Gravitational radiation in black-hole collisions at the speed of light. III. Results and conclusions

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This paper summarizes results following from the two preceding papers, I and II, on the gravitational radiation emitted in the head-on collision of two black holes, each with energy μ, at or near the speed of light. The radiation (in the speed-of-light case) near the forward and backward directions $\hat{\theta}=0, \pi$, where $\hat{\theta}$ is the angle from the symmetry axis in the center-of-mass frame, is given by the series $c_0(\hat{\tau},\hat{\theta})=\sum_{n=0}^{\infty}a_{2n}(\hat{\tau}/\mu)\sin^{2n}\hat{\theta}$ for the news function c_0 of retarded time $\hat{\tau}$ and angle $\hat{\theta}$; successive terms can in principle be found from a perturbation treatment. Here the form of $a_2(\hat{\tau}/\mu)$ is presented. Knowledge of a_2 allows the new mass-loss formula of paper I to be applied, giving a calculation of the mass of the (assumed) final Schwarzschild black hole. Since the "final mass" resulting from the calculation exceeds 2μ, the assumptions of the new mass-loss formula must not all hold. The most likely explanation is that there is a "second burst" of radiation present in the space-time, centered for small angles $\hat{\theta}$ on retarded times roughly $|8\mu\ln\hat{\theta}|$ later than the "first burst" described above. A more realistic crude estimate of the energy emitted in gravitational waves is given by the Bondi expression, taking only the first two terms a_0 and a_2 in c_0; this gives an efficiency of 16.4% for gravitational wave generation.

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I. INTRODUCTION

In the two preceding papers, papers I and II [1,2], the axisymmetric collision of two black holes was studied in the case that the black holes approach at the speed of light, each with energy μ in the center-of-mass frame. A perturbation approach was used, in which a large Lorentz boost is applied to the incoming data, so that the energies λ, ν of the resulting incoming shock waves obey $\lambda \ll \nu$. The subsequent evolution is described by a singular perturbation problem, with small parameter (λ/ν). When one boosts back to the center-of-mass frame, successive terms of the perturbation series provide information on gravitational radiation which propagates at small angles $\hat{\theta}$ from the symmetry axis $\hat{\theta}=0$, near the curved shock 2 which has been distorted and deflected in the collision. (By symmetry, the same holds for radiation near the backward direction $\hat{\theta}=\pi$.) The gravitational radiation is described by the news function [3], which is expected to have the convergent expansion

$$c_0(\hat{\tau},\hat{\theta})=\sum_{n=0}^{\infty}a_{2n}(\hat{\tau}/\mu)\sin^{2n}\hat{\theta}\ , \tag{1.1}$$

(I1.3′)

where $\hat{\tau}$ is a retarded time coordinate. First-order perturbation theory gave $a_0(\hat{\tau}/\mu)$ in paper I, in agreement with the results of a calculation [4] of radiation emitted near the forward direction when two black holes approach, each with large Lorentz factor γ in the center-of-mass frame, and collide. In that case, the news function has an asymptotic expansion

$$c_0(\bar{\tau},\hat{\theta}=\gamma^{-1}\psi)\sim\sum_{n=0}^{\infty}\gamma^{-2n}Q_{2n}(\bar{\tau},\psi)\ , \tag{1.2}$$

(I1.1′)

valid as $\gamma\rightarrow\infty$ with $\bar{\tau},\psi$ fixed. The function $a_0(\hat{\tau}/\mu)$ is found from the limiting form of $Q_0(\bar{\tau},\psi)$ as $\psi\rightarrow\infty$.

For the speed-of-light collision, the higher-order perturbation theory in (λ/ν) can be considerably simplified following the analysis in paper II. There is a conformal symmetry at each order in perturbation theory, and hence all metric perturbations can be expressed in terms of functions of two variables, rather than the three variables which would be expected given only axisymmetry. This leads to an integral expression for $a_2(\hat{\tau}/\mu)$, using second-order perturbation theory, which is numerically tractable.

In Sec. II of the present paper we evaluate the time integral $\int_{-\infty}^{\infty}a_2(\hat{\tau}/\mu)d\hat{\tau}$, which appears in the new mass-loss formula of Sec. VI, paper I. This formula gives an expression for the mass of the final black hole resulting from the collision, assuming that at late times there is only one Schwarzschild black hole at rest, and that the gravitational radiation obeys a certain uniformity condition [Eq. (I6.12)]. The new mass-loss formula gives a "final mass" exceeding 3.5 μ. The most probable explanation is that the uniformity condition fails. This is expected to happen if, as seems likely, there is a "second burst" of gravitational radiation, generated near the center of the space-time, which will be delayed relative to the "first burst," described by Eq. (1.1), by an amount $\Delta\hat{\tau}\simeq|8\mu\ln\hat{\theta}|$ for small angles $\hat{\theta}$. The requirement of matching of the late-time radiation pattern of the "first burst" with the early-time pattern of the "second burst" shows that the radiation at such times is given by a sum of terms proportional to $e^{j\hat{\tau}/4\mu}\sin^{2n}\hat{\theta}$, where j and n are positive integers. Thus the presence of a "second burst" will be signaled by exponential growth of this type at late times in the news function (1.1), where the "first burst" occurs near $\hat{\tau}=0$. Such exponential behavior is indeed liable to occur because of the singular nature of the perturbation problem, in which the initial data become large at late times on the characteristic initial surface just to the future of the strong shock 1. It only fails to occur in $a_2(\hat{\tau}/\mu)$ because of the cancellation between exponentially growing volume and surface contributions to a_2.

In Sec. III the result of the numerical calculation of $a_2(\hat{\tau}/\mu)$ is presented (Fig. 4), following the analytical simplifications of paper II. A crude estimate of the mass loss in gravitational waves can be found by keeping only the first two terms in the series (1.1), i.e., approximating $c_0(\hat{\tau},\hat{\theta})$ by $a_0(\hat{\tau}/\mu)+a_2(\hat{\tau}/\mu)\sin^2\hat{\theta}$. With this truncation, an energy loss of 0.328 μ was found from the conventional Bondi formula [3], corresponding to an efficiency of 16.4% for gravitational wave generation. Further discussion is given in Sec. IV.

