Gravitational radiation in black-hole collisions at the speed of light. I. Perturbation treatment of the axisymmetric collision

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In this and the two following papers II and III we study the axisymmetric collision of two black holes at the speed of light, with a view to understanding the more realistic collision of two black holes with a large but finite incoming Lorentz factor γ. The curved radiative region of the space-time, produced after the two incoming impulsive plane-fronted shock waves have collided, is treated using perturbation theory, following earlier work by Curtis and Chapman. The collision is viewed in a frame to which a large Lorentz boost has been applied, giving a strong shock with energy ν off which a weak shock with energy $\lambda \ll \nu$ scatters. This yields a singular perturbation problem, in which the Einstein field equations are solved by expanding in powers of λ/ν around flat space-time. When viewed back in the center-of-mass frame, this gives a good description of the regions of the space-time in which gravitational radiation propagates at small angles $\hat{\theta}$ but a large distance from the symmetry axis, near each shock as it continues to propagate, having been distorted and deflected in the initial collision. The news function $c_0(\hat{\tau},\hat{\theta})$ describing the gravitational radiation is expected to have a convergent series expansion $c_0(\hat{\tau},\hat{\theta})=\sum_{n=0}^{\infty} a_{2n}(\hat{\tau})\sin^{2n}\hat{\theta}$, where $\hat{\tau}$ is a retarded time coordinate. First-order perturbation theory gives an expression for $a_0(\hat{\tau})$ in agreement with that found previously by studying the finite-γ collisions. Second-order perturbation theory gives $a_2(\hat{\tau})$ as a complicated integral expression. A new mass-loss formula is derived, which shows that if the end result of the collision is a single Schwarzschild black hole at rest, plus gravitational radiation which is (in a certain precise sense) accurately described by the above series for $c_0(\hat{\tau},\hat{\theta})$, then the final mass can be determined from knowledge only of $a_0(\hat{\tau})$ and $a_2(\hat{\tau})$. This leads to an interesting test of the cosmic censorship hypothesis. The numerical calculation of $a_2(\hat{\tau})$ is made practicable by analytical simplifications described in the following paper II, where the perturbative field equations are reduced to a system in only two independent variables. Results are presented in the concluding paper III, which discusses the implications for the energy emitted and the nature of the radiative space-time.

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I. INTRODUCTION

In the many years since general relativity was originally formulated by Einstein no one has found, owing to the complexity and nonlinearity of the field equations, any physically realistic analytic solution which does not possess a large number of simplifying symmetries. However, if one is interested in studying the generation of gravitational radiation by realistic physical sources, then one must of necessity consider isolated gravitating systems that are time dependent and which can have no simplifying features apart from axisymmetry.

An exact treatment of such problems is, at present, quite out of the question and one must therefore seek recourse to approximation procedures [1]. There are two alternatives. The first is numerical simulation, whereby one replaces the space-time continuum by a discrete grid and the differential field equations by a finite difference scheme. One sets up appropriate boundary conditions on some initial surface, and then one "constructs" the space-time to the future of this surface by evolving the initial data on a computer. This approach has been used to study, amongst other problems, the axisymmetric collision of two black holes starting from rest at a finite separation [2].

The other method of treatment is perturbation theory. Here one assumes that the space-time metric differs only very slightly from some fixed background (which is taken to be one of the highly symmetric exact solutions mentioned above). The field equations for the metric perturbations are linear to lowest order, and often prove mathematically tractable, owing to the (relatively) simple nature of the background metric.

However, since the time-dependent perturbations must be small, the gravitational radiation produced is almost always correspondingly weak (in the sense that the energy carried off by the waves is only a small fraction of the total energy of the system). To deduce the behavior of gravitating systems when the perturbations are not small, one is obliged to extrapolate from the weak-field limit, which can provide physical insight, but no strict quantitative results.

In fact, there is only one physical process in which perturbation methods have proved successful in describing

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truly strong-field gravitational radiation: namely, the high-speed collision of two black holes. The success of perturbation theory in these space-times is due to certain special features of their geometry, which we now briefly describe.

Owing to special-relativistic effects, the gravitational field of a black hole traveling close to the speed of light becomes concentrated in the vicinity of its trajectory, which lies close to a null plane in the surrounding nearly Minkowskian space-time. At precisely the speed of light, the black hole turns into a particular sort of impulsive gravitational plane-fronted wave [3] (as the speed increases one must scale down the rest mass appropriately, in order that the energy be finite). The curvature is then zero except on the null plane of its trajectory, and there is a massless particle traveling along the axis of symmetry at the center of this null plane.

An important property of this sort of gravitational shock wave is that geodesics crossing it are not only bent inwards, but also undergo an instantaneous translation along the null surface that describes the trajectory of the wave. The nature of this translation is such that geodesics crossing the shock close to the axis of symmetry are delayed relative to those which cross the shock far out from the axis. Hence, when two such waves pass through each other in a head-on collision, the far-field region of each wave (i.e., the region far from the axis of symmetry) is given a large head start over its near-field counterpart, in addition to being bent slightly inwards. Because of this, the self-interaction of the far field of each wave as it propagates out towards null infinity takes place without interference from the highly nonlinear region near the axis of symmetry; and because gravity is weak in the far-field region, perturbation theory can be used to study this process. However, the radiation produced in the forward and backward null directions is not weak, for although the far fields contain only a fraction of the total energy, the solid angle into which they are focused is small, and so the energy flux per unit solid angle in these directions is not small (i.e., the news function [4], which characterizes the gravitational radiation, is of order unity in dimensionless units). Thus perturbation methods can successfully describe the generation of truly strong-field gravitational radiation in these space-times [5,6].

The physics is practically identical in collisions at just less than the speed of light—the main difference being that everything is slightly smoothed out, since the incoming shock waves are no longer impulsive [7].

For a range of impact parameters in the speed-of-light collision, Penrose [8] has found an apparent horizon on the union of the two null planes that describe the trajectories of the incoming waves. If one makes the cosmic censorship hypothesis (see, for example, Ref. [9]), then the area of the initial apparent horizon can be used to put a lower bound on the areas of the event horizons of any black holes formed by the collision. In this way, Penrose has shown that the total rest mass of the black hole (or holes) formed by the axisymmetric collision must be more than $100/\sqrt{2}\%$ of the initial energy. Conversely, if too much energy is carried off by gravitational waves then the hypothesis must be wrong. These high-speed collisions thus provide an interesting test of cosmic censorship.

There are two different perturbation methods that one can use to treat these high-speed collisions. In one approach [7], the collision was studied by large but finite γ, where γ is the Lorentz factor of the incoming holes. It was shown that the metric of a single high-speed hole, and hence also the precollision metric in the high-speed collision, can be expressed as a perturbation series in γ^{-1}. A method of matched asymptotic expansions could then be used to investigate the space-time geometry to the future of the collision. It is necessary to use a number of different asymptotic expansions to allow for the various length and time scales characteristic of the gravitational field in different parts of the space-time. One expects that expansions holding in adjacent regions will match smoothly on to each other; the regions to the past thereby providing boundary conditions for those neighboring regions to the future.

Following this approach, one may calculate the radiation on angular scales of $O(\gamma^{-1})$ produced by the focusing of the far fields of the waves as they pass through each other during the collision. It was found [7] that in this region the news function has an asymptotic expansion of the form

$$c_0(\bar{\tau},\hat{\theta}=\gamma^{-1}\psi)\sim\sum_{n=0}^{\infty}\gamma^{-2n}Q_{2n}(\bar{\tau},\psi) \tag{1.1}$$

valid as $\gamma\to\infty$ with $\bar{\tau},\psi$ fixed, where $\bar{\tau}$ is a suitable retarded time coordinate and $\hat{\theta}$ is the angle from the symmetry axis in the center-of-mass frame. In Ref. [7] the leading term $Q_0(\bar{\tau},\psi)$ was calculated; this does not vanish and is a regular function of $\bar{\tau}$. Since $Q_0(\bar{\tau},\psi)$ is not damped by any power of γ^{-1}, the news function is of order 1, and so describes truly strong-field gravitational radiation (the square of the news function is $4\pi\times$ power radiated/unit solid angle). $Q_0(\bar{\tau},\psi)$ and its first angular derivative, $\partial Q_0(\bar{\tau},\psi)/\partial\psi$, both vanish at $\psi=0$, as they must if the space-time is to be regular [4]. What is most interesting is that, as ψ tends to infinity, $Q_0(\bar{\tau},\psi)$ approaches a nonzero limiting form, which is such that 25% of the incident energy would be carried off by gravitational waves if the radiation were emitted isotropically with the limiting power/solid angle.

It was further shown in Ref. [7] that on angular scales of order 1 the news function should have an asymptotic expansion of the form

$$c_0(\hat{\tau},\hat{\theta})\sim\sum_{n=0}^{\infty}\gamma^{-n}S_n(\hat{\tau},\hat{\theta}) \tag{1.2}$$

valid as $\gamma\to\infty$ with $\hat{\tau},\hat{\theta}$ fixed. [The retarded time variables used in Eqs. (1.1) and (1.2) are not the same, owing to the varying time delays suffered by different parts of the shocks when they collide.] Here $S_0(\hat{\tau},\hat{\theta})$ must be the news function for the collision at the speed of light ($\gamma=\infty$). If the two asymptotic expansions (1.1) and (1.2) both hold in the intermediate region where $\gamma^{-1}\ll\hat{\theta}\ll 1$, then matching enables one to gain information about the angular dependence of $S_0(\hat{\tau},\hat{\theta})$ near the axis $\hat{\theta}=0$. Moreover, if $S_0(\hat{\tau},\hat{\theta})$ is sufficiently regular then it will possess a convergent series of the form

$$S_0(\hat{\tau},\hat{\theta})=\sum_{n=0}^{\infty} a_{2n}(\hat{\tau})\sin^{2n}\hat{\theta}\ , \tag{1.3}$$

since it is symmetrical about $\hat{\theta}=\pi/2$ (in the center-of-mass frame). Since $\hat{\theta}=\gamma^{-1}\psi$ in Eq. (1.1), the $\hat{\theta}^{2n}$ part of (1.3) will be found from the $(\gamma^{-1}\psi)^{2n}=\gamma^{-2n}\psi^{2n}$ part of (1.1), and thus finding $Q_{2n}(\bar{\tau},\psi)$ enables one to determine the coefficient $a_{2n}(\hat{\tau})$ of $\sin^{2n}\hat{\theta}$ in (1.3). In this way $a_0(\hat{\tau})$ was found, given by the limiting form of $Q_0(\bar{\tau},\psi)$ as $\psi\to\infty$. If these matching ideas are correct, then perturbation methods can be used to determine the entire news function of the highly nonlinear speed-of-light collision. But to calculate higher-order $Q_{2n}(\bar{\tau},\psi)$ requires the solution of inhomogeneous flat-space wave equations with extremely complicated source terms, and it is not a technically feasible way of determining the nonisotropic part of $S_0(\hat{\tau},\hat{\theta})$.

But there is another way of calculating $S_0(\hat{\tau},\hat{\theta})$ using perturbation methods, which deals with the collision at precisely the speed of light. The method was used by Curtis [10], following a suggestion by Penrose. Curtis examined the result of scattering a weak shock wave off a fully nonlinear one, using twistor methods. The perturbation parameter here is the ratio of the energies of the two waves. Curtis derived an expression for the radiation pattern at lowest order in perturbation theory, valid over the whole celestial two-sphere except very near $\hat{\theta}=\pi$. However, he did not use this expression to derive quantitative answers. It was pointed out in Ref. [7] that results in perturbation theory concerning the radiation pattern in this weak-shock–strong-shock system translate, when one makes a Lorentz boost to a center-of-mass frame, to a description of the gravitational radiation in the neighborhood of $\hat{\theta}=0$ in the fully nonlinear space-time formed by the collision of two shocks with equal energy. One can then match the expressions one derives for the news function close to $\hat{\theta}=0$ with Eq. (1.3) in order to find the entire news function $S_0(\hat{\tau},\hat{\theta})$, just as in the finite -γ collision.

