{1}{4}\rho}(k_2+\tfrac{1}{2}\rho^{1/2}+\tfrac{1}{8}\rho^{3/2}) . \tag{66}$$

(Note that the spinless Salpeter model does not generate a ξ=const solution like that of the Todorov model.) In general these solutions cannot satisfy the last equality in Eq. (62). Thus, no acceptable canonical transformation (which goes to the identity transformation when $e_1e_2\to0$) can bring the spinless Salpeter model into agreement with the results of classical field theory. We point out that the spinless Salpeter equation results from approximating the Green's function $\delta([z_1(\tau_1)-z_2(\tau_2)]^2)$ of the classical field theory by $\delta(t_1-t_2)/|\mathbf{r}_1-\mathbf{r}_2|$. Thus, it is not surprising that the Salpeter equation contains no retarded (or advanced) effects. In contrast, the Darwin Hamiltonian, Eq. (28), does include lowest-order retardation (and advancement) since in its derivation the $O(1/c^2)$ terms in the Green's function are not dropped.

For the Lichtenberg model [34], we find the relations

$$2\eta+1=-\frac{1}{4}(\rho+2\xi\rho)=\frac{\zeta\rho}{2}+1+\frac{(m_1-m_2)^2}{M^2} \tag{67}$$

differing only from (62) for the spinless Salpeter model for ζ by a constant that vanishes for equal masses. Thus, the ξ solution is altered but again is not compatible with the right-hand side of Eq. (54).

Finally, we examine the simplified Schrödinger model. We restrict our attention to the equal-mass case $m_1=m_2=\frac{1}{2}M=2\mu\equiv m$. Then we have

$$2\eta+1=-1-\tfrac{1}{4}(\rho+2\xi\rho)=-3+\tfrac{1}{4}\zeta\rho . \tag{68}$$

The expression for η' is again just Eq. (55), unchanged so that the ζ is that in Eq. (58). Once again, the differential equation for ξ possesses no ξ=const solution. In general the solution to the differential equation

$$\xi(1-\tfrac{3}{4}\rho)+\xi'(2\rho-\tfrac{1}{2}\rho)=\tfrac{3}{4}(1+\tfrac{1}{2}\rho) , \tag{69}$$

i.e.,

$$\xi=\frac{\zeta^{1/2}}{1-\frac{1}{4}\rho}(k_2+\rho^{1/2}+\tfrac{1}{8}\rho^{3/2}) \tag{70}$$

and the ζ solution are not compatible with the right-hand side of Eq. (54).

Thus, only the Todorov form has the $O(1/c^2)$ dynamical recoil terms generated by weak, linear classical field theory. None of the other forms examined in Ref. [13] possesses the correct $O(1/c^2)$ field-theoretic relativistic dynamics for either the scalar or vector interactions.

But, what is the practical significance of our investigation, since the equations considered in Ref. [13] were to be used only for phenomenological description of the $q\bar{q}$ meson system? Our argument shows that of these relativistic wave equations, only the minimal Todorov equation can be used to treat spinless electrodynamics. This is due to the fact that for the vector interaction, only the Todorov equation contains the dynamical equivalent of the field-theoretic Darwin interaction of Eq. (26). Moreover, this important structure persists for spinning particles. We have found such equations for spinor electrodynamics (coupled 16-component "two-body Dirac equations" [38]), which we were able to solve exactly for parapositronium solutions [39], that duplicate the quantum field-theoretic spectrum through $O(\alpha^4)$. In the Dirac γ-matrix representation, the squares of our first-order Dirac equations produce a second-order wave equation for the upper-upper component of the wave function that contains the basic spin-independent potentials of the minimal Todorov equation [Eq. (14)], elaborated by additional spin-dependent recoil terms of electrodynamics. Thus the results of this paper tie the structures of those equations, as well as the classical and quantum-mechanical spinless Todorov equations, to classical field theory. Now, while the long-distance part of the chromodynamic interaction may be imperfectly known (and subject to phenomenological manipulation) the short-distance part shares the Darwin-like structure of Eq. (39) (see Ref. [26]). We consider it highly unlikely that the weak vector structure missing from Eqs. (36)–(38) will emerge in some indirect fashion. Without that structure, the S states (susceptible to short-distance corrections) are liable to be distorted relative to the other angular momentum states [49]. We have recently carried out a treatment of the meson spectrum that retains the vector structure of Todorov (and of classical field theory) [9,14]. We suspect that some of the successful features of the resulting meson spectrum are due in fact to our incorporation of Todorov's version of the relativistic vector interaction.