II. PREDICTION OF NEW MASS-LOSS FORMULA

In this section we insert the calculated values of the relevant quantities into the mass-loss formula in the limit $\gamma\to\infty$:

$$\begin{aligned} m_{\text{final}} &= -\int_{-\infty}^{\infty}[a_0(\hat{\tau}/\mu)]^2 d\hat{\tau} \\ &\quad +\int_{-\infty}^{\infty}[4a_2(\hat{\tau}/\mu)-a_0(\hat{\tau}/\mu)]d\hat{\tau}+o(1) \\ &= -\frac{\mu}{2}+\int_{-\infty}^{\infty}[4a_2(\hat{\tau}/\mu)-a_0(\hat{\tau}/\mu)]d\hat{\tau}+o(1)\,, \end{aligned} \tag{2.1}$$

(I6.20′)

which was derived in Sec. VI of paper I for the high-speed axisymmetric black-hole collision. Here $a_0(\hat{\tau}/\mu)$ and $a_2(\hat{\tau}/\mu)$ are the first two coefficients in the $\sin^2\hat{\theta}$ expansion for the news function on angular scales of order 1 in the center-of-mass frame:

$$c_0(\hat{\tau},\hat{\theta})=\sum_{n=0}^{\infty}a_{2n}(\hat{\tau}/\mu)\sin^{2n}\hat{\theta}+O(\gamma^{-1})\,. \tag{2.2}$$

The metric function $c(\hat{\tau},\hat{\theta})$, whose retarded time derivative is the news function, has a corresponding expansion

$$c(\hat{\tau},\hat{\theta})=\sum_{n=0}^{\infty}b_{2n}(\hat{\tau}/\mu)\sin^{2n}\hat{\theta} + \text{time-independent terms}+O(\gamma^{-1})\,, \tag{2.3}$$

where clearly

$$b_{2n}(x)=\mu\int_{-\infty}^{x}a_{2n}(y)dy+\text{const}\,. \tag{2.4}$$

Hence

$$m_{\text{final}}=\frac{-\mu}{2}+[4b_2(\hat{\tau}/\mu)-b_0(\hat{\tau}/\mu)]|_{-\infty}^{\infty}+o(1)\,, \tag{2.5}$$

where, again, the $o(1)$ correction denotes a term tending to zero as $\gamma\to\infty$. For convenience, as will be seen shortly, we have left b_2 and b_0 in the combination $(4b_2-b_0)$, although it was shown in Sec. VI of paper I that $b_0|_{-\infty}^{\infty}=-2\mu$.

In the boosted frame we have computed $\hat{d}^{(2)}+\hat{e}^{(2)}$, as described in paper II, from which the notation is taken, where

$$\begin{aligned} h_{\phi\phi}^{(2)} &= r^2\sin^2\theta(D^{(2)}+E^{(2)}) \\ &= r^2\sin^2\theta\left[\frac{\hat{d}^{(2)}+\hat{e}^{(2)}}{q^3\rho^4}\right] \end{aligned} \tag{2.6}$$

(here r is the radial coordinate). As described in Sec. VIII of paper II, it is assumed that the $(\ln r)/r$ part of the transverse second-order metric perturbations has been subtracted off by a gauge transformation, before $(\hat{d}^{(2)}+\hat{e}^{(2)})$ is evaluated. By $(\hat{d}^{(2)}+\hat{e}^{(2)})$ we mean simply the remaining order-1 part of this quantity as $r\to\infty$. Since $c=-\frac{1}{2}\lim_{r\to\infty}(h_{\phi\phi}/r\sin^2\theta)$ (in a Bondi gauge), the $e^{-4\alpha}$ term in c in the boosted frame is [using Eq. (II6.26)]

$$c^{(2)}=-\frac{\nu e^{-4\alpha}}{\sqrt{2}}\tan^2\frac{\theta}{2}\sec^2\frac{\theta}{2}[\hat{d}^{(2)}(\xi)+\hat{e}^{(2)}(\xi)]\,, \tag{2.7}$$

where

$$\xi=(\tau/\nu)\sec^2(\theta/2)-8\ln\left[\frac{2\tan\theta/2}{\nu}\right]+8\ln 8-8\,.$$

Further, as explained in Secs. VI and VIII of paper II, we choose the supertranslation state so that $\hat{d}^{(2)}(\xi)+\hat{e}^{(2)}(\xi)$ contains no $\ln|\xi|$ term in an expansion about $\xi=0$. Using Eqs. (I4.24), (I5.3) and $c=\hat{c}/K^2$, where $K(\theta)=\cosh\alpha+\sinh\alpha\cos\theta$ (see Ref. [3]), it is easy to show that $b_0(\hat{\tau}/\mu)+b_2(\hat{\tau}/\mu)\sin^2\hat{\theta}$ transforms to

$$e^{-\alpha}\sec^2\frac{\theta}{2}b_0\left[\frac{\tau}{\nu}\sec^2\frac{\theta}{2}\right]+e^{-3\alpha}\tan^2\frac{\theta}{2}\sec^2\frac{\theta}{2}\left[4b_2\left[\frac{\tau}{\nu}\sec^2\frac{\theta}{2}\right]-b_0\left[\frac{\tau}{\nu}\sec^2\frac{\theta}{2}\right]-\frac{\tau}{\nu}\sec^2\frac{\theta}{2}b_0'\left[\frac{\tau}{\nu}\sec^2\frac{\theta}{2}\right]\right] \tag{2.8}$$

in the boosted frame. Hence

$$4b_2\left[\frac{\tau}{\nu}\sec^2\frac{\theta}{2}\right]-b_0\left[\frac{\tau}{\nu}\sec^2\frac{\theta}{2}\right]-\frac{\tau}{\nu}\sec^2\frac{\theta}{2}b_0'\left[\frac{\tau}{\nu}\sec^2\frac{\theta}{2}\right]=-\frac{\nu e^{-\alpha}}{\sqrt{2}}\left[\hat{d}^{(2)}\left[\frac{\tau}{\nu}\sec^2\frac{\theta}{2}\right]+\hat{e}^{(2)}\left[\frac{\tau}{\nu}\sec^2\frac{\theta}{2}\right]\right]. \tag{2.9}$$

