PHYSICAL REVIEW D VOLUME 46, NUMBER 2 15 JULY 1992

Twistorlike $D = 10$ superparticle action with manifest $N = 8$ world-line supersymmetry

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(Received 19 March 1992)

We propose a new formulation of the $D = 10$ Brink-Schwarz superparticle which is manifestly invariant under both the target-space super-Poincaré group and the world-line local $N = 8$ superconformal group. This twistorlike construction naturally involves the sphere S^8 as a coset space of the $D = 10$ Lorentz group. The action contains only a finite set of auxiliary fields, but they appear in unusual trilinear combinations. The origin of the on-shell $D = 10$ fermionic κ symmetry of the standard Brink-Schwarz formulation is explained. The coupling to a $D = 10$ super-Maxwell background requires a new mechanism, in which the electric charge appears only on shell as an integration constant.

PACS number(s): 11.30.Pb

I. INTRODUCTION

In supersymmetric string theory there are two essentially different approaches [1]. The fundamental underlying symmetry of the Neveu, Schwarz, and Ramond (NSR) approach is the world-sheet superconformal group, but there is no supersymmetry in the target space. In the Green and Schwarz (GS) construction it is just the other way around: Target-space supersymmetry is manifest, while there is no supersymmetry on the world sheet. Instead, in the "magic" dimensions $D = 3, 4, 6$, and 10 the GS superstring possesses a mysterious fermionic κ symmetry [2] of unclear geometric origin. This symmetry is crucial for the consistency of the GS superstring. At the same time, it has proven to be a very serious obstacle for the Lorentz-covariant quantization of the theory.

Remarkably enough, under certain conditions the above theories are equivalent in *the light-cone gauge*, which means with broken manifest Lorentz covariance. The natural question arises: If the NSR and GS approaches are just the two faces of the same theory, is it possible to find a new formulation that would combine the attractive features of both? In other words, it should be invariant under *both* the world-sheet superconformal group and target-space Poincaré supersymmetry. At the same time, it should not have any κ invariance [3]. Clearly, in such a hypothetic formulation the light-cone gauge linking the GS and NSR approaches should be properly Lorentz covariantized.

The same question can be posed in the analogous, but simpler problem of the Brink-Schwarz (BS) superparticle [5] moving in a $D = 3, 4, 6$, or 10-dimensional superspace. The BS superparticle is the infinite tension limit of the GS superstring; being much simpler, it still preserves some features of the GS theory, notably its κ symmetry. Recently, there has been an important development in this direction. The pioneering step was made by Sorokin, Tkach, Volkov, and Zheltukhin [6] (STVZ), who proposed a new action for the $D = 3$ superparticle with κ symmetry replaced by $N = 1$ world-line superconformal symmetry. The bosonic part of the STVZ action

$$S^{\rm bos}_{\rm STVZ} = \int d\tau\, p_\mu[\dot{x}^\mu - \psi\gamma^\mu\psi] \tag{1}$$

is different from the usual massless particle action

$$S = \int d\tau[p_\mu\dot{x}^\mu - \tfrac{1}{2}ep^2]. \tag{2}$$

An essential ingredient of (1) is the "twistorlike" variable ψ which is a commuting spinor of the $D = 3$ Lorentz group. It is only on shell that one can show the classical equivalence of the two theories. The Chern-Simons nature of the STVZ action, i.e., the absense of the world-line metric e in it, was emphasized by Howe and Townsend [7].

It is natural to try to generalize the STVZ action for the cases $D = 4, 6$, and 10, where the $D - 2$ parameters of κ symmetry could be replaced by $N = 2, 4, 8$ world-line supersymmetries. Delduc and Sokatchev [8] made a step further by constructing $D = 4, 6$ twistorlike superparticle actions with $N = 2, 4$ world-line superconformal symmetry, respectively [9]. Independently, Ivanov and Kapustnikov [11] proposed a superstring action of this type for $N = 2, D = 4$. As in (1), these new actions contain a lightlike vector $v^\mu = \psi\gamma^\mu\psi$ made out of the

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(commuting) Majorana spinor ψ with 4 and 8 real components in $D=4$ and $D=6$.

However, the attempts to generalize the STVZ approach to the $D=10$ case have hit upon an obstacle. While $D=10$ superparticle (or superstring) actions with $N=1$ or 2 world-line supersymmetry have been successfully constructed [12–14], it seemed very hard to imagine an action with full $N=8$ supersymmetry. The reason for this difficulty can be explained as follows. In $D=10$, like in the lower-dimensional cases $D=3,4,6$ there also exists a twistorlike representation of a lightlike vector v^μ, $v^2=0$ in terms of a (commuting) Majorana-Weyl spinor ψ^α, $\alpha=1,2,\ldots,16$: $v^\mu=\psi\gamma^\mu\psi$. The correspondence between v^μ and ψ^α is, however, not one to one: the null vector has $10-1=9$ components, while the spinor has 16. The 7 extra components of the spinor cannot be attributed to any gauge *group* acting on ψ, in contrast with the cases $D=4$ and $D=6$, where the extra components can be accounted for as U(1) and SU(2) gauge *groups*, respectively. Instead, the 7 gauge transformations in the case $D=10$ form an algebra with field-dependent structure constants [15]. Therefore, it seemed impossible to construct a $D=10$ STVZ-like superparticle action with a linear realization of all the symmetries.

A way to overcome this difficulty was found by Galperin, Howe, and Stelle [16] (GHS). They gave a group-theoretical analysis of the above $D=3,4$ and 6 actions and identified the lightlike vectors as the parameters of the cosets of the corresponding Lorentz groups $\mathrm{SO}(1,D-1)$ by their *maximal* subgroups. Then the Majorana spinors (with their gauge groups) naturally arise as the parameters of the same cosets in the spinor representation. However, for the case $D=10$ the GHS construction gives a different recipe how to construct lightlike vectors in terms of spinors. Instead of taking *one* MW spinor ψ^α one should take *eight* MW spinors ψ^α_a, $a=1,\ldots,8$, which are, however, properly *constrained* by the SO(1,9) conditions. This allows for *linear* realizations of all the symmetries and defines the "twistorlike" variables appropriate for $D=10$.

The main open problem after the work of Ref. [16] was how to find an action, where all those symmetries are manifest. The answer is given in this paper. We generalize the results of Refs. [6,16] and propose an essentially new action for the most interesting case of the $D=10$ superparticle. In some sense our action is the most straightforward generalization of the $N=1, D=3$ STVZ superfield action to the $N=8, D=10$ case. At first sight it might even seem too naive a generalization to have any chances to work. Miraculously, however, the various dangers are avoided and it does work. Moreover, the group-theoretical description of the $D=10$ lightlike vectors in terms of spinors developed in Ref. [16] comes out automatically from that action. The action possesses manifest $N=8$ world-line superconformal invariance as well as $D=10$ target-space Poincaré supersymmetry. In addition to the usual target-superspace coordinates X^μ and Θ^α it also involves a Lagrange multiplier with its own Abelian gauge group. The action contains a finite number of auxiliary fields, some of which appear in trilinear combinations. This allows us to break the "$N=8$ barrier" that one might anticipate in such a theory.

We emphasize that only the classical theory of the superparticle is discussed here. The quantum theory will be the subject of a future publication.

The paper is organized as follows. In Sec. II we introduce the action and discuss its symmetries and kinematics. In Sec. III we study the equations of motion for X and Θ, their group-theoretical meaning and their solution in the light-cone gauge. We show that the on-shell dynamics of the superparticle naturally gives rise to a Lorentz-harmonic (twistorlike) description of the sphere S^8. The large $N=8$ superfields X and Θ are reduced on-shell and in a conformal supersymmetry gauge to the usual dynamical variables of the superparticle. In Sec. IV we analyze the equations of motion for the Lagrange multiplier and fix a Wess-Zumino gauge for its Abelian gauge invariance, in which only a single constant vector (the particle momentum) survives. We also demonstrate the on-shell equivalence to the BS superparticle, and discuss the origin of κ symmetry in the latter. In Sec. V we couple the superparticle to a $D=10$ supergravity and super-Maxwell background. The Maxwell coupling might seem impossible by dimensional reasons. Its existence is due to an unusual phenomenon: the electric charge appears in the theory only as an integration constant. The consistency of the coupling puts constraints on the $D=10$ background. Finally, in Sec. VI we compare the $D=4$ and $D=6$ analogues of our action with the results of Ref. [8] (in the case $D=3$ our action coincides with that of STVZ [6]). It turns out that in our formulation the complex structures of the $D=4,6$ theories, which were put in the basis of the construction of Ref. [8], manifest themselves only on shell. We conclude the paper by formulating further problems that may be solved along the lines developed here.