In this paper we describe a similar calculation, based in part on the work of Chapman [11]. Starting with the speed-of-light collision of two shocks which each have energy μ, one can then make a large Lorentz boost away from the center-of-mass frame, so that one shock becomes much more energetic than the other. The metric describing the scattering of the weak shock off the strong one possesses a perturbation expansion in powers of λ/ν, where λ and ν are the energies of the weak and strong shock, respectively. The news function can be found to lowest order in λ/ν in the boosted frame, and then matched to obtain an expression for $a_0(\hat{\tau})$. Pleasingly, the resulting expression is identical to that derived in Ref. [7] for the isotropic part of the news function in the finite-γ collision on angular scales of order 1. By solving the second-order field equations in the boosted frame, which take the form of inhomogeneous flat-space wave equations with complicated source terms, one can go on to derive an integral expression for the first nonisotropic term $a_2(\hat{\tau})$ in Eq. (1.3). In papers II and III following [12,13], we will show how the computation of $a_2(\hat{\tau})$ can be simplified by reducing the perturbation field equations to equations in two independent variables, and will discuss the implications for the energy emitted in gravitational radiation and the nature of the radiative space-time.

One expects the apparent and event horizons in the finite-γ collision to be very similar to those in the speed-of-light encounter. A plausible scenario for each process is that at the collision there is a burst of radiation accompanied by the formation of a black hole, which settles down asymptotically to a Schwarzschild geometry. However, since $a_0(\hat{\tau})$ is nonzero, the speed-of-light news function does not vanish on the axis of symmetry $\hat{\theta}=0,\pi$, which indicates that this space-time is not smooth on the axis at future null infinity [4,7]. The logarithmic singularity [7] in the news function is another indication that, strictly speaking, this space-time is not asymptotically flat. Further, the speed-of-light space-time is certainly not asymptotically flat in the past, since the null shocks extend to infinity. These properties show that one should be careful about considering the speed-of-light collision as an isolated radiating system, and that it is better to think of it as the limit of the perfectly regular finite-γ collisions; and of the speed-of-light news function $S_0(\hat{\tau},\hat{\theta})$ as describing the radiation in the finite-γ collisions on angular scales of order 1. This will be our attitude here; that is, we are principally interested in the speed-of-light collision as a calculational tool which we use to find the higher-order moments in the off-axis news function in the finite-γ encounters. Whenever we loosely refer to future null infinity $\mathcal{I}^+$ for the speed-of-light collision, the argument can always be rephrased in terms of limiting properties of the finite-γ space-times, which are expected to have a regular $\mathcal{I}^+$.

It has been conjectured [14,15] that the radiation pattern in the high-speed collision is isotropic, apart from the detailed structure near the axis of symmetry. This conjecture was motivated by the zero-frequency limit (ZFL) calculation of Smarr [14], who found that the zero-frequency limit of the gravitational energy spectrum does have this angular distribution. If valid, it would mean that all the $a_{2n}(\hat{\tau})$ vanish for $n\geq 1$, and that the relative mass loss is 25%. It has been shown in Ref. [16] that Smarr's ZFL calculation is in fact a linearized approximation valid only when the gravitational radiation is weak, so that it cannot be applied to the head-on collision of two black holes. As will be seen from the results presented in paper III, $a_2(\hat{\tau})$ is certainly nonzero, and a complicated angular distribution is expected for the gravitational radiation.

In Sec. II of this paper we review the geometry of a single black hole moving at the speed of light, giving an impulsive plane-fronted wave. The axisymmetric collision of two such shock waves is then studied (Sec. III). After the initial collision, the shocks lie on curved surfaces to the future of regions of Minkowski space-time. To the future of both curved shocks lies the curved interaction region of the space-time, which contains the gravitational radiation. As already described, in the approach used here a large Lorentz boost is applied such that one incoming wave has an energy ν much greater than the energy λ of the other wave. The scattering of the weak shock off the strong shock is regarded as a characteristic-initial value problem for the perturbed space-time, with characteristic initial data known just to the future of the strong shock, which in lowest approximation appears as a null hyperplane between two portions of Minkowski space-time. In Sec. III the characteristic initial data is described, and the region of validity of the perturbation theory is studied. As with many strong-field problems in general relativity, when subjected to perturbation methods, the perturbation series is expected to be nonuniform, corresponding to a singular perturbation problem.

In Sec. IV the first-order field equations are studied and the first-order news function found. On boosting back to the center-of-mass frame, one finds $a_0(\hat{r})$. The second-order field equations are solved in terms of integrals in Sec. V, leading to an integral expression for the second-order news function and hence ultimately for $a_2(\hat{r})$. This expression is however intractable numerically in this particular form. Sections III–V draw heavily on the unpublished work of Chapman [11].

In Sec. VI we derive a new mass-loss formula for the axisymmetric finite-γ collision, which shows that if the product of the collision is a single black hole, and the burst of radiation whose form we are calculating is the only gravitational radiation present in the space-time, then the final mass of this hole is determined, with fractional error tending to zero as $\gamma\rightarrow\infty$, by only the first two coefficients, $a_0(\hat{r})$ and $a_2(\hat{r})$, in Eq. (1.3). This provides further motivation for computing $a_2(\hat{r})$. Section VII summarizes the paper.

In paper II following, we then go on to show that in the speed-of-light collision, in the boosted frame of reference, the metric possesses a conformal symmetry at each order in perturbation theory. This, along with the obvious axisymmetry, enables us to reduce all the field equations from four dimensions to two. This reduction to two dimensions makes the numerical computation of the second-order metric practicable. We analyze the reduced field equations, show that they are hyperbolic partial differential equations, and find their Green's functions. The numerical calculation of the second-order metric perturbation entails extensive computation, and is not entirely straightforward. We also show that there are $\ln r/r$ terms present in the second-order metric. The existence of such terms in second-order perturbation theory in harmonic gauges is well known [17]. We eliminate them by an explicit transformation to a Bondi coordinate system [4].

The numerical results are presented in paper III of the series. We show that our mass-loss formula makes the unphysical prediction that the final mass of the residual hole will be approximately twice the initial energy of the colliding waves, indicating either that the collision product is not a single black hole, or else that there is some other gravitational radiation present in the space-time. The latter possibility is more likely, and we indicate how this radiation might be generated. We also give an estimated upper bound on the final mass of the residual hole, using the conventional formula of Bondi, van der Burg, and Metzner [4].

II. THE BOOSTED METRIC

Consider a Schwarzschild metric with rest mass m, written in isotropic coordinates:

$$ds^2 \equiv -\left[\frac{1-m/2r}{1+m/2r}\right]^2 dt^2 + \left[1+\frac{m}{2r}\right]^4 [dr^2+r^2(d\theta^2+\sin^2\theta\, d\phi^2)] . \tag{2.1}$$

Make an appropriate coordinate transformation so that the black hole moves with speed $\beta=(1-\gamma^{-2})^{1/2}$ in the $-z$ direction (our units are such that the speed of light c is 1). The energy of the black hole is then $\mu=m\gamma$. Aichelburg and Sexl [3] have shown that as $\beta\rightarrow 1$ with μ fixed, the metric approaches a limit, which can be put in the form

$$ds^2 = du\, dv + dx^2 + dy^2 - 4\mu \ln(x^2+y^2)\delta(u) du^2 , \tag{2.2}$$

where $u=z+t$ and $v=z-t$. The axisymmetric space-time described by Eq. (2.2) is flat, except on the null hypersurface $u=0$ where the curvature has a δ-function singularity. For $x^2+y^2>0$ the space-time is purely radiative since the infinite Lorentz boost has changed the algebraic type of the Weyl tensor from type D, for the Schwarzschild metric (2.1), to type N for the metric (2.2). It can be shown that the energy-momentum tensor has the form $T^{ab}=\mu\delta(u)\delta(x)\delta(y)l^a l^b$, where the null vector l^a is orthogonal to the hypersurface $u=0$. Hence there is a massless pointlike particle of energy μ traveling at the speed of light along the axis of symmetry in $u=0$. The impulsive plane-fronted shock wave represented by the metric (2.2) is the gravitational field generated by the null particle. Owing to the infinite boost the black-hole properties have been lost and there is no event horizon present.

Making the discontinuous coordinate transformation [7]

$$\begin{aligned} x &= \hat{x} - 4\mu\hat{u}\,\theta(\hat{u})\frac{\hat{x}}{\hat{x}^2+\hat{y}^2} , \\ y &= \hat{y} - 4\mu\hat{u}\,\theta(\hat{u})\frac{\hat{y}}{\hat{x}^2+\hat{y}^2} , \\ u &= \hat{u} , \\ v &= \hat{v} + 4\mu\theta(\hat{u})\ln(\hat{x}^2+\hat{y}^2) - \frac{16\mu^2\hat{u}\,\theta(\hat{u})}{\hat{x}^2+\hat{y}^2} , \end{aligned} \tag{2.3}$$

where $\theta(\hat{u})$ is the Heaviside step function, the metric (2.2) becomes

$$ds^2 = d\hat{u}\, d\hat{v} + [1+4\mu\hat{u}\,\theta(\hat{u})\hat{\rho}^{-2}]^2 d\hat{\rho}^2 + \hat{\rho}^2[1-4\mu\hat{u}\,\theta(\hat{u})\hat{\rho}^{-2}]^2 d\phi^2 , \tag{2.4}$$

where $\hat{\rho}^2=\hat{x}^2+\hat{y}^2$ and $\phi=\arctan(\hat{y}/\hat{x})$. The half-space $u>0$ in Eq. (2.2) has been mapped into the region $\hat{u}>0$, $4\mu\hat{u}\leq\hat{\rho}^2$ by (2.3). The "boundary" $4\mu\hat{u}=\hat{\rho}^2$ is in fact the axis of symmetry $\rho=0$. The metric (2.4) is continuous but there is the disadvantage that the metric form is no longer Minkowskian behind the shock (in $\hat{u}>0$) [18].

Apart from a discontinuity at $\hat{u}=0$, the Christoffel symbols Γ^a_{bc} are well-behaved functions of $\hat{u},\hat{v},\hat{x},\hat{y}$. From the geodesic equation

$$\frac{d^2x^a}{ds^2}+\Gamma^a_{bc}\frac{dx^b}{ds}\frac{dx^c}{ds}=0\ , \tag{2.5}$$

we immediately see that in the careted coordinate system a geodesic crossing the shock $\hat{u}=0$ will be continuous and have a continuous tangent vector. Returning to the uncareted coordinates u,v,x,y and using Eq. (2.3), we see that the value of v will change discontinuously by $8\mu\ln\rho$ on crossing the shock, and that the geodesic will simultaneously be bent inwards.

The metric (2.2) transforms very simply under Lorentz boosts along the z axis. Define u',v',x',y' by

$$\begin{aligned} &x=x',\quad y=y'\ ,\\ &u=e^{\alpha}u',\quad v=e^{-\alpha}v'\ , \end{aligned} \tag{2.6}$$

where $e^{\alpha}=(1+\beta)^{1/2}/(1-\beta)^{1/2}$ (the primed system is therefore moving with speed β in the $+z$ direction with respect to the unprimed system). Written in terms of u',v',x',y', Eq. (2.2) becomes

$$ds^2=du'dv'+dx'^2+dy'^2-4\mu e^{\alpha}\ln(x'^2+y'^2)\delta(u')du'^2\ . \tag{2.7}$$

Thus the effect of the Lorentz boost is simply to scale the energy by a factor e^{α}, which from the Doppler formula

$$E'=E\left[\frac{1+\beta}{(1-\beta^2)^{1/2}}\right] \tag{2.8}$$

is exactly how we expect the energy of a massless particle to transform. It is obvious that in the boosted frame the metric in its continuous form will be

$$\begin{aligned} ds^2=&d\hat{u}'d\hat{v}'+[1+4\mu e^{\alpha}\hat{u}'\theta(\hat{u}')\hat{\rho}'^{-2}]^2d\hat{\rho}'^2\\ &+\hat{\rho}'^2[1-4\mu e^{\alpha}\hat{u}'\theta(\hat{u}')\hat{\rho}'^{-2}]^2d\phi^2\ . \end{aligned} \tag{2.9}$$

To obtain the metric (in its C^0 form) describing an identical wave traveling in the opposite direction, we merely replace $\hat{z}$ by $-\hat{z}$ in Eq. (2.4); or equivalently $\hat{u}$ by $-\hat{v}$ and $\hat{v}$ by $-\hat{u}$. Thus the metric will be

$$\begin{aligned} ds^2=&d\hat{u}\,d\hat{v}+[1-4\mu\hat{v}\theta(-\hat{v})\hat{\rho}^{-2}]^2d\hat{\rho}^2\\ &+\hat{\rho}^2[1+4\mu\hat{v}\theta(-\hat{v})\hat{\rho}^{-2}]^2d\phi^2\ . \end{aligned} \tag{2.10}$$