Note also, following the remarks above, that this implies that $b_2(\hat{\tau}/\mu)$ contains no $\ln|\hat{\tau}/\mu|$ term in an expansion about $\hat{\tau}=0$. Therefore

$$\left[4b_2\left[\frac{\hat{\tau}}{\mu}\right]-b_0\left[\frac{\hat{\tau}}{\mu}\right]\right]\Bigg|_{-\infty}^{\infty} = \frac{-\mu}{\sqrt{2}}[\hat{d}^{(2)}(\xi)+\hat{e}^{(2)}(\xi)]|_{-\infty}^{\infty} \quad (2.10)$$

As $\xi\rightarrow-\infty, \hat{d}^{(2)}+\hat{e}^{(2)}\rightarrow 0$. This may be verified by noting that the integration region in the source integral (II5.20) tends to zero as $\xi\rightarrow-\infty$. Inspection of (II5.19) shows that the contribution from the surface term is also zero in this limit. The asymptotic behavior of $\hat{d}^{(2)}+\hat{e}^{(2)}$ for large positive ξ is shown in Figs. 1(a) and 1(b). It is expected to be of the form

$$(\hat{d}^{(2)}+\hat{e}^{(2)})\sim\kappa+\sum_{n=1}^{\infty}\frac{R_n(\ln|\xi|)}{\xi^n} \quad (2.11)$$

as $\xi\rightarrow\infty$, where the R_n are polynomials. Terms of this form appear in the first-order news function, as one can check by a detailed asymptotic analysis of the integral expression (I4.22) for $a_0(\hat{\tau}/\mu)$, there denoted by $H_0(\hat{\tau}/\mu)$. One can reasonably expect them here too, in which case $\lim_{\xi\rightarrow\infty}(\hat{d}^{(2)}+\hat{e}^{(2)})=\kappa$. We have only computed $\hat{d}^{(2)}+\hat{e}^{(2)}$ out to $\xi=49.5$, since, as explained in Sec. IX of paper II, it is not possible to go to larger values of ξ because there are large cancellations between the source and surface term contributions to $\hat{d}^{(2)}+\hat{e}^{(2)}$. In fact the source and surface contributions to $\hat{d}^{(2)}+\hat{e}^{(2)}$ separately have parts which grow exponentially at late retarded times, at rates $\exp(l\hat{\tau}/8\mu)$ for suitable integer l. In the case of the surface contribution, these arise from the exponential growth of the initial data for the second-order metric perturbations at large negative coordinate $\hat{v}$ on the characteristic initial surface $\hat{u}=0$ (the strong shock 1). This late-time growth of the characteristic initial data is responsible for the singular nature of the perturbation theory for this space-time, as discussed at the end of Sec. III of paper I; in particular, see Eq. (I3.27). The very accurate numerical cancellation between the source and surface contributions [as seen in Figs. 1(a) and 1(b)] leads one to expect the late-time behavior of Eq. (2.11) for $\hat{d}^{(2)}+\hat{e}^{(2)}$. To determine κ accurately, we must do a least-squares fit of the form (2.11) to the computed $\hat{d}^{(2)}+\hat{e}^{(2)}$. However, because there is no way of telling which terms are actually present in Eq. (2.11), this method proves incapable of determining κ very accurately. We find $\kappa\simeq-6.3$, with an estimated error of 5%.

Fortunately, a rough estimate is quite sufficient for our needs. Since the change in $\hat{d}^{(2)}+\hat{e}^{(2)}$ in going from $\xi=25$ to $\xi=50$ will be of roughly the same magnitude as in going from $\xi=50$ to $\xi=\infty$, inspection of Fig. 1(b) shows that κ will certainly be less than $-4\sqrt{2}$. Hence

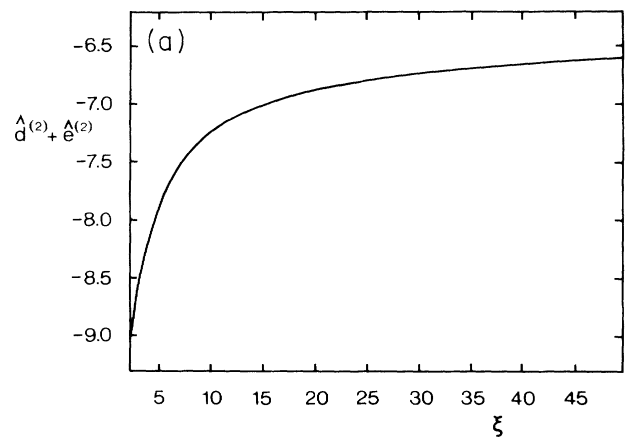
$$\left[4b_2\left[\frac{\hat{\tau}}{\mu}\right]-b_0\left[\frac{\hat{\tau}}{\mu}\right]\right]\Bigg|_{-\infty}^{\infty}>4\mu\ , \quad (2.12)$$

and substituting this into Eq. (2.5) we find

$$m_{\text{final}}>3.5\mu\ . \quad (2.13)$$

Thus Eq. (2.1) predicts that the final mass will be considerably greater than the initial energy 2μ. This result cannot, of course, be correct—energy must be conserved—and therefore one (or more) of the assumptions that went into the derivation of Eq. (2.1) must be wrong.