II. THE ACTION AND ITS INVARIANCES

We propose the following action for a superparticle moving in D=10 Minkowski superspace:

$$S=\int d\tau\, d^8\eta\, P_{a\mu}(D_aX^\mu - iD_a\Theta\gamma^\mu\Theta). \tag{3}$$

It is an integral over the $N=8$ world-line superspace $(\tau,\ \eta_a)$ $(a=1,2,\ldots,8)$, which contains one even and eight odd coordinates. D_a are the $N=8$ supercovariant derivatives

$$D_a=\partial_a+i\eta_a\partial_\tau,\quad \{D_a,\ D_b\}=2i\delta_{ab}\partial_\tau\quad (\partial_a=\partial/\partial\eta^a). \tag{4}$$

Note the natural $\mathrm{SO}^\uparrow(1,1)\times \mathrm{O}(8)$ automorphism of the D algebra. The dynamical variables $P(\tau,\eta)$, $X(\tau,\eta)$, and $\Theta(\tau,\eta)$ are $N=8$ superfields. The anticommuting superfield $P_{a\mu}$ (which we shall call a Lagrange multiplier) carries a world-line O(8) index a and a ten-dimensional Lorentz vector index μ. X^μ and Θ^α are the coordinates of $N=1, D=10$ target superspace, Θ^α is a Majorana-Weyl spinor, which has 16 real components in $D=10$.

The action (3) has a number of symmetries. First of all, it is manifestly invariant under $N=1$ $D=10$ *rigid* target-space supersymmetry:

$$\delta\Theta^\alpha = \epsilon^\alpha, \quad \delta X^\mu = i\Theta\gamma^\mu\epsilon, \quad \delta P_{a\mu} = 0 \tag{5}$$

and under the orthocronous ten-dimensional Lorentz group $O^\uparrow(1,9)$.

The action also has two types of *gauge* invariances. The first one is the $N=8$ world-line superconformal group which can be defined as the subgroup of the general diffeomorphism group $\tau \to \tau'(\tau,\eta)$, $\eta \to \eta'(\tau,\eta)$ that transforms the flat world-line spinor derivatives D_a (4) homogeneously. The defining constraint and the transformation law are

$$D'_a\tau - i\eta_b D'_a\eta_b = 0 \;\Rightarrow\; D'_a = (D'_a\eta_b)D_b. \tag{6}$$

For infinitesimal transformations the constraint in (6) can be solved in terms of an unconstrained world-line superfunction $\Lambda(\tau,\eta)$:

$$\delta\tau \equiv \tau' - \tau = \Lambda - \frac{1}{2}\eta_a D_a\Lambda, \;\; \delta\eta_a \equiv \eta'_a - \eta_a = -\frac{i}{2}D_a\Lambda. \tag{7}$$

Then the superconformal transformation law of the covariant derivatives can be rewritten in the suggestive form

$$\delta D_a \equiv D'_a - D_a = -\frac{1}{2}\partial_\tau\Lambda\; D_a + \frac{i}{4}[D_a, D_b]\Lambda\; D_b, \tag{8}$$

where the first term corresponds to scale transformations and the second to O(8) ones. The Weyl parameter $\partial_\tau\Lambda$ appears also in the world-line supervolume transformations

$$\delta(d\tau\, d^8\eta) = -3\;\partial_\tau\Lambda(d\tau\, d^8\eta). \tag{9}$$

The ten-dimensional coordinates X^μ and Θ^α are world-line scalars

$$X'(\tau',\eta') = X(\tau,\eta), \quad \Theta'(\tau',\eta') = \Theta(\tau,\eta). \tag{10}$$

The superconformal invariance of the action (3) is achieved by the following transformation of the Lagrange multiplier:

$$\delta P^\mu_a = \frac{7}{2}\partial_\tau\Lambda\; P^\mu_a + \frac{i}{4}[D_a, D_b]\Lambda\; P^\mu_b, \tag{11}$$

which compensates for the derivative and volume transformations (8) and (9).

Second, the Lagrange multiplier P has a large *Abelian gauge invariance*:

$$\delta P^\mu_a = D_b(\xi_{abc}\gamma^\mu D_c\Theta), \tag{12}$$

where the parameter $\xi^\alpha_{abc}(\tau,\eta)$ is a (commuting) Majorana-Weyl spinor and it is totally symmetric and traceless with respect to the O(8) indices a,b,c. To check this invariance one integrates D_b by parts and uses the O(8) properties of ξ_{abc} and the ten-dimensional γ-matrix identity

$$(\gamma^\mu)_{(\alpha\beta}(\gamma_\mu)_{\gamma)\delta} = 0. \tag{13}$$

As we shall see in what follows, this Abelian gauge invariance of P is crucial for the consistency of the action. Since it relies on the γ-matrix identity (13), which is valid in 3,4,6, and 10 space-time dimensions, it is clear that superparticle actions similar to (3) can be written down in all those special dimensions. In fact, in the case $D=3$ the action (3) is just the action of STVZ [6]. Note, however, that the case $D=3$ is exceptional, because there one has $N=1$ world-line supersymmetry and the right-hand side of (12) automatically vanishes. The cases $D=4$ and 6 will be discussed in Sec. VI.

It should be emphasized that all the above symmetries, the rigid $D=10$ super-Poncaré group, the local $N=8$ superconformal group, and the Lagrange multiplier invariance, are off-shell, independent of each other and linearly realized.

To complete the definition of the action one should postulate the following kinematical properties: the vector $\partial_\tau X^\mu$ is strictly nonvanishing and the matrix $D_a\Theta^\alpha$ is of the maximal rank, i.e.,

$$\partial_\tau X^\mu \neq 0, \quad \text{rank}\,||D_a\Theta^\alpha|| = 8. \tag{14}$$

Geometrically, this means that the superparticle trajectory $X^\mu = X^\mu(\tau,\eta)$, $\Theta^\alpha = \Theta^\alpha(\tau,\eta)$ is a nondegenerate (1,8) surface in ten-dimensional Minkowski superspace. At the same time, it means that the superconformal group (7) is spontaneously broken; i.e., it can be compensated by the superfields X and Θ. This requirement will also make possible choosing a light-cone gauge, where one component of X^μ and eight components of Θ^α will be identified with τ, η_a (see the next section). In addition to (14), one should require that the tensor of the gauge groups (11) and (12),

$$p^\mu \equiv \epsilon^{a_1\cdots a_8} D_{a_1}\cdots D_{a_7} P^\mu_{a_8}, \tag{15}$$

do not vanish. As we shall see, this $D=10$ vector plays the role of the superparticle momentum, which should be strictly nonvanishing in Chern-Simons-type actions such as (1) and (3).

In Secs. III and IV we will prove the classical on-shell equivalence of the action (3) to the Brink-Schwarz action.

III. GEOMETRY OF THE SUPERPARTICLE MOTION

In this section we shall study the superparticle equation of motion following from the variation of the action (3) with respect to the Lagrange multiplier P^μ_a:

$$D_a X^\mu - iD_a\Theta\gamma^\mu\Theta = 0. \tag{16}$$

We shall show that this equation, together with the superconfromal group (7) allows one to express all the components of the $N=8$ superfields $X^\mu(\tau,\eta)$ and $\Theta^\alpha(\tau,\eta)$ in terms of their lowest-order components $x^\mu(\tau) = X^\mu|_{\eta=0}$, $\theta^\alpha(\tau) = \Theta^\alpha|_{\eta=0}$, which are the usual dynamical variables of the superparticle [17]. We shall also exhibit the geometric meaning of the motion of the superparticle on an $N=8$ super world line. In particular, we shall emphasize the double role of the commuting spinors

$\psi_a^\alpha = D_a\Theta^\alpha|_{\eta=0}$ as superpartners of θ^α under $N = 8$ world-line conformal supersymmetry, on the one hand, and as twistorlike variables parametrizing the sphere S^8, on the other hand.

