Kinetic approach to the initial-value problem in ϕ^4 field theory

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A time-dependent projection technique is used to develop kinetic equations for simple observables in the context of ϕ^4 field theory. A mean-field expansion can be written for these equations which are numerically tractable in the few lower orders. The procedure is applied to the case of the spatially uniform system in $1+1$ dimensions, including numerical solutions.

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I. INTRODUCTION

Interest in the initial-value problem for quantum-field-theoretical models over the last decade stems mainly from two different areas of physics: on the one hand, the inflationary scenario of the early Universe involves the control of the time evolution of a driving scalar field [1]; on the other hand, properties of hadronic matter manifest themselves through transient phenomena in globally off-equilibrium situations in high-energy collisions [2]. In either of the two contexts nonperturbative methods must be employed, and any sufficiently realistic microscopic model will involve a set of mutually interacting quantum fields, which can be thought of as interacting subsystems forming a larger, possibly autonomous system. The quantum state of each of the different subsystems can be described in terms of a density operator which will in general evolve nonunitarily on account of correlation effects involving different subsystems [3,4]. The nonunitary effects will manifest themselves through the dynamical evolution of the eigenvalues of the subsystem density matrices, so that individual subsystems behave in general in a nonisoentropic fashion [3].

The overwhelming complexity of such a picture is considerably reduced whenever one is able to find physical grounds to motivate the mean-field-like approximation which consists in assuming isoentropic subsystem evolution under effective, time-dependent Hamiltonian operators for each subsystem [4]. In this case, in fact, the dynamics of the subsystem density matrix can be formulated in terms of a Liouville–von Neuman equation governed by an effective Hamiltonian, e.g., from the point of view of the functional field-theoretical Schrödinger picture, as proposed by Jackiw [4]. Since, however, the resulting problem involves in general nonlinear Hamiltonians, it still cannot be handled without further approximation. In the field-theoretical context, this has been implemented through the use of a Gaussian ansatz for the subsystem density functional in the framework of a time-dependent variational principle supplying the appropriate dynamical information.

It is not difficult to see that this last approximation amounts to a second mean-field approximation, now at the microscopic level of the single-field, nonlinear, isoentropic effective dynamics. Actually, the Gaussian ansatz, having the form of an exponential of a quadratic form in the field operators, implies that many-point correlation functions can in fact be factored in terms of two-point functions, as is well known in the context of the derivation of the Hartree-Fock approximation to the nonrelativistic many-body problem [5]. This factorization has been in fact assumed by Chang [6] to implement the Gaussian approximation in the context of φ^4 theory. The dynamics of the reduced two-point density becomes then itself isoentropic, as a result of irreducible higher-order correlation effects being neglected.

The point to which we address ourselves in this paper is a reevaluation and improvement of this second mean-field approximation. In order to avoid the complications of the many-fields problem we consider the simplest case of a single, self-interacting real scalar field. It will be taken moreover as a closed quantum system, whose dynamics is governed by a time-independent Hamiltonian (see Sec. IV below). In order to reach our goal, we follow a time-dependent projection approach developed earlier in the context of nuclear many-body dynamics [7,8]. This approach allows for the formulation of a mean-field *expansion* for the dynamics of the two-point correlation function from which one recovers the results of the Gaussian mean-field approximations in lowest order. In particular, we recover the well-known Gaussian approximation to the effective potential from constrained static solutions of the lowest-order equations. Beyond this, we are able to explicitly include and evaluate higher dynamical correlation effects through suitable memory integrals added to the mean-field dynamical equations. The resulting dynamical equations acquire then the structure of kinetic equations, with the memory integrals performing as collision terms which eliminate the isoentropic mean-field constraint.

Using our projection approach we derive microscopically exact formal expressions for the collision integrals. An energy-conserving (for closed systems) systematic mean-field expansion scheme for these integrals given by Buck, Feldmeier, and Nemes [9] is adopted, with a few useful modifications, for the purpose of producing a numerically tractable approximation of the formal result. We give numerical solutions of the approximate kinetic

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equations in $0+1$ dimensions (the anharmonic oscillator) and for spatially uniform field configurations in $1+1$ dimensions, adopting nonequilibrium Gaussian states as initial conditions. In the case of $0+1$ dimensions exact numerical results are also given. We find that the elimination of the isoentropic constraint associated with the usual Gaussian approximations has an important effect on the dynamics of the Gaussian parameters which substantially improves agreement with the exact solutions in $0+1$ dimensions. In the case of $1+1$ dimensions the effect of the collision integrals is similar to that obtained in $0+1$ dimensions both qualitatively and quantitatively.

In Sec. II we describe the groundwork for the implementation of our projection technique. This technique and the approximation scheme for the collision integrals are described in Sec. III. In Sec. IV we give explicit expressions for the dynamical equations in the mean-field (isoentropic) approximation. The adopted renormalization scheme and the static solutions of these equations, which lead to the well-known [6,10] Gaussian effective potential, are also discussed there. In Sec. V the full kinetic equations are given explicitly for the case of spatially uniform field configurations. Numerical solutions of the dynamical equations are given and discussed in Sec. VI, and points of a more technical nature concerning the construction of projection operators and the adopted numerical procedure are discussed in the Appendix.

II. KINETICS OF A SELF-INTERACTING QUANTUM FIELD

In this section, we shall describe a formal treatment of the kinetics of a self-interacting quantum field. Although the procedure is quite general, we will adopt the simplest context of a single scalar field in $1+1$ dimensions and assume spatial uniformity. This will illustrate all the relevant points of the approach and cut down inessential technical complications. Features of more general contexts are discussed in Ref. [8] and briefly outlined in Sec. VI.

The general idea of our approach [8] is to focus on the time development of observables which involve the field either linearly [i.e., $\phi(x),\pi(x)$] or in bilinear forms such as $\phi(x)\phi(x')$, $\phi(x)\pi(x')$, etc. These are in fact the observables which are kept under direct control when one works variationally using a Gaussian functional ansatz. In order to keep as close as possible to the formulation appropriate for the many-body problem, we work instead with expressions which are linear or bilinear in the creation and annihilation parts of the fields in momentum space, defined in terms of an expansion mass parameter μ. We begin therefore by expanding the Heisenberg field operators $\phi(x)$ and $\pi(x)$ as

$$\begin{aligned}\phi(x)&=\sum_k [v_k(x)\gamma_k(t)+v_k^*(x)\gamma_k^\dagger(t)]\ ,\\ \pi(x)&=-i\sum_k k_0[v_k(x)\gamma_k(t)-v_k^*(x)\gamma_k^\dagger(t)]\ ,\end{aligned}\tag{2.1}$$

so that γ_k, $\gamma_k^\dagger$ are annihilation and creation parts satisfying boson commutation relations at equal times:

$$[\gamma_k(t)\ ,\gamma_{k'}^\dagger(t)]=\delta_{kk'}\ .\tag{2.2}$$

The $v_k(x)$ are the periodic boundary condition plane waves

$$v_k(x)=e^{ik\cdot k}/(2Lk_0)^{1/2}\ ,\tag{2.3}$$

L being the length of the periodicity box. Here x is the spatial coordinate only and

$$k_0^2=k^2+\mu^2\ .\tag{2.4}$$

The parameter μ will be fixed later in a convenient way.

The state of the system (assumed spatially uniform) is given in terms of a density matrix F in the Heisenberg picture. F is therefore Hermitian, time independent, and has a unit trace. The corresponding mean value of the field operators is then given in terms of the quantities

$$\Gamma_k(t)=\mathrm{Tr}\gamma_k(t)F\ ,\tag{2.5}$$

and their complex conjugates. Using them, we can furthermore define the shifted boson operators

$$\beta_k(t)=\gamma_k(t)-\Gamma_k(t)\ ,\tag{2.6}$$

which have vanishing F-expectation values.

Next, in order to handle bilinear expressions in the field operators we consider expectation values of pairs of shifted bosons β_k, $\beta_k^\dagger$:

$$R_{kk'}(t)=\mathrm{Tr}\beta_{k'}^\dagger(t)\beta_k(t)F\ \xrightarrow{\text{uniform system}}\ p_k(t)\delta_{kk'}\ ,\tag{2.7a}$$

$$\Pi_{kk'}(t)=\mathrm{Tr}\beta_{k'}(t)\beta_k(t)F\ \xrightarrow{\text{uniform system}}\ r_k(t)\delta_{k,-k'}\ .\tag{2.7b}$$

The Hermitian matrix **R** and the symmetric matrix **Π** are in fact the one-boson density matrix and the pairing density for the shifted bosons. The corresponding matrices for the γ bosons are of course easily expressed in terms of **R**, **Π**, and of the $\Gamma_k(t)$.

An important point is that the plane waves (2.3) diagonalize the uniform one-boson density $\mathbb{R}$, Eq. (2.7a). They correspond, in other words, to what is known as the natural orbitals of this one-boson density. The form of the pairing density Π in the uniform case [Eq. (2.7b)], on the other hand, follows from a zero-momentum condition. This latter object can actually be eliminated by redefining the shifted creation and annihilation parts β_k, $\beta_k^\dagger$ through a Bogoliubov canonical transformation allowing for linear combinations of β_k and $\beta_{-k}^\dagger$ [11]. As will be shown explicitly below, this amounts to a momentum-dependent redefinition of the expansion parameter μ. To implement this in a compact way we first set up the extended density matrix (for a uniform system)

$$\mathcal{R}_k(t)=\begin{bmatrix}R_k(t) & \Pi_k(t)\\ \Pi_{-k}^*(t) & 1+R_{-k}(t)\end{bmatrix}\ \text{(uniform system)}\ ,\tag{2.8}$$

and obtain from it generalized natural orbitals which incorporate the pairing effects by solving the eigenvalue problem

$$\mathbf{G}\mathcal{R}_k\mathbf{X}_k=\mathbf{X}_k\mathbf{G}\mathbf{N}_k \ , \tag{2.9}$$

where

$$\mathbf{G}=\begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix}, \quad \mathbf{X}_k=\begin{bmatrix}x_k & y_k^*\\ y_k & x_k^*\end{bmatrix}, \quad \mathbf{N}_k=\begin{bmatrix}\nu_k & 0\\ 0 & 1+\nu_k\end{bmatrix}. \tag{2.10}$$

The eigenvalues ν_k can be interpreted as shifted boson occupation numbers for the paired natural orbitals described by $\mathbf{X}_k$. From reflection symmetry one must have $\nu_k=\nu_{-k}$.