Finally, we point out to the reader that each of the equations that we have examined in this section is fully relativistic, possessing exact relativistic two-body kinematics along with covariant potentials. In this paper, we have used expansion in powers of $1/c^2$ merely as a device to detect dynamical inequivalence among wave equations. When such equations are applied phenomenologically to mesons formed from light quarks moving with high velocities, we have to solve them numerically to make such comparisons [9,14]. But we find it extremely unlikely that structures missing from equations expanded in low orders of $1/c^2$ (and known to be present for slowly moving heavy quarks) play no role in fully relativistic calculations for light quarks.

III. DERIVATION OF THE TODOROV EQUATION FROM THE SEMIRELATIVISTIC LIMIT OF CLASSICAL FIELD THEORY

In Sec. II, we treated each of the relativistic wave equations studied in Ref. [13] as given, formed the semirelativistic limit of each, and found that of these equations, only the Todorov equation possesses a semirelativistic limit that is canonically equivalent to that of the corresponding classical field theory. But, suppose that we proceed in the opposite direction, that is, treating the semirelativistic limit of classical field theory as given, we derive the quasipotential Φ of the corresponding relativistic wave equation. If we should end up with the Φ of Todorov [Eq. (15)], we will have discovered the origin in classical field theory of the peculiar minimal structures of the Todorov equation (as well as the roles played in them by the kinematic relativistic reduced mass m_w and energy ϵ_w).

We begin with the semirelativistic equation for the total c.m. energy w of classical field theory for combined vector and scalar interactions [Eq. (28)], and canonically transform it using the Schwinger transformation to Eq. (34) in order to remove all dependence on $\mathbf{p}\cdot\mathbf{r}$ through order $1/c^2$. On the other hand, we arrange the generic constraint form Eq. (5) into its (implicit) solution for w as the sum of two square roots [Eq. (12)]. For slow motion (small $\mathbf{p}^2$) and a weak potential (small Φ) compared with each mass we expand the square roots to obtain

$$w=M+\frac{\mathbf{p}^2+\Phi}{2\mu}-\frac{1}{8}\left[\frac{1}{m_1^3}+\frac{1}{m_2^3}\right](\mathbf{p}^2+\Phi)^2+\cdots . \tag{71}$$

Thus far, we have assumed only a particular form for classical field theory (additive interactions of charged point particles through massless vector and scalar fields) and the generic relativistic constraint form [Eq. (5)] shared (as the correspondence limit) by all the relativistic wave equations of Ref. [13]. We now assume that the quasipotential Φ is an analytic function of the total c.m. energy w through the relative internal energy

$$\Delta=\frac{w-M}{M}, \tag{72}$$

so that

$$\Phi(x_\perp,\Delta)=\Phi(x_\perp,0)+\Delta\Phi'(x_\perp,0)+\cdots . \tag{73}$$

We assume that the quasipotential Φ has a well-defined nonrelativistic limit in Eq. (71) and rewrite the first term of Eq. (73) in terms of it:

$$\Phi(x_\perp,0)=\Phi_{\rm NR}(x_\perp)+\tilde{\Phi}(x_\perp) . \tag{74}$$

Thus, Eq. (71) may contain three different relativistic corrections to the nonrelativistic Hamiltonian: (i) that arising from the second-order $(1/m^3)$ term in the square-root expansion, (ii) that arising from the first-order relative internal energy (Δ) correction to the quasipotential in Eq. (73), and (iii) that arising from the difference between the zeroth order of Eq. (73) and the nonrelativistic limit of the quasipotential.