The first of those assumptions was that the final mass aspect of the system is isotropic. This enabled us to replace $M(\infty,\hat{\theta})$ by m_{final} on the left-hand side of Eq. (I6.9) and in all the equations derived from it, such as Eq. (2.1). This assumption seems very reasonable. If cosmic censorship holds, the event horizon that is formed in the collision cannot bifurcate [5], and hence there must be a connected "object," symmetrical about the equator $\hat{\theta}=\pi/2$ in the center-of-mass frame, at the center of the space-time, which is bounded by a horizon that always has an area greater than $32\pi\mu^2$, the area of the apparent horizon found by Penrose [6] (as mentioned in Sec. I of



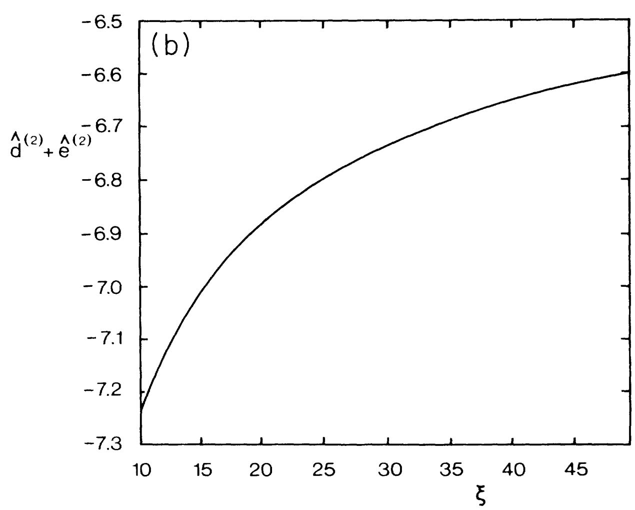


FIG. 1. (a) and (b) The asymptotic second-order metric function $\hat{d}^{(2)}(\xi)+e^{(2)}(\xi)$ is shown for moderately large values of ξ, where $\mu\xi$ is a measure of retarded time. Numerical problems prevented its accurate evaluation for larger values of ξ. The limiting value of $\hat{d}^{(2)}(\xi)+\hat{e}^{(2)}(\xi)$ as $\xi\rightarrow\infty$ yields $-(\sqrt{2}/\mu)$ times the change in the quantity $[4b_2(\hat{\tau}/\mu)-b_0(\hat{\tau}/\mu)]$ from early to late retarded times, and hence leads to an expression for the final mass in Eq. (2.5), following the assumptions of the new mass-loss formula.

paper I). This object should shed its nonspherical perturbations and decay to a Schwarzschild geometry. Assuming all this to be true, the only way the final mass aspect can be nonisotropic is for there to be other bodies present, and they must be moving relative to the center-of-mass frame. For example, in addition to the central black hole, one could (just) envisage two small black holes being formed by the focusing of the incoming waves, which might then fly off in opposite directions along the axis of symmetry, thereby concentrating the mass aspect around this axis, so that $M(\hat{\tau}=\infty,\hat{\theta}=0,\pi)>m_{\text{final}}$. But this seems very unlikely, since any focusing that is strong enough to produce horizons should take place within the central trapped region.

Second, we assumed that each $Q_{2n}(\hat{\tau},\psi)$, in the asymptotic expansion (1.2) for the news function at angles $\hat{\theta}=\gamma^{-1}\psi$ close to the axis in the large finite-γ collision, tends to a limiting form given by $a_{2n}(\hat{\tau}/\mu)$ as $\psi\rightarrow\infty$. In other words, the radiation pattern in the high-speed collision tends to that produced by the speed-of-light collision as $\gamma\rightarrow\infty$. This seems intuitively obvious, has been explicitly shown for $n=0$, by combining the arguments of Ref. [4] and of the end of Sec. IV of paper I, and we see no reason to doubt it.

Instead, the reason for the nonphysical answer in Eq. (2.13) is probably that at late times, close to the axis of symmetry, there is another burst of radiation, making an additional contribution to the integral $\int_{-\infty}^{\infty}c_0(\hat{\tau},\psi)d\hat{\tau}$, which when substituted into Eq. (I6.9) will modify Eq. (2.1) and bring m_{final} down to some sensible value below 2μ. In other words, we are suggesting that $c(\hat{\tau},\psi)$ will not in fact satisfy the "uniformity" condition (I6.12) which, we recall, had to be assumed in order to derive Eq. (2.1).

One can see how this might happen. At both finite and infinite γ one uses perturbation theory to calculate the radiation in the neighborhood of the axis of symmetry (in the center-of-mass frame) produced by the focusing of the far fields (where $\rho\gg\mu$) of the incident shock waves as they pass through each other during the collision, and one then matches out to larger angular scales. But if one is close to the axis $\hat{\theta}=0,\pi$, as is necessary when doing perturbation theory, then one must wait a long time after the peak of the first burst of radiation has come by before one can see nonzero initial data at $\rho\sim\mu$ (see Figs. 2 and 3 in paper I). The effectiveness of perturbation theory is due to this delay by an amount proportional to $\ln(\rho_0)$, where $(\rho_0)^2=(x_0)^2+(y_0)^2$ gives the distance from the axis at the moment when the shocks collide, as described in Secs. I and II of paper I. This gives the far field of each shock a large head start over its near-field counterpart, enabling one to study its essentially wavelike self-interaction without having to concern oneself with the details of the highly nonlinear region at the center of the space-time. And yet it is in this nonlinear region where the shocks come through each other at $\rho\sim\mu$, and in which we expect a black hole to be formed, that additional radiation might be produced. This central black hole would certainly not be formed "cleanly," and could shed its nonspherical perturbations only by emitting gravitational waves.

Using the parametric representation (I3.20) for the weak shock, it is easy to show that in the speed-of-light collision in the center-of-mass frame the two bursts would be separated by a time interval $\Delta\hat{\tau}\simeq 8\mu\ln[\mu/2\tan(\theta/2)]$ when $\hat{\theta}\ll 1$. In the finite-γ collision the time delay $\Delta\hat{\tau}$ has the same form so long as $\gamma^{-1}\ll\hat{\theta}\ll 1$, but when $\hat{\theta}$ becomes of $O(\gamma^{-1})$ it flattens off to a value $\Delta\hat{\tau}\simeq 8\mu\ln(\gamma/\mu)$ (i.e., $\Delta\hat{\tau}\not\rightarrow\infty$ as $\hat{\theta}\rightarrow 0$), reflecting the fact that the radiation has detailed angular structure in this region. Since $\Delta\hat{\tau}$ is large, one would expect the "influence" of the second burst on the first to be small. However, the series (1.1) and (1.2) provide exact or asymptotic descriptions of their respective space-time metrics in the vicinity of the first burst of radiation. One would expect to be able to find traces of any second burst somewhere in these series. We shall demonstrate this for the speed-of-light collision; there is a completely analogous argument at finite γ.