Since (2.9) is a non-Hermitian eigenvalue problem, it is useful to consider also the adjoint equation

$$\mathcal{R}_k\mathbf{G}\tilde{\mathbf{X}}_k=\tilde{\mathbf{X}}_k\mathbf{G}\mathbf{N}_k \tag{2.11}$$

from which one finds that

$$\tilde{\mathbf{X}}_k=\mathbf{G}\mathbf{X}_k \ . \tag{2.12}$$

The adjoint vectors $\tilde{\mathbf{X}}_k$ satisfy biorthogonality relations with the $\mathbf{X}_k$ which allow one to introduce the normalization condition

$$\tilde{\mathbf{X}}_k^\dagger\mathbf{X}_k=\mathbf{X}_k^\dagger\mathbf{G}\mathbf{X}_k=\mathbf{G} \ , \tag{2.13}$$

and the completeness relation

$$\mathbf{X}_k\mathbf{G}\mathbf{X}_k^\dagger=\mathbf{G} \ . \tag{2.14}$$

Furthermore, one can use the paired natural orbitals to construct new boson operators η_k, $\eta_k^\dagger$ and shift amplitudes A_k, A_k^* as

$$\begin{bmatrix}\eta_k\\ \eta_{-k}^\dagger\end{bmatrix}=\mathbf{X}_k^\dagger\begin{bmatrix}\beta_k\\ \beta_{-k}^\dagger\end{bmatrix}, \quad \begin{bmatrix}A_k\\ A_{-k}^*\end{bmatrix}=\mathbf{X}_k^\dagger\begin{bmatrix}\Gamma_k\\ \Gamma_{-k}^*\end{bmatrix} \tag{2.15}$$

which can be inverted with the help of Eq. (2.14).

With the help of Eqs. (2.5), (2.6), and (2.15) (actually of its inverse) it is an easy task to express $\phi(x)$, $\pi(x)$ [Eqs. (2.1)] in terms of η_k, $\eta_k^\dagger$, and A_k. In doing so one finds that the plane waves $v_k(x)$ are modified by what amounts in general to a complex, momentum-dependent redefining of μ involving the Bogoliubov parameters x_k, y_k. The complex character of these parameters is actually crucial in dynamical situations, where the imaginary parts will allow the description of time-odd (i.e., velocitylike) properties.

What we have achieved so far amounts to an expansion of the fields $\phi(x)$, $\pi(x)$ such that the mean values in F of linear and bilinear observables are parametrized in terms of the x_k, y_k, A_k and of the occupation numbers $\nu_k=\mathrm{Tr}\,\eta_k^\dagger\eta_kF$. The last two objects are mean values of operators defined in terms of a natural orbital basis specified by the Bogoliubov parameters x_k, y_k. In general, all these quantities are time dependent under the Heisenberg dynamics of the field operators, and we now proceed to write the corresponding equations of motion. For the $\Gamma_k(t)$ one finds immediately

$$i\dot{\Gamma}_k=\mathrm{Tr}[\gamma_k,H]F=x_k\mathrm{Tr}[\eta_k,H]F-y_k^*\mathrm{Tr}[\eta_{-k}^*,H]F \ , \tag{2.16}$$

H being the field Hamiltonian. As for the remaining quantities, we first rewrite the eigenvalue equation (2.9), using Eq. (2.13), as

$$\mathbf{X}_k^\dagger\mathcal{R}_k\mathbf{X}_k=\mathbf{N}_k \tag{2.17}$$

from which it follows that

$$\mathbf{X}_k^\dagger\dot{\mathcal{R}}_k\mathbf{X}_k=\dot{\mathbf{N}}_k-\dot{\mathbf{X}}_k^\dagger\mathcal{R}_k\mathbf{X}_k-\mathbf{X}_k^\dagger\mathcal{R}_k\dot{\mathbf{X}}_k \ , \tag{2.18}$$

we now evaluate the left-hand side of this equation using the Heisenberg equation of motion to obtain

$$i\mathbf{X}_k^\dagger\dot{\mathcal{R}}_k\mathbf{X}_k=\begin{bmatrix}\mathrm{Tr}[\eta_k^\dagger\eta_k,H]F & \mathrm{Tr}[\eta_k\eta_{-k},H]F\\ \mathrm{Tr}[\eta_{-k}^\dagger\eta_k^\dagger,H]F & \mathrm{Tr}[\eta_k^\dagger\eta_k,H]F\end{bmatrix}. \tag{2.19}$$

The right-hand side of Eq. (2.18) can also be evaluated explicitly using Eqs. (2.9) and (2.10). Equating the result to (2.19) yields

$$i\dot{\nu}_k=\mathrm{Tr}[\eta_k^\dagger\eta_k,H]F \tag{2.20}$$

and

$$i(\dot{x}_y y_k-x_k\dot{y}_k)(1+2\nu_k)=\mathrm{Tr}[\eta_{-k}^\dagger\eta_k^\dagger,H]F \ . \tag{2.21}$$

Equations (2.16), (2.20), and (2.21), together with the normalization condition (2.13), determine the time rate of change of the relevant quantities in terms of expectation values of appropriate commutators. They are, however, clearly not closed equations when the Hamiltonian H involves self-interacting fields, since in this case the time derivatives of the selected quantities are given in the terms of traces which are in general not expressible in terms of the quantities themselves. One handy way to obtain a closed approximation to the equation of motion is by means of a Hartree-Fock factorization of traces involving more than two field operators. This is what we refer to as the mean-field approximation to the equation of motion. It can be implemented in a particularly compact and convenient way by replacing the full density F by a truncated ansatz F_0 having the form of an exponential of a bilinear, Hermitian expression in the fields normalized to unit trace [5]. In the momentum basis, Eq. (2.3), it reads as

$$F_0=\frac{\exp\left[\sum_{k_1k_2}(N_{k_1k_2}\beta_{k_1}^\dagger\beta_{k_2}+P_{k_1k_2}\beta_{k_1}^\dagger\beta_{k_2}^\dagger+P_{k_1k_2}^*\beta_{k_1}\beta_{k_2})\right]}{\mathrm{Tr}\left[\exp\left[\sum_{k_1k_2}(N_{k_1k_2}\beta_{k_1}^\dagger\beta_{k_2}+P_{k_1k_2}\beta_{k_1}^\dagger\beta_{k_2}^\dagger+P_{k_1k_2}^*\beta_{k_1}\beta_{k_2})\right]\right]} \ . \tag{2.22}$$

This amounts to adopting a Gaussian functional ansatz and is usually done in a Schrödinger picture in which the field operators are time independent, but conversion to a unitarily equivalent Heisenberg picture is trivial. The parameters in Eq. (2.22) are fixed by requiring that mean values in F_0 of expressions that are linear or bilinear in the fields reproduce the corresponding F averages [see Eq. (3.2) below]. F_0 is therefore in general a time-dependent object, which acquires a particularly simple form when expressed in terms of the boson operators transformed to the natural orbital basis.

An important fact about Eq. (2.22) is that this density can be written as a time-dependent projection of the full density F. In the following section we use this fact to derive formally exact, closed equations of motion which are equivalent to Eqs. (2.16), (2.20), and (2.21), on the basis of which we develop a numerically tractable, closed approximation for correcting the mean-field approximation.

III. PROJECTION TECHNIQUE AND APPROXIMATION SCHEME

In order to develop our treatment of the equations of motion (2.16), (2.20), and (2.21) we begin by decomposing the full density F as

$$F=F_0(t)+F'(t)\ , \tag{3.1}$$

where $F_0(t)$ is chosen as the exponential of a bilinear expression in the fields, Eq. (2.22). It is convenient to express this part of the density of terms of the operators η_k as

$$F_0=\prod_k \frac{1}{1+\nu_k(t)}\left[\frac{\nu_k(t)}{1+\nu_k(t)}\right]^{\eta_k^\dagger(t)\eta_k(t)}\ , \tag{3.2}$$

which has unit trace and is such that

$$\mathrm{Tr}\gamma_k(t)F_0(t)=\Gamma_k(t)=\mathrm{Tr}\gamma_k(t)F\ ,$$
$$\mathrm{Tr}\eta_k^\dagger(t)\eta_{k'}(t)F_0(t)=\delta_{kk'}\nu_k(t)=\mathrm{Tr}\eta_k^\dagger(t)\eta_{k'}(t)F\ ,$$
$$\mathrm{Tr}\eta_k(t)\eta_{k'}(t)F_0(t)=0=\mathrm{Tr}\eta_k(t)\eta_{k'}(t)F\ .$$

It follows that $F'(t)$, defined by Eq. (3.1), is a traceless, pure correlation density. As already remarked, a crucial point is to observe that $F_0(t)$ can be written as a time-dependent projection of F, i.e.,

$$F_0=\mathbb{P}(t)F\ ,\quad \mathbb{P}(t)^2=\mathbb{P}(t)\ . \tag{3.3}$$

It is important to keep in mind that $\mathbb{P}$ is an operator acting on a linear space of densities, sometimes called super-space. Such operators are correspondingly called super- operators. The scalar product for any two vectors of this space is defined as

$$(X,Y)=\mathrm{Tr}(X^\dagger Y)\ . \tag{3.4}$$

In order to construct the projector $\mathbb{P}(t)$ we require that, in addition to Eqs. (3.3), it satisfies

$$i\dot{F}_0(t)=[\mathbb{P}(t),\mathbb{L}]F=[F_0(t),H]+\mathbb{P}(t)[H,F]\ , \tag{3.5}$$

where $\mathbb{L}$ is the superoperator defined as

$$\mathbb{L}=[H,\cdot]\ , \tag{3.6}$$

H being the Hamiltonian of the field. Equation (3.5) is the Heisenberg picture counterpart of the equation $[\partial_t\mathbb{P}(t)]F=0$ which has been used to define $\mathbb{P}(t)$ in the Schrödinger picture [9]. It is possible to prove that conditions (3.3) and (3.4) make $\mathbb{P}(t)$ unique.

The explicit construction of $\mathbb{P}(t)$ is a lengthy but straightforward algebraic exercise, the relevant steps of which are given in Appendix A. What one obtains is

$$\mathbb{P}(t)\cdot=\left\{\left[1-\sum_k\frac{\eta_k^\dagger\eta_k-\nu_k}{1+\nu_k}\right]\mathrm{Tr}(\cdot)+\sum_{k_1k_2}\frac{\eta_{k_1}^\dagger\eta_{k_2}-\nu_{k_2}\delta_{k_1k_2}}{\nu_{k_2}(1+\nu_{k_2})}\mathrm{Tr}(\eta_{k_2}^\dagger\eta_{k_1}\cdot)+\sum_k\left[\frac{\eta_k}{\nu_k}\mathrm{Tr}(\eta_k^\dagger\cdot)+\frac{\eta_k^\dagger}{1+\nu_k}\mathrm{Tr}(\eta_k\cdot)\right]\right.$$
$$\left.+\sum_k\left[\frac{\eta_k\eta_{-k}}{2\nu_k\nu_{-k}}\mathrm{Tr}(\eta_{-k}^\dagger\eta_k^\dagger\cdot)+\frac{\eta_{-k}^\dagger\eta_k^\dagger}{2(1+\nu_{-k})(1+\nu_k)}\mathrm{Tr}(\eta_k\eta_{-k}\cdot)\right]\right\}F_0(t)\ . \tag{3.7}$$

Using the scalar product (3.4), one can also obtain $\mathbb{P}^\dagger(t)$ which does not coincide with Eq. (3.7), i.e., $\mathbb{P}(t)$ is not an orthogonal projection (see Appendix A).