III. THE AXISYMMETRIC COLLISION

We now consider the head-on collision of two such plane-fronted waves. Initially, we work in the center-of-mass frame and denote the energy of each wave by μ. The region ahead of each shock is flat and so, before the waves collide, they propagate freely, each unaware of the other's presence. We shall choose the origin of coordinates so that the trajectories of the waves before the collision are given by $\hat{u}=0$ (call this shock 1) and $-\hat{v}=0$ (shock 2), respectively. In addition to the region ahead of the waves, there will be two further flat regions in the space-time; one behind shock 1 before shock 2 comes by, and its "mirror image" behind shock 2 before shock 1 makes its presence felt. The metric in the union of these various regions will be simply the "sum" of the individual metrics for each wave. In its C^0 form this is

$$\begin{aligned} ds^2=&d\hat{u}\,d\hat{v}+[1+4\mu\hat{u}\theta(\hat{u})\hat{\rho}^{-2}]^2d\hat{\rho}^2\\ &+[-8\mu\hat{v}\theta(-\hat{v})\hat{\rho}^{-2}+16\mu^2\hat{v}^2\theta(-\hat{v})\hat{\rho}^{-4}]d\hat{\rho}^2\\ &+\hat{\rho}^2[1-4\mu\hat{u}\theta(\hat{u})\hat{\rho}^{-2}]^2d\phi^2\\ &+\hat{\rho}^2[8\mu\hat{v}\theta(-\hat{v})\hat{\rho}^{-2}+16\mu^2\hat{v}^2\theta(-\hat{v})\hat{\rho}^{-4}]d\phi^2\ . \end{aligned} \tag{3.1}$$

Before the collision, the null generators of shock 2 (say) are the lines

$$\hat{u}=\Lambda,\quad -\hat{v}=0,\quad \hat{x}=\xi,\quad \hat{y}=\eta\ , \tag{3.2}$$

where the affine parameter Λ is negative. These null geodesics will intersect shock 1 at the spacelike collision surface $\hat{u}=-\hat{v}=0$. Their continuation into $\hat{u}>0$ will mark the future boundary of one of the flat space-time regions mentioned above, the metric having the form (3.1) to its past. Since geodesics are C^1 when viewed in careted coordinates, the null generators of shock 2 will still have $d\hat{v}/d\Lambda=d\hat{x}/d\Lambda=d\hat{y}/d\Lambda=0$ at $\hat{u}=0^+$. Inspection of Eq. (3.1) now shows that their continuation into $\hat{u}>0$ is simply given by Eq. (3.2) with $\Lambda>0$. A similar result holds for the null generators of shock 1. Thus the metric form (3.1) is valid in the regions denoted by I, II, and III in Fig. 1. Region IV to the future of the shocks will be

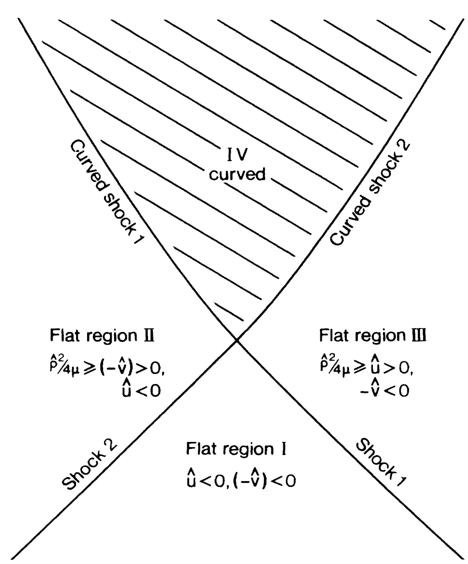

FIG. 1. A schematic space-time diagram for the speed-of-light collision. At the collision ($\hat{u}=\hat{v}=0$), the null generators of each shock will acquire shear, and will thereafter be both shearing and converging. Hence to the future of the collision the shocks will appear to lie on curved null surfaces when viewed in flat coordinates from the flat regions II and III (see Fig. 2). Region IV of space-time to the future of the curved shocks 1 and 2 will be curved.

curved, and the metric there can be found only by integrating Einstein's equations in some appropriate way.

The geometry of the shocks can be most easily understood if we view them from regions II and III when using flat coordinates. Therefore let us transform to coordinates defined by Eq. (2.3). In this coordinate system the null generators of shock 2 are clearly parametrized by

$$\begin{aligned} u &= \Lambda , \\ v &= 4\mu\theta(\Lambda)\ln(\xi^2+\eta^2) - \frac{16\mu^2\Lambda\theta(\Lambda)}{\xi^2+\eta^2} , \\ x &= \xi \left[1 - \frac{4\mu\Lambda\theta(\Lambda)}{\xi^2+\eta^2} \right] , \\ y &= \eta \left[1 - \frac{4\mu\Lambda\theta(\Lambda)}{\xi^2+\eta^2} \right] . \end{aligned} \tag{3.3}$$

Thus, viewed in these coordinates, a geodesic generator of shock 1 crossing shock 2 suffers an instantaneous translation of magnitude $4\mu\ln(\xi^2+\eta^2)$ along the hypersurface $u=0$ and is simultaneously bent inwards and backwards. Following this geodesic into $\Lambda>0$ we see that after a finite affine distance $[\Lambda=(\xi^2+\eta^2)/4\mu]$ it crosses the axis of symmetry $\rho=0$. This means that shock 2 intersects itself at a caustic, given by

$$\rho=0, \quad v=4\mu\ln(4\mu u)-4\mu . \tag{3.4}$$

The geometrical configuration is shown in Fig. 2. An analogous (inverted) picture can, of course, be drawn for

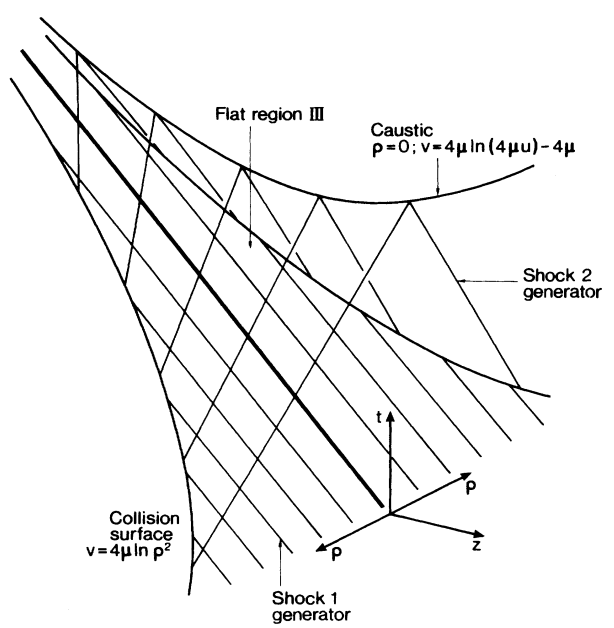

FIG. 2. The curved shock 2, as viewed from the flat region III, which lies between the curved shock 2 and the incoming plane shock 1. The curved shock 2 originates at the collision surface $v=4\mu\ln(\rho^2)$; its generators intersect one another at the caustic $\rho=0, v=4\mu\ln(4\mu u)-4\mu$. After passing through the caustic, the generators enter the curved region IV of the space-time.

shock 1. These results were first worked out by Penrose [8]. In the finite-γ collision the gross features of the geometry will be similar to those shown in Fig. 2; the main difference being that the shock and caustic structures will taper off at distances of $O(\gamma)$ from the axis of symmetry on the initial surface and so will not extend out to infinity.

In the finite-γ collision it was shown [7] that the curvature within each shock is $O(\gamma)$ before it reaches the caustic. In the limit $\gamma\to\infty$ this $O(\gamma)$ structure becomes the δ-function profile of the impulsive speed-of-light shock. It was also shown in Ref. [7] that in the caustic region itself the curvature rises to $O(\gamma^{3/2})$. This means that in the speed-of-light collision there will be a curvature singularity at each caustic, this singularity being, in some sense, worse than just a δ function. This has in fact been shown by Corkill [19] and Stewart [19], who by integrating Einstein's equations have, in addition to deriving the metric form (3.1), shown that the δ-function part of the Weyl tensor is singular at $4\mu\hat{u}=\hat{\rho}^2$ on $-\hat{v}=0$ and at $-4\mu\hat{v}=\hat{\rho}^2$ on $\hat{u}=0$ (the former being simply the caustic equation (3.4) written in terms of careted coordinates, and the latter its equivalent for shock 1). One might have expected a breakdown in predictability to the future of each singularity, the Cauchy horizon coinciding with the continuation of the shock generators beyond the caustic. However, at finite γ it was found [7] that although the shock's profile is altered by its wavelike self-interaction in the caustic region, it continues to travel in a nearly null direction beyond the caustic into the analogue of the curved region IV. This indicates that the singularities in the speed-of-light collision should be quite harmless, the space-time having a natural extension through the supposed Cauchy horizons alluded to above. The fact that we can do perturbation theory exactly as if such a continuation does exist and obtain sensible results using it (results that agree exactly with the radiation calculation of Ref. [7] at finite γ) supports this view, and we shall assume it is the case.

In the finite-γ collision the curvature of each shock has the form $\gamma^{3/2}/(1+Ap^2)^{5/2}$ just before the caustic, where γp measures the distance from the peak of the shock and A is some constant [7]. As mentioned above, this shock profile is the finite-γ analogue of the δ function in the speed-of-light collision. Owing to the shock's self-interaction, its curvature profile just beyond the caustic is approximately $\gamma^{3/2}/q$ when q is large, where again γq measures the distance from the center of the shock [7]. This indicates that the impulsive structure of the shocks in the speed-of-light collision will be destroyed at the singularities, the curvature having a rather odd $1/s$ form to leading order near the continuation of the shocks beyond their respective caustics. That this $1/s$ form continues all the way out to $\mathscr{I}^+$ can be seen from Ref. [7], for $a_0(\hat{\tau})$ has a logarithmic singularity and so the curvature, which is given by the time derivative of the news function, has a $1/(\hat{\tau}-\hat{\tau}_0)$ term.

One consequence of the self-intersection of each shock is that any point on a null shock generator to the future of that shock's caustic will be timelike connected with points on the same shock to the past of the caustic [20], and will lie within the curved space-time region IV in Fig. 1. This means that the null surfaces on which the shocks lie cannot be used as characteristic initial surfaces (for solving Einstein's equations) beyond their caustics, since null data cannot be freely given there. This is consistent with Eq. (3.1), which provides initial data only for $4\mu\hat{u} \le \hat{\rho}^2$ on $-\hat{v}=0$ and for $-4\mu\hat{v} \le \hat{\rho}^2$ on $\hat{u}=0$ (i.e., up to the caustics but not beyond). This difficulty with caustics is a common feature of the characteristic initial value problem in general relativity (for a general discussion and possible resolution see Ref. [21]), but we shall see that it does not affect our perturbation problem.

One should note that the portion of each null hypersurface on which data can be given [and which is given by Eq. (3.1)] does not intersect future null infinity. This is easily seen for shock 2 (say), since Eq. (3.4) shows that the caustic does not reach $\mathcal{J}^+(-v\to-\infty$ as $u\to\infty$ on the caustic), and therefore neither does the precollision shock (see Fig. 2). This would appear to create problems if one were to try to construct the whole spacetime numerically by evolving the initial data off the null shock surfaces (see Ref. [22] for a general discussion of the numerical construction of space-times from characteristic initial surfaces), since one cannot reach $\mathcal{J}^+$ by making a series of finite jumps off these surfaces. But perhaps one could discover all the essential features of the space-time without going too far out from the center.

Penrose [8] has found an apparent horizon on the union of the two null planes that describe the trajectories of the incoming shock waves in the speed-of-light collision. The horizon is formed by the union of two flat discs whose common boundary is a circle $\rho=4\mu$ in the collision surface $\hat{u}=-\hat{v}=0$, having area $32\pi\mu^2$. If cosmic censorship holds, there will be an event horizon outside this apparent horizon [9], and its area cannot decrease, so that if the space-time eventually settles down to the Schwarzschild geometry, as seems likely, the area of the final black hole must be greater than $32\pi\mu^2$; or in other words it must have a mass greater than $(1/\sqrt{2})2\mu$. (A figure $\frac{1}{2}\times 2\mu$ was wrongly quoted in Ref. [7].) We expect a horizon of a very similar nature to be formed in the finite-γ collision, and the lower bound on the mass of the final hole there should be the same, to within a relative error tending to zero as $\gamma\to\infty$. It is interesting that the lower bound which cosmic censorship places on the ratio (mass of final black hole)/(initial energy) should be the same for the ultrarelativistic encounter and the collision of two black holes starting from rest at infinity.