Suppose that the supertranslation state in the speed-of-light collision is such that the first burst is centered around

$$\hat{\tau}=8\mu\ln\left[\frac{2\tan\hat{\theta}/2}{\mu}\right]$$

when $\hat{\theta}$ is small. [We are therefore keeping the logarithmic term which appears quite naturally in the argument of the news function in Sec. IV of paper I; see, in particular, Eq. (I4.26) and lines following.] From the discussion in the previous paragraph it is clear that the second burst would be centered near $\hat{\tau}=0$. Let us restrict attention to angles $\hat{\theta}\ll 1$. One might, at first sight, guess that the fall-off of the news function as one moves back in time, away from the center of the second burst, is given by an asymptotic expansion of a type similar to Eq. (2.11), going as

$$\sin^2\hat{\theta}\left[\sum_{i=1\text{ or }2}^{\infty}\frac{R_i[\ln(-\hat{\tau}/\mu)]}{(\hat{\tau}/\mu)^i}\right]$$

$$+\text{analogous terms proportional to }\sin^4\hat{\theta},\ \sin^6\hat{\theta},\ \ldots,\qquad(2.14)$$

where each R_i is polynomial. [There will be no $\sin^0\hat{\theta}$ term in the second burst: it is only the very special focusing of the far fields of the incoming shocks that causes the $a_0\sin^0\hat{\theta}$ term in Eq. (1.1).] Define $\hat{u}$ by

$$\hat{u}=\hat{\tau}-8\mu\ln\left[\frac{2\tan\hat{\theta}/2}{\mu}\right],$$

and note that the right-hand side (RHS) of Eq. (1.1) has the form $\sum_{n=0}^{\infty}a_{2n}(\hat{u}/\mu)\sin^{2n}\hat{\theta}$ in the supertranslation state we have chosen. Then expressing (2.14) in terms of $\hat{u}$ we obtain

$$\sin^2\hat{\theta}\left[\sum_{i=1\text{ or }2}^{\infty}\frac{R_i\left\{\ln\left[-\frac{\hat{u}}{\mu}-8\ln\left[\frac{2\tan\hat{\theta}/2}{\mu}\right]\right]\right\}}{\left[\frac{\hat{u}}{\mu}+8\ln\left[\frac{2\tan\hat{\theta}/2}{\mu}\right]\right]^i}\right]+\text{analogous terms proportional to}\sin^4\hat{\theta},\ \sin^6\hat{\theta},\ldots \tag{2.15}$$

If we now fix $\hat{u}$ and let $\hat{\theta}\to 0$ then all the resulting terms in Eq. (2.15) should appear in Eq. (1.1) (with $\hat{\tau}$ replaced by $\hat{u}$), since perturbation theory is certainly valid in this limit. The first term in Eq. (2.15) becomes

$$\sin^2\hat{\theta}\sum_i\frac{R_i\left[\ln(-8\tan\hat{\theta}/2)Q_i\left[\frac{\hat{u}+\text{const}}{8\mu\ln(\tan\hat{\theta}/2)}\right]\right]}{[8\ln(\tan\hat{\theta}/2)]^i}V_i\left[\frac{\hat{u}+\text{const}}{8\mu\ln(\tan\hat{\theta}/2)}\right], \tag{2.16}$$

where Q_i and V_i are appropriate convergent power series. This certainly does not match with the $\sin^2\hat{\theta}$ expansion (1.1) (again with $\hat{\tau}$ replaced by $\hat{u}$) for the news function, and therefore the second burst cannot have a tail with an inverse power-law decay, as described by Eq. (2.14).

We must instead find a suitable form for the behavior of the news function at times early compared to the second burst, but late compared to the first burst. The form $8\mu\ln[\mu/2\tan(\hat{\theta}/2)]$ of the time-delay between the first and second bursts, together with the property (1.1) that the radiation admits a convergent series expansion in powers of $\sin^2\hat{\theta}$, shows that the news function, at times early compared to the second burst, should have the $\hat{\tau}\to-\infty$ fall-off

$$\sin^2\hat{\theta}\sum_{j=1}^{\infty}A_je^{j\hat{\tau}/4\mu}+\text{analogous terms proportional to }\sin^2\hat{\theta},\sin^6\hat{\theta},\ldots, \tag{2.17}$$

where the A_j are constants. The particular form of the exponentials in Eq. (2.17) can be seen as follows to be necessary. First, note that $\tan(\hat{\theta}/2)$ can be expressed as a power series in $\sin\hat{\theta}$, as $\tan(\hat{\theta}/2)=\sin\hat{\theta}\sum_{k=0}^{\infty}B_k\sin^{2k}\hat{\theta}$. Hence the first term in Eq. (2.17), which can be written as $\sin^2\hat{\theta}\sum_{j=1}^{\infty}A_je^{j\hat{u}/4\mu}(\mu^{-1}\tan(\hat{\theta}/2))^{2j}$, can be rewritten as

$$\sin^2\hat{\theta}\left[\sum_{j=1}^{\infty}A_je^{j\hat{u}/4\mu}\sin^{2j}\hat{\theta}\left[\mu^{-1}\sum_{k=0}^{\infty}B_k\sin^{2k}\hat{\theta}\right]^{2j}\right]. \tag{2.18}$$

This matches with the form (1.1) of the radiation, as viewed relative to the first burst using retarded coordinate $\hat{u}$, as $\hat{u}\to\infty$. Indeed, only the particular exponential fall-off of Eq. (2.17) as $\hat{\tau}\to-\infty$ will "unscramble" the logarithmic time delay $8\mu\ln[\mu/2\tan(\hat{\theta}/2)]$ between the two bursts, so as to give an expression such as Eq. (2.18) in a power series in $\sin^2\hat{\theta}$. Exactly the same applies to the terms in Eq. (2.17) with higher powers of $\sin^2\hat{\theta}$.

In summary, we are led to conclude that the presence of a second burst of radiation will be indicated by exponentially growing terms at late times in the $a_{2n}(\hat{u}/\mu)$ in Eq. (1.1) (with $\hat{\tau}$ replaced by $\hat{u}$) beginning at third, or perhaps some higher, order. [We recall that there were such terms in the source and surface contributions to $a_2(\hat{u}/\mu)$ which fortunately canceled.] The important point is that the second burst cannot be detected directly at either first or second order in perturbation theory; only indirectly through Eq. (2.1) and an examination of the assumptions under which it holds.