We begin by deriving an important consequence of Eq. (16) [19]. It is obtained by hitting it by another spinor derivative D_b and taking the symmetric traceless part in a, b:

$$D_a\Theta\gamma^\mu D_b\Theta = \frac{1}{8}\delta_{ab}D_c\Theta\gamma^\mu D_c\Theta. \tag{17}$$

There is a remarkable analogy between (16) and (17) and the *finite* $N = 8$ superconformal transformations. Hitting the defining constraint (6) by D_c and taking the symmetric and traceless part in a, c, one finds

$$D'_a\eta_b\, D'_c\eta_b = \frac{1}{8}\delta_{ac}D'_d\eta_b\, D'_d\eta_b. \tag{18}$$

This means that the matrix $D'_a\eta_b$ takes its values in $SO^\uparrow(1,1)\times O(8)$, where $SO^\uparrow(1,1)$ is represented by the positive root $\lambda = (D'_d\eta_b\; D'_d\eta_b/8)^{1/2}$ and $O(8)$ by $D'_a\eta_b/\lambda$.

There is a clear similarity between (16), (17) and (6), (18). In fact, Eqs. (17) and (18), together with the nondegeneracy condition (14) and the superconformal transformation law [see (6), (10)]

$$D'_a\Theta'^\alpha = D'_a\eta_b\, D_b\Theta^\alpha \tag{19}$$

have an important geometric meaning. It turns out that the rank-8 matrix $D_a\Theta^\alpha$, constrained by (17) and considered modulo the $SO^\uparrow(1,1)\times O(8)$ gauge transformations (18), (19) parametrizes the sphere S^8. This is not surprising, since the lightlike velocity $\dot{x}^\mu$ of a massless $D = 10$ particle does describe an eight-sphere. Indeed, $\dot{x}^\mu\dot{x}_\mu = 0 \;\rightarrow\; (\dot{x}^1)^2 + \cdots + (\dot{x}^9)^2 = (\dot{x}^0)^2$. This, together with the physical assumption $\dot{x}^0 \neq 0$ and up to $SO^\uparrow(1,1)$ (scale) transformations, is the definition of S^8. To be more precise, in our case the eight-sphere emerges as the following coset space of the $D = 10$ Lorentz group [21]

$$S^8 = \frac{\mathrm{Pin}(1,9)}{[SO^\uparrow(1,1)\times O(8)] ⋉ K^i}. \tag{20}$$

Here Pin(1, 9) is the double covering group for the orthocronous Lorentz group $O^\uparrow(1,9)$ [analogously, Spin(1, 9) covers the proper orthocronous group $SO^\uparrow(1,9)$]. The denominator group is its *maximal* subgroup, K^i are the conformal boost transformations [in fact, $D_a\Theta^\alpha$ is that half of the Pin(1, 9) 16×16 matrix which transforms only under $SO^\uparrow(1,1)\times O(8)$ and is K^i inert, see Ref. [16] for details].

To see why the matrix $D_a\Theta^\alpha$, subject to the algebraic constraint (17) and considered modulo the transformations (19), corresponds to the sphere S^8, it is convenient to use light-cone coordinates. Given a D=10 vector $v^\mu = (v^0, v^i, v^9)$ one defines $v^{\pm\pm} = v^0 \pm v^9 \;\rightarrow v^\mu v_\mu = -v^{++}v^{--} + (v^i)^2$; a Majorana-Weyl spinor Θ^α is decomposed into a pair $(\Theta^-_A,\; \Theta^+_{\dot{A}})$, where A and $\dot{A}$ are indices of the $\mathbf{8}_s$ and $\mathbf{8}_c$ representations of the O(8) subgroup of the Lorentz group and the weights $\pm$ correspond to the $SO^\uparrow(1,1)$ subgroup (in what follows we shall often omit the $SO^\uparrow(1,1)$ weights, if this will not cause confusion). Using a suitable representation for the γ matrices [22], we can write down the three light-cone projections of (17) as

$$D_a\Theta_A\; D_b\Theta_A = \frac{1}{8}\delta_{ab}D_c\Theta_A\; D_c\Theta_A, \tag{21}$$

$$D_a\Theta_A\gamma^i_{A\dot{B}}D_b\Theta_{\dot{B}} + (a\leftrightarrow b) = \frac{1}{4}\delta_{ab}D_c\Theta_A(\gamma^i)_{A\dot{B}}D_c\Theta_{\dot{B}}, \tag{22}$$

$$D_a\Theta_{\dot{A}}D_b\Theta_{\dot{A}} = \frac{1}{8}\delta_{ab}D_c\Theta_{\dot{A}}\; D_c\Theta_{\dot{A}}. \tag{23}$$

The nondegeneracy of $D_a\Theta^\alpha$ (14) implies that the right-hand sides of (21) and (23) cannot vanish simultaneously. This corresponds to the two charts needed to cover S^8: on the first chart

$$D_c\Theta_A D_c\Theta_A \neq 0; \tag{24}$$

on the other chart

$$D_c\Theta_{\dot{A}} D_c\Theta_{\dot{A}} \neq 0. \tag{25}$$

Suppose that we deal with the first chart (24) and consider the superconformal transformation (19):

$$D'_a\Theta'_A = D'_a\eta_b\; D_b\Theta_A. \tag{26}$$

Clearly, the parameter $D'_a\eta_b$ has the same content as $D_a\Theta_A$, see (18) (this means that the superconformal group is spontaneously broken). The conclusion is that on the first chart the matrix $D_a\Theta_A$ can be gauged into the unit matrix

$$D_a\Theta_A = \delta_{aA}. \tag{27}$$

This gauge identifies the O(8) and $SO^\uparrow(1,1)$ subgroups of the superconformal group with the O(8) and $SO^\uparrow(1,1)$ subgroups of the Lorentz group $O^\uparrow(1,9)$. In particular, the indices $a, b, \ldots$ now correspond to the $\mathbf{8}_s$ representaion of O(8) and the world-line coordinates and derivatives carry the following $SO^\uparrow(1,1)$ weights: $\tau \rightarrow \tau^{--}$, $\eta_a \rightarrow \eta^-_a$, $D_a \rightarrow D^+_a$, $\partial_\tau \rightarrow \partial^{++}_\tau$.

Next, we substitute the gauge (27) in the transverse projection (22):

$$\gamma^i_{a\dot{B}}D_b\Theta_{\dot{B}} + (a\leftrightarrow b) = \frac{1}{4}\delta_{ab}\gamma^i_{c\dot{B}}D_c\Theta_{\dot{B}}. \tag{28}$$

The matrix $D_a\Theta_{\dot{A}}$ describes a reducible O(8) representation [23], $\mathbf{8}_s\times\mathbf{8}_c = \mathbf{8}_v + \mathbf{56}_v$ or $D_a\Theta_{\dot{A}} = \gamma^i_{a\dot{A}}Y^i + \gamma^{[ijk]}_{a\dot{A}}Y^{ijk}$. Only the $\mathbf{8}_v$ part solves (28), so we find

$$D_a\Theta_{\dot{A}} = \gamma^i_{a\dot{A}}Y^i. \tag{29}$$

Inserting this into (23) we see that it is solved as well.

Thus, on the first chart the general solution to (21)–(23) (modulo gauge transformations) is parametrized by eight coordinates Y^i. A similar analysis for the second chart (25) leads to another set of coordinates:

$$D_a\Theta_A = \gamma^i_{aA}Z^i, \quad D_a\Theta_{\dot{A}} = \delta_{a\dot{A}}. \tag{30}$$

Now we should relate these two sets of coordinates in the overlapping area, i.e., when both conditions (24) and (25) are satisfied. To this end consider the vector appearing in the right-hand side (RHS) of (17),

$$v^\mu = \frac{1}{8}D_a\Theta\gamma^\mu D_a\Theta. \tag{31}$$

This is a lightlike vector, $v^\mu v_\mu = 0$ due to the identity (13) and Eq. (17). It is O(8) invariant and has a nonvanishing $SO^\uparrow(1,1)$ weight. Hence the ratio of any two of its components is gauge invariant. The two charts described above correspond to $v^{--} \neq 0$ and $v^{++} \neq 0$, respectively. Using (27), (29)–(31) one finds

$$Y^i = \frac{v^i}{v^{--}}, \quad Z^i = \frac{v^i}{v^{++}}. \tag{32}$$

In the overlapping area $Y^i = Z^i/(Z^j)^2$, so Y^i and Z^i can be considered as stereographic coordinates of the sphere S^8.