The next step is to obtain a differential equation of $F'(t)$, which follows immediately from Eqs. (3.1) and (3.5). It reads as

$$\left[i\frac{d}{dt}-\mathbb{P}(t)\mathbb{L}\right]F'(t)=\mathbb{Q}(t)\mathbb{L}F_0(t)\ , \tag{3.8}$$

where we introduced the complementary projector $\mathbb{Q}(t)=1-\mathbb{P}(t)$. This equation has the formal solution

$$F'(t)=\mathbb{G}(t,0)F'(0)-i\int dt'\mathbb{G}(t,t')\mathbb{Q}(t')\mathbb{L}F_0(t')\ . \tag{3.9}$$

The first term accounts for initial correlations. In the second term $\mathbb{G}(t,t')$ is the time-ordered Green's function

$$\mathbb{G}(t,t')=T\exp\left[i\int_{t'}^{t}d\tau\,\mathbb{P}(\tau)\mathbb{L}\right]\ . \tag{3.10}$$

We see thus that $F'(t)$, and therefore also F [see Eq. (3.1)], can be formally expressed in terms of $F_0(t')$ (for $t'\le t$) and of initial correlations $F'(0)$. This allows us also to express the dynamical equations (2.16), (2.20), and (2.21) as traces over functionals of $F_0(t')$ and of the initial correlations. Since, on the other hand, the reduced density $F_0(t')$ is expressed in terms of the relevant variables alone, we see that the resulting equations are now essentially closed equations. Note, however, that the complicated time dependence of the field operators is explicitly probed through the memory effects present in the expression (3.9) for $F'(t)$. Approximations are therefore needed for the actual evaluation of this object.

A systematic expansion scheme for the memory effects has been discussed in Ref. [9] in the Schrödinger picture. An important feature of this scheme is that the mean energy is conserved to all orders, i.e.,

$$\frac{\partial}{\partial t}\langle H\rangle_n=0\ , \tag{3.11}$$

where

$$\langle H\rangle_n=\mathrm{Tr}HF_0^{(n)}(t)+\mathrm{Tr}HF'^{(n)}(t)\ ,$$

$F_0^{(n)}$ and $F'^{(n)}$ being the approximation of order n to $F_0(t)$ and $F'(t)$, respectively. Here we implement a modified version of the lowest-order approximation given in Ref. [9]. It consists in approximating the actual time evolution of the field operators, when evaluating memory effects, by the simpler mean-field evolution given by

$$\begin{aligned} i\dot{\eta}_k=&[\eta_k,H_0(t)]-i\dot{A}_k+i(\dot{x}_k^*x_k-\dot{y}_k^*y_k)\eta_k\\ &-i(\dot{x}_k^*y_k^*-\dot{x}_k^*y_k^*)\eta_k^\dagger\ . \end{aligned}\tag{3.12}$$

The last three terms account for the (explicit) time dependence of the $\eta_k(t)$ related to the shift amplitudes $A_k(t)$ and to the pairing effects. $H_0(t)$ is taken as the effective mean-field Hamiltonian

$$\begin{aligned} H_0=&\mathbb{P}^\dagger H+\sum_k\eta_k^\dagger\mathrm{Tr}[\eta_k\ ,H]F'(t)-\sum_k\eta_k\mathrm{Tr}[\eta_k^\dagger\ ,H]F'(t)\\ &+\sum_k\frac{\eta_{-k}^\dagger\eta_k^\dagger}{2(1+2\nu_k)}\mathrm{Tr}[\eta_k\eta_{-k}\ ,H]F'(t)\\ &-\sum_k\frac{\eta_{-k}\eta_k}{2(1+2\nu_k)}\mathrm{Tr}[\eta_k^\dagger\eta_{-k}^\dagger\ ,H]F'(t)\ . \end{aligned}\tag{3.13}$$

The lowest approximation according to Ref. [9] corresponds to taking just the first term in this expression. The remaining terms, included here, represent correlation contributions to the effective mean field. Consistent with this approximation, the Green's function (3.10) is also replaced by

$$\mathbb{G}_0(t,t')=T\exp\left[i\int_{t'}^t d\tau\,\mathbb{P}(\tau)\mathbb{L}_0(\tau)\right]\ , \tag{3.14}$$

where

$$\mathbb{L}_0\cdot=[H_0,\cdot]\ , \tag{3.15}$$

so that the correlation density is written as

$$\begin{aligned} F'(t)\cong&\mathbb{G}_0(t,t')F'(0)-i\int_0^t dt'\,\mathbb{G}_0(t,t')\mathbb{Q}(t')\mathbb{L}F_0(t')\\ =&\mathbb{G}_0(t,t')F'(0)-i\int_0^t dt'\,\mathbb{Q}(t')\mathbb{L}F_0(t')\ , \end{aligned}\tag{3.16}$$

since it is easy to see that Eqs. (3.12) and (3.14) imply that $\mathbb{G}_0$ acts as the unit operator in the memory integral. According to the approximation scheme just described the basic dynamical equations to be solved are Eqs. (2.16), (2.20), and (2.21), where F is expressed in terms of Eqs. (3.2) and (3.16). Furthermore, for the purpose of evaluating Eq. (3.16) the field operators are time evolved according to Eq. (3.12). The resulting scheme can be interpreted as follows. The dynamical evolution of the field is split into a pure mean-field part, related to the contributions to the dynamical equations involving the projected density $F_0(t)$, plus correlation contributions, approximated by the contributions involving the adopted form for $F'(t)$. These are nonunitary, in the sense that they change the coherence properties of $F_0(t)$ through the time evolution of the occupation numbers $\nu_k(t)$ [see Eq. (2.20)]. In fact, replacing F by just $F_0(t)$ in this equation gives $\dot{\nu}_k(t)=0$. Consequently, the entropy function associated with $F_0(t)$ will change in time as a result of the correlation contributions, which therefore perform as collision terms from the point of view of the one-boson density. Moreover, the correlation contributions will also modify the pure mean-field evolution in Eqs. (2.16) and (2.21). The adopted approximation amounts to restricting correlation effects to second order in H [as can be seen by substituting Eq. (3.15) in the dynamical equations] while taking full account of the effective mean field [see Eqs. (3.12) and (3.13)].

IV. MEAN-FIELD APPROXIMATION: RENORMALIZATION AND EFFECTIVE POTENTIAL

We now discuss the actual evaluation of the general expressions obtained in the preceding sections for the Hamiltonian

$$H=\int dx\,\mathcal{H}\ , \tag{4.1}$$

$$\mathcal{H}=\frac{\pi^2}{2}+\frac{1}{2}(\partial_x\phi)^2+\frac{m^2}{2}\phi^2+\frac{g}{4!}\phi^4+\frac{\delta m^2}{2}\phi^2+\lambda\phi\ . \tag{4.2}$$

In the section we consider only the lowest (mean-field) approximation, which amounts to assuming $F'(t)\equiv 0$. Collisional correlations will be discussed in Sec. V.

In Eq. (4.2), m stands for the renormalized mass, and a prescription for the mass counterterm δm^2 will be given below. The last term is an external linear coupling which will be used to allow for constraining the expectation value of ϕ at equilibrium in an evaluation of the effective potential.

Consistent with the orthogonality and completeness relations (2.13) and (2.14), we parametrize the elements of the transformation matrix $\mathbf{X}_k$ [see Eq. (2.15)] as

$$x_k=\cosh\sigma_k+\tfrac{1}{2}i\tau_k\ , \tag{4.3}$$

$$y_k=\sinh\sigma_k+\tfrac{1}{2}i\tau_k\ , \tag{4.4}$$

with σ_k and τ_k real. It is also convenient to associate σ_k with a dynamic mass parameter $\mu_k(t)$ through

$$[k^2+\mu_k^2(t)]^{1/2}=e^{2\sigma_k(t)}(k^2+\mu^2)^{1/2}\ , \tag{4.5}$$

as shown below in Eq. (4.8), τ_k is related to $\dot{\mu}_k(t)$. This

$\dot{\mu}_k(t)$ can be seen as an effective mass incorporating momentum-dependent mean-field interaction of the uniform system.

The mean-field approximation to the dynamical equations (2.16), (2.20), and (2.21) amounts to replacing F by just $F_0(t)$. Introducing the ingredients described above, one obtains, assuming uniform condensates only, i.e., $\Gamma_k(t)=\Gamma_0(t)\delta_{k0}$,

$$\partial_t^2\langle\phi\rangle=-\left[\lambda+m^2\langle\phi\rangle+\frac{g}{3!}\langle\phi\rangle^3\right]-\frac{g}{4L}\langle\phi\rangle\sum_k\frac{1}{(k^2+\mu_k^2)^{1/2}}-\delta m^2\langle\phi\rangle\ , \tag{4.6}$$

$$\dot{\nu}_k=0\ , \tag{4.7}$$

$$\dot{\mu}_k=-2(k^2+\mu^2)^{1/4}\frac{(k^2+\mu_k^2)^{5/4}}{\mu_k}\tau_k\ , \tag{4.8}$$

$$\left[\frac{k^2+\mu^2}{k^2+\mu_k^2}\right]^{1/4}\left[\frac{1}{2}\frac{\mu_k\dot{\mu}_k}{k^2+\mu_k^2}\tau_k+\dot{\tau}_k\right]=(k^2+\mu_k^2)^{1/2}-(k^2+\mu^2)^{1/2}\tau_k^2-\frac{k^2+m^2}{(k^2+\mu_k^2)^{1/2}}-\frac{g}{2}\frac{\langle\phi\rangle^2}{(k^2+\mu_k^2)^{1/2}}$$
$$-\frac{1}{(k^2+\mu_k^2)^{1/2}}\left[\frac{g}{4L}\sum_{k'}\frac{1}{(k'^2+\mu_{k'}^2)^{1/2}}+\delta m^2\right]\ . \tag{4.9}$$

Equation (4.7), in particular, shows that the reduced occupation numbers ν_k are constant in the mean-field approximation.

It is interesting to look at the static solutions $\langle\phi\rangle=\phi_0$ of the mean-field equations. They are given as the solutions of

$$\lambda+m^2\phi_0+\frac{g}{3!}\phi_0^3+\frac{g\phi_0}{4L}\sum_{k'}\frac{1}{(k'^2+\mu_{k'}^2)^{1/2}}+\delta m^2\phi_0=0 \tag{4.10}$$

and

$$\mu_k^2=m^2+\frac{g}{2}\phi_0^2+\frac{g}{4L}\sum_{k'}\frac{1}{(k'^2+\mu_{k'}^2)^{1/2}}+\delta m^2\ . \tag{4.11}$$

Equation (4.11) shows that μ_k^2 is in fact independent of k in the static case. The logarithmic divergences of the sums in Eqs. (4.10) and (4.11) are controlled by adjusting the mass counterterm as

$$\delta m^2=-\frac{g}{4L}\sum_k\frac{1}{(k^2-m^2)^{1/2}}\ . \tag{4.12}$$

The mean-field energy density, on the other hand, is easily evaluated as

$$\left\langle\frac{E}{L}\right\rangle=\frac{1}{L}\operatorname{Tr}HF_0=\frac{\langle\pi\rangle^2}{2}+\lambda\langle\phi\rangle+\frac{m^2}{2}\langle\phi\rangle^2+\frac{g}{4!}\langle\phi\rangle^4+\frac{g\langle\phi\rangle^2}{8L}\sum_k\left[\frac{1}{(k^2+\mu_k^2)^{1/2}}-\frac{1}{(k^2+m^2)^{1/2}}\right]$$
$$+\frac{g}{32L^2}\left[\sum_k\left[\frac{1}{(k^2+\mu_k^2)^{1/2}}-\frac{1}{(k^2+m^2)^{1/2}}\right]\right]^2-\frac{g}{32L^2}\left[\sum_k\frac{1}{(k^2+m^2)^{1/2}}\right]^2$$
$$+\frac{1}{2L}\sum_k\left[(k^2+\mu_k^2)^{1/2}-\frac{m^2-\mu_k^2}{2(k^2+\mu_k^2)^{1/2}}\right]+\frac{1}{4L}\sum_k(k^2+\mu^2)^{1/2}\tau_k^2\ , \tag{4.13}$$

which is rendered finite after subtracting a (divergent) vacuum energy (cf. Ref. [10]). It is straightforward to check that this energy density is conserved under the mean-field equations of motion (4.6)–(4.9).