As mentioned above, a similar analysis may be done for the finite-γ collision, leading to an identical result for the perturbation expansion (1.2) in powers of γ^{-2}: that the imprints of the second burst will be exponentially growing terms beginning at third $[Q_4(\bar{\tau},\psi)]$, or some higher, order. This does, we feel, provide a satisfactory explanation of the unphysical result in Eq. (2.13), since if the $Q_{2n}(\bar{\tau},\psi)$ contain exponentially growing terms then Eq. (I6.12) will be violated, and so the chain of reasoning leading to Eq. (2.13) will be broken.

To find the true loss of mass (assuming final isotropy) it would only be necessary to compute the time integral of the coefficient of $\sin^2\hat{\theta}$ in the news function describing the second burst. This would, presumably, make a negative contribution to Eq. (I6.9), thus modifying Eq. (2.1), and lead to a sensible value for m_{final}. However, since no perturbation theory will be able to describe the strong-field region from which the second burst emanates, the task of calculating the complete $\sin^2\hat{\theta}$ term is rather formidable.

As already outlined in this section, the time delay in the speed-of-light collision between the centers of the two bursts will be

$$\Delta\hat{\tau}\simeq 8\mu\left|\ln\left[\frac{2\tan\hat{\theta}/2}{\mu}\right]\right|$$

when $\hat{\theta}$ is small. At finite γ the form of the time delay will be $\Delta\hat{\tau}\simeq 8\mu\ln(\gamma/\mu)$ if $\hat{\theta}$ is $O(\gamma^{-1})$, and $\Delta\hat{\tau}\simeq 8\mu[\ln\gamma-\ln(\psi/\mu)]$ when ψ^{-1} and $\gamma^{-1}\psi$ are $o(1)$ (i.e., in the matching region where $\gamma^{-1}\ll\hat{\theta}\ll 1$). This means that the "two" bursts of radiation will be truly separate only in the limit $\hat{\theta}\to 0$ in the speed-of-light collision, and for $\gamma\to\infty$ with $\hat{\theta}=o(1)$ in the "finite"-γ collision. Away from the axis of symmetry they will merge, and cannot be thought of as physically distinct. So long as Eq. (1.1) is convergent it should provide the exact speed-of-light news function—even if the perturbation theory has broken down near the initial surface to the past of the point in question.

Because of the possible exponentially growing terms in $a_4(\hat{u}/\mu)$, $a_6(\hat{u}/\mu),\ldots,$ one might doubt whether the series (1.1) for the speed-of-light news function (with $\hat{\tau}$ replaced by $\hat{u}$) could converge at late retarded times $\hat{u}$. However, at least if one considers the time-integrated quantity $c(\hat{u},\hat{\theta})$, which is expected to be a continuous function for $0\le\hat{\theta}\le\pi$, symmetrical about $\hat{\theta}=\pi/2$, then by the Stone-Weierstrass theorem [7] it will admit the convergent expansion

$$c(\hat{u},\hat{\theta})=\sum_{n=0}^{\infty} b_{2n}(\hat{u}/\mu)\sin^{2n}\hat{\theta} \tag{2.19}$$

as a power series in $\sin^2\hat{\theta}$. The analogous expansion should also hold for the function c when regarded as a function of $\hat{\tau}$ and $\hat{\theta}$, where the retarded-time coordinates $\hat{\tau}$ and $\hat{u}$ are related by the supertranslation $\hat{u}=\hat{\tau}-8\mu\ln[2\tan(\hat{\theta}/2)/\mu]$. Thus

$$c(\hat{u}=\hat{\tau}-8\mu\ln[2\tan(\hat{\theta}/2)/\mu],\hat{\theta}) = \sum_{n=0}^{\infty} D_{2n}(\hat{\tau}/\mu)\sin^{2n}\hat{\theta}\,, \tag{2.20}$$

where the b_{2n} and D_{2n} are of course different functions. It is only the simultaneous validity of Eqs. (2.19) and (2.20), for this particular supertranslation, which is powerful enough to lead us to the form (2.17) of the early-$\hat{\tau}$ expansion of $c_0(\hat{\tau},\hat{\theta})$.

III. THE SECOND-ORDER NEWS FUNCTION

In this section, we return to the convention that the supertranslation state is such that the radiation is centered on the retarded time $\hat{\tau}=0$.

From Eq. (2.9) we have

$$4b_2\left[\frac{\hat{\tau}}{\mu}\right]=\frac{-\mu}{\sqrt{2}}\left[\hat{d}^{(2)}\left[\frac{\hat{\tau}}{\mu}\right]+\hat{e}^{(2)}\left[\frac{\hat{\tau}}{\mu}\right]-\frac{\sqrt{2}}{\mu}b_0\left[\frac{\hat{\tau}}{\mu}\right]-\frac{\sqrt{2}}{\mu}\left[\frac{\hat{\tau}}{\mu}\right]b_0'\left[\frac{\hat{\tau}}{\mu}\right]\right], \tag{3.1}$$

where the b_{2n} are the functions appearing in Eq. (2.3). Using Eqs. (II6.29) and (2.8), we see that

$$-2\sec^2\frac{\theta}{2}b_0\left[\frac{\tau}{\nu}\sec^2\frac{\theta}{2}\right]=\frac{\mu}{\sqrt{2}}\sec^2\frac{\theta}{2}e_1\left[\frac{\tau}{\nu}\sec^2\frac{\theta}{2}\right]. \tag{3.2}$$