In conclusion we can say that the eight commuting spinors ("twistor variables") $D_a\Theta^\alpha|_{\eta=0}$ have a double role. On the one hand, they are the superpartners of the Grassmann coordinates of target superspace, $\theta^\alpha = \Theta^\alpha|_{\eta=0}$ with respect to $N = 8$ world-line supersymmetry. On the other hand, they are Lorentz-harmonic coordinates on the sphere S^8, regarded as the coset (20). The requirement of double supersymmetry (worldline and targetspace) establishes a natural link between these two concepts [6,11,8,16]. We stress also that the sphere S^8 and the related lightlike vector (31) appear on shell only, while off shell the eight Majorana spinors $D_a\Theta^\alpha|_{\eta=0}$ parametrize a larger manifold.

Having clarified the geometric meaning of the superparticle equation of motion (16) and of its gauge group (18), (19), now we come back to the analysis of the content of that equation. In what follows we shall deal with the first chart on S^8, so the gauge (27) is implied below. This gauge completely fixes the superconformal group, up to a constant translation and a supertranslation:

$$\delta\tau = \rho + i\eta_a\epsilon_a, \quad \delta\eta_a = \epsilon_a. \tag{33}$$

The general solution to (27) is given by

$$\Theta_A = \eta_A, \tag{34}$$

where we have used the constant world supersymmetry parameter (33) to fix the possible constant term in (34), thus identifying the rigid $N = 8$ world-line supersymmetry with one half of ten-dimensional supersymmetry.

The above results allow us to considerably simplify the original equation (16). In the light-cone gauge it has the three projections

$$D_aX^{--} - 2i\eta_a = 0, \tag{35}$$

$$D_aX^i - i\gamma^i_{a\dot{B}}\Theta_{\dot{B}} - iD_a\Theta_{\dot{B}}\gamma^i_{\dot{B}B}\eta_B = 0, \tag{36}$$

$$D_aX^{++} - i(D_a\Theta_{\dot{B}})\Theta_{\dot{B}} = 0. \tag{37}$$

The first one, Eq. (35), has the obvious solution

$$X^{--} = 2\tau^{--}, \tag{38}$$

where we have fixed the possible constant term in the *RHS* by means of the translation from (33). Hitting (36) with D_a and using (27), (29) one obtains

$$Y^i = \frac{1}{2}(\dot{X}^i - i\dot{\Theta}\gamma^i\eta), \tag{39}$$

where the overdot denotes $\partial/\partial\tau^{--}$. Combining (29), (36), and (39) one finds

$$D_a\Theta_{\dot{B}} = \frac{1}{2}\gamma^i_{a\dot{B}}(\dot{X}^i - i\dot{\Theta}\gamma^i\eta), \tag{40}$$

$$D_aX^i = i\gamma^i_{a\dot{B}}\Theta_{\dot{B}} + \frac{i}{2}\gamma^k_{a\dot{B}}\gamma^i_{b\dot{B}}\eta_b(\dot{X}^k - i\dot{\Theta}\gamma^k\eta). \tag{41}$$

This allows us to conclude that all the components of X^i and $\Theta_{\dot{A}}$ are expressed in terms of their lowest-order components $x^i(\tau) = X^i|_{\eta=0}$ and $\theta_{\dot{A}}(\tau) = \Theta_{\dot{A}}|_{\eta=0}$. It is clear from (37) that the same applies to X^{++} [the solution to (37) will be given later, see (66)]. It is rather remarkable that the content of Eq. (16) is so simple, given the original complexity of the $N = 8$ superfields under consideration.

The careful reader may have noticed that (16) does not restrict the τ dependence of the world-line fields $x^i(\tau)$ and $\theta_{\dot{A}}(\tau)$. The on-shell equations for these fields will be found in the next section.

IV. EQUATIONS OF MOTION AND GAUGE FIXING FOR THE LAGRANGE MULTIPLIER

In the previous section we studied the equation of motion (16) obtained from the action (3) by varying with respect to the Lagrange multiplier P^μ_a. Usually, employing a Lagrange multiplier is not a safe trick, since in addition to the desired propagating modes in X and Θ, it could also give rise to a number of extra ones, coming from the equations for the Lagrange multiplier itself. One of the most unexpected features of the action (3) is that this danger is miraculously avoided. Indeed, now we shall demonstrate that the equations of motion for the Lagrange multiplier P^μ_a following from the variation of X^μ and Θ^α, together with the powerful abelian gauge invariance (12) reduce P^μ_a to a single constant $D = 10$ vector p^μ [see (15)], which is just the on-shell momentum of the superparticle. They also imply the correct dynamical equations for $x^i(\tau)$ and $\theta_{\dot{A}}(\tau)$.

The equations under consideration are

$$D_aP^\mu_a = 0, \tag{42}$$

$$(\gamma_\mu D_a\Theta)_\alpha P^\mu_a = 0. \tag{43}$$

Because of the complexity of the $N = 8$ superfields the analysis of Eqs. (42) and (43) and of the gauge transformation (12) is rather involved and it is worthwhile to first sketch the procedure. We decompose these equations in the light-cone frame and use the light-cone gauge (27).

Then we study the $(--)$ projections of (42) and (15) to show that P_a^{--} is reduced to a single constant component in a suitable Wess-Zumino (WZ) gauge for the invariance (12). Next we consider the transverse projections of (42) and (15) and the $\alpha = \dot{A}$ projection of (43) to find the WZ gauge of P_a^i on shell. Finally, we take the $(++)$ projections of (42), (15) and the $\alpha = A$ projection of (43) to find an expression for P_a^{++}.

Before proceeding further, we mention that as a consequence of (42) the invaraint of the Abelian gauge transformations $p^\mu = \epsilon^{a_1 \cdots a_8} D_{a_1} \cdots D_{a_7} P^\mu_{a_8}$ (15) is a constant,

$$D_a p^\mu = 0 \;\rightarrow\; p^\mu = \text{const.} \tag{44}$$

This follows from the relation

$$D_a p^\mu = \epsilon_{ab_1 \cdots b_7} D_{b_1} \cdots D_{b_7} D_c P^\mu_c, \tag{45}$$

which can be proved using the identities

$$\begin{aligned} D_{a_1} D_{[a_2} \cdots D_{a_k]} = {} & D_{[a_1} \cdots D_{a_k]} \\ & + i(k-1)\delta_{a_1[a_2} D_{a_3} \cdots D_{a_k]} \partial_\tau, \end{aligned} \tag{46}$$

$$\begin{aligned} D_{[a_2} \cdots D_{a_k]} D_{a_1} = {} & D_{[a_2} \cdots D_{a_k} D_{a_1]} \\ & + i(k-1) D_{[a_2} \cdots D_{a_{k-1}} \delta_{a_k]a_1} \partial_\tau. \end{aligned}$$

Let us first study the $(--)$ light-cone projection of (42). Using the light-cone gauge condition (27) one finds the following gauge transformation (12) for P_a^{--}:

$$\delta P_a^{--} = D_b(\xi^D_{abc} D_c \Theta_D) \equiv D_b \xi_{ab}, \tag{47}$$

where $\xi_{ab} = \xi^c_{abc}$ is an arbitrary symmetric traceless superfield. On shell, the *only* invariant of the gauge transformation (47) is the p^{--} component of p^μ. This can be proved as follows. The component content of an $N = 8$ world-line superfield can be extracted by the derivatives D_a, $D_{[a_1} D_{a_2]}$, $\ldots, D_{[a_1} D_{a_2} \cdots D_{a_8]}$ at $\eta = 0$. Now, the lowest-order component P_a^{--} is shifted by an arbitrary parameter $D_b \xi_{ab}$ and can therefore be completely gauged away. The next component, $D_c P_a^{--}$ is shifted by $D_c D_b \xi_{ab}$, which is an arbitrary traceless parameter and hence only the trace $D_a P_a^{--}$ survives, but the latter vanishes on shell (42). Proceeding in this manner we find that up to the level of six derivatives the only gauge invariant components are the derivatives of the tensor $D_a P_a^{--}$, i.e., $D_a P_a^{--}$, $D_b D_a P_a^{--}$, $D_{[b_1} D_{b_2]} D_a P_a^{--}, \ldots, D_{[b_1} \cdots D_{b_5]} D_a P_a^{--}$, but all of them vanish on shell (42). At the level of seven derivatives the component $D_{[b_1} \cdots D_{b_7]} P_a$ contains the O(8) representations $\mathbf{8}_s \times \mathbf{8}_s = \mathbf{1}_s + \mathbf{28} + \mathbf{35}_s$, while the parameter $D_{[b_1} \ldots D_{b_7]} D_b \xi_{ab}$ contains $\mathbf{35}_s$ only. The surviving representations $\mathbf{1}_s$ and $\mathbf{28}$ correspond to p^{--} and $D_{[b_1} \ldots D_{b_6]} D_a P_a^{--} = 0$, respectively. So, we conclude that on shell and in the ξ Wess-Zumino gauge described above the superfield P_a^{--} contains one constant component only:

$$P_a^{--} = \frac{1}{7!8!} \epsilon_{ab_1 \cdots b_7} \eta_{b_1} \cdots \eta_{b_7} p^{--}, \quad p^{--} = \text{const.} \tag{48}$$