We determine the three parts of Φ by equating Eq. (71) with the field-theoretic Eq. (34). That is,

$$\frac{\mathbf{p}^2}{2\mu}+\mathcal{V}+S+\frac{1}{c^2}\left[\frac{-(\mathbf{p}^2)^2}{8}\left[\frac{1}{m_1^3}+\frac{1}{m_2^3}\right]+\frac{\mathbf{p}^2\mathcal{V}}{m_1m_2}-\frac{S\mathbf{p}^2}{2}\left[\frac{1}{m_1^2}+\frac{1}{m_2^2}\right]+\frac{(\mathcal{V}+S)^2}{2M}\right]$$
$$=\frac{\mathbf{p}^2+\Phi_{\rm NR}(x_\perp)}{2\mu}+\frac{\Delta\Phi'(x_\perp,0)+\tilde{\Phi}(x_\perp)}{2\mu}-\frac{1}{8}\left[\frac{1}{m_1^3}+\frac{1}{m_2^3}\right][\mathbf{p}^2+\Phi_{\rm NR}(x_\perp)]^2 , \quad (75)$$

in which

$$\mathcal{V}=\frac{e_1e_2}{r},\quad S=\frac{-g_1g_2}{r} . \quad (76)$$

Since the terms of lowest order in $1/c^2$ on the right and left must be equal, we find that

$$\Phi_{\rm NR}(x_\perp)=2\mu(\mathcal{V}+S) , \quad (77)$$

and, thus, that $\Phi_{\rm NR}$ is independent of $\mathbf{p}^2$. In turn, Eq. (77) determines Δ $[=(\mathbf{p}^2/2\mu+\mathcal{V}+S)/M]$ to lowest order, sufficient for evaluation of the remaining terms of Eq. (75). Since Φ' and $\tilde{\Phi}$ can only contain $\mathbf{p}^2$ as still higher-order relativistic corrections to the dynamics above $O(1/c^2)$, each must be independent of $\mathbf{p}^2$ in lowest order. Thus, the momentum-independent parts and coefficients of $\mathbf{p}^2$ on the right and left of Eq. (75) must separately be equal. This equality implies the simultaneous equations

$$2M\mathcal{V}-\Phi'(x_\perp,0)=2\mu(\mathcal{V}+S) \quad (78)$$

and

$$M(S+\mathcal{V})^2-\Phi'(x_\perp,0)(S+\mathcal{V})-M\tilde{\Phi}(x_\perp)=2\mu(\mathcal{V}+S)^2 , \quad (79)$$

with solutions

$$\Phi'(x_\perp,0)=2M\mathcal{V}-2\mu(\mathcal{V}+S) \quad (80)$$

and

$$\tilde{\Phi}(x_\perp)=S^2-\mathcal{V}^2 . \quad (81)$$

Thus our constraint equation (5) becomes

$$\mathbf{p}^2+2S\mu(1-\Delta)+2\mathcal{V}[\mu+\Delta(M-\mu)]+S^2-\mathcal{V}^2=b^2 . \quad (82)$$

Therefore, the requirement that the relativistic constraint (wave equation) agree with classical field theory in the semirelativistic approximation, plus the assumption that the relativistic quasipotential be analytic in w through the relative internal energy Δ, are sufficient to force a structure for Φ that is (to this order of approximation) quadratic in the potentials S and $\mathcal{V}$ with no $S\mathcal{V}$ cross term. Furthermore, if we examine the coefficients of the terms linear in the potentials, we find that they are just Todorov's kinematic reduced mass and energy of a particle of relative motion evaluated in lowest order in Δ. That is,

$$m_w\equiv\frac{m_1m_2}{w}=\frac{\mu}{1+\Delta}\approx\mu(1-\Delta) , \quad (83)$$

$$\epsilon_w\equiv\frac{w^2-m_1^2-m_2^2}{2w}=\frac{M^2(1+\Delta)^2-m_1^2-m_2^2}{2M(1+\Delta)}$$
$$\approx\frac{2M^2\Delta+2\mu M}{2M}(1-\Delta)$$
$$\approx\mu+\Delta(M-\mu) . \quad (84)$$

Then, since Todorov's variables obey the Einstein condition

$$\epsilon_w^2-m_w^2=b^2(w) \quad (85)$$

by construction, Eq. (82) is the Todorov constraint equation

$$\mathbf{p}^2+(m_w+S)^2+(\epsilon_w-\mathcal{V})^2=0 , \quad (14')$$

correct to our order of approximation.