The null particles lie within the apparent horizon and are therefore trapped, and presumably will eventually run into an unpleasant space-like singularity, which should be hidden from infinity by the event horizon, assuming cosmic censorship holds. The inner portions of each shock (certainly for $\rho\le 4\mu$) should fold up into the same singularity. It is likely that this singularity will be formed at the point where the particles collide $(\hat{u}=-\hat{v}=\hat{\rho}=0)$. This would certainly be the most satisfactory outcome, since it would remove any ambiguities concerning the particle-particle interaction. It is reasonable as the limit of the finite-γ collision, in which the small, fast-moving, black holes should stop each other when they collide.

In order to calculate the form of the gravitational radiation in the speed-of-light space-time, we make a large Lorentz boost and observe the collision in a frame of reference moving with velocity β [where $(1-\beta)\ll 1$] in the $+z$ direction with respect to the center-of-mass frame. From the discussion leading up to Eq. (2.9) it is clear that in the boosted frame the precollision metric takes the form

$$ds^2=d\hat{u}'d\hat{v}'+[1+4\nu\hat{u}'\theta(\hat{u}')\hat{\rho}'^{-2}]^2d\hat{\rho}'^2+[-8\lambda\hat{v}'\theta(-\hat{v}')\hat{\rho}'^{-2}+16\lambda^2\hat{v}'^2\theta(-\hat{v}')\hat{\rho}'^{-4}]d\hat{\rho}'^2$$
$$+\hat{\rho}'^2[1-4\nu\hat{u}'\theta(\hat{u}')\hat{\rho}'^{-2}]^2d\phi^2+\hat{\rho}'^2[8\lambda\hat{v}'\theta(-\hat{v}')\hat{\rho}'^{-2}+16\lambda^2\hat{v}'^2\theta(-\hat{v}')\hat{\rho}'^{-4}]d\phi^2 , \tag{3.5}$$

where $\nu=\mu e^{\alpha}$, $\lambda=\mu e^{-\alpha}$, and $e^{\alpha}=(1+\beta)^{1/2}/(1-\beta)^{1/2}$. The collision appears to be between a weak shock wave of energy λ and a strong shock of energy ν, where $\lambda/\nu\ll 1$.

We now consider the evolution of the weak shock in the region behind the strong shock. For convenience we drop the prime on coordinates in the boosted frame, since we will be working exclusively in this frame of reference. The boundary data on $\hat{u}=0$ for the characteristic initial value problem whose solution describes the propagation of the weak shock in $\hat{u}>0$ is

$$g_{ab}=\eta_{ab}+\lambda\hat{h}^{(1)}_{ab}+\lambda^2\hat{h}^{(2)}_{ab} , \tag{3.6}$$

where the only nonzero components of $\hat{h}^{(1)}_{ab}$ and $\hat{h}^{(2)}_{ab}$ are

$$\hat{h}^{(1)}_{\hat{\rho}\hat{\rho}}=-\hat{\rho}^{-2}\hat{h}^{(1)}_{\phi\phi}=-8\lambda\hat{v}\theta(-\hat{v})\hat{\rho}^{-2} ,$$
$$\hat{h}^{(2)}_{\hat{\rho}\hat{\rho}}=\hat{\rho}^{-2}\hat{h}^{(2)}_{\phi\phi}=16\lambda^2\hat{v}^2\theta(-\hat{v})\hat{\rho}^{-4} . \tag{3.7}$$

On the initial surface $\hat{u}=0$, in the region where $\hat{v}$ and $\hat{\rho}$ are of $O(\nu)$ (call this region R_ν), the contribution to Eq. (3.6) from the weak shock appears as a small perturbation of $O(\lambda/\nu)$ to the "background" metric of the strong shock. Therefore, in that region of space-time to the future of the initial surface which can be influenced only by R_ν, the metric should possess a perturbation expansion in powers of λ/ν. We shall demonstrate this explicitly below.

The geometry is easier to visualize if we transform to a coordinate system (u,v,x,y) in which the background metric of the strong shock in $\hat{u}>0$ is manifestly Minkowskian. An appropriate coordinate transformation is given by Eq. (2.3) with μ replaced by ν:

$$x=\hat{x}[1-4\nu\hat{u}\theta(\hat{u})\hat{\rho}^{-2}] ,$$
$$y=\hat{y}(1-4\nu\hat{u}\theta(\hat{u})\hat{\rho}^{-2}) ,$$
$$u=\hat{u} ,$$
$$v=\hat{v}+8\nu\theta(\hat{u})\ln\hat{\rho}-16\nu^2\hat{u}\theta(\hat{u})\hat{\rho}^{-2} . \tag{3.8}$$

The metric transforms as

$$g_{ab}=\left[\frac{\partial\hat{x}^{c}}{\partial x^{a}}\right]\left[\frac{\partial\hat{x}^{d}}{\partial x^{b}}\right]\hat{g}_{cd}\ . \tag{3.9}$$

On $u=0^{+}$ one easily finds

$$\frac{\partial\hat{x}^{a}}{\partial x^{b}}=\begin{bmatrix}1 & 0 & 0 & 0\\ -16v^2\rho^{-2} & 1 & -8vx\rho^{-2} & -8vy\rho^{-2}\\ 4vx\rho^{-2} & 0 & 1 & 0\\ 4vy\rho^{-2} & 0 & 0 & 1\end{bmatrix}, \tag{3.10}$$

where $\hat{x}^{a}=(\hat{u},\hat{v},\hat{x},\hat{y})$ and $x^{b}=(u,v,x,y)$. If we rescale the coordinates by a factor ν (to exhibit more clearly the perturbative behavior mentioned above), letting

$$X^{a}_{\rm old}=\nu X^{a}_{\rm new} \tag{3.11}$$

and redefine u and v through $u=(z+t)/\sqrt{2}$ and $v=(z-t)/\sqrt{2}$, then the form of the metric in the unhatted coordinate system on $u=0^{+}$ is

$$g_{ab}=\nu^2\left[\eta_{ab}+\left[\frac{\lambda}{\nu}\right]h^{(1)}_{ab}+\left[\frac{\lambda}{\nu}\right]^2h^{(2)}_{ab}\right]. \tag{3.12}$$

Here

$$\begin{aligned}&h^{(1)}_{uu}=A,\ \ h^{(1)}_{vv}=0,\ \ h^{(1)}_{xx}=(y^2-x^2)\rho^{-2}E\ ,\\ &h^{(1)}_{yy}=(x^2-y^2)\rho^{-2}E,\ \ h^{(1)}_{uv}=0,\ \ h^{(1)}_{vx}=0\ ,\\ &h^{(1)}_{xy}=-2xy\rho^{-2}E,\ \ h^{(1)}_{ux}=x\rho^{-1}B\ ,\\ &h^{(1)}_{vy}=0,\ \ h^{(1)}_{uy}=y\rho^{-1}B\ ,\end{aligned} \tag{3.13}$$

where

$$\begin{aligned}&A=32\rho^{-4}f(8\ln(\nu\rho)-\sqrt{2}v)\ ,\\ &B=4\sqrt{2}\rho^{-3}f(8\ln(\nu\rho)-\sqrt{2}v)\ ,\\ &E=-\rho^{-2}f(8\ln(\nu\rho)-\sqrt{2}v)\ ,\end{aligned} \tag{3.14}$$

and $f(x)=8x\theta(x)$. Also

$$\begin{aligned}&h^{(2)}_{uu}=H^{(2)},\ \ h^{(2)}_{vv}=0,\ \ h^{(2)}_{xx}=D^{(2)},\ \ h^{(2)}_{yy}=D^{(2)}\ ,\\ &h^{(2)}_{uv}=0,\ \ h^{(2)}_{vx}=0,\ \ h^{(2)}_{xy}=0\ ,\\ &h^{(2)}_{ux}=x\rho^{-1}I^{(2)},\ \ h^{(2)}_{vy}=0,\ \ h^{(2)}_{uy}=y\rho^{-1}I^{(2)}\ ,\end{aligned} \tag{3.15}$$

where

$$\begin{aligned}&H^{(2)}=32\rho^{-6}g(8\ln(\nu\rho)-\sqrt{2}v)\ ,\\ &I^{(2)}=4\sqrt{2}\rho^{-5}g(8\ln(\nu\rho)-\sqrt{2}v)\ ,\\ &D^{(2)}=\rho^{-4}g(8\ln(\nu\rho)-\sqrt{2}v)\ ,\end{aligned} \tag{3.16}$$

and $g(x)=16x^2\theta(x)$. The flat background metric η_{ab} is given by

$$ds^2=\eta_{ab}dx^a dx^b=2\,du\,dv+dx^2+dy^2\ . \tag{3.17}$$

To the future of $u=0$, the metric will possess the perturbation expansion

$$g_{ab}\sim\nu^2\left[\eta_{ab}+\sum_{i=1}^{\infty}\left[\frac{\lambda}{\nu}\right]^{i}h^{(i)}_{ab}\right]. \tag{3.18}$$

In principle we can find $h^{(1)}_{ab},h^{(2)}_{ab},\ldots$ by successively solving the linearized field equations at first, second, . . . order in λ/ν.

We now comment on the region of the validity of this expansion. In Fig. 3 we show the locus of intersection of the past null cone of a point $P=(u,v,\rho,\phi)$ with the initial surface $(0,v',\rho',\phi')$. It is a paraboloid, given by

$$v'=v+\frac{(\rho')^2-2\rho\rho'\cos(\phi-\phi')+\rho^2}{2u}\ . \tag{3.19}$$

We shall see in the next section that, as might be expected, the gravitational radiation in the space-time is concentrated in the region surrounding the continuation of the weak shock generators beyond the caustic. From Eq. (3.3) it is clear that these generators are parametrized by

$$\begin{aligned}&u=\sqrt{2}\Lambda\ ,\\ &v=4\sqrt{2}\ln(\nu\sqrt{\xi^2+\eta^2})-\frac{16\sqrt{2}}{\xi^2+\eta^2}\Lambda\ ,\\ &x=\xi\left[1-\frac{8\Lambda}{\xi^2+\eta^2}\right],\\ &y=\eta\left[1-\frac{8\Lambda}{\xi^2+\eta^2}\right].\end{aligned} \tag{3.20}$$

Therefore let u,v, and ρ be given by

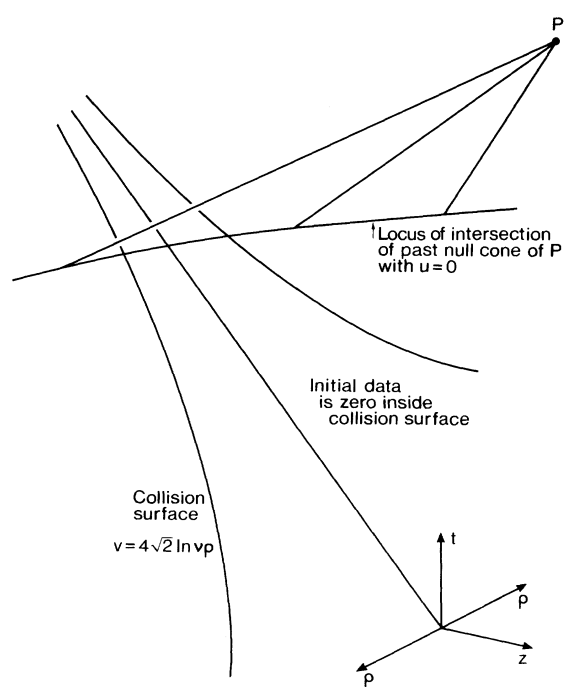

FIG. 3. This shows the locus of intersection of the past null cone of a point P with the initial surface $u=0$, on which the incoming shock 1 lies. The initial data for the perturbation problem is zero inside the collision surface $v=4\sqrt{2}\ln(\nu\rho)$.