For the transverse projection P_a^i the gauge transformation (12) can be written in a way similar to (47):

$$\delta P_a^i = D_b \xi^i_{ab}, \tag{49}$$

where

$$\begin{aligned} \xi^i_{ab} &= (\xi^i_{ab})_{\text{add}} + \xi^A_{abc} \gamma^i_{A\dot{B}} D_c \Theta_{\dot{B}}, \\ (\xi^i_{ab})_{\text{add}} &\equiv \xi^{\dot{B}}_{abc} \gamma^i_{c\dot{B}}. \end{aligned} \tag{50}$$

Note that ξ^i_{ab} consists of two terms, of which only the first one is field independent, i.e., can be used to gauge away parts of P_a^i. This additive term is constrained as follows:

$$\gamma^i_{a\dot{A}} (\xi^i_{ab})_{\text{add}} = \delta_{\dot{A}\dot{B}} \xi^{\dot{B}}_{aba} = 0. \tag{51}$$

It is easy to see that this is the only condition on the additive part. Indeed, the parameter ξ^i_{ab} is symmetric and traceless with respect to the O(8) indices a, b, therefore it contains $\mathbf{35}_s \times \mathbf{8}_v = \mathbf{56}_v + \mathbf{224}_{sv}$. At the same time, the *LHS* of (51) contains $\mathbf{8}_s \times \mathbf{8}_c = \mathbf{8}_v + \mathbf{56}_v$, so (51) kills the $\mathbf{56}_v$ in $(\xi^i_{ab})_{\text{add}}$. Thus the latter corresponds to the irreducible representation $\mathbf{224}_{sv}$, which cannot be restricted any more.

The restriction (51) implies the existence of another "tensor"

$$T_{\dot{A}} = \gamma^i_{a\dot{A}} P_a^i. \tag{52}$$

Though noninvariant under the second, field-dependent term in (51), $T_{\dot{A}}$ is invariant under the $(\xi^i_{ab})_{\text{add}}$ transformations, so it cannot be gauged away. Instead, it is expressed in terms of the other fields by means of the second equation for the Lagrange multiplier (43). In the light-cone frame the latter reads

$$-\frac{1}{2}(\gamma^{++} D_a \Theta)_\alpha P_a^{--} - \frac{1}{2}(\gamma^{--} D_a \Theta)_\alpha P_a^{++} + (\gamma^i D_a \Theta)_\alpha P_a^i = 0. \tag{53}$$

The $\alpha = \dot{A}$ projection of this equation is given by

$$\gamma^i_{a\dot{A}} (P_a^i - Y^i P_a^{--}) = 0 \;\rightarrow\; T_{\dot{A}} = \gamma^i_{a\dot{A}} Y^i P_a^{--}. \tag{54}$$

Studying the WZ gauge for P_a^i, one finds that up to the sixth order all the derivatives $D_{[a_1} \cdots D_{a_k]} P_b^i$, $(k \leq 6)$ are expressed in terms of derivatives of $D_a P_a^i$ and $T_{\dot{A}}$ and therefore vanish on shell and in the WZ gauge (48) for P_a^{--}. The seventh order is less trivial and we give some details for the inquiring reader. First, the additive part of the gauge parameter at this order is $D_{[a_1} \cdots D_{a_7]} D_b \xi^i_{ab}$ and it contains $\mathbf{8}_s \times \mathbf{56}_c = \mathbf{8}_v + \mathbf{56}_v + \mathbf{160}_v + \mathbf{224}_{sv}$ [with the restriction (51) taken into account]. However, among the latter $\mathbf{28} \times \mathbf{8}_v = \mathbf{8}_v + \mathbf{56}_v + \mathbf{160}_v$ vanish due to the relation $D_{[a_1} \cdots D_{a_6]} D_a D_b \xi^i_{ab} = 0$. Hence the seventh-order parameter contains $\mathbf{224}_{sv}$ only. Second, the seventh order in P_a^i, i.e., $D_{[a_1} \cdots D_{a_7]} P_b^i$ contains $\mathbf{8}_s \times \mathbf{8}_s \times \mathbf{8}_v = (8_v)^2 + (56_v)^2 + \mathbf{160}_v + \mathbf{224}_{sv}$, where the $\mathbf{224}_{sv}$ is a pure gauge. Third, at the seventh order there are the following tensor components: $D_{[a_1} \cdots D_{a_6]} D_b P^i_b \rightarrow \mathbf{28} \times \mathbf{8}_v = \mathbf{8}_v + \mathbf{56}_v + \mathbf{160}_v$, $p^i \rightarrow \mathbf{8}_v$, $D_{[a_1} \cdots D_{a_7]} T_{\dot{A}} \rightarrow \mathbf{8}_s \times \mathbf{8}_c = \mathbf{8}_v + \mathbf{56}_v$. This matches the gauge-invariant content of $D_{[a_1} \cdots D_{a_7]} P^i_b$ except for one extra $\mathbf{8}_v$. Hence there should exist a relation among the three $\mathbf{8}_v$ in $D_{[a_1} \cdots D_{a_6]} D_b P^i_b$, p^i and $D_{[a_1} \cdots D_{a_7]} T_{\dot{A}}$. It is given by

$$p^i = \gamma^i_{b\dot{A}} \epsilon_{a_1 \cdots a_7 b} D_{[a_1} \cdots D_{a_7]} T_{\dot{A}} + \frac{7}{2} \gamma^i_{b\dot{A}} \gamma^j_{c\dot{A}} \epsilon_{bca_1 \cdots a_6} D_{[a_1} \cdots D_{a_6]} D_d P^j_d, \tag{55}$$

up to some fifth-order terms, which were already shown to vanish on shell and in the WZ gauge. Using (42) and (54) we find

$$p^i = \gamma^i_{b\dot{A}} \epsilon_{a_1 \cdots a_7 b} D_{[a_1} \cdots D_{a_7]} (\gamma^k_{c\dot{A}} Y^k P^{--}_c). \tag{56}$$

Substituting (39) and (48) in (56), we arrive at the important equations

$$p^i = \frac{1}{2} \dot{x}^i p^{--}, \quad p^{--} \dot{\theta}_{\dot{A}} = 0, \tag{57}$$

which will be discussed later on, see (65). To finish with P^i_a, its on-shell expression in the WZ gauge contains only one constant O(8) vector:

$$P^i_a = \frac{1}{7!8!} \epsilon_{ab_1 \cdots b_7} \eta_{b_1} \cdots \eta_{b_7} p^i, \quad p^i = \text{const}. \tag{58}$$

The $(++)$ projection of P^μ_a is an auxiliary superfield. This can be seen from the $\alpha = A$ projection of (53):

$$P^{++}_A - \gamma^i_{A\dot{A}} \gamma^k_{b\dot{A}} Y^k P^i_b = 0. \tag{59}$$

With the help of (39), (57), and (58) this implies

$$P^{++}_a = \frac{1}{7!8!} \epsilon_{ab_1 \cdots b_7} \eta_{b_1} \cdots \eta_{b_7} p^{++}, \tag{60}$$

where

$$p^{++} = \frac{1}{2} \dot{x}^i p^i = \frac{1}{4} (\dot{x}^i)^2 p^{--} = \text{const}. \tag{61}$$

Combining Eqs. (48), (57), (58), (60), and (61) we find

$$P^\mu_a = \frac{1}{7!8!} \epsilon_{ab_1 \cdots b_7} \eta_{b_1} \cdots \eta_{b_7} p^\mu, \tag{62}$$

where p^μ is a constant $D = 10$ vector [see (44)] with the light-cone projections

$$p^\mu = (p^{--},\ p^i,\ p^{++}) = p^{--}(1,\ \tfrac{1}{2}\dot{x}^i,\ \tfrac{1}{4}(\dot{x}^i)^2). \tag{63}$$

This is the momentum of the massless superparticle:

$$p^\mu p_\mu = -p^{--} p^{++} + (p^i)^2 = 0. \tag{64}$$

Now we are in a position to justify the kinematical requirement (15). On the chart (24) of S^8 the condition $p^\mu \neq 0$ is equivalent to $p^{--} \neq 0$. Given this, from (57) one derives the on-shell equations for the superparticle variables:

$$\ddot{x}^i = 0, \quad \dot{\theta}_{\dot{A}} = 0. \tag{65}$$

The conclusion is that (15) is a nondegeneracy condition for the Chern-Simons action (3), similar to the condition $e \neq 0$ on the einbein in the ordinary relativistic particle theory (2).