The function $e_1(\xi)$ was defined in Eq. (II6.11), and from Eq. (II6.10) has the explicit form

$$e_1(\xi)=\frac{8\sqrt{2}}{\pi}\int_0^{\infty}\int_0^{\pi}dx\,d\phi' x^{-2}\cos\phi' \times\theta(8\ln x+\xi+8+8x\cos\phi')-4\sqrt{2}\,, \tag{3.3}$$

where we have used the gauge conditions (II6.18). Hence $b_0(\hat{\tau}/\mu)=-(\mu/2\sqrt{2})e_1(\hat{\tau}/\mu)$ and

$$b_2\left[\frac{\hat{\tau}}{\mu}\right]=\frac{-\mu}{4\sqrt{2}}[\hat{d}^{(2)}(\xi)+\hat{e}^{(2)}(\xi)+\tfrac{1}{2}e_1(\xi)+\tfrac{1}{2}\xi e_1'(\xi)]|_{\xi=\hat{\tau}/\mu}\,. \tag{3.4}$$

Therefore $a_2(\hat{\tau}/\mu)$, the coefficient of $\sin^2\hat{\theta}$ in Eq. (2.2) or Eq. (1.1), is

$$a_2\left[\frac{\hat{\tau}}{\mu}\right]=\frac{-1}{4\sqrt{2}}\left[\frac{d}{d\xi}[\hat{d}^{(2)}(\xi)+\hat{e}^{(2)}(\xi)+\tfrac{1}{2}e_1(\xi)+\tfrac{1}{2}\xi e_1'(\xi)]\right]\Bigg|_{\xi=\hat{\tau}/\mu}\,. \tag{3.5}$$

The function we have computed, at a number of discrete values of ξ, is $\hat{d}^{(2)}+\hat{e}^{(2)}$, and so to calculate $a_2(\hat{\tau}/\mu)$ we shall have to do some numerical differentiation. A graph of the computed $\hat{d}^{(2)}(\xi)+\hat{e}^{(2)}(\xi)+\frac{1}{2}[e_1(\xi)+\xi e_1'(\xi)]$ is shown in Fig. 2. It has two stationary points, in contrast with the first-order metric function $e_1(\xi)$, shown in Fig. 3, which has only one. We cannot compute $\hat{d}^{(2)}+\hat{e}^{(2)}$ or e_1 too close to the singular point $\xi=0$, and so there is a gap there.

In each region $\xi<0$ and $\xi>0$, we interpolate $\hat{d}^{(2)}(\xi)+\hat{e}^{(2)}(\xi)+\frac{1}{2}[e_1(\xi)+\xi e_1'(\xi)]$ using the cubic spline that passes exactly through all the data points (a cubic spline is a $C^{(2)}$ function made up of cubic polynomial segments). It is the continuity of its derivatives that is the great advantage of the spline here; in addition it does not develop nasty fluctuations between data points near the end points of the region of interpolation, as ordinary polynomial interpolants are apt to do (for the unpleasant things that can happen with polynomials, see Ref. [8]). To check the accuracy of interpolation, we compare each of our splines with another that has only 2/3 as many cubic segments (this second spline is a best least-squares fit, since it cannot pass exactly through all the data points). The maximum difference between the pairs of splines is about 5×10^{-5}, giving a conservative estimate for the true accuracy of interpolation.

To find $a_2(\hat{\tau}/\mu)$ we differentiate the spline interpolant and divide by $-4\sqrt{2}$. The result of this is shown in Fig. 4. The singularity at $\hat{\tau}=0$ is of the form

$$\sum_{n=0}^{\infty}\left[\frac{\hat{\tau}}{\mu}\right]^n\left[E_n\left[\ln\left|\frac{\hat{\tau}}{\mu}\right|\right]^2+F_n\ln\left|\frac{\hat{\tau}}{\mu}\right|+G_n\right]. \tag{3.6}$$

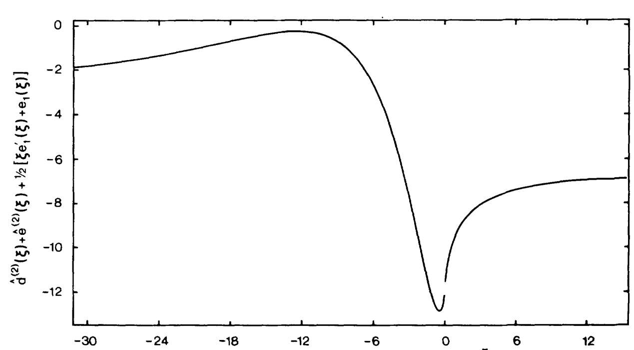


FIG. 2. The asymptotic metric function $\hat{d}^{(2)}(\xi)+\hat{e}^{(2)}(\xi)+\frac{1}{2}[\xi e_1'(\xi)+e_1(\xi)]$ is shown. Its ξ derivative, evaluated at $\xi=\hat{\tau}/\mu$, gives $-4\sqrt{2}a_2(\hat{\tau}/\mu)$, where the news function is given by $c_0(\hat{\tau},\hat{\theta})=\sum_{n=0}^{\infty}a_{2n}(\hat{\tau}/\mu)\sin^{2n}\hat{\theta}+O(\gamma^{-1})$. There is a gap in the computational results near the singular point $\xi=0$.

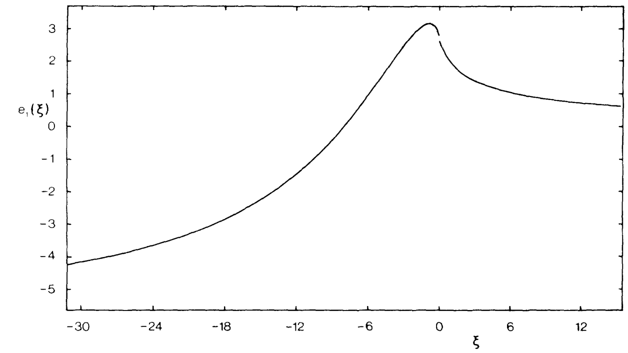


FIG. 3. The asymptotic metric function $e_1(\xi)$, whose ξ derivative, evaluated at $\xi=\hat{\tau}/\mu$, gives $-2\sqrt{2}a_0(\hat{\tau}/\mu)$. There is again a gap in the computational results near the singular point $\xi=0$.