$$u=\sqrt{2}\Lambda\ ,$$
$$v=4\sqrt{2}\ln(\nu\rho_0)-\frac{16\sqrt{2}}{\rho_0^2}\Lambda-\frac{K}{\sqrt{2}}\ , \qquad (3.21)$$
$$\rho=\left[\frac{8\Lambda}{\rho_0^2}-1\right]\rho_0\ ,$$

where future null infinity is reached by letting $\Lambda\to\infty$. (As K increases, the paraboloid sweeps up the initial surface in the $-v$ direction.) Then

$$\tan\theta\equiv\frac{\rho}{z}=\frac{2(4/\rho_0)}{1-(4/\rho_0)^2}+O(\Lambda^{-1})\ , \qquad (3.22)$$

and so $\tan(\theta/2)=(4/\rho_0)+O(\Lambda^{-1})$, from the standard trigonometric formula. For the point (3.21), Eq. (3.19) reduces to

$$v'=4\sqrt{2}\left[\ln(\nu\rho_0)-1-\frac{\rho'}{\rho_0}\cos(\phi-\phi')\right]-\frac{K}{\sqrt{2}}\ . \qquad (3.23)$$

Using Eqs. (3.13)–(3.16), we find that, at this locus of intersection,

$$h_{ab}^{(1)}\propto S\theta(S)(\rho')^{-m},\quad m=2,3,\text{ or }4\ ,$$
$$h_{ab}^{(2)}\propto S^2\theta(S)(\rho')^{-n},\quad n=4,5,\text{ or }6\ , \qquad (3.24)$$

where S is defined by

$$S=8\ln\left[\frac{\rho'}{\rho_0}\right]+8+8\left[\frac{\rho'}{\rho_0}\right]\cos(\phi-\phi')+K\ . \qquad (3.25)$$

Because of the form of the initial data (3.13)–(3.16), the metric perturbations become small at early retarded times or equivalently as $K\to-\infty$. For simplicity then, let us consider the worst case in which K is large and positive. Differentiating Eq. (3.24) we find that $|h_{ab}^{(1)}|$ and $|h_{ab}^{(2)}|$ take their respective maximum values when

$$\frac{8}{\rho'}+\frac{8}{\rho_0}\cos(\phi-\phi')-\frac{mS}{\rho'}=0\ ,$$
$$\frac{16}{\rho'}+\frac{16}{\rho_0}\cos(\phi-\phi')-\frac{nS}{\rho'}=0\ . \qquad (3.26)$$

At these points

$$h_{ab}^{(1)}\propto\left[\tan\frac{\theta}{2}e^{K/8}\right]^m\ ,$$
$$h_{ab}^{(2)}\propto\left[\tan\frac{\theta}{2}e^{K/8}\right]^n\ . \qquad (3.27)$$

Since, roughly speaking, the metric perturbations decrease as we move away from the initial surface along the past null cone of P, we see that the perturbation series (3.18) will converge everywhere in this null cone only if $(\lambda/\nu)[\tan(\theta/2)\exp(K/8)]^4$ is small. (If desired, this property can be checked more carefully using the integral representation of the metric perturbations, as employed in Secs. IV and V.) But since λ/ν can be made arbitrarily small, our perturbation theory is valid in the region of interest surrounding $K=0$.

Note that these rough estimates indicate the possibility that the metric perturbations may grow exponentially with time at late retarded times. Such nonuniform behavior is typical of singular perturbation theory, encountered in a variety of strong-field problems in general relativity. In fact it seems that $h_{ab}^{(1)}$ and $h_{ab}^{(2)}$ are well behaved at late times, at least as far as is borne out by the behavior of the corresponding parts $a_0(\hat{\tau})$ and $a_2(\hat{\tau})$ of the news function, computed in Secs. IV and V and in papers II and III. However it is very likely that the higher-order perturbations $h_{ab}^{(3)},\ldots$ do grow exponentially at late times, for reasons to be discussed further in paper III.

It is easy to show that the caustic in the initial surface $u=0$ is at

$$8\ln(\nu\rho)-\sqrt{2}v=\frac{\rho^2}{4(\lambda/\nu)}\ . \qquad (3.28)$$

This lies well beyond the region in which perturbation theory is valid, and so can be ignored in our present calculation.

IV. THE FIRST-ORDER CALCULATION

The field equations for $h_{ab}^{(1)}$ in $u\geq0$ are

$$-\bar{h}^{(1)}{}_{ab,c}{}^{c}-\eta_{ab}\bar{h}^{(1)}{}_{cd,}{}^{cd}+\bar{h}^{(1)}{}_{ac,}{}^{c}{}_{b}+\bar{h}^{(1)}{}_{bc,}{}^{c}{}_{a}=0\ , \qquad (4.1)$$

where $\bar{h}_{ab}^{(1)}=h_{ab}^{(1)}-\frac{1}{2}\eta_{ab}h^{(1)}{}_c{}^c$, $\partial/\partial x^c$ is denoted by ${}_{,c}$, and indices are raised and lowered using η_{ab}. We solve Eq. (4.1) in the usual way by making a gauge transformation $x^a=x^{Na}+(\lambda/\nu)\xi^a$ such that the new first-order perturbation $h_{ab}^{N(1)}=h_{ab}^{(1)}+2\xi_{(a,b)}$ satisfies the de Donder gauge condition

$$\bar{h}^{N(1)}{}_{ab,}{}^{b}=0 \qquad (4.2)$$

in $u\geq0$. Then Eq. (4.1) reduces to

$$\Box\bar{h}_{ab}^{N(1)}\equiv\left[2\frac{\partial^2}{\partial u\,\partial v}+\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right]\bar{h}_{ab}^{N(1)}=0\ . \qquad (4.3)$$

We now prove that for Eq. (4.2) to hold it is sufficient that $h_{ab}^{N(1)}$ satisfy

$$\bar{h}^{N(1)}{}_{ab,}{}^{b}{}_{v}|_{u=0}=0\ . \qquad (4.4)$$

The general solution to the wave equation $\Box F=0$ in $u\geq0$, with F given on $u=0$ (subject to the restriction $F\to0$ sufficiently rapidly as $v\to\infty$), is [23,24]

$$F(u,v,x,y)=\frac{-1}{2\pi u}\int_0^\infty\int_0^{2\pi}\rho'd\rho'd\phi'\left[\frac{\partial}{\partial v'}F(0,v',x',y')\right], \qquad (4.5)$$

where $x=\rho\cos\phi$, $y=\rho\sin\phi$, $x'=\rho'\cos\phi'$, $y'=\rho'\sin\phi'$, and v' is determined as a function of x' and y' through

$$v'=v+[\rho^2-2\rho\rho'\cos(\phi-\phi')+\rho'^2]/2u\ . \qquad (4.6)$$

The functional form of Eq. (4.6) is such that $(0,v',x',y')$ lies in the past null cone of (u,v,x,y). It can be seen from Eq. (4.5) that if $\partial F/\partial v|_{u=0}=0$, then $F=0$ everywhere in $u \geq 0$. Now if $\Box\bar{h}^{N(1)}_{ab}=0$, then also $\Box\bar{h}^{N(1)}{}_{ab,}{}^{b}=0$. Hence $\bar{h}^{N(1)}{}_{ab,}{}^{b}{}_{v}|_{u=0}=0$ will ensure that Eq. (4.2) holds [our initial data (3.13) and (3.14) do go to zero sufficiently rapidly as $v \to \infty$]. Thus any solution to the combined system $\Box\bar{h}^{N(1)}_{ab}=0$, $\bar{h}^{N(1)\,b}_{ab,\ \ v}|_{u=0}=0$ will also be a solution to the field equations (4.1).

On rewriting Eq. (4.4) in terms of ξ_a and $h^{(1)}_{ab}$ we find that ξ_a must satisfy

$$-\Box\xi_{[a,v]}-\tfrac{1}{2}\eta_{av}\Box\xi_{c,}{}^{c}=\bar{h}^{(1)}_{ax,xv}+\bar{h}^{(1)}_{ay,yv}+\bar{h}^{(1)}_{au,vv}-\tfrac{1}{2}(\bar{h}^{(1)}_{av,xx}+\bar{h}^{(1)}_{av,yy})\ , \qquad (4.7)$$

where the $\bar{h}^{(1)}_{av,uv}$ term has been eliminated using Eq. (4.3). For $a=v$ both sides of Eq. (4.7) vanish identically; the right-hand side by virtue of Eq. (4.1) with $a=b=v$. Substituting the initial data (3.13) and (3.14) we find that the other three equations (4.7) for ξ_a on $u=0$ are

$$\Box\xi_{[x,v]}=0,\quad \Box\xi_{[y,v]}=0\ ,$$
$$-\tfrac{1}{2}\Box(\xi_{x,x}+\xi_{y,y}+2\xi_{u,v})=128\rho^{-4}\theta(8\ln(\nu\rho)-\sqrt{2}v)\ . \qquad (4.8)$$

We look for a solution which has a power series expansion near $u=0$:

$$\xi_a(u,v,x,y)=\xi^{(0)}_a(v,x,y)+u\xi^{(1)}_a(v,x,y)+\cdots\ . \qquad (4.9)$$

One possibility is

$$\xi^{(0)}_a=0,\quad \xi^{(1)}_v=\xi^{(1)}_x=\xi^{(1)}_y=0\ ,$$
$$\xi^{(1)}_u=-16\rho^{-4}(8\ln(\nu\rho)-\sqrt{2}v)^2\theta(8\ln(\nu\rho)-\sqrt{2}v)\ . \qquad (4.10)$$

The gauge transformation (4.10) retains the form (3.13) for the metric coefficients on $u=0$, but (3.14) becomes

$$A=32\rho^{-4}f(8\ln(\nu\rho)-\sqrt{2}v)-4\rho^{-4}(8\ln(\nu\rho)-\sqrt{2}v)f(8\ln(\nu\rho)-\sqrt{2}v)\ ,$$
$$B=4\sqrt{2}\rho^{-3}f(8\ln(\nu\rho)-\sqrt{2}v)\ , \qquad (4.11)$$
$$E=-\rho^{-2}f(8\ln(\nu\rho)-\sqrt{2}v)\ .$$

It can be verified that Eq. (4.4) is satisfied in this gauge.

The metric perturbation $h^{N(1)}_{ab}$ is traceless on the initial surface:

$$h^{N(1)}|_{u=0}\equiv\eta^{ab}h^{N(1)}_{ab}|_{u=0}=0\ , \qquad (4.12)$$

and it is therefore traceless everywhere in $u\geq 0$ by virtue of Eq. (4.3). Thus Eq. (4.3) reduces to

$$\Box h^{N(1)}_{ab}=0\ . \qquad (4.13)$$

It is clear that (4.13) preserves the metric form (3.13) in $u>0$.

Bondi, Metzner, and van der Burg [4] have shown that the gravitational radiation in an axisymmetric, reflection-symmetric space-time is described by a single real function $c_0(\tau,\theta)$ of retarded time and polar angle, known as the news function. The news function is an invariant quantity, but for convenience we use the definition

$$c_0(\tau,\theta)=-\tfrac{1}{2}\lim_{r\to\infty}\left[r^{-1}(\sin\theta)^{-2}\frac{\partial g_{\phi\phi}}{\partial\tau}\right]. \qquad (4.14)$$

[Strictly speaking, Eq. (4.14) is valid only when the metric is written in Bondi coordinates in the radiation zone. However, it gives the correct answer here even for our Minkowskian coordinates. We defer an explicit verification of this technical point to the following paper II.] Here (τ,r,θ,ϕ) are defined in terms of (u,v,x,y) through

$$r^2=\nu^2(x^2+y^2+z^2)\ ,$$
$$\tau=\nu t-r\ ,$$
$$\theta=\arctan\left[\frac{\rho}{z}\right],\quad \phi=\arctan\left[\frac{y}{x}\right]. \qquad (4.15)$$

Using Eq. (3.13) we find that the first-order news function in the boosted frame is

$$c^{(1)}_0(\tau,\theta)=-\frac{1}{2}\left[\frac{\lambda}{\nu}\right]\lim_{r\to\infty}\left[r\frac{\partial E}{\partial\tau}\right]. \qquad (4.16)$$

Since $\Box h^{(1)}_{xx}=\Box h^{(1)}_{yy}=0$, the metric function $E(u,v,\rho)$ satisfies

$$\Box(e^{2i\phi}E)=0\ ,$$

where

$$e^{2i\phi}E|_{u=0}=8\rho^{-2}e^{2i\phi}[8\ln(\nu\rho)-\sqrt{2}v]\theta(8\ln(\nu\rho)-\sqrt{2}v)\ . \qquad (4.17)$$