We mention that Eq. (37) can now be solved too:

$$X^{++} = x^{++} + i\eta_a \gamma^i_{a\dot{A}} \theta_{\dot{A}} \dot{x}^i, \quad \dot{x}^{++} = \frac{1}{2} (\dot{x}^i)^2. \tag{66}$$

The final expressions for the other world-line superfields are

$$X^{--} = 2\tau^{--}, \quad X^i = x^i + i\eta_a \gamma^i_{a\dot{A}} \theta_{\dot{A}}, \qquad \Theta_A = \eta_A, \quad \Theta_{\dot{A}} = \theta_{\dot{A}} + \frac{1}{2} \eta_a \gamma^i_{a\dot{A}} \dot{x}^i. \tag{67}$$

The superparticle velocity is given by (77)

$$\dot{x}^\mu = (\dot{x}^{--}, \dot{x}^i, \dot{x}^{++}) = (2, \dot{x}^i, \tfrac{1}{2}(\dot{x}^i)^2). \tag{68}$$

Comparing (63) and (68) we find

$$p^\mu = \frac{1}{2} p^{--} \dot{x}^\mu. \tag{69}$$

Note that the right-hand side of (69) is proportional to $D_a \Theta \gamma^\mu D_a \Theta|_{\eta=0}$ when computed in the light-cone gauge. In fact, the relation

$$p^\mu \sim D_a \Theta \gamma^\mu D_a \Theta \tag{70}$$

is gauge independent. It can be derived from the gauge-invariant equation

$$p^\mu (\gamma_\mu D_a \Theta)_\alpha = 0, \tag{71}$$

which is a consequnce of the equations of motion (17), (42), and (43).

So, we have solved all the equations of motion following from the action (3). The results should be compared with those from the Brink-Schwarz action

$$S_{\text{BS}} = \int d\tau [p_\mu (\dot{x}^\mu - i\dot{\theta}\gamma^\mu \theta) - \tfrac{1}{2} e p_\mu p^\mu]. \tag{72}$$

The latter is invariant under world-line diffeomorphisms $\tau \rightarrow \tau'(\tau)$ and under the κ transformations

$$\delta p_\mu = 0, \quad \delta\theta^\alpha = p_\mu (\gamma^\mu)^{\alpha\beta} \kappa_\beta, \qquad \delta \dot{x}^\mu = -i\theta\gamma^\mu \delta\theta, \quad \delta e = -4i\dot{\theta}^\alpha \kappa_\alpha. \tag{73}$$

The equations of motion following from (72),

$$\dot{p}_\mu = 0, \quad p^2 = 0, \quad \dot{x}^\mu - i\dot{\theta}\gamma^\mu\theta - ep^\mu = 0, \quad p_\mu (\gamma^\mu \dot{\theta})_\alpha = 0, \tag{74}$$

can be easily solved in the reparametrization and κ-symmetry gauges $x^{--} = 2\tau$ and $\theta_A = 0$. Thus one can identify the physical modes corresponding to the $N = 8$ supersymmetric action (3) and the Brink-Schwarz action (72).

As we have pointed out earlier, in the action (3) κ symmetry is completely replaced by local world-line $N = 8$

supersymmetry. Now we can show the on-shell relation between these two symmetries. Suppose that we do not fix the superconformal group completely, as in (27), (33), but instead keep only the $N=8$ supersymmetry parameter local, $\epsilon_a = \delta\eta_a|_{\eta=0}$. The $N=8$ supersymmetry transformation of $\theta^\alpha = \Theta^\alpha|_{\eta=0}$ is given by

$$\delta\theta^\alpha = \epsilon_a \psi_a^\alpha, \tag{75}$$

where $\psi_a^\alpha = D_a\Theta^\alpha|_{\eta=0}$. Let us substitute in (75) the field-dependent parameter $\epsilon_a = p^{--}\psi_a^\alpha \kappa_\alpha(\tau)$ and make use of the relations

$$\psi_a^\alpha \psi_a^\beta = \frac{1}{16}(\gamma_\mu)^{\alpha\beta}\psi_a\gamma^\mu\psi_a, \tag{76}$$

$$\dot{x}^\mu - i\dot{\theta}\gamma^\mu\theta = \frac{1}{8}\psi_a\gamma^\mu\psi_a. \tag{77}$$

Equation (76) can be proved with the help of the gauge (27) and Eq. (29) (but it is valid in any gauge), Eq. (77) is obtained from (16) by hitting it with D_a. Putting all this in (75) we find the κ transformation of θ^α (73). The conclusion is that κ symmetry emerges as a result of an almost complete gauge fixing of the $N=8$ superconformal group and a partial use of the equations of motion.

V. COUPLING TO A D=10 SUPERGRAVITY AND MAXWELL BACKGROUND

The action (3) describes a free superparticle theory. Therefore it is of interest to see how it can be coupled to background $D=10$ fields. Introducing a supergravity background is straightforward. It is sufficient to replace the flat D=10 superspace vielbeins in (3) by curved ones, $E_M^{\hat{A}}(Z^N)$. Here $Z^M = (X^\mu, \Theta^\alpha)$ and $\hat{A} = (\hat{\mu}, \hat{\alpha})$ are the tangent-space vector and spinor Lorentz indices. After that (3) becomes

$$S_{\rm SG} = i\int d\tau d^8\eta\, P_{a\hat{\mu}} D_a E^{\hat{\mu}}, \tag{78}$$

where we used the notation $D_a E^{\hat{A}} \equiv (D_a Z^M) E_M^{\hat{A}}$. Note that the Lagrange multiplier is now a tangent-space Lorentz vector. The invariance of the action (78) under $N=8$ world-line conformal supersymmetry, target-space diffeomorphisms and tangent-space Lorentz transformations is manifest. As we saw earlier, the consistency of the superparticle action crucially depended on the Abelian gauge invariance (12) of the Lagrange multiplier. This invariance is not automatic in (78), and we have to make sure it still works. The obvious generalization of (12) to the curved case is

$$\delta P_{a\hat{\mu}} = (\delta_{\hat{\mu}}{}^{\hat{\nu}} D_b + D_b E^{\hat{A}} \omega_{\hat{A}\hat{\mu}}{}^{\hat{\nu}}) \left[\xi_{abc}^{\hat{\alpha}} (\gamma_{\hat{\nu}})_{\hat{\alpha}\hat{\beta}} D_c E^{\hat{\beta}}\right]. \tag{79}$$

It is not hard to obtain the variation of (78) with respect to (79):

$$\begin{aligned}\delta S_{\rm SG} = \frac{i}{2}\int d\tau\, d^8\eta\, (\xi_{abc}\gamma_{\hat{\mu}})_{\hat{\alpha}} D_c E^{\hat{\alpha}} \\ \times[-D_a E^{\hat{\beta}} D_b E^{\hat{\gamma}} T^{\hat{\mu}}_{\hat{\gamma}\hat{\beta}} - 2 D_a E^{\hat{\beta}} D_b E^{\hat{\nu}} T^{\hat{\mu}}_{\hat{\nu}\hat{\beta}} \\ + D_a E^{\hat{\lambda}} D_b E^{\hat{\nu}} T^{\hat{\mu}}_{\hat{\nu}\hat{\lambda}}].\end{aligned} \tag{80}$$

Here $T^{\hat{C}}_{\hat{A}\hat{B}}$ and $\omega_{\hat{A}\hat{B}}{}^{\hat{C}}$ are the background supergravity torsion and Lorentz connection. It is clear that the second and the third terms in (80) can be compensated for by the following variation of the Lagrange multiplier:

$$\delta P_{b\hat{\nu}} = (\xi_{abc}\gamma_{\hat{\mu}} D_c E)[D_a E^{\hat{B}} T^{\hat{\mu}}_{\hat{\nu}\hat{B}} - \tfrac{1}{2} D_a E^{\hat{\rho}} T^{\hat{\mu}}_{\hat{\nu}\hat{\rho}}]. \tag{81}$$

As to the first term in (80), it vanishes due to the γ-matrix identity (13) and the symmetry of ξ_{abc}, provided we impose the following $D=10$ supergravity torsion constraint [24]:

$$T^{\hat{\mu}}_{\hat{\alpha}\hat{\beta}} = (\gamma^{\hat{\mu}})_{\hat{\alpha}\hat{\beta}}. \tag{82}$$

The conclusion is that the consistency of the superparticle action requires constraints on the background. This phenomenon is well known [25]. In the Brink-Schwarz action one demands compatibility of the background with κ symmetry, and that leads to the constraint (82). In our case κ symmetry is replaced by world-line conformal supersymmetry, which is manifest in (78). Instead, we had to make the background compatible with the Lagrange multiplier gauge invariance (79), which lead us to the same constraint.