The mean-field effective potential $V_{\text{eff}}(\phi_0)$ is now easily obtained from Eq. (4.13) evaluated in the static case. As shown by Eq. (4.10), the equilibrium value $\langle\phi\rangle$ can be adjusted through the external coupling parameter λ (which acts as a Lagrange multiplier) so that

$$V_{\text{eff}}(\phi_0)=\left\langle\frac{E}{L}\right\rangle_{\langle\phi\rangle=\phi_0}-\lambda\phi_0-\text{ vacuum energy density}\ , \tag{4.14}$$

yielding, in the continuum limit ($L\to\infty$),

$$V_{\text{eff}}(\phi_0)=\frac{m^2}{2}\phi_0^2+\frac{g}{4!}\phi_0^4+\frac{g}{16\pi}\phi_0^2\ln\frac{m^2}{\mu^2}+\frac{g}{128\pi}\left[\ln\frac{m^2}{\mu^2}\right]^2+\frac{1}{8\pi}(\mu^2-m^2)+\frac{m^2}{8\pi}\ln\frac{m^2}{\mu^2}\ , \tag{4.15}$$

which reproduces the well-known effective potential obtained in the Gaussian approximation [10].

V. COLLISIONAL DYNAMICS FOR HOMOGENEOUS SYSTEMS

In order to calculate the collision terms of Eqs. (2.16), (2.20), and (2.21), one must evaluate traces of the type

$$\mathrm{Tr}[\hat{O}(t),H]\int_0^t dt'\,\mathbb{Q}(t')[H,F_0(t')]\,, \tag{5.1}$$

where $\hat{O}(t)$ can be η, $\eta^\dagger\eta$, or $\eta\eta$. The density $\mathbb{Q}(t')[H,F_0(t')]$ can be evaluated in a straightforward way using Eqs. (3.7), (4.2), and (3.2). One obtains

$$\begin{aligned}
\mathbb{Q}(t')[H,F_0(t')]=&\frac{g}{96L}\sum_{k_1k_2k_3k_4}\frac{\exp[-(\sigma_{k_1}+\sigma_{k_2}+\sigma_{k_3}+\sigma_{k_4})_{t'}]}{(k_{01}k_{02}k_{03}k_{04})^{1/2}}\delta_{k_1+k_2+k_3+k_4,0}\\
&\times\left[\frac{\eta^\dagger_{k_1}\eta^\dagger_{k_2}\eta^\dagger_{k_3}\eta^\dagger_{k_4}}{(1+\nu_{k_1}(1+\nu_{k_2})(1+\nu_{k_3})(1+\nu_{k_4})}-\frac{\eta_{k_1}\eta_{k_2}\eta_{k_3}\eta_{k_4}}{\nu_{k_1}\nu_{k_2}\nu_{k_3}\nu_{k_4}}\right]_{t'}\\
&\times\left[1+\sum_i^4\nu_{k_i}+\sum_{i<j}^4\nu_{k_i}\nu_{k_j}+\sum_{i<j<l}^4\nu_{k_i}\nu_{k_j}\nu_{k_l}\right]_{t'}F_0(t')\\
&+\frac{g}{24L}\sum_{k_1k_2k_3k_4}\frac{\exp[-(\sigma_{k_1}+\sigma_{k_2}+\sigma_{k_3}+\sigma_{k_4})_{t'}]}{(k_{01}k_{02}k_{03}k_{04})^{1/2}}\delta_{k_1+k_2+k_3-k_4,0}\\
&\times\left[\frac{\eta^\dagger_{k_1}\eta^\dagger_{k_2}\eta^\dagger_{k_3}\eta_{k_4}}{(1+\nu_{k_1})(1+\nu_{k_2})(1+\nu_{k_3})\nu_{k_4}}-\frac{\eta^\dagger_{k_4}\eta_{k_1}\eta_{k_2}\eta_{k_3}}{\nu_{k_1}\nu_{k_2}\nu_{k_3}(1+\nu_{k_4})}\right]_{t'}\\
&\times\left[\nu_{k_4}-\nu_{k_1}\nu_{k_2}\nu_{k_3}+\nu_{k_4}\left[\sum_i^3\nu_{k_i}\right]+\nu_{k_4}\left[\sum_{i<j}^3\nu_{k_i}\nu_{k_j}\right]\right]_{t'}F_0(t')\\
&+\frac{g}{16L}\sum_{k_1k_2k_3k_4}\frac{\exp[-(\sigma_{k_1}+\sigma_{k_2}+\sigma_{k_3}+\sigma_{k_4})_{t'}]}{(k_{01}k_{02}k_{03}k_{04})^{1/2}}\delta_{k_1+k_2-k_3-k_{4,0}}\\
&\times\left[1-\frac{\nu_{k_1}}{1+\nu_{k_1}}\frac{\nu_{k_2}}{1+\nu_{k_2}}\frac{1+\nu_{k_3}}{\nu_{k_3}}\frac{1+\nu_{k_4}}{\nu_{k_4}}\right]_{t'}\eta^\dagger_{k_1}\eta^\dagger_{k_2}\eta_{k_3}\eta_{k_4}F_0(t')\\
&+\frac{g}{12\sqrt{2L}}\langle\phi\rangle_{t'}\sum_{k_1k_2k_3}\frac{\exp[-(\sigma_{k_1}+\sigma_{k_2}+\sigma_{k_3})_{t'}]}{(k_{01}k_{02}k_{03})^{1/2}}\delta_{k_1+k_2+k_{3,0}}\\
&\times\left[\frac{\eta^\dagger_{k_1}\eta^\dagger_{k_2}\eta^\dagger_{k_3}}{(1+\nu_{k_1})(1+\nu_{k_2})(1+\nu_{k_3})}-\frac{\eta_{k_1}\eta_{k_2}\eta_{k_3}}{\nu_{k_1}\nu_{k_2}\nu_{k_3}}\right]_{t'}\left[1+\sum_i^3\nu_{k_i}+\sum_{i<j}^3\nu_{k_i}\nu_{k_j}\right]_{t'}F_0(t')\\
&+\frac{g}{4\sqrt{2L}}\langle\phi\rangle_{t'}\sum_{k_1k_2k_3}\frac{\exp[-(\sigma_{k_1}+\sigma_{k_2}+\sigma_{k_3})_{t'}]}{(k_{01}k_{02}k_{03})^{1/2}}\delta_{k_1+k_2+k_{3,}0}\\
&\times\left[\frac{\eta^\dagger_{k_1}\eta^\dagger_{k_2}\eta_{k_3}}{(1+\nu_{k_1})(1+\nu_{k_2})\nu_{k_3}}-\frac{\eta^\dagger_{k_1}\eta_{k_2}\eta_{k_3}}{\nu_{k_1}\nu_{k_2}(1+\nu_{k_3})}\right]_{t'}[\nu_{k_3}(1+\nu_{k_1}+\nu_{k_2})-\nu_{k_1}\nu_{k_2}]_{t'}F_0(t')\\
&+\frac{g}{4L}\sum_{k_1k_2}\frac{\exp[-2(\sigma_{k_1}+\sigma_{k_2})_{t'}]}{k_{01}k_{02}}\left[\frac{\eta^\dagger_{-k_2}\eta^\dagger_{k_2}}{(1+\nu_{k_1})(1+\nu_{k_2})}-\frac{\eta_{k_2}\eta_{-k_2}}{\nu_{k_1}\nu_{k_2}}\right]_{t'}\nu_{k_1}(1+\nu_{k_1}+\nu_{k_2})_{t'}F_0(t')\\
&-\frac{g\langle\phi\rangle_{t'}}{\sqrt{8L\mu}}e^{-\sigma_0}\sum_k\frac{e^{-2\sigma_k(t')}\nu(t')}{k_0}\left[\frac{\eta_0^\dagger}{1+\nu_0}-\frac{\eta_0}{\nu_0}\right]_{t'}F_0(t')\,.
\end{aligned}\tag{5.2}$$

The traces in Eq. (5.1) still cannot be taken directly, since the operators in Eq. (5.2) and in the first commutator are at different times. To overcome this we adopt the approximation discussed in Sec. III and describe the time evolution of the operators by Eq. (3.12). Using also Eqs. (3.13), (4.3) and (4.4), one obtains

$$i\dot{\eta}_k=\left\{\left[\frac{k^2+m^2}{2}+\frac{k^2+\mu_k^2}{2}+\frac{1}{2}(k^2+\mu^2)^{1/2}(k^2+\mu_k^2)^{1/2}\tau_k^2+\frac{g}{4}\langle\phi\rangle^2 +\frac{g}{4L}\sum_{k'}\frac{\nu_{k'}}{(k'^2+\mu_{k'}^2)^{1/2}}+\frac{g}{8L}\sum_{k'}\left[\frac{1}{(k'^2+\mu_{k'}^2)^{1/2}}-\frac{1}{(k^2+m_k^2)^{1/2}}\right]\right]\frac{1}{(k^2+\mu_k^2)^{1/2}} +\frac{1}{2}\left[\frac{k^2+\mu^2}{k^2+\mu_k^2}\right]^{1/4}\left[\frac{1}{2}\frac{\mu_k\dot{\mu}_k}{k^2+\mu_k^2}\tau_k+\dot{\tau}_k\right]\right\}\eta_k=f_k(t)\eta_k\ . \tag{5.3}$$