Thus, we learn two important facts from our examination of the coefficients of the first-order potential terms along with the quadratic structure. First, the semirelativistic dynamics of field theory implies the minimal structure of the potentials in Eq. (14′) to this order of approximation. Second, the unapproximated Todorov equation (14) merely extrapolates this semirelativistic structure to all orders in $1/c^2$ by replacing the approximate forms for m_w and ϵ_w in the coefficients of the first-order potential terms by their unapproximated forms on the left sides of Eqs. (83) and (84), so as to extend the minimal structures of Eq. (14′) to all orders in Δ [Eq. (14)].

As we have seen, the structures of Eq. (14) to semirelativistic order are consequences of the semirelativistic structure of classical field theory. This means that for the vector interaction the minimal structure is a consequence of the interaction structure of a gauge-invariant theory evaluated in a fixed gauge, that in which the Darwin interaction appears—in our work, the Lorentz gauge (in Ref. [41], the Coulomb gauge). Nevertheless, the potentials in Eq. (14) exhibit an extra invariance, an effective gauge transformation on the system potential. To see how this occurs in our equations, we note that we can rewrite (14) in an effective one-particle generalized mass shell or Klein-Gordon form if we define

$$\mathcal{P}^\mu=p^\mu+\epsilon_w\hat{P}^\mu , \quad (86)$$

$$M_w=m_w+S , \quad (87)$$

$$\pi^\mu=\mathcal{P}^\mu-\mathcal{V}\hat{P}^\mu=p^\mu+\hat{P}^\mu(\epsilon_w-\mathcal{V}) . \quad (88)$$

Then the effective Hamiltonian corresponding to the minimal Todorov form (11) or (14) is

$$\mathcal{H}=\lambda(\pi^2+M_w^2)=0 , \quad (89)$$

with λ determining the scale of the parametric time. The equations of motion are

$$\dot{x}^{\mu}=\{x^{\mu},\mathcal{H}\}=2\lambda\pi^{\mu} , \tag{90a}$$

$$\dot{\pi}^{\mu}=\{\pi^{\mu},\mathcal{H}\}=2\lambda\{\pi^{\mu},\pi^{\nu}\}\pi_{\nu}-2\lambda M_{w}\partial^{\mu}S \equiv 2\lambda F^{\mu\nu}\pi_{\nu}-2\lambda M_{w}\partial^{\mu}S, \tag{90b}$$

where

$$F^{\mu\nu}=\{p^{\mu},-\hat{P}^{\mu}\mathcal{V}\}+\{-\hat{P}^{\mu}\mathcal{V},p^{\mu}\}=\partial^{\mu}\hat{P}^{\nu}\mathcal{V}-\partial^{\nu}\hat{P}^{\mu}\mathcal{V} \equiv\partial^{\mu}\mathcal{V}^{\nu}-\partial^{\nu}\mathcal{V}^{\mu} . \tag{91}$$

The effective field-strength tensor $F^{\mu\nu}$ is invariant under the change of "gauge":

$$\mathcal{V}^{\mu}\rightarrow\mathcal{V}^{\mu}+\partial^{\mu}\chi(x_{\perp}) . \tag{92}$$

Further, note that because of the $x_{\perp}$ dependence the vector potential $\mathcal{V}^{\mu}=\hat{P}^{\mu}\mathcal{V}(x_{\perp})$ satisfies the Lorentz gauge condition $\partial_{\mu}\mathcal{V}^{\mu}=\hat{P}^{\mu}\partial_{\mu}\mathcal{V}(x_{\perp})=0$. Furthermore, any gauge transformation of the form (92) can be absorbed by a canonical transformation on p_{μ} of the form $p_{\mu}\rightarrow p_{\mu}+\partial_{\mu}\chi$, $x_{\mu}\rightarrow x_{\mu}$. These facts show how the effective gauge invariance is embodied in the minimal Todorov constraint equation. In contrast the transformation $\mathcal{V}^{\mu}\rightarrow\mathcal{V}^{\mu}+\hat{P}^{\mu}\chi(x_{\perp})$ (which would correspond to the change of the static part of the interaction $\mathcal{V}\rightarrow\mathcal{V}+\chi$) is not a gauge transformation and would change entirely the dynamics. That is,

$$F^{\mu\nu}\rightarrow F^{\mu\nu}+\partial^{\mu}\chi\hat{P}^{\nu}-\partial\chi\hat{P}^{\mu}\neq F^{\mu\nu} . \tag{93}$$