Using the integral representation (4.5) we find

$$e^{2i\phi}E=\frac{-4\sqrt{2}}{\pi u}\int_0^\infty\int_0^{2\pi}\frac{d\rho'}{\rho'}d\phi' e^{2i\phi'}\theta\left[8\ln(\nu\rho')-\sqrt{2}\left[v+\frac{\rho^2-2\rho\rho'\cos(\phi-\phi')+\rho'^2}{2u}\right]\right]. \qquad (4.18)$$

Eliminating u,v,ρ using Eq. (4.15), we find

$$E(\tau,r,\theta)=\frac{-8\nu}{\pi[\tau+2r\cos^2(\theta/2)]}\int_0^\infty\int_0^{2\pi}\frac{d\rho'}{\rho'}d\omega\cos(2\omega)\theta\left[8\ln(\nu\rho')+\frac{\tau}{\nu}\sec^2(\theta/2)+2\tan(\theta/2)\rho'\cos\omega+O(1/r)\right]. \qquad (4.19)$$

Defining $s=2\rho'\tan(\theta/2)$ and letting $r\to\infty$, the first-order news function is found to be

$$c_0^{(1)}(\tau,\theta)=\left[\frac{\lambda}{\nu}\right]\frac{2}{\pi}\sec^4(\theta/2)\int_0^\infty\int_0^{2\pi}\frac{ds}{s}d\omega\cos(2\omega)\delta\left[8\ln s+s\cos\omega+\frac{\tau}{\nu}\sec^2(\theta/2)-8\ln\left[\frac{2\tan(\theta/2)}{\nu}\right]\right]. \quad (4.20)$$

Performing the angular integration, Eq. (4.20) reduces to

$$c_0^{(1)}(\tau,\theta)=\left[\frac{\lambda}{\nu}\right]\sec^4(\theta/2)H_0(T)\ , \quad (4.21)$$

where $T=(\tau/\nu)\sec^2(\theta/2)-8\ln[2\tan(\theta/2)/\nu]$ and

$$H_0(T)=\frac{4}{\pi}\int_D\frac{ds}{s^2}\left[2\left[\frac{T+8\ln s}{s}\right]^2-1\right]\times\left[1-\left[\frac{T+8\ln s}{s}\right]^2\right]^{-1/2}. \quad (4.22)$$

Here D is the domain in which

$$-s\le(8\ln s+T)\le s\ . \quad (4.23)$$

The reason for the appearance of the $8\ln[2\tan(\theta/2)/\nu]$ term in the expression for T can be traced to the logarithmic delay across the strong shock. In fact, if we look at the null geodesic generators of the weak shock in $u\ge 0$, as given by Eq. (3.20), and let the affine parameter $\Lambda\to\infty$, then we find that at their intersection with $\mathcal{I}^+, T=8-8\ln 8$. The news function has a logarithmic singularity at this value of T and is significantly nonzero only in the surrounding region, dying away asymptotically on either side of the weak shock.

We now transform back to the center-of-mass frame to see what Eq. (4.21) tells us about the $\sin^2\hat{\theta}$ series (1.3) for the news function. Let $\hat{\tau},\hat{r},\hat{\theta}$ denote the center-of-mass coordinates. Then [4]

$$\tau=\hat{\tau}/K,\quad r=K\hat{r}\ ,$$
$$\tan(\theta/2)=e^\alpha\tan(\hat{\theta}/2)\ , \quad (4.24)$$

while the news function transforms as

$$\hat{c}_0=c_0/K^2\ . \quad (4.25)$$

Here $K(\hat{\theta})=\cosh\alpha-\sinh\alpha\cos\hat{\theta}$. Using Eqs. (4.24), (4.25), we find that (4.21) transforms to

$$\hat{c}_0(\hat{\tau},\hat{\theta})=\frac{2}{1+\cos\hat{\theta}}H_0(T)\ , \quad (4.26)$$

where now $T=[2\hat{\tau}/\mu(1+\cos\hat{\theta})]-8\ln\{2[\tan(\hat{\theta}/2)]/\mu\}$. By making a supertranslation, $\hat{\tau}=\hat{\tau}'+4\ln[2\tan(\hat{\theta}/2)/\mu]/\mu(1+\cos\hat{\theta})$, which leaves the news function invariant [4], we may eliminate the logarithmic term from T. In terms of $\hat{\tau}'$, one has $T=2\hat{\tau}'/\mu(1+\cos\hat{\theta})$, but for convenience we shall omit the prime on $\hat{\tau}'$ and merely write $T=2\hat{\tau}/\mu(1+\cos\hat{\theta})$ in what follows.

The expression (4.21) for the first-order news function in the boosted frame is valid only for values of θ not too close to π [i.e., for $(\pi-\theta)\sim 1$; but not, for example, for $(\pi-\theta)=O(\lambda/\nu)$]. It is easy to see this from the parametric representation (3.20) for the weak shock generators; for if $(\pi-\theta)\ll 1$ out near $\mathcal{I}^+$ then $(\xi^2+\eta^2)\ll 1$ on $u=0$, and the initial data (3.13) will then not be a small perturbation. This implies that in the center-of-mass frame Eq. (4.26) is valid only in the neighborhood of $\hat{\theta}=0$ (and by symmetry near $\hat{\theta}=\pi$). Thus the right-hand side of Eq. (4.26) should really be written as

$$H_0\left[\frac{\hat{\tau}}{\mu}\right]+\sin^2\hat{\theta}\left[\frac{1}{4}H_0\left[\frac{\hat{\tau}}{\mu}\right]+\frac{1}{4}\left[\frac{\hat{\tau}}{\mu}\right]H_0'\left[\frac{\hat{\tau}}{\mu}\right]\right]+\cdots. \quad (4.27)$$

Hence all Eq. (4.26) tells us is that the isotropic term $a_0(\hat{\tau})$ in Eq. (1.3) is $H_0(\hat{\tau}/\mu)$. We cannot say at this stage what the $\sin^2\hat{\theta}$, $\sin^4\hat{\theta}$, ... terms are, since there will be contributions to these from the second, third, ... order news functions in the boosted frame.

The expression $H_0(\hat{\tau}/\mu)$ for $a_0(\hat{\tau})$ agrees with that derived previously in Ref. [7] as the isotropic part of the news function in the finite-γ collisions on angular scales of order 1. This is pleasing, for it indicates that the matching ideas of Ref. [7] outlined in the Introduction are working. The form of $a_0(\hat{\tau})$ is shown in Fig. 4. The singularity at $\hat{\tau}=0$ is logarithmic. The magnitude of $a_0(\hat{\tau})$ is such that if the radiation were isotropic [i.e., if $a_{2n}(\hat{\tau})=0$, $\forall n\ge 1$] then the total energy carried off by gravitational waves would be 25% of the initial energy 2μ.

V. THE SECOND-ORDER CALCULATION

We now show that, by finding the news function to second order in λ/ν in the boosted frame, we may determine the coefficient $a_2(\hat{\tau})$ of $\sin^2\hat{\theta}$ in Eq. (1.3), and thereby get some idea of the angular dependence of the total news function.

Near the axis, the series expansion for the news func-

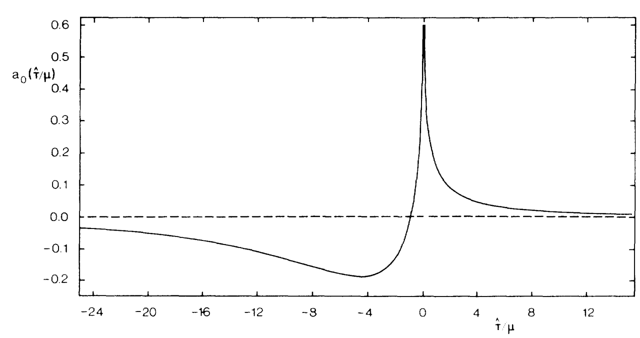

FIG. 4. The isotropic part $a_0(\hat{\tau})$ of the news function, appearing in the expansion (1.3) about the symmetry axis for the gravitational radiation. The singularity at $\hat{\tau}=0$ is logarithmic. For convenience, the origin of retarded time $\hat{\tau}$ has been shifted (supertranslated) by a constant amount $(8-8\ln 8)\mu$.

tion in the center-of-mass frame is

$$\hat{c}_0(\hat{\tau},\hat{\theta})=a_0\left[\frac{\hat{\tau}}{\mu}\right]+a_2\left[\frac{\hat{\tau}}{\mu}\right]\sin^2\hat{\theta}+O(\sin^4\hat{\theta}) \ , \quad (5.1)$$

in which we have so far found the first term $a_0(\hat{\tau}/\mu)=H_0(\hat{\tau}/\mu)$. [For convenience we write $\hat{\tau}/\mu$ instead of $\hat{\tau}$; and in fact $\hat{\tau}$ and μ *will* appear in this combination at every order. That is, in this section we make the replacement $a_{2n}(\hat{\tau})\rightarrow a_{2n}(\hat{\tau}/\mu)$.] In the boosted frame (5.1) becomes

$$c_0(\tau,\theta)=K^{-2}\left[a_0\left[\frac{\tau}{\mu K}\right]+\frac{\sin^2\theta}{K^2}a_2\left[\frac{\tau}{\mu K}\right]+O\left[\frac{\sin^4\theta}{K^4}\right]\right] , \quad (5.2)$$

where

$$K(\theta)=\cosh\alpha+\sinh\alpha\cos\theta$$
$$=e^{\alpha}\cos^2(\theta/2)[1+e^{-2\alpha}\tan^2(\theta/2)] \ . \quad (5.3)$$

Combining Eq. (5.3) with Eq. (5.2) we find

$$c_0(\tau,\theta)=e^{-2\alpha}\sec^4(\theta/2)H_0[(\tau/\nu)\sec^2(\theta/2)]$$
$$+e^{-4\alpha}\{4\tan^2(\theta/2)\sec^4(\theta/2)a_2[(\tau/\nu)\sec^2(\theta/2)]-2\tan^2(\theta/2)\sec^4(\theta/2)H_0[(\tau/\nu)\sec^2(\theta/2)]$$
$$-(\tau/\nu)\sec^6(\theta/2)\tan^2(\theta/2)H_0'[(\tau/\nu)\sec^2(\theta/2)]\}+\cdots \ . \quad (5.4)$$

Thus the $e^{-4\alpha}$ term depends solely on $H_0(\hat{\tau}/\mu)$ and $a_2(\hat{\tau}/\mu)$. Conversely, by finding the news function to second order in the boosted frame we may determine the first two coefficients in Eq. (5.1).

To find the field equations satisfied by $h_{ab}^{(2)}$, write Eq. (3.18) as

$$g_{ab}\sim\nu^2(\eta_{ab}+e^{-2\alpha}h_{ab}^{(1)}+e^{-4\alpha}h_{ab}^{(2)}+\cdots) \ . \quad (5.5)$$

Then

$$g^{ab}\sim\frac{1}{\nu^2}[\eta^{ab}-e^{-2\alpha}h^{(1)ab}-e^{-4\alpha}(h^{(2)ab}-h^{(1)ad}h^{(1)}{}_d{}^b)+\cdots] \quad (5.6)$$

(where indices on the right-hand side are raised and lowered with η_{ab}). Using the properties $h^{(1)}{}_c{}^c=h^{(1)}{}_{ab,}{}^b=0$, it is straightforward to show that the $e^{-4\alpha}$ term in the Ricci tensor is

$$R_{ab}^{(2)}=\tfrac{1}{2}(h^{(2)c}{}_{a,bc}+h^{(2)c}{}_{b,ac}-h^{(2)}{}_{ab,}{}^c{}_c-h^{(2)}{}_c{}^c{}_{,ab})$$
$$-\tfrac{1}{2}h^{(1)cd}(h_{ca,bd}^{(1)}+h_{cb,ad}^{(1)}-h_{ab,cd}^{(1)}-h_{cd,ab}^{(1)})$$
$$+\tfrac{1}{4}h^{(1)cd}{}_{,a}h_{cd,b}^{(1)}-\tfrac{1}{2}h^{(1)}{}^c{}_{a,d}(h^{(1)}{}^d{}_{b,c}-h^{(1)}{}_{cb,}{}^d) \ . \quad (5.7)$$

Rearranging terms slightly, we may write the second-order field equation $R_{ab}^{(2)}=0$ as

$$\Box h_{ab}^{(2)}-\bar{h}^{(2)}{}_{ac,}{}^c{}_b-\bar{h}^{(2)}{}_{bc,}{}^c{}_a=T_{ab}(h_{cd}^{(1)}) \ , \quad (5.8)$$

where $\bar{h}_{ab}^{(2)}=h_{ab}^{(2)}-\frac{1}{2}\eta_{ab}h^{(2)}{}_c{}^c$ and T_{ab} is equal to twice the sum of all the $h_{ab}^{(1)}$ terms on the right side of Eq. (5.7). Equations (5.8) are similar in form to those of linearized theory with a source term; the first-order metric perturbations $h_{ab}^{(1)}$ giving rise to an effective energy-momentum tensor T_{ab}. It may be easily verified that T_{ab} satisfies the conservation equation

$$\bar{T}_{ab,}{}^b=0 \ . \quad (5.9)$$

As in the first-order case, we look for a gauge transformation $x^a=x^{Na}+e^{-4\alpha}\delta^a$ such that $h_{ab}^{N(2)}=h_{ab}^{(2)}+2\delta_{(a,b)}$ satisfies the de Donder condition $\bar{h}^{N(2)}{}_{ab,}{}^b=0$ in $u\geq 0$. As before, a sufficient condition for this to hold is $\bar{h}^{N(2)}{}_{ab,}{}^b{}_v|_{u=0}=0$. The argument is identical to that used in Sec. IV, since by Eq. (5.9) we again have $\Box\bar{h}^{N(2)}{}_{ab,}{}^b=0$.