The operators η_k at different times are thus related as

$$\eta_k(t)=e^{i\varphi_k(t,t')}\eta_k(t')\ , \tag{5.4}$$

the phase $\varphi_k(t,t')$ being given by

$$\varphi_k(t,t')=-\int_{t'}^{t}d\tau f_k(\tau)\ . \tag{5.5}$$

The derivation of the proposed approximation to the collisional dynamics is now a lengthy but straightforward algebraic exercise. The resulting equations of motion are

$$\partial_t^2\langle\phi\rangle=-m^2\langle\phi\rangle-\frac{g}{3!}\langle\phi\rangle^3-\frac{g}{4L}\langle\phi\rangle\sum_k\left[\frac{1}{(k^2+\mu_k^2)^{1/2}}-\frac{1}{(k^2+m^2)^{1/2}}\right]-\frac{g}{2L}\langle\phi\rangle\sum_k\frac{\nu_k}{(k^2+\mu_k^2)^{1/2}}-\Gamma_{\langle\phi\rangle}(t)\ , \tag{5.6}$$

$$\dot{\nu}_k=\Gamma_\nu(t)\ , \tag{5.7}$$

$$\dot{\mu}_k=-2(k^2+\mu^2)^{1/4}\frac{(k^2+\mu_k^2)^{5/4}}{\mu_k}\tau_k+\frac{2(k^2+\mu_k^2)\dot{\nu}_k}{(1+2\nu_k)\mu_k}\ , \tag{5.8}$$

$$\left[\frac{k^2+\mu^2}{k^2+\mu_k^2}\right]^{1/4}\left[\frac{1}{2}\frac{\mu_k\dot{\mu}_k}{k^2+\mu_k^2}\tau_k+\dot{\tau}_k\right] =(k^2+\mu_k^2)^{1/2}-(k^2+\mu_k^2\tau_k^2)^{1/2}-\frac{k^2+m^2}{(k^2+\mu_k^2)^{1/2}}-\frac{g}{2}\frac{\langle\phi\rangle^2}{(k^2+\mu_k^2)^{1/2}} -\frac{g}{2L}\frac{1}{(k^2+\mu_k^2)^{1/2}}\sum_{k'}\frac{\nu_{k'}}{(k'^2+\mu_{k'}^2)^{1/2}}-\frac{g}{4L}\frac{1}{(k^2+\mu_k^2)^{1/2}}\sum_{k'}\left[\frac{1}{(k'^2+\mu_{k'}^2)^{1/2}}-\frac{1}{(k'^2+m^2)^{1/2}}\right]-\frac{\Gamma_\mu(t)}{1+2\nu_k}\ , \tag{5.9}$$

where the collision integrals $\Gamma(t)$ are

$$\Gamma_{\langle\phi\rangle}(t)=\frac{g^2}{24L^2}\sum_{k_1k_2k_3}\left[\frac{1}{(k_1^2+\mu_{k_1}^2)^{1/2}}\frac{1}{(k_2^2+\mu_{k_2}^2)^{1/2}}\frac{1}{(k_3^2+\mu_{k_3}^2)^{1/2}}\right]^{1/2} \times(\delta_{k_1+k_2+k_3,0}I^{(4)}_{k_1k_2k_3}+3\delta_{k_1+k_2-k_3,0}I^{(5)}_{k_1k_2k_3})\ , \tag{5.10}$$

$$\Gamma_\nu(t)=\frac{g^2}{48L^2}\sum_{k_1k_2k_3}\left[\frac{1}{(k_1^2+\mu_{k_1}^2)^{1/2}}\frac{1}{(k_2^2+\mu_{k_2}^2)^{1/2}}\frac{1}{(k_3^2+\mu_{k_3}^2)^{1/2}}\frac{1}{(k^2+\mu_k^2)^{1/2}}\right]^{1/2} \times(\delta_{k_1+k_2+k_3+k,0}J^{(1)}_{k_1k_2k_3k}+3\delta_{k_1+k_2+k-k_3,0}J^{(2)}_{k_1k_2kk_3} -\delta_{k_1+k_2+k_3-k,0}J^{(2)}_{k_1k_2k_3k}+3\delta_{k+k_1-k_2-k_3,0}J^{(3)}_{kk_1k_2k_3}) +\frac{g^2}{8L}\langle\phi\rangle\sum_{k_1k_2}\left[\frac{1}{(k_1^2+\mu_{k_1}^2)^{1/2}}\frac{1}{(k_2^2+\mu_{k_2}^2)^{1/2}}\frac{1}{(k^2+\mu_k^2)^{1/2}}\right]^{1/2} \times(\delta_{k_1+k_2+k,0}J^{(4)}_{k_1k_2k}+2\delta_{k_1+k-k_2,0}J^{(5)}_{k_1kk_2}-\delta_{k_1+k_2-k,0}J^{(5)}_{k_1k_2k})\ , \tag{5.11}$$

$$\Gamma_\mu(t)=\frac{g^2}{24L^2}\sum_{k_1k_2k_3}\left[\frac{1}{(k_1^2+\mu_{k_1}^2)^{1/2}}\frac{1}{(k_2^2+\mu_{k_2}^2)^{1/2}}\frac{1}{(k_3^2+\mu_{k_3}^2)^{1/2}}\frac{1}{(k^2+\mu_k^2)^{1/2}}\right]^{1/2}$$
$$\times(\delta_{k_1+k_2+k_3+k,0}I^{(1)}_{k_1k_2k_3k}+3\delta_{k_1+k_2+k-k_3,0}I^{(2)}_{k_1k_2kk_3}$$
$$+\delta_{k_1+k_2+k_3-k,0}I^{(2)}_{k_1k_2k_3k}+\delta_{k+k_1-k_2-k_3,0}I^{(3)}_{kk_1k_2k_3})$$
$$+\frac{g^2}{4L}\langle\phi\rangle\sum_{k_1k_2}\left[\frac{1}{(k_1^2+\mu_{k_1}^2)^{1/2}}\frac{1}{(k_2^2+\mu_{k_2}^2)^{1/2}}\frac{1}{(k^2+\mu_k^2)^{1/2}}\right]^{1/2}$$
$$\times(\delta_{k_1+k_2+k,0}I^{(4)}_{k_1k_2k}+2\delta_{k_1+k-k_2,0}I^{(5)}_{k_1kk_2}+\delta_{k_1+k_2-k,0}I^{(5)}_{k_1k_2k}\ . \tag{5.12}$$

The energy density is

$$\left\langle\frac{E}{L}\right\rangle=\frac{\langle\Pi\rangle^2}{2}+\frac{m^2}{2}\langle\phi\rangle^2+\frac{g}{4!}\langle\phi\rangle^4+\frac{g}{8L}\langle\phi\rangle^2\sum_k\left[\frac{1}{(k^2+\mu_k^2)^{1/2}}-\frac{1}{(k^2+m^2)^{1/2}}\right]$$
$$+\frac{g}{4L}\langle\phi\rangle^2\sum_k\frac{\nu_k}{(k^2+\mu_k^2)^{1/2}}+\frac{g}{8L^2}\sum_{k_1}\frac{\nu_{k_1}}{(k_1^2+\mu_{k_1}^2)^{1/2}}\sum_{k_2}\left[\frac{1}{(k_2^2+\mu_{k_2}^2)^{1/2}}-\frac{1}{(k_2^2+m^2)^{1/2}}\right]$$
$$+\frac{g}{8L^2}\left[\sum_k\frac{\nu_k}{(k^2+\mu_k^2)^{1/2}}\right]^2+\frac{g}{32L^2}\left[\sum_k\left[\frac{1}{(k^2+\mu_k^2)^{1/2}}-\frac{1}{(k^2+m^2)^{1/2}}\right]\right]^2$$
$$+\frac{1}{4L}\sum_k\left[\frac{k^2+m^2}{(k^2+\mu_k^2)^{1/2}}+(k^2+\mu_k^2)^{1/2}+(k^2+\mu^2)^{1/2}\tau_k^2\right]-\frac{1}{2L}\sum_k(k^2+m^2)^{1/2}$$
$$+\frac{g^2}{192L^3}\sum_{k_1k_2k_3k_4}\left[\frac{1}{(k_1^2+\mu_{k_1}^2)^{1/2}}\frac{1}{(k_2^2+\mu_{k_2}^2)^{1/2}}\frac{1}{(k_3^2+\mu_{k_3}^2)^{1/2}}\frac{1}{(k_4^2+\mu_{k_4}^2)^{1/2}}\right]^{1/2}$$
$$\times(\delta_{k_1+k_2+k_3+k_4,0}I^{(1)}_{k_1k_2k_3k_4}+4\delta_{k_1+k_2+k_3-k_4,0}I^{(2)}_{k_1k_2k_3k_4}+3\delta_{k_1+k_2-k_3-k_4,0}I^{(3)}_{k_1,k_2k_3k_4})$$
$$+\frac{g^2}{24L^2}\langle\phi\rangle\sum_{k_1k_2k_3}\left[\frac{1}{(k_1^2+\mu_{k_1}^2)^{1/2}}\frac{1}{(k_2^2+\mu_{k_2}^2)^{1/2}}\frac{1}{(k_3^2+\mu_{k_3}^2)^{1/2}}\right]^{1/2}$$
$$\times(\delta_{k_1+k_2+k_3,0}I^{(4)}_{k_1k_2k_3}+\delta_{k_1+k_2-k_3,0}I^{(5)}_{k_1k_2k_3})\ . \tag{5.13}$$

In these equations use was made of the abbreviations

$$I^{(1)}_{k_1k_2k_3k_4}(t)=\int_0^t dt'\prod_{i=1}^4\left[\frac{1}{(k_i^2+\mu_{k_1}^2)^{1/2}}\right]^{1/2}_{t'}\left[1+\sum_i^4\nu_{k_i}+\sum_{i<j}^4\nu_{k_i}\nu_{k_j}+\sum_{i<j<l}^4\nu_{k_i}\nu_{k_j}\nu_{k_l}\right]_{t'}$$
$$\times\sin[\varphi_{k_1}(t,t')+\varphi_{k_2}(t,t')+\varphi_{k_3}(t,t')+\varphi_{k_4}(t,t')]\ , \tag{5.14}$$

$$I^{(2)}_{k_1k_2k_3k_4}(t)=\int_0^t dt'\prod_{i=1}^4\left[\frac{1}{(k_i^2+\mu_{k_1}^2)^{1/2}}\right]^{1/2}_{t'}\left[\nu_{k_4}-\nu_{k_1}\nu_{k_2}\nu_{k_3}+\nu_{k_4}\sum_i^3\nu_{k_i}+\nu_{k_4}\sum_{i<j}^3\nu_{k_i}\nu_{k_j}\right]_{t'}$$
$$\times\sin[\varphi_{k_1}(t,t')+\varphi_{k_2}(t,t')+\varphi_{k_3}(t,t')-\varphi_{k_4}(t,t')]\ , \tag{5.15}$$

$$I^{(3)}_{k_1k_2k_3k_4}(t)=\int_0^t dt'\prod_{i=1}^4\left[\frac{1}{(k_i^2+\mu_{k_1}^2)^{1/2}}\right]^{1/2}_{t'}[\nu_{k_3}\nu_{k_4}(1+\nu_{k_1})(1+\nu_{k_2})-\nu_{k_1}\nu_{k_2}(1+\nu_{k_3})(1+\nu_{k_4})]_{t'}$$
$$\times\sin[\varphi_{k_1}(t,t')+\varphi_{k_2}(t,t')-\varphi_{k_3}(t,t')-\varphi_{k_4}(t,t')]\ , \tag{5.16}$$

$$I^{(4)}_{k_1k_2k_3}(t)=\int_0^t dt'\prod_{i=1}^3\left[\frac{1}{(k_i^2+\mu_{k_1}^2)^{1/2}}\right]^{1/2}_{t'}\langle\phi\rangle_{t'}\left[1+\sum_i^3\nu_{k_i}+\sum_{i<j}^3\nu_{k_i}\nu_{k_j}\right]_{t'}\sin[\varphi_{k_1}(t,t')+\varphi_{k_2}(t,t')+\varphi_{k_3}(t,t')]\ , \tag{5.17}$$

$$I^{(5)}_{k_1k_2k_3}(t)=\int_0^t dt'\prod_{i=1}^3\left[\frac{1}{(k_i^2+\mu_{k_1}^2)^{1/2}}\right]^{1/2}_{t'}\langle\phi\rangle_{t'}[\nu_{k_3}(1+\nu_{k_1}+\nu_{k_2})-\nu_{k_1}\nu_{k_2}]_{t'}$$
$$\times\sin[\varphi_{k_1}(t,t')+\varphi_{k_2}(t,t')-\varphi_{k_3}(t,t')]\ . \tag{5.18}$$

The $J^{(i)}$ are identical to $I^{(i)}$ with the sine functions replaced by cosines in the integrand. Energy conservation $\partial_t \langle E/L \rangle = 0$ can also be checked directly by using the dynamical equations.