Note that our argument is limited only by the form of the interaction provided by classical relativistic field theory and our analytic ability to construct a canonical transformation like that of Schwinger [37]. Thus, this method is not limited in principle to the Coulombic interactions (and their relativistic corrections generated by standard weak-coupling classical vector and scalar field theory) that we have treated in this paper. If one knows a relativistic exact or approximate solution for the field produced by a point particle, one can construct a Fokker-Tetrode action in which the electromagnetic kernel $\delta([z_1(\tau_1)-z_2(\tau_{23})]^2)$ of Wheeler and Feynman is replaced by a more general form with a non-Coulombic static part. One can then use a transformation analogous to Eqs. (29) and (30) to bring the resultant order-of-$1/c^2$ Hamiltonian to a form independent of $\mathbf{L}^2$. This in turn leads through the method of this section to a generalization of Todorov's equation [Eq. (14)] appropriate for that non-Coulombic interaction.

Finally, note that although the minimal Todorov equation depends on the relativistic scalar $1/r$ (arising from the static Coulomb potential) in a simple way, this simple structure contains all the classical field-theoretic dynamics correct through order $1/c^2$ including the crucial Darwin interaction (due to the spacelike pieces of the vector potential of the classical field theory). The only restriction we have imposed is that the field elimination to a relativistic dynamics of point particles be done using half-advanced, half-retarded Green's functions (here in Lorentz gauge). Expansion of the Green's function in powers of $1/c$ ($1/c^2$) around $t=t'$ then rewrites the fully relativistic dynamics with retardation (and advancement) as an infinite series of "instantaneous" terms. We retain whatever terms contribute to the desired order $[O(1/c^2)]$ in this paper. Then we convert the structure through the methods of this section (Schwinger canonical transformation) into the simple relativistic potential structure of the corresponding Todorov equation. Even though the Todorov system contains no spatial part in the c.m. system (proportional to $\mathbf{p}$) [42] it contains the dynamical effects of the field-theoretic Darwin interaction.

IV. CONCLUSIONS

In this paper, we have investigated the restrictions imposed on the interaction structures of relativistic wave equations by the requirement that their relativistic corrections agree with those of classical field theory. This requirement serves as a double-edged sword. As we have used it in this paper, its application in the semirelativistic approximation serves to single out the minimal Todorov equation as that relativistic wave equation that correctly encapsulates the field-theoretic semirelativistic results [43]. On the other hand, should we enforce agreement between the field-theoretic expansion and the constraint structure to higher orders in $1/c^2$, we will be led to new relativistic wave equations [new quasipotentials in Eq. (5)] that successively improve the minimal Todorov equation. One starting point for such a procedure [44] would be the use of Fokker-Tetrode dynamics provided by the elimination of the classical field from the action, in the role played in this paper by classical field theory.

APPENDIX: THE CONSTRAINT EQUATIONS AND RELATIVISTIC POTENTIAL MODELS

Dirac's constraint mechanics provides a manifestly covariant description of two relativistic particles given by two simultaneous generalized mass-shell constraints:

$$\mathcal{H}_i=p_i^2+m_i^2+\Phi_i(x,p_1,p_2)\approx 0, \quad i=1,2 . \tag{A1}$$

Conservation of the constraints by the dynamics implies the compatibility condition $\{\mathcal{H}_1,\mathcal{H}_2\}\approx 0$, which becomes

$$2p_1\cdot\partial\Phi_2+2p_2\cdot\partial\Phi_1+\{\Phi_1,\Phi_2\}\approx 0 . \tag{A2}$$

This is satisfied by the relativistic counterpart of Newton's third law [19–23]:

$$\Phi_1=\Phi_2=\Phi(x_{\perp},p_1,p_2) . \tag{A3}$$

This condition follows since the only independent nonzero invariants involving $x_{\perp}$ are $x_{\perp}^2$ and $x_{\perp}\cdot p$ ($=x_{\perp}\cdot p_{\perp}$), so that $P\cdot\partial\Phi$ is proportional to $P\cdot x_{\perp}\equiv 0$ while $P\cdot p_{\perp}\equiv 0$.