The next issue we would like to discuss is the coupling of the superparticle to a Maxwell supersymmetric background. This can be done in the framework of a Kaluza-Klein scenario. Namely, we extend the action (78) by adding two new terms:

$$S_{{\rm SG}+M} = i\int d\tau\, d^8\eta\, \left(P_{a\hat{\mu}} D_a E^{\hat{\mu}} + P_a D_a A + P_a D_a X\right). \tag{83}$$

Here $D_a A \equiv D_a Z^M A_M(Z)$, where $A_M = (A_\mu, A_\alpha)$ are the Maxwell background connections defined modulo Abelian gauge transformations:

$$\delta A_M = \partial_M \lambda(Z). \tag{84}$$

We emphasize the absence of a Maxwell coupling constant (electric charge) in the action (83). Instead, there we find a new world-line superfield $P_a(\tau,\eta)$, which plays the same role of a Lagrange multiplier for the Maxwell term as $P_{a\hat{\mu}}$ plays for the first term. In what follows we shall show that on shell the only surviving component of P_a is just the electric charge of the superparticle. The superflied $X(\tau,\eta)$ can be interpreted as an additional Kaluza-Klein bosonic coordinate of target superspace. Indeed, the action (83) is invariant under the Maxwell transformations (84) provided

$$\delta P_a = 0, \quad \delta X = -\lambda. \tag{85}$$

Then the Maxwell superfield A_M can be treated as part of the supervielbeins $E_M^{\hat{A}}$ in a $D=11$ target space, according to the standard Kaluza-Klein philosophy. The action is also world-line superconformally invariant if we take X to be a scalar, and P_a to transform in the same way as $P_{a\mu}$ in (11).

As in the case of supergravity, we have to make sure that the Maxwell terms in (83) are invariant under the following Abelian gauge transformations of the Lagrange multipler P_a [cf. (47)]:

$$\delta P_a = D_b \xi_{ab}, \tag{86}$$

where $\xi_{ab}(\tau,\eta)$ is an arbitrary symmetric and traceless parameter. The variation of (83) is easily shown to be

$$\delta S_{\mathrm{SG}+M} = \frac{i}{2}\int d\tau\, d^8\eta\, \xi_{ab}\Big(D_a E^{\hat\alpha} D_b E^{\hat\beta} F_{\hat\beta\hat\alpha} +2 D_a E^{\hat A} D_b E^{\hat\nu} F_{\hat\nu\hat A}\Big), \tag{87}$$

where $F_{\hat A\hat B}$ is the Maxwell field-strength. Once again, the second term in (87) can be compensated for by

$$\delta P_{b\hat\nu} = -\xi_{ab} D_a E^{\hat A} F_{\hat\nu\hat A}, \tag{88}$$

and the first term vanishes after imposing the $D = 10$ Maxwell constraint [25]

$$F_{\hat\alpha\hat\beta} = 0. \tag{89}$$

Note that (89) is at the same time the integrability condition for the equation obtained from varying (83) with respect to P_a:

$$D_a X = -D_a A. \tag{90}$$

In other words, because of (89) Eq. (90) allows us to solve for $X(\tau,\eta)$ in terms of the other fields (up to a constant), without imposing any new restrictions on them.

Now we come to the important point about the origin of the electric charge in the action (83). Varying it with respect to the Kaluza-Klein field X, one gets $D_a P_a = 0$. Repeating the arguments of Sec. IV, we see that this equation, together with the gauge invariance (86) leave just one constant component in P_a [cf. (47) and (48)]:

$$P_a = \frac{1}{7!8!}\epsilon_{ab_1\cdots b_7}\eta_{b_1}\cdots\eta_{b_7} e, \quad e = \text{const.} \tag{91}$$

Substituting this into (83) gives, in particular, the component term

$$\int d\tau e \dot x^\mu A_\mu(x), \tag{92}$$

which is the usual Maxwell coupling for a charged particle. Thus, we conclude that the integration constant e is indeed the electric charge of the superparticle [26].

We would like to point out that the existence of the Maxwell coupling (83), which is invaraint under the $N = 8$ superconformal group and under $D = 10$ superdiffeomorphisms, is a highly nontrivial phenomenon. Suppose that we were given the electric charge e, the Maxwell superfields A_μ, A_α, the superparticle coordinates $X^\mu(\tau,\eta), \Theta^\alpha(\tau,\eta)$ and their derivatives. Then by simple dimensional arguments we would immediately conclude that it is impossible to construct any off-shell $N = 8$ invariant coupling *bilinear* in the Maxwell fields and the target-space coordinates [as required by (92)]. The striking property of the Maxwell coupling (83) is the absense of an off-shell electric charge. Instead, it is *trilinear* in the fields, and only *on shell*, where one of the fields (the Lagrange multiplier P_a) reduces to the constant e, the action becomes bilinear.

The mechanism explained above may suggest a loophole in the various "no-go" theorems [27] that forbid the existence of off-shell supersymmetric actions for theories such as $D = 10$, $N = 1$ or $D = 4$, $N = 4$ super-Yang-Mills, etc. The point is that those theorems always assume *bilinearity* in the fields. As we have seen, this assumption could be wrong off shell, although it should definitely hold on shell (or rather upon elimination of the auxiliary fields) [28].

VI. COMPLEX STRUCTURES AND GRASSMANN ANALYTICITY IN THE CASES $N = 2, D = 4$ AND $N = 4, D = 6$

The analysis of the superparticle action (3) carried out in the most complex case $N = 8, D = 10$ can easily be adapted to the simpler cases of the superparticle moving in $D = 3, 4$, and 6 superspaces [where the identity (13) holds]. To this end one should consider Θ^α as *real* (Majorana) spinors in those dimensions, and the world line should have $N = 1, 2$, and 4 conformal supersymmetry, correspondingly [29]. As we already mentioned, the $N = 1, D = 3$ analogue of (3) is just the STVZ action [6]. An essential feature of all these actions is the absense of any complex structure: all superfields and gauge parameters are *real* functios of *real* variables. However, the $N = 2, D = 4$ and $N = 4, D = 6$ cases allow for a different treatment [8], which utilizes the concept of *complex* structures inherent in four and six dimensions, and the related concept of Grassmann analyticity. In this section we shall show how those two alternative treatments, the real and the complex ones, are related to each other.

In four dimensions the Majorana spinor Θ^α has four real components. However, one can think of it as a complex two-component Weyl spinor (and its conjugate). Indeed, using the matrix γ^5 [which is an example of a complex structure, $(\gamma^5)^2 = -1$] one can construct projection operators that split Θ^α into two complex halves. In other words, one can introduce the well-known two-component spinor notation

$$(X^\mu,\ \Theta^\alpha) \ \rightarrow\ (X^{\alpha\dot\alpha},\ \Theta^\alpha,\ \bar\Theta^{\dot\alpha}), \tag{93}$$

where now α and $\dot\alpha$ take two values. Note also that the matrix γ^5 generates a U(1) automorphism of the $D = 4$ supersymmetry algebra, which *commutes* with the Lorentz group $\mathrm{SO}(1,3) \sim \mathrm{SL}(2,C)$. In the two-component formalism Eq. (16) has the form

$$D_a X^{\alpha\dot\alpha} - i(D_a\Theta^\alpha)\bar\Theta^{\dot\alpha} - i(D_a\bar\Theta^{\dot\alpha})\Theta^\alpha = 0. \tag{94}$$

Here $a = 1, 2$ (we recall that in the case $D = 4$ the world line has $N = 2$ supersymmetry). Further, Eq. (17) now becomes

$$D_a\Theta^\alpha D_b\bar{\Theta}^{\dot\alpha} + D_b\Theta^\alpha D_a\bar{\Theta}^{\dot\alpha} = \delta_{ab}D_c\Theta^\alpha D_c\bar{\Theta}^{\dot\alpha}. \quad (95)$$