VI. NUMERICAL RESULTS AND CONCLUDING REMARKS

In this section, we give numerical solutions of the equations of motion (5.6)–(5.9). A useful technique to

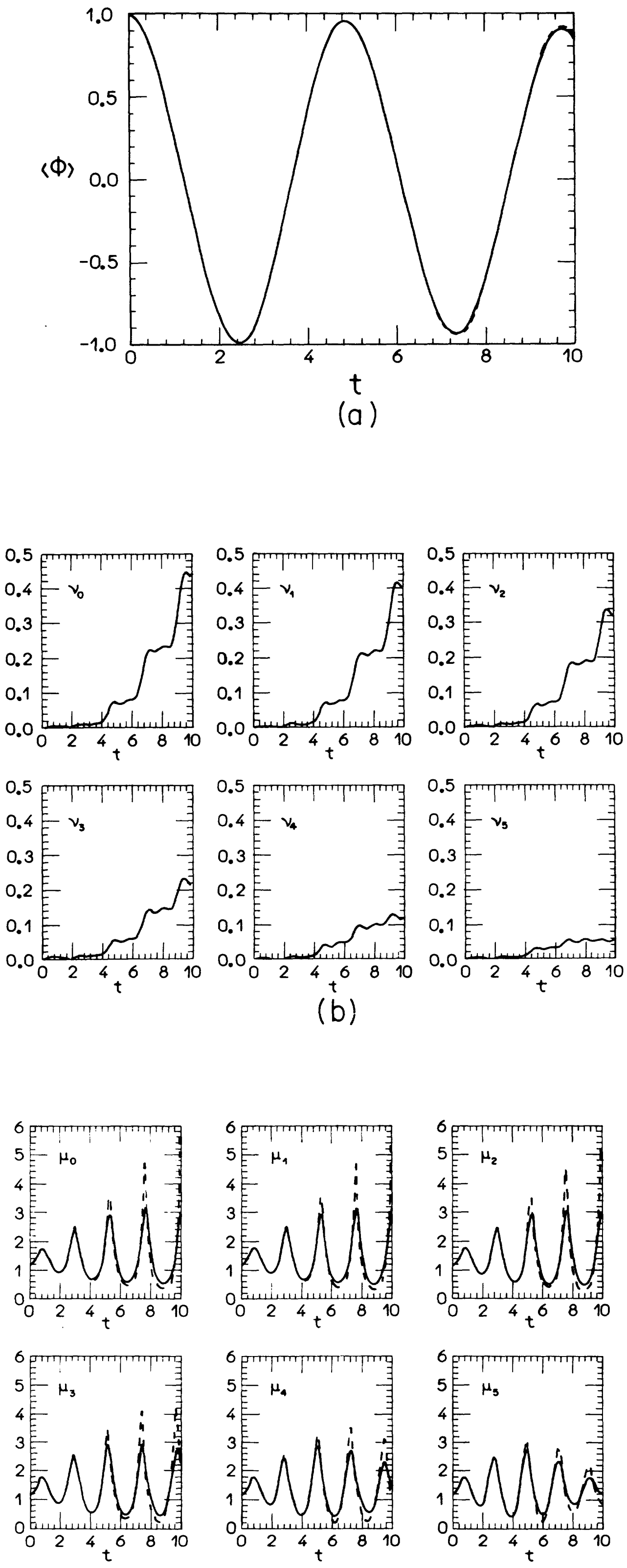

FIG. 1. Time evolution of the expectation value of the field operator $\langle \phi \rangle$ (A), mean-occupation number ν_k (B) and dynamical effective mass μ_k (C). Solid line: collisional approximation; dashed line: mean-field approximation. See text for parameter values.

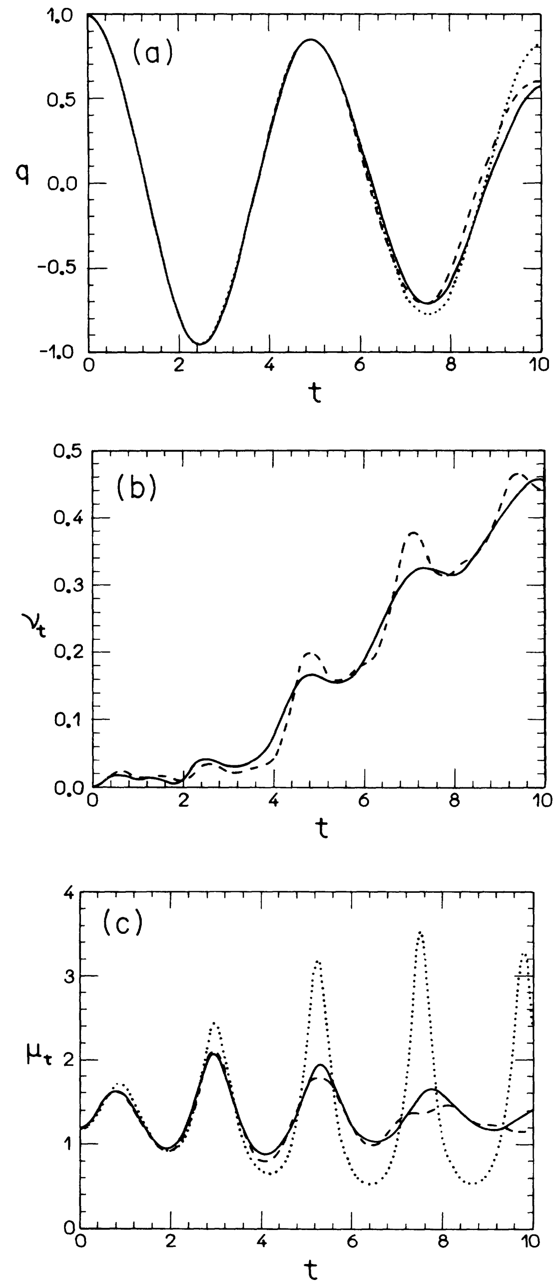

FIG. 2. Corresponding results to Fig. 1 for the case of 0+1 dimensions. Parameter values: $m = \mu = 1.191\,487\,9$ $(m_B^2 = 1)$ and $g = 2$; q is the mean position, μ_t is the dynamical effective mass and ν_t is the occupation number. Solid line: exact solution; dashed line: collisional approximation; dotted line: mean-field approximation.

treat the memory integrals in these equations is described in Appendix B. In order to control the domain of validity of the approximations involved in the derivation of the equations of motions (see Secs. IV and V), it is useful to inspect also corresponding solutions for quantum mechanics (i.e., 0+1 dimensions) [12]. We find, by comparison with the exact numerical solutions which are available in this case, that the collisional approximation

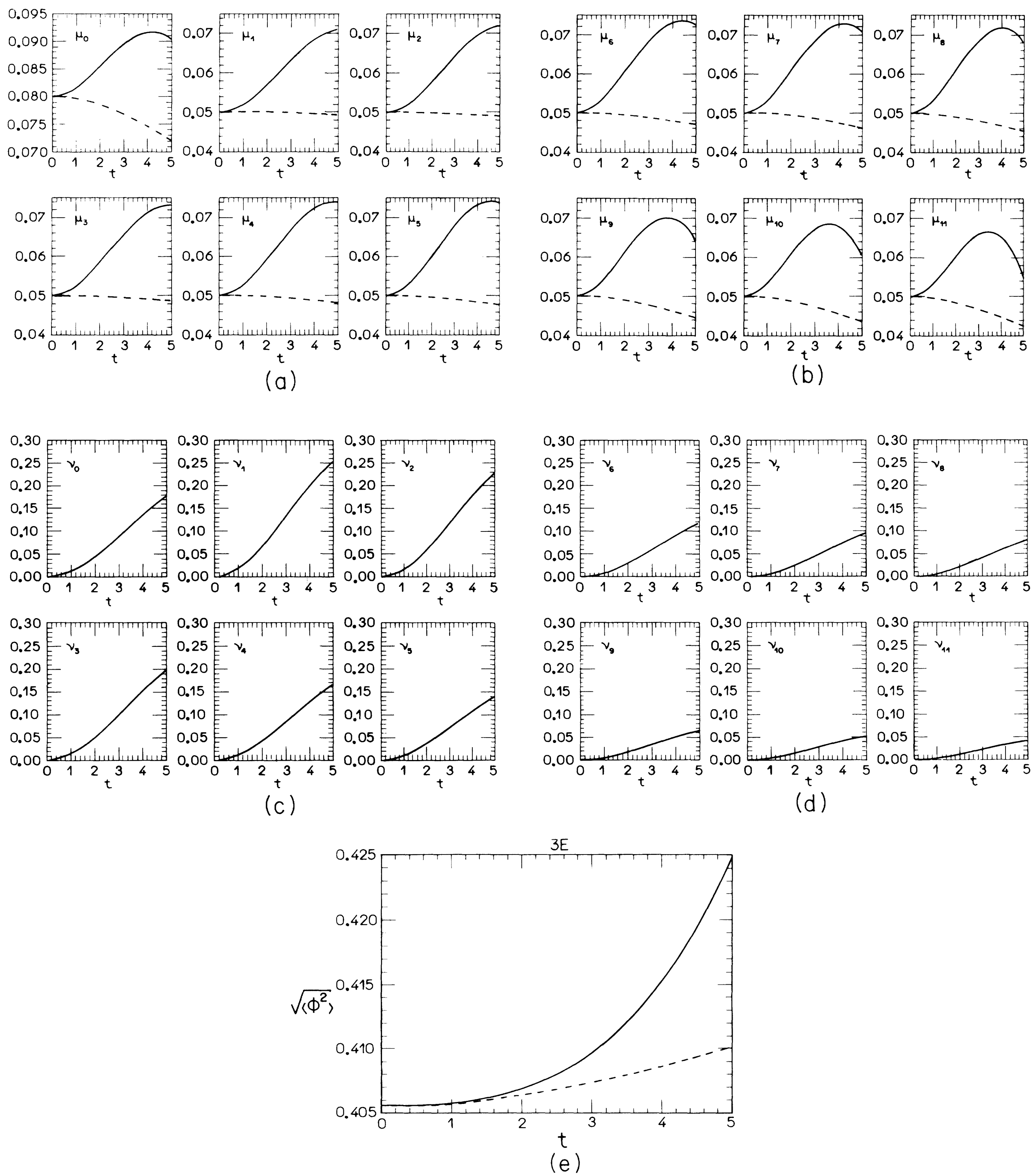

FIG. 3. Time evolution of the dynamical effective mass μ_k [(A) and (B)], mean-occupation number ν_k [(C) and (D)], and root-mean-square field (E) in broken-symmetry potential with $m=0.05$ and $g=0.155$. Conventions are the same as in Fig. 1.

improves qualitatively the dynamical description of field variables.