Before proceeding, we must first find the change in the second-order initial data induced by the first-order gauge transformation with parameter ξ^a. This takes the form

$$2\xi^d{}_{,(a}h_{b)d}^{(1)}+2\xi_{d,(a}\xi^d{}_{,b)}+\xi_{d,a}\xi^d{}_{,b} \ . \quad (5.10)$$

However, from Eq. (4.10) we see that the only nonzero $\xi^c{}_{,d}$ on $u=0$ is

$$\xi^v{}_{,u}=-16\rho^{-4}[8\ln(\nu\rho)-\sqrt{2}v]^2\theta(8\ln(\nu\rho)-\sqrt{2}v) \ . \quad (5.11)$$

Thus although the first-order gauge transformation will change $h_{ab}^{(2)}$ on $u=0$, it will not affect the radiative components of the field, $h_{xx}^{(2)}$, $h_{yy}^{(2)}$, and $h_{xy}^{(2)}$.

The equations that δ^a must satisfy on $u=0$ are identical to those in the first-order case, except for an extra term in T_{ab}:

$$-\Box\delta_{[a,v]}-\tfrac{1}{2}\eta_{av}\Box\delta_{c,}{}^c=\bar{h}_{ax,xv}^{(2)}+\bar{h}_{ay,yv}^{(2)}+\bar{h}_{au,vv}^{(2)}-\tfrac{1}{2}(\bar{h}_{av,xx}^{(2)}+\bar{h}_{av,yy}^{(2)})+\tfrac{1}{2}\bar{T}_{av} \ . \quad (5.12)$$

As before, we look for a solution

$$\delta_a=uf_a(\rho,v)+O(u^2) \ . \quad (5.13)$$

When $a=v$ both sides of Eq. (5.12) vanish identically. We therefore choose $f_v=0$. For $a=x$ and $a=y$, Eq. (5.12) has the form

$$\frac{\partial^2}{\partial v^2}f_x(\rho,v)=g_x(\rho,v) \ ,$$
$$\frac{\partial^2}{\partial v^2}f_y(\rho,v)=g_y(\rho,v) \ , \quad (5.14)$$

which can be integrated directly to find f_x and f_y. Using these functions, the remaining equation can be written $\partial^2 f_u(\rho,v)/\partial v^2 = g_u(\rho,v)$, determining f_u. Thus a solution of the form (5.13) does exist. However, there is no need to calculate it explicitly since it will not alter the initial data for the radiative components of the field. In this gauge the field equations for $h_{ab}^{(2)}$ reduce to

$$\Box h_{ab}^{(2)} = T_{ab} \ . \tag{5.15}$$

The most general form that $h_{ab}^{(2)}$ can have in $u \geq 0$ is

$$\begin{aligned} &h_{tt}^{(2)} = A^{(2)}, \quad h_{tx}^{(2)} = \rho^{-1} x B^{(2)} , \\ &h_{ty}^{(2)} = \rho^{-1} y B^{(2)}, \quad h_{tz}^{(2)} = C^{(2)} , \\ &h_{zz}^{(2)} = G^{(2)}, \quad h_{zx}^{(2)} = \rho^{-1} x F^{(2)} , \\ &h_{zy}^{(2)} = \rho^{-1} y F^{(2)}, \quad h_{xx}^{(2)} = D^{(2)} + (y^2 - x^2)\rho^{-2} E^{(2)} , \\ &h_{xy}^{(2)} = -2xy\rho^{-2} E^{(2)}, \quad h_{yy}^{(2)} = D^{(2)} + (x^2 - y^2)\rho^{-2} E^{(2)} . \end{aligned} \tag{5.16}$$

The initial data for the radiative part of the field are given in Eqs. (3.15) and (3.16):

$$\begin{aligned} &D^{(2)}|_{u=0} = \rho^{-4} g [8 \ln(\nu\rho) - \sqrt{2} v] , \\ &E^{(2)}|_{u=0} = 0 . \end{aligned} \tag{5.17}$$

From the definition (4.14) of the news function we find

$$c_0^{(2)}(\tau,\theta) = -\frac{1}{2} \left[\frac{\lambda}{\nu} \right]^2 \lim_{r\to\infty} \left[r \frac{\partial}{\partial \tau} (D^{(2)} + E^{(2)}) \right] . \tag{5.18}$$

[We shall see in paper II that this is not quite correct. All the information about the second-order news function *is* contained in $D^{(2)} + E^{(2)}$, but in addition there are some spurious gauge terms which must be eliminated by transforming to Bondi coordinates. This will be carried out in paper II.] A straightforward, though tedious, analysis of Eq. (5.15) shows that $D^{(2)}$ satisfies

$$\begin{aligned} \Box D^{(2)} = {} & \rho^{-1} B E_{,v} + B E_{,v\rho} - \tfrac{1}{2}(B_{,v})^2 + 2 E_{,u} E_{,v} \\ & + 4\rho^{-2} E^2 + 2\rho^{-1} E E_{,\rho} + (E_{,\rho})^2 + E_{,v} E_{,\rho} , \end{aligned} \tag{5.19}$$

and $E^{(2)}$ satisfies

$$\begin{aligned} \Box (e^{2i\phi} E^{(2)}) = e^{2i\phi} [& A E_{,vv} + \rho^{-1} B E_{,v} + B E_{,v\rho} - E_{,v} B_{,\rho} \\ & + \tfrac{1}{2}(B_{,v})^2 - E E_{,\rho\rho} - \rho^{-1} E E_{,\rho} \\ & + 4\rho^{-2} E^2] . \end{aligned} \tag{5.20}$$

These equations are each of the form

$$\Box F = H, \quad F(0,v,x,y) \text{ known} . \tag{5.21}$$

The solution to Eq. (5.21) at P with coordinates (u,v,x,y) may be expressed as the sum of two integrals [24]: (1) a surface term, given by Eq. (4.5), which arises from integrating over the two-surface where the past null cone of P intersects $u=0$; and (2) a volume term which comes from integrating down the past null cone of P to the initial surface $u=0$. It is given by

$$F_{\text{vol}}(t,\mathbf{r}) = \frac{1}{4\pi} \int_{u' \geq 0} \frac{H(t - |\mathbf{r} - \mathbf{r}'|, \mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|} d^3\mathbf{r}' . \tag{5.22}$$

The contribution to $\partial D^{(2)}/\partial \tau$ in the far field from the surface integral is easily found to be

$$\frac{\partial D_{\text{surf}}^{(2)}}{\partial \tau} = \frac{16 \sec^2(\theta/2) \tan^2(\theta/2)}{\pi r} \int_0^\infty \int_0^{2\pi} \frac{ds}{s^3} dw\, \theta \left[8 \ln s + s \cos w + \frac{\tau}{\nu} \sec^2(\theta/2) - 8 \ln \left[\frac{2 \tan(\theta/2)}{\nu} \right] \right] . \tag{5.23}$$

The volume terms, however, are by no means easy to compute. If the source terms in Eqs. (5.19) and (5.20) are inserted into Eq. (5.22), then we see that to find the functional forms of $D^{(2)}$ and $E^{(2)}$ would require the evaluation of triple integrals whose integrands are themselves products of single integrals. To do this analytically is clearly out of the question, since the single integrals are themselves not analytically tractable. On the other hand, any numerical computation would suffer from two difficulties. First, one would have to identify the regions of integration which contribute most to the total integral. Second, and perhaps more importantly, the accurate numerical computation of what are essentially four-dimensional integrals with infinite ranges of integration would require an exorbitant amount of computer time: indeed it is quite impractical. However we shall see in paper II that the perturbative field equations can be reduced to equations in only two independent variables rather than three, and that this allows an accurate numerical computation of the second-order news function, or equivalently of $a_2(\hat{\tau})$. The results are presented in paper III.

VI. A NEW MASS-LOSS FORMULA FOR THE AXISYMMETRIC COLLISION

To provide further motivation for the computation of the function $a_2(\hat{\tau})$ in Eq. (1.3), in this section we shall show that if the product of the high-speed black-hole collision at finite γ is a single black hole at rest, plus outgoing gravitational radiation whose form close to the axis of symmetry is fully described by Eq. (1.1), then the final mass of this residual black hole is determined by $Q_0(\bar{\tau},\psi)$ and $Q_2(\bar{\tau},\psi)$, up to small corrections of $O(\gamma^{-2})$. Further, if Eqs. (1.1) and (1.2) match smoothly then an alternative, more useful, formula relating the final mass to the first two coefficients $a_0(\hat{\tau})$ and $a_2(\hat{\tau})$ in Eq. (1.3) can be obtained. It is the latter that we derive first. The arguments used are similar to those employed in Ref. [16] for studying Smarr's zero-frequency limit.

Since this system is axisymmetric and reflection symmetric, we may use the results of Bondi, van der Burg, and Metzner [4] concerning the form of the geometry near $\mathcal{I}^+$. In particular, we shall make use of the supplementary condition (Eq. (35) in Ref. [4])

$$\frac{\partial M}{\partial \hat{\tau}} = -\left[\frac{\partial c}{\partial \hat{\tau}}\right]^2 + \frac{1}{2}\frac{\partial}{\partial \hat{\tau}}\left[\frac{\partial^2 c}{\partial \hat{\theta}^2} + 3\cot\hat{\theta}\frac{\partial c}{\partial \hat{\theta}} - 2c\right] . \tag{6.1}$$

Here $M(\hat{\tau},\hat{\theta})$ is the mass aspect of the system, and $\partial c(\hat{\tau},\hat{\theta})/\partial\hat{\tau}$ is the news function. In the Bondi metric, M appears in

$$g_{\hat{\tau}\hat{\tau}} = 1 - \frac{2M(\hat{\tau},\hat{\theta})}{r} + \cdots , \tag{6.2}$$

while

$$g_{\hat{\theta}\hat{\theta}} = r^2\left[1 + \frac{2c}{r} + \cdots\right] , \quad g_{\hat{\phi}\hat{\phi}} = r^2\sin^2\hat{\theta}\left[1 - \frac{2c}{r} + \cdots\right] . \tag{6.3}$$

M is thus a generalized mass suitable for nonstatic systems. The mass aspect of a particle of rest mass m moving with speed v [and Lorentz factor $\gamma = (1-v^2)^{-1/2} = \cosh\lambda$] in the $\hat{\theta} = \pi$ direction is (Ref. [4], Appendix 3).

$$M(\hat{\theta}) = \frac{m}{(\cosh\lambda + \cos\hat{\theta}\sinh\lambda)^3} . \tag{6.4}$$

Reexpressing this in terms of v, we find

$$M = \frac{m(1-v^2)^{3/2}}{(1+v\cos\hat{\theta})^3} . \tag{6.5}$$

Let the rest mass of each body now be m. Before the collision one particle moves with speed v in the $\hat{\theta} = 0$ direction, while the other moves with the same speed in the $\hat{\theta} = \pi$ direction. The respective mass aspects of these two particles are thus

$$M_1 = \frac{m(1-v^2)^{3/2}}{(1-v\cos\hat{\theta})^3}, \quad M_2 = \frac{m(1-v^2)^{3/2}}{(1+v\cos\hat{\theta})^3} . \tag{6.6}$$

In the distant past the total mass aspect of the system is simply the linear superposition of M_1 and M_2:

$$M(-\infty,\hat{\theta}) = M_1(\hat{\theta}) + M_2(\hat{\theta}) . \tag{6.7}$$

If the final product of the collision is a single black hole, then

$$M(\hat{\tau} = \infty,\hat{\theta}) = m_{\text{final}} . \tag{6.8}$$

Integrating Eq. (6.1), over $\hat{\tau}$ we find that

$$m_{\text{final}} = M(-\infty,\hat{\theta}) - \int_{-\infty}^{\infty}(c_0)^2 d\hat{\tau} + \frac{1}{2}\left[\frac{\partial^2}{\partial\hat{\theta}^2} + 3\cot\hat{\theta}\frac{\partial}{\partial\hat{\theta}} - 2\right](c|_{-\infty}^{\infty}) , \tag{6.9}$$

under the assumption (6.8). We shall examine Eq. (6.9) in the region where (1.1) is presumed to match with (1.2); that is, where $\gamma^{-1} \ll \hat{\theta} \ll 1$.