Using the conventional formula for the energy emitted in gravitational radiation [3],

$$\Delta m = \tfrac{1}{2}\int_{-\infty}^{\infty}\int_0^{\pi}(c_0)^2 \sin\hat{\theta}\, d\hat{\tau}\, d\hat{\theta} \ , \quad (3.7)$$

we can derive an estimate for the mass loss, assuming that the total news function is given only by $a_0(\hat{\tau}/\mu)+a_2(\hat{\tau}/\mu)\sin^2\hat{\theta}$. From the computed $a_0(\hat{\tau}/\mu)$ and $a_2(\hat{\tau}/\mu)$ we find

$$\begin{aligned} &\int_{-\infty}^{\infty}[a_0(x)]^2dx=0.500 \ , \\ &\int_{-\infty}^{\infty}a_0(x)a_2(x)dx=-0.586 \ , \\ &\int_{-\infty}^{\infty}[a_2(x)]^2dx=1.14 \ . \end{aligned} \quad (3.8)$$

Therefore, if the news function were

$$c_0=a_0\left[\frac{\hat{\tau}}{\mu}\right]+a_2\left[\frac{\hat{\tau}}{\mu}\right]\sin^2\hat{\theta} \ , \quad (3.9)$$

then the mass loss would be

$$\Delta m = \frac{1}{2}\int_{-\infty}^{\infty}\int_0^{\pi}\left[a_0\left[\frac{\hat{\tau}}{\mu}\right]+a_2\left[\frac{\hat{\tau}}{\mu}\right]\sin^2\hat{\theta}\right]^2 \times \sin\hat{\theta}\, d\hat{\tau}\, d\hat{\theta}=0.328\mu \quad (3.10)$$

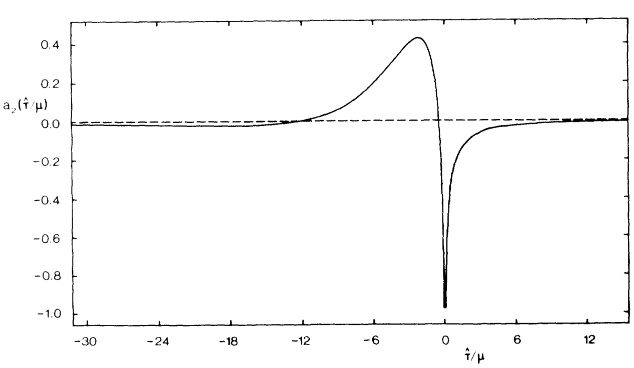


FIG. 4. The contribution $a_2(\hat{\tau}/\mu)$ to the series $c_0(\hat{\tau},\hat{\theta})=\sum_{n=0}^{\infty}a_{2n}(\hat{\tau}/\mu)\sin^{2n}\hat{\theta}$ for the speed-of-light news function.

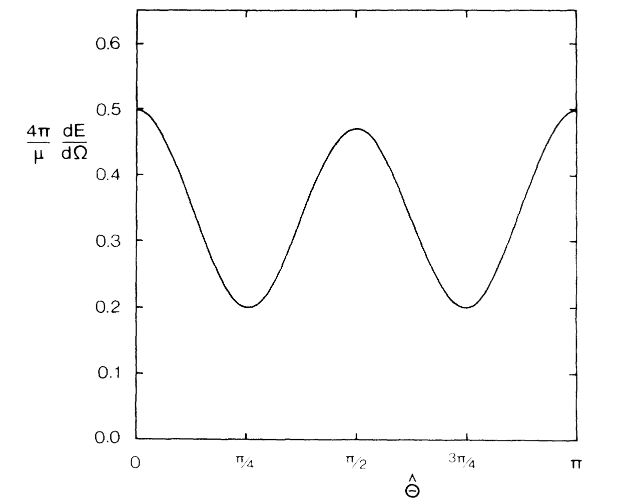


FIG. 5. The energy $dE/d\Omega$ (in units of $\mu/4\pi$) which would be radiated per unit solid angle, if the news function had the truncated form $c_0(\hat{\tau},\hat{\theta})=a_0(\hat{\tau}/\mu)+a_2(\hat{\tau}/\mu)\sin^2\hat{\theta}$.

or about 16.4% of the initial energy (this may be compared with the 25% one obtains when using just the isotropic term). The angular dependence of $dE/d\Omega$, the energy radiated per unit solid angle, for the news function (3.9) is shown in Fig. 5. It is interesting to note that the magnitude of $dE/d\Omega$ is never greater than its isotropic part $(1/4\pi)\int_{-\infty}^{\infty}[a_0(\hat{\tau}/\mu)]^2d\hat{\tau}$.

Of course this estimate must be taken with several grains of salt, since we have truncated the series (1.1). It is nevertheless useful as an order-of-magnitude estimate.

IV. DISCUSSION

In previous papers [4,9–11] on high-speed black-hole collisions, both at finite and infinite γ, it has been tacitly assumed that the product of the collision is a single black hole plus out-going gravitational radiation, where the burst of radiation is that produced by the focusing of the far fields of the colliding holes– and its continuation to larger angular scales. If this were the case, then by solving the perturbation theory to all orders one could determine the entire news function.

Using the mass-loss formula (2.1), or Eq. (I6.9) from which it was derived, we have shown that the true picture is not so simple. We have argued that there will be an additional burst of radiation produced during the decay to equilibrium (i.e., Schwarzschild) of the central black hole formed by the collision, and that this will be manifested by the appearance at high orders in perturbation theory of terms growing exponentially with time. The two bursts are truly separate only in the limit $\hat{\theta}\to 0$ or π, merging together away from this axis. Because of this, and because the "second" burst cannot be treated using perturbation theory, originating as it does in a highly nonlinear part of the space-time, we cannot calculate the news function away from the axis of symmetry, except at early times. The expressions that have been derived for the news function, by previous authors and in this series of paper, are only valid either in the vicinity of the first burst close to the axis of symmetry, or at early times away from this axis.

It is likely that further analytical progress on this problem will be limited. Computational limitations will prevent higher-order calculations in perturbation theory—and there is little point in attempting them, at least within the framework presented here. The speed-of-light space-times are algebraically general [9] (as are those at finite γ), and given the complexity of just the news function one certainly cannot expect to find an exact solution. The combined analytical-numerical approach used here must be regarded as complementary to the numerical construction of these speed-of-light space-times [12].

ACKNOWLEDGMENTS

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