In what follows we use natural units. As a first case, we take the parameters of the Hamiltonian (4.2) as $m=1.2$, $g=2$. The static mean-field solution is then $\langle \pi \rangle=0$, $\langle \phi \rangle=0$, $\nu_k=0$, and $\mu_k=\mu=m$. Figure 1 shows the mean-field and the collision approximations to the time evolution of the various dynamical variables. For the initial condition $f'(0)=0$, $\langle \phi \rangle(0)=1$, $\langle \pi \rangle(0)=0$, $\nu_k(0)=0$, and $\mu_k(0)=\mu$. Periodic boundary conditions were implemented as $k=(2\pi/L)N$ with $L=40$ and $-6\leq N \leq 6$. A comparison with a calculation involving a larger dimensionality indicated substantial convergence for the variables shown. Although the mean-field and collision approximations to $\langle \phi \rangle(t)$ [Fig. 1(a)] do not differ much, the former does not show the damping which is present in collisional approximation. More dramatic differences show up, however, in the case of the time evolution of $\nu_k(t)$ and $\mu_k(t)$, as shown in Figs. 1(b) and 1(c). A natural way to interpret these results is that the $\nu_k=0$ constraint imposed by the mean-field approximation strongly distorts the dynamical behavior of the extended density $\mathcal{R}_k$, as revealed by $\mu_k(t)$. This effect can be noted also in the results for 0+1 dimensions as shown in Figs. 2(a)–2(c). The exact numerical solution is also shown in this case. It shows in fact that the collisional effects are necessary to describe properly the dynamics of $\mu_k(t)$. However, the collisional approximation is seen here also to fail for large enough times, leading in particular to an overestimation of $\nu_k(t)$. Numerical checks in the code (involving, e.g., energy conservation) demonstrate that this is not due to numerical failures, but should be seen as a limitation of the adopted approximation for the collision terms.

Figure 3 refers to results in the case of broken-symmetry and initial conditions $\langle \phi \rangle(0)=0$, $\langle \pi \rangle(0)=0$, $\nu_k(0)=0$, $\mu_k(0)=0.08$, and $\mu_{k\neq 0}(0)=0.05=\mu$. The k sum is now cut off at $|N|=12$ with $L=420$, which indicated sufficient convergence. Figures 3(a) and 3(b) show dramatic collisional effects on the time evolution of $\mu_k(t)$: In fact, $\mu_k(t)$ initially decreases in the mean-field approximation while it increases when collisional terms are turned on. In the mean-field approximation, the $\mu_k(t)$ are the only degrees of freedom affecting the root-mean-square field, evaluated here simply as

$$\langle \phi^2 \rangle - \langle \phi \rangle^2 = \sum_k \frac{1+2\nu_k}{k^2+\mu_k^2} ,$$

and it is natural to expect that they approach the equilibrium value, $\mu=0.05$. In the collisional calculation, the root-mean-square field evolves also due to the change in time of the occupation $\nu_k(t)$. We see again, therefore, that the mean-field constraint $\nu_k=0$ strongly distorts the dynamical behavior of mass parameters. Figure 3(e) shows the root-mean-square field $\langle \phi(t)^2 \rangle^{1/2}$ as a function of time. It shows that the increase of $\mu_k(t)$ is overcompensated by the change of the $\nu_k(t)$, resulting in the positive evolution $(\langle \phi^2 \rangle)^{1/2}$ in the case of collisional approximation. Figures 4(a)–4(c) show the corresponding results in 0+1 dimensions. It can be seen that again the collisional approximation is able to reproduce qualitatively the exact time evolution of ν_t and μ_t, until it fails again due to overestimation of ν_t for larger times.

We conclude, from these examples, that the mean-field approximation fails qualitatively in the description of certain field variables. These failures are partially corrected by the collisional integrals used here. However, improvements of the simplest approximation to the collisional effects, as implemented here, are needed if one wishes a quantitatively reliable description for larger times. Attempts along this line are underway.

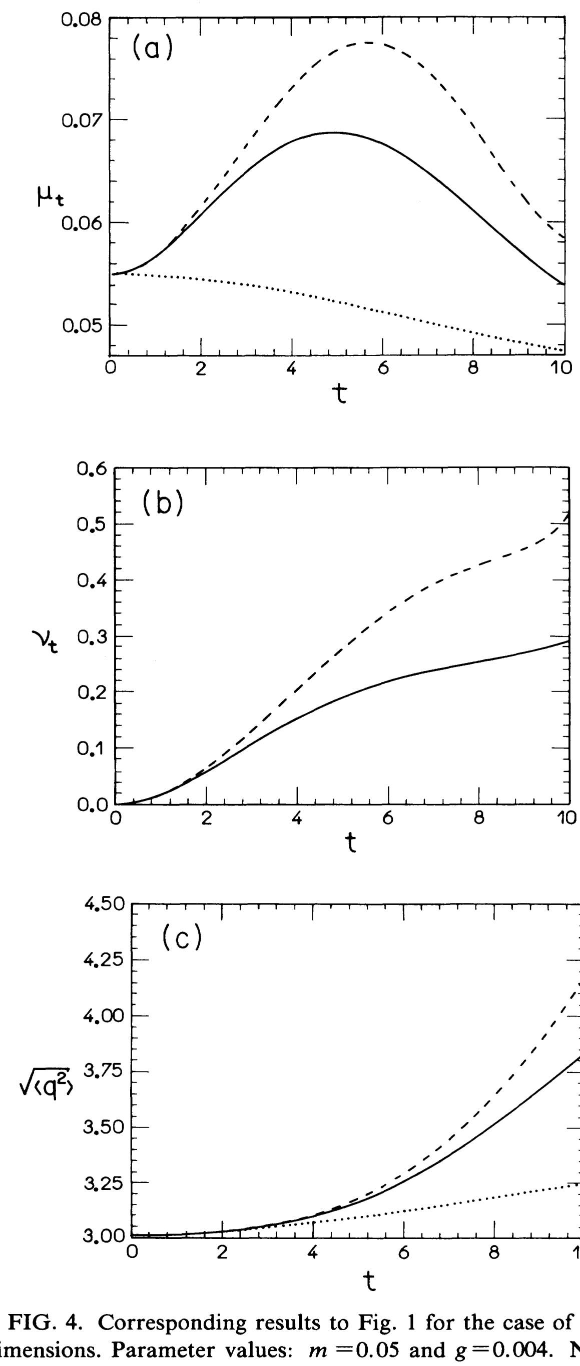

FIG. 4. Corresponding results to Fig. 1 for the case of 0+1 dimensions. Parameter values: $m=0.05$ and $g=0.004$. Notations and conventions are as in Fig. 2.

Finally, we comment on the extension of our treatment to nonuniform field configurations. In this case, the spatial dependence of the field operator is expanded in the general natural orbitals of the extended density (2.7). These orbitals can be given in terms of a momentum expansion which will also evolve in time according to additional dynamical equations which are in this case obtained from the Heisenberg equation of motion for $\phi(x)$, again in close analogy with the nonrelativistic many-body treatment. Further details on this point are given elsewhere [8].

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APPENDIX A: CONSTRUCTION OF THE PROJECTORS P AND P$^+$

In order to simplify the presentation, we shall show the technique for the case of 0+1 dimensions. The same general procedure applies also to the case of 1+1 or higher dimensions.

In Sec. II, we have stated the conditions to be satisfied by $\mathbb{P}$ as

$$F_0(t)=\mathbb{P}(t)F \ , \tag{A1}$$

$$\mathbb{P}(t)\mathbb{P}(t)=\mathbb{P}(t) \ , \tag{A2}$$

$$i\dot{F}_0(t)=[F_0 \,,H]+\mathbb{P}(t)[H,F] \ , \tag{A3}$$

where $F_0(t)$ in 0+1 dimensions is

$$F_0(t)=\frac{1}{1+\nu(t)}\left[\frac{\nu(t)}{1+\nu(t)}\right]^{\eta^\dagger(t)\eta(t)} . \tag{A4}$$

The derivative of $F_0(t)$ with respect to time is first written as

$$i\dot{F}_0=\frac{\eta^\dagger\eta-\nu}{\nu(1+\nu)}F_0\mathrm{Tr}\eta^\dagger\eta[H,F]+\frac{i}{1+\nu}\frac{d'}{dt}\left[\frac{\nu}{1+\nu}\right]^{\eta^\dagger\eta} , \tag{A5}$$

where in the last term d'/dt acts only on the operators η and $\eta^\dagger$. In order to evaluate this term, rewrite the exponential as

$$\left[\frac{\nu(t)}{1+\nu(t)}\right]^{\eta^\dagger(t)\eta(t)}=e^{m(t)\eta^\dagger(t)\eta(t)} , \tag{A6}$$

so that

$$i\frac{d'}{dt}e^{m\eta^\dagger\eta}=\sum_{n=0}^{\infty}\frac{m^n}{n!}\left[i\left[\frac{d}{dt}\eta^\dagger\eta\right](\eta^\dagger\eta)^{n-1}+\cdots+(\eta^\dagger\eta)^{n-1}i\frac{d}{dt}(\eta^\dagger\eta)\right] . \tag{A7}$$

Using Eq. (4.5) and the Heisenberg equation of motion, one finds

$$i\frac{d}{dt}\eta^\dagger\eta=[\eta^\dagger\eta,H]-i\dot{A}^*\eta-i\dot{A}\eta-i(\dot{x}y-x\dot{y})\eta\eta-i(\dot{x}^*y^*-x^*\dot{y}^*)\eta^\dagger\eta^\dagger , \tag{A8}$$

so that the last term of (A5) becomes

$$\begin{aligned}\frac{i}{1+\nu}\frac{d'}{dt}e^{m\eta^\dagger\eta}=&\frac{1}{1+\nu}\sum_{n=0}^{\infty}\frac{m^n}{n!}\{[\eta^\dagger\eta,H](\eta^\dagger\eta)^{n-1}+\cdots+(\eta^\dagger\eta)^{n-1}[\eta^\dagger\eta,H]\}\\&-\frac{1}{1+\nu}\sum_{n=0}^{\infty}\frac{m^n}{n!}\dot{A}\{i\eta(\eta^\dagger\eta)^{n-1}+\cdots+(\eta^\dagger\eta)^{n-1}i\eta\}\\&-\frac{1}{1+\nu}\sum_{n=0}^{\infty}\frac{m^n}{n!}\dot{A}\{i\eta^\dagger(\eta^\dagger\eta)^{n-1}+\cdots+(\eta^\dagger\eta)^{n-1}i\eta^\dagger\}\\&-\frac{1}{1+\nu}\sum_{n=0}^{\infty}\frac{m^n}{n!}(\dot{x}y-x\dot{y})\{i\eta\eta(\eta^\dagger\eta)^{n-1}+\cdots+(\eta^\dagger\eta)^{n-1}i\eta\eta\}\\&-\frac{1}{1+\nu}\sum_{n=0}^{\infty}\frac{m^n}{n!}(\dot{x}^*y^*-x^*\dot{y}^*)\{i\eta+\eta^\dagger(\eta^\dagger\eta)^{n-1}+\cdots+i(\eta^\dagger\eta)^{n-1}\eta^\dagger\eta^\dagger\} .\end{aligned} \tag{A9}$$