Equation (A3) forms the difference of constraints into the purely kinematical constraint

$$\mathcal{H}_1-\mathcal{H}_2=2P\cdot p+(\epsilon_2-\epsilon_1)w+m_1^2-m_2^2\approx 0 . \tag{A4}$$

In the c.m. frame $P\cdot p=-wp^0=0$ since, in that frame,

$$p^0=\frac{p_1^0\epsilon_2-\epsilon_1 p_2^0}{w}=\frac{\epsilon_1\epsilon_2-\epsilon_2\epsilon_1}{w}=0 . \tag{A5}$$

Thus, on the constraint hypersurface

$$\epsilon_1-\epsilon_2=\frac{m_1^2-m_2^2}{w}. \quad (A6)$$

Equation (A6) and $\epsilon_1+\epsilon_2=w$ give ϵ_1 and ϵ_2 in terms of w on the constraint hypersurface

$$\epsilon_1=(w^2+m_1^2-m_2^2)/2w, \quad \epsilon_2=(w^2+m_2^2-m_1^2)/2w . \quad (A7)$$

Note that alternatively, we could assume (A6) as a strong equality and conclude that $P\cdot p\approx 0$ is a constraint. Since

$$\mathcal{H}_1=p^2-\epsilon_1^2+m_1^2+\Phi+2\epsilon_1\hat{P}\cdot p , \quad (A8a)$$

$$\mathcal{H}_2=p^2-\epsilon_2^2+m_2^2+\Phi-2\epsilon_2\hat{P}\cdot p , \quad (A8b)$$

the remaining independent combination of the constraints (not involving $\hat{P}\cdot p$) then becomes

$$\mathcal{H}\equiv\frac{\epsilon_2}{w}\mathcal{H}_1+\frac{\epsilon_1}{w}\mathcal{H}_2=p^2-b^2(w)+\Phi$$
$$\approx p_\perp^2-b^2(w)+\Phi(x_\perp^2,p_\perp^2,l^2,w) . \quad (A9)$$

Equation (A9), when evaluated in the c.m. frame, for arbitrary $\Phi(x_\perp)$, takes the form of Eq. (5). In that frame, the invariant $x_\perp^2$ reduces to $\mathbf{r}^2$, while the constraint $P\cdot p\approx 0$ is automatically satisfied.

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[41] J. D. Jackson, *Classical Electrodynamics,* 2nd ed. (Wiley, 1975, New York), pp. 593–595.

[42] In Ref. [36], Crater and Yang have shown that at the next order $(1/c^4)$ the corresponding canonical transformation transforms these higher-order retarded and advanced effects into spatial as well as temporal interactions [with $\mathcal{V}^\mu=(V^0(\mathbf{r}),\mathbf{p}\Gamma(\mathbf{r}))$ in the c.m. system].

[43] We have assumed, of course, that S and $\mathcal{V}$ are identified as the nonrelativistic static form of the potentials and do not already include the relativistic corrections.

[44] Since the Todorov equation contains exact relativistic kinematics and is canonically equivalent to field theory in the semirelativistic approximation through $O(1/c^2)$ (given $\mathcal{V}$ and S from a nonrelativistic match with field theory), it provides a covariant extrapolation of the noncovariant Hamiltonian [Eq. (28)] to *all orders* of $1/c^2$. Higher-order $1/c^2$ "elastic truncations" of classical field theory, arising from the Fokker-Tetrode action of Wheeler-Feynman dynamics, would modify the Todorov equation (see Ref. [36]). Each successively derived Todorov equation would have exact relativistic kinematics accompanied by successively corrected relativistic interactions. Thus, in analogy to Padé approximants (which provide a sequence of extrapolations of finite truncations of a Taylor series to all orders), the successive Todorov equations would provide a sequence of "Todorov approximants" of the field-theoretic interactions to all orders.