Introducing the complex notation

$$D = D_1 + iD_2, \quad \bar{D} = D_1 - iD_2, \quad (96)$$

one can rewrite (95) as

$$D\Theta^\alpha D\bar{\Theta}^{\dot\alpha} = 0. \quad (97)$$

This equation has two possible solutions: $D\Theta^\alpha = 0$ or $D\bar{\Theta}^{\dot\alpha} = 0$. Both of them cannot vanish at the same time, since this would contradict the basic nondegeneracy condition (14). Without loss of generality we can choose the solution

$$D\bar{\Theta}^{\dot\alpha} = 0 \quad \rightarrow \quad \bar{D}\Theta^\alpha = 0. \quad (98)$$

These are nothing but chirality conditions for the superfields Θ^α and $\bar{\Theta}^{\dot\alpha}$. Thus we see that the natural complex structure of the $D = 4$ spinors (represented by γ^5), together with the Lorentz-harmonic defining condition (17) induce a complex structure in the world-line superspace of the superparticle. Further, Eq. (94) implies chirality of X as well. Indeed, introducing the notation

$$X_L^{\alpha\dot\alpha} = X^{\alpha\dot\alpha} + i\Theta^\alpha\bar{\Theta}^{\dot\alpha}, \quad X_R^{\alpha\dot\alpha} = X^{\alpha\dot\alpha} - i\Theta^\alpha\bar{\Theta}^{\dot\alpha}, \quad (99)$$

we can rewrite (94) as a chirality condition:

$$\bar{D}X_L^{\alpha\dot\alpha} = 0 \quad \text{or} \quad DX_R^{\alpha\dot\alpha} = 0. \quad (100)$$

So, in the case $N = 2, D = 4$ the physical content of the superparticle is encoded in the chirality conditions (98), (100) and the definition (99). Actually, as it was explained in Ref. [8], one can reverse the argument as follows. Off shell one does not impose the relations (99). Solving the chirality conditions (98), (100) in suitable chiral bases in the world-line superspace, one considers $X_L^{\alpha\dot\alpha}$ and Θ^α as *unconstrained* chiral superfields. They become the basic dynamical variables. Then one treats Eq. (99) as a definition of the real coordinate $X^{\alpha\dot\alpha} = \frac{1}{2}(X_L^{\alpha\dot\alpha} + X_R^{\alpha\dot\alpha})$, as well as an equation of motion,

$$\frac{i}{2}(X_L^{\alpha\dot\alpha} - X_R^{\alpha\dot\alpha}) + \Theta^\alpha\bar{\Theta}^{\dot\alpha} = 0. \quad (101)$$

The latter can be obtained from the action

$$S = \int d\tau\, d^2\eta\, P_{\alpha\dot\alpha}\left(\frac{i}{2}(X_L^{\alpha\dot\alpha} - X_R^{\alpha\dot\alpha}) + \Theta^\alpha\bar{\Theta}^{\dot\alpha}\right). \quad (102)$$

This is an alternative form of the $N = 2, D = 4$ action. The essential difference between the two actions is that (3) is substantially real, while (102) makes use of the concept of chirality (holomorphicity) inherent in this case. Note also that the Lagrange multiplier $P_{\alpha\dot\alpha}$ in (102) has the same dimension as the superparticle momentum and it does not possess any Abelian gauge invariance such as (12). The on-shell contents of these theories are, however, identical.

The case $N = 4, D = 6$ can be treated in a similar way. There the Majorana spinors have eight real components. At the same time they can be considered as four-component Weyl spinors with an extra SU(2)-doublet index and a pseudoreality condition [30]. This SU(2) group is an automorphism group of the $D = 6$ supersymmetry algebra which commutes with the $D = 6$ Lorentz group $SO(1,5) \sim SU^*(4) = SL(2,H)$. Thus in this case there are three complex structures (in other words, a quaternionic structure) corresponding to the three generators of the SU(2) group. The world-line Grassmann coordinates η_a $(a = 1, \ldots, 4)$ are now $SO(4) = SU(2)\times SU(2)$ spinors, so they also have a natural $SU(2)$ structure. Using all this, one can show that the basic superparticle equation (16) and its consequence (17) lead to the concept of SU(2) harmonic analyticity [31]. From that one can derive the alternative $N = 4, D = 6$ superparticle action proposed in Ref. [8]. Note, however, that the harmonic action of Ref. 8 uses infinite sets of auxiliary fields (coming from the harmonic expansions of the world-line superfields), whereas the new action (3) involves only finite sets. Also, SU(2) harmonic analyticity appears in the new formulation on shell only, while in the approach of Ref. [8] all the off-shell dynamical variables are by definition analytic. Again, the on-shell content of the real and the SU(2)-analytic formulations coincide.

A remarkable feature of the action (3) is its universality: it equally well describes all the magic cases $D = 3, 4, 6, 10$ with the corresponding maximal $N = 1, 2, 4, 8$ world-line supersymmetry. In the lower-dimensional cases $D = 4$ and 6 it reproduces the specific properties of analyticity on shell. The latter reflect the existence of complex structures there, which are in turn related to the automorphisms of the supersymmetry algebra in D dimensions commuting with the Lorentz group $SO(1, D-1)$. At the same time, the action (3) does not require any analyticity and/or complex structures *off shell*.

In ten dimensions the real 16 component spinors are both Majorana and Weyl. There exists no automorphism of the supersymmetry algebra $\{D_\alpha, D_\beta\} = \gamma^\mu_{\alpha\beta}\partial_\mu$ that commutes with the Lorentz group (except for the irrelevant scale transformations). This probably explains the failure of many attempts to generalize the notion of complex structure to the case $N = 8, D = 10$ and to find "octonionic analyticity." The action (3) escapes from this problem in a very simple way: it does not refer to any complex structures at all.

VII. CONCLUSIONS

In this paper we have presented a new formulation of the $D = 10$ superparticle theory. The new action (3) propagates the same modes as the Brink-Schwarz one (72). At the same time they have essentially different symmetries. While the κ symmetry of the BS action forms an algebra only modulo the equations of motion, all the symmetries of the new action are realized linearly and close off shell. In other words, the problem of finding auxiliary fields for $D = 10$ BS superparticle was solved. The κ symmetry of the BS action (72) can now be understood as an on-shell and partially gauge-fixed form of conformal world-line supersymmetry.

Note that in the new formulation the specific role of κ

symmetry in constraining the Yang-Mills or supergravity background is played not by the superconformal group (which is manifest), but by the Abelian gauge group of the Lagrange multiplier (12). The latter, however, is not directly related to κ symmetry. For instance, in the $N = 1, D = 3$ case the right-hand side of (12) vanishes, while κ symmetry is nontrivial.

It should be emphasized that in this paper we studied the classical theory of the superparticle. After having understood the structure of its symmetries, it should now be possible to attack the problem of Lorentz-covariant quantization of this theory.

In a future publication an $N = 8, D = 10$ superstring action of the type (3) will be presented. The specific Wess-Zumino term in the superstring action is in many respects analogous to the Maxwell coupling of the superparticle, with a two-form Abelian gauge field instead of the Maxwell field. This term will involve a separate Lagrange multiplier, which will produce the string tension on shell as an integration constant (such as the electric charge in Sec. V; see also references cited in [26]).

Another direction of possible development is related to constructing off-shell formulations of the $D = 10$ super-Yang-Mills and supergravity theories. The nonlinear structure of the Maxwell coupling (87) may shed new light on this old problem.

ACKNOWLEDGMENTS

The authors would like to thank E. Bergshoeff, N. Berkovits, E. Ivanov, and V.I. Ogievetsky and S. Shnider for stimulating discussions and Professor J. Ellis for hospitality at the Theory Division of CERN, where part of this work has been done.

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[22] We use the following conventions:

$$\eta_{\mu\nu} = \mathrm{diag}(-1, 1, \ldots, 1),$$

$$\gamma^0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \gamma^9 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \gamma^i = \begin{pmatrix} 0 & \gamma^i \\ (\gamma^i)^T & 0 \end{pmatrix},$$

where $\gamma^i_{A\dot{B}}$ are 8×8 matrices of O(8), the indices i, A and $\dot{B}$ correspond to the representations $\mathbf{8}_v$, $\mathbf{8}_s$, and $\mathbf{8}_c$ of O(8), respectively.

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