In order to reobtain the exponential (A6), one moves the operators η, $\eta^\dagger$, $\eta\eta$, $\eta^\dagger\eta^\dagger$, to the right and obtains, after some algebra,

$$\begin{aligned}\frac{1}{1+\nu}\frac{d'}{dt}e^{m\eta^\dagger\eta}=&[F_0,H]+i\dot{A}^*\frac{\eta}{\nu}F_0+i\dot{A}\frac{\eta^\dagger}{1+\nu}F_0+i(\dot{x}y-x\dot{y})(1-2\nu)\frac{\eta\eta}{2\nu^2}F_0\\&+i(\dot{x}^*y^*-x^*\dot{y}^*)(1+2\nu)\frac{\eta^\dagger\eta^\dagger}{2(1+\nu)^2}F_0 .\end{aligned} \tag{A10}$$

Using the dynamic equations for the extended density, Eqs. (2.20) and (2.21), one gets

$$i\dot{F}_0=\frac{\eta^\dagger\eta-\nu}{\nu(1+\nu)}F_0\mathrm{Tr}\eta^\dagger\eta[H,F]+[F_0,H]+\frac{\eta}{\nu}F_0\mathrm{Tr}\eta^\dagger[H,F]+\frac{\eta^\dagger}{1+\nu}F_0\mathrm{Tr}\eta[H,F]+\frac{\eta\eta}{2\nu^2}F_0\mathrm{Tr}\eta^\dagger\eta^\dagger[H,F]$$
$$+\frac{\eta^\dagger\eta^\dagger}{2(1+\nu)^2}F_0\mathrm{Tr}\eta\eta[H,F]\ . \tag{A11}$$

Equation (A3) is therefore satisfied by

$$\bar{P}\cdot=\frac{\eta^\dagger\eta-\nu}{\nu(1+\nu)}F_0\mathrm{Tr}(\eta^\dagger\eta\cdot)+\frac{\eta}{\nu}F_0\mathrm{Tr}(\eta^\dagger\cdot)$$
$$+\frac{\eta^\dagger}{\nu(1+\nu)}F_0\mathrm{Tr}(\eta\cdot)+\frac{\eta\eta}{2\nu^2}F_0\mathrm{Tr}(\eta^\dagger\eta^\dagger\cdot)$$
$$+\frac{\eta^\dagger\eta^\dagger}{2(1+\nu)^2}F_0\mathrm{Tr}(\eta\eta\cdot)\ . \tag{A12}$$

This object, however, fails to satisfy Eqs. (A1) and (A2). In fact,

$$\bar{\mathbb{P}}F=\frac{\eta^\dagger\eta-\nu}{1+\nu}F_0\ . \tag{A13}$$

The projector $\mathbb{P}$ is, however, immediately obtained by adding to $\bar{\mathbb{P}}$ terms which guarantee the validity of Eq. (A1), i.e.,

$$\mathbb{P}\cdot=\bar{\mathbb{P}}\cdot-\frac{\eta^\dagger\eta-\nu}{1+\nu}F_0\mathrm{Tr}(\cdot)+F_0\mathrm{Tr}(\cdot)\ . \tag{A14}$$

The full expression for the projector is thus

$$\mathbb{P}\cdot=\left\{\left[1-\frac{\eta^\dagger\eta-\nu}{1+\nu}\right]\mathrm{Tr}(\cdot)+\frac{\eta^\dagger\eta-\nu}{\nu(1+\nu)}\mathrm{Tr}(\eta^\dagger\eta\cdot)\right.$$
$$+\frac{\eta}{\nu}\mathrm{Tr}(\nu^\dagger\cdot)+\frac{\eta^\dagger}{1+\nu}\mathrm{Tr}(\eta\cdot)+\frac{\eta\eta}{2\nu^2}\mathrm{Tr}(\eta\eta^\dagger\cdot)$$
$$\left.+\frac{\eta^\dagger\eta^\dagger}{2(1+\nu)^2}\mathrm{Tr}(\eta\eta\cdot)\right\}F_0\ . \tag{A15}$$

The construction of $\mathbb{P}^+$ follows immediately from the definition of the scalar product:

$$(y,\mathbb{P}x)=\mathrm{Tr}[y^\dagger(\mathbb{P}x)]=\mathrm{Tr}\left[y^\dagger\left[1-\frac{\eta^\dagger\eta-\nu}{1+\nu}\right]F_0\right]\mathrm{Tr}(x)+\mathrm{Tr}\left[y^\dagger\frac{\eta^\dagger\eta-\nu}{\nu(1+\nu)}F_0\right]\mathrm{Tr}(\eta^\dagger\eta x)+\mathrm{Tr}\left[y^\dagger\frac{\eta}{\nu}F_0\right]\mathrm{Tr}(\eta^\dagger x)$$
$$+\mathrm{Tr}\left[y^\dagger\frac{\eta^\dagger}{1+\nu}F_0\right]\mathrm{Tr}(\eta x)+\mathrm{Tr}\left[y^\dagger\frac{\eta\eta}{2\nu^2}F_0\right]\mathrm{Tr}(\eta^\dagger\eta^\dagger x)$$
$$+\mathrm{Tr}\left[y^\dagger\frac{\eta^\dagger\eta^\dagger}{2(1+\nu)^2}F_0\right]\mathrm{Tr}(\eta\eta x)=\mathrm{Tr}[(P^\dagger y)^\dagger x]\ . \tag{A16}$$

From this, one obtains immediately

$$\mathbb{P}^+\cdot=\mathrm{Tr}\left[F_0\left[1-\frac{\eta^\dagger\eta-\nu}{1+\nu}\right]\cdot\right]+\frac{\eta^\dagger\eta}{\nu(1+\nu)}\mathrm{Tr}[F_0(\eta^\dagger\eta-\nu)\cdot]$$
$$+\frac{\eta}{\nu}\mathrm{Tr}[F_0\eta^\dagger\cdot]+\frac{\eta^\dagger}{1+\nu}\mathrm{Tr}[F_0\eta\cdot]+\frac{\eta\eta}{2\nu^2}\mathrm{Tr}[F_0\eta^\dagger\eta^\dagger\cdot]+\frac{\eta^\dagger\eta^\dagger}{2(1+\nu)^2}\mathrm{Tr}[F_0\eta\eta\cdot]\ . \tag{A17}$$

APPENDIX B: NUMERICAL TREATMENT OF MEMORY INTEGRALS

In order to obtain numerical solutions for the equations of motion, we need to evaluate a memory integral of the type

$$I(t)=\int_0^t dt'\left[\frac{1}{(k_1^2+\mu_{k_1}^2)^{1/2}}\frac{1}{(k_2^2+\mu_{k_2}^2)^{1/2}}\frac{1}{(k_3^2+\mu_{k_3}^2)^{1/2}}\right]_{t'}^{1/2}$$
$$\times\left[1+\sum_i \nu_{k_i}+\sum_{i<j}\nu_{k_i}\nu_{k_j}\right]_{t'}\langle\phi\rangle_{t'}\sin[\phi_{k_1}(t,t')+\phi_{k_2}(t,t')+\phi_{k_3}(t,t')]\ . \tag{B1}$$

Using the phase equation (5.6) explicitly, this appears as

$$I(t)=-\sin\left\{\int_0^t dt'[f_{k_1}(t')+f_{k_2}(t')+f_{k_3}(t')]\right\}$$
$$\times\int_0^t dt'\left[\frac{1}{(k_1^2+\mu_{k_1}^2)^{1/2}}\frac{1}{(k_2^2+\mu_{k_2}^2)^{1/2}}\frac{1}{(k_3^2+\mu_{k_3}^2)^{1/2}}\right]_{t'}^{1/2}\left[1+\sum_i^3 \nu_{k_i}+\sum_{i<j}^3 \nu_{k_i}\nu_{k_j}\right]_{t'}\langle\phi\rangle_{t'}$$
$$\times\cos\int_0^{t'} dt''[f_{k_1}(t'')+f_{k_2}(t'')+f_{k_3}(t'')]$$
$$+\cos\left\{\int_0^t dt'[f_{k_1}(t')+f_{k_2}(t')+f_{k_3}(t')]\right\}$$
$$\times\int_0^t dt'\left[\frac{1}{(k_1^2+\mu_{k_1}^2)^{1/2}}\frac{1}{(k_2^2+\mu_{k_2}^2)^{1/2}}\frac{1}{(k_3^2+\mu_{k_3}^2)^{1/2}}\right]_{t'}^{1/2}\left[1+\sum_i^3 \nu_{k_i}+\sum_{i<j}^3 \nu_{k_i}\nu_{k_j}\right]_{t'}\langle\phi\rangle_{t'}$$
$$\times\sin\int_0^{t'} dt''[f_{k_1}(t'')+f_{k_2}(t'')+f_{k_3}(t'')]\ . \tag{B2}$$

In order to evaluate this a useful trick is to write a differential equation for the integral appearing in (B2). Thus

$$\dot I_f=f_{k_1}(t)+f_{k_2}(t)+f_{k_3}(t)\ , \tag{B3}$$

$$\dot I_c=\left[\frac{1}{(k_1^2+\mu_{k_1}^2)^{1/2}}\frac{1}{(k_2^2+\mu_{k_2}^2)^{1/2}}\frac{1}{(k_3^2+\mu_{k_3}^2)^{1/2}}\right]^{1/2}\left[1+\sum_i^3 \nu_{k_i}+\sum_{i<j}^3 \nu_{k_i}\nu_{k_j}\right]\langle\phi\rangle\cos(I_f)\ , \tag{B4}$$

$$\dot I_s=\left[\frac{1}{(k_1^2+\mu_{k_1}^2)^{1/2}}\frac{1}{(k_2^2+\mu_{k_2}^2)^{1/2}}\frac{1}{(k_3^2+\mu_{k_3}^2)^{1/2}}\right]^{1/2}\left[1+\sum_i^3 \nu_{k_i}+\sum_{i<j}^3 \nu_{k_i}\nu_{k_j}\right]\langle\phi\rangle\sin(I_f)\ , \tag{B5}$$

so that

$$I(t)=-\sin(I_f)I_c+\cos(I_f)I_s\ . \tag{B6}$$

The differential equations (B3), (B4), and (B5) can be integrated easily by standard numerical methods together with the remaining dynamical equations.

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