From Eqs. (6.4) and (6.7) we have

$$M(-\infty,\hat{\theta}) = \mu\gamma^{-4}\{[1+\cos\hat{\theta}(1-\gamma^{-2})^{1/2}]^{-3} + [1-\cos\hat{\theta}(1-\gamma^{-2})^{1/2}]^{-3}\} , \tag{6.10}$$

with $\mu = m\gamma$. One can easily show from the above that $M(-\infty,\hat{\theta}) = O(\gamma^{-1})$ when $\hat{\theta}$ is $O(\gamma^{-1/2})$.

The news function is given by Eq. (1.1), which, reformulated, says that

$$c_0(\bar{\tau},\psi=\gamma\hat{\theta}) - \sum_{n=0}^{m}\gamma^{-2n}Q_{2n}(\bar{\tau},\psi) = O[\gamma^{-(2m+2)}] , \tag{6.11}$$

$\forall m$ as $\gamma \to \infty$ with $\bar{\tau},\psi$ fixed. It can be seen from Eq. (6.9) that $\int_{-\infty}^{\infty} c_0(\bar{\tau},\psi)d\bar{\tau}$ and its first two angular derivatives must exist, otherwise $M(\infty,\hat{\theta})$ will not be well defined. In order to make use of Eq. (6.9), we must assume that Eq. (1.1) satisfies a kind of uniformity condition, by which we mean that in addition to Eq. (6.11) it will be assumed that the news function satisfies (for each fixed ψ)

$$\left|\gamma^{2m+2}\left[c_0(\bar{\tau},\psi) - \sum_{n=0}^{m}\gamma^{-2n}Q_{2n}(\bar{\tau},\psi)\right]\right| \le f_m(\bar{\tau},\psi) , \tag{6.12}$$

$\forall m$, $\forall \gamma > \Gamma$ where Γ is some constant, for some functions $f_m(\bar{\tau},\psi)$, where every $\int_{-\infty}^{\infty} f_m(\bar{\tau},\psi)d\bar{\tau}$ exists. If this is the case then Eq. (6.11) may be integrated to yield

$$\int_{-\infty}^{\infty} c_0(\bar{\tau},\psi)d\bar{\tau} - \sum_{n=0}^{m}\gamma^{-2n}\int_{-\infty}^{\infty} Q_{2n}(\bar{\tau},\psi)d\bar{\tau} = O[\gamma^{-(2m+2)}] , \tag{6.13}$$

$\forall m$ as $\gamma \to \infty$ with ψ fixed. Clearly, $\int_{-\infty}^{\infty} Q_{2n}(\bar{\tau},\psi)d\bar{\tau}$ has to exist if Eq. (6.12) is to hold, so that each $Q_{2n}(\bar{\tau},\psi)$ must $\to 0$ as $\bar{\tau} \to \infty$ [presumably according to some inverse power law, if $Q_0(\bar{\tau},\psi)$ is any guide]. But more importantly, in order that the news function satisfy Eq. (6.12) it is necessary that the burst of radiation described by Eq. (6.11), and its continuation to large angular scales, be the only gravitational radiation in the space-time, the system approaching isotropy asymptotically after it comes by. Another way of looking at this is to say that if Eq. (6.12) is to be valid, the news function must have the form (1.1) even in the limit $\bar{\tau} \to \infty$ (with γ large but fixed) when the perturbation theory breaks down near the initial surface. It is plausible only if the $Q_{2n}(\bar{\tau},\psi)$ do all fall off according to various inverse power laws as $\bar{\tau} \to \infty$.

We also assume that (1.1) matches smoothly to (1.2), so that (1.1) is still a good asymptotic expansion in the intermediate region where $1 \ll \psi \ll \gamma$. The analogue of Eq. (6.13) when ψ is $O(\gamma^{1/2})$ will then be

$$\int_{-\infty}^{\infty} c_0(\bar{\tau},\psi)d\bar{\tau} - \sum_{n=0}^{m}\gamma^{-2n}\int_{-\infty}^{\infty} Q_{2n}(\bar{\tau},\psi) = O(\gamma^{-(m+1)}) , \tag{6.14}$$

$\forall m$ as $\gamma \to \infty$ with $\gamma^{-1/2}\psi$ fixed. The reason why

$\gamma^{-(m+1)}$ appears on the right-hand side instead of $\gamma^{-(2m+2)}$ is that the $Q_{2n}(\bar{\tau},\psi)$ must grow as ψ^{2n} when $\psi\to\infty$, in order to match to the $\sin^{2n}\hat{\theta}$ terms in Eq. (1.3), and so $Q_{2n}(\bar{\tau},\psi)$ will be $O(\gamma^n)$ when ψ is $O(\gamma^{1/2})$.

Indeed, if matching works, $Q_{2n}(\bar{\tau},\psi)$ will have the form given by

$$\gamma^{-2n}Q_{2n}(\bar{\tau}=\tau+8\mu\ln(\psi/\mu),\psi) \sim \sin^{2n}\hat{\theta}\left[a_{2n}(\tau)+\frac{fn(\tau)}{\psi}+\frac{fn(\tau)}{\psi^2}+\cdots\right] \quad (6.15)$$

as $\gamma\to\infty$ with $\tau,\gamma^{1/2}\psi$ fixed (the origin of the $8\mu\ln(\psi/\mu)$ term can be traced to the logarithmic delay across the shocks [7]), and

$$\gamma^{-2n}\int_{-\infty}^{\infty}Q_{2n}(\bar{\tau},\psi)d\bar{\tau} \sim \sin^{2n}\hat{\theta}\left[\int_{-\infty}^{\infty}a_{2n}(\tau)d\tau+\frac{\lambda_{n1}}{\psi}+\frac{\lambda_{n2}}{\psi^2}+\cdots\right] \quad (6.16)$$

as $\gamma\to\infty$ with $\gamma^{-1/2}\psi$ fixed. This is consistent with the calculation of D'Eath [15], who showed that $\int_{-\infty}^{\infty}Q_0(\bar{\tau},\psi)d\bar{\tau}=[\psi^2/(1+\psi^2)]\int_{-\infty}^{\infty}a_0(\tau)d\tau$.

It is clear from the form of Eq. (6.16) that only

$$\int_{-\infty}^{\infty}Q_0(\bar{\tau},\psi)d\bar{\tau}=\left[\int_{-\infty}^{\infty}a_0(\tau)d\tau\right]\left[1-\frac{1}{\psi^2}+\cdots\right] \quad (6.17)$$

and

$$\gamma^{-2}\int_{-\infty}^{\infty}Q_2(\bar{\tau},\psi)d\bar{\tau}\sim\sin^2\hat{\theta}\left[\int_{-\infty}^{\infty}a_2(\tau)d\tau+\frac{\lambda_{21}}{\psi}+\cdots\right] \quad (6.18)$$

can make an order 1 contribution to the last term in Eq. (6.9) as $\gamma\to\infty$ with $\gamma^{-1/2}\psi$ fixed. Differentiating these expressions with respect to $\hat{\theta}$ we find

$$\frac{1}{2}\left[\frac{\partial^2}{\partial\hat{\theta}^2}+3\cot\hat{\theta}\frac{\partial}{\partial\hat{\theta}}-2\right](c|_{-\infty}^{\infty}) = \int_{-\infty}^{\infty}[4a_2(\tau)-a_0(\tau)]d\tau+O(\gamma^{-1/2}) \quad (6.19)$$

as $\gamma\to\infty$ with $\gamma^{-1/2}\psi$ constant. [Note that it is fortunate, or perhaps significant, that λ_{01} vanishes, since it would otherwise give a γ^2/ψ^3 contribution to Eq. (6.19).] Clearly $\int_{-\infty}^{\infty}(c_0)^2d\hat{\tau}=\int_{-\infty}^{\infty}a_0(\tau)^2d\tau+O(\gamma^{-1/2})$ in the same limit, so collecting the various results we find that

$$m_{\text{final}}=-\int_{-\infty}^{\infty}[a_0(\tau)]^2d\tau + \int_{-\infty}^{\infty}[4a_2(\tau)-a_0(\tau)]d\tau+O(\gamma^{-1/2}) \quad (6.20)$$

as $\gamma\to\infty$ with $\gamma^{-1/2}\psi$ fixed, proving our assertion at the beginning of this section.

One derives the equation relating m_{final} to $Q_0(\bar{\tau},\psi)$ and $Q_2(\bar{\tau},\psi)$ by examining Eq. (6.9) in the limit $\psi\to0$. It is

$$m_{\text{final}}=\mu(8\gamma^2-2)+\frac{1}{2}\lim_{\psi\to0}\left[\gamma^2\frac{\partial^2}{\partial\psi^2}+\frac{\gamma}{\psi}\frac{\partial}{\partial\psi}\right]\left[\int_{-\infty}^{\infty}[Q_0(\bar{\tau},\psi)+\gamma^{-2}Q_2(\bar{\tau},\psi)]d\bar{\tau}\right]+O(\gamma^{-2}) . \quad (6.21)$$

As noted previously, it is rather hard to calculate $Q_2(\bar{\tau},\psi)$, which is why we shall use Eq. (6.20), rather than Eq. (6.21), to calculate m_{final}.

Clearly a formula similar to Eq. (6.21) could be derived for any process in which the initial momenta are known and the final product is a single body at rest.

The first term in Eq. (6.20) has been calculated numerically [7,11] and found to be $\mu/2$ to three significant figures (here 2μ is the initial energy). The term $\int_{-\infty}^{\infty}a_0(\tau)d\tau$ can be calculated exactly. From Eqs. (4.22) and (4.27) we have

$$a_0(\tau)=\frac{4}{\pi}\int_0^{\infty}\int_0^{\pi}\frac{ds}{s}dw\,\cos(2w)\delta\left[8\ln s+s\cos w+\frac{\tau}{\mu}\right] . \quad (6.22)$$

Hence

$$\int_{-\infty}^{\infty}a_0(\tau)d\tau=\frac{4\mu}{\pi}\int_0^{\infty}\int_0^{\pi}\frac{ds}{s}dw\,\cos(2w)[\theta(8\ln s+s\cos w+x)]|_{x=-\infty}^{x=\infty} = \frac{4\mu}{\pi}\left[\int_{|\cos\phi_0|\le1}\frac{ds}{s}\sin\phi_0\cos\phi_0\right]\Bigg|_{x=-\infty}^{x=\infty} , \quad (6.23)$$

where

$$\cos\phi_0=-\left[\frac{8\ln s+x}{s}\right] . \quad (6.24)$$

When $x\ll-1$, the domain in which $|\cos\phi_0|\le1$ is connected, bounded below by $S_L\simeq-x$, and unbounded above. Hence

$$\lim_{x\to-\infty}\int_{|\cos\phi_0|\le1}\frac{ds}{s}\sin\phi_0\cos\phi_0 = \lim_{x\to-\infty}\int_{-x}^{\infty}\frac{ds}{s}\left[1-\left[\frac{x}{s}\right]^2\right]^{1/2}\left[\frac{-x}{s}\right] = \int_1^{\infty}\frac{dw}{w^3}\sqrt{w^2-1}=\frac{\pi}{4} . \quad (6.25)$$

When $x \gg 1$, the range of integration has two disconnected regions. One is bounded below by $S_L \simeq x$ and unbounded above, and clearly contributes $-\pi/4$ to the total integral. The other is approximately $[e^{-x/8}(1-\frac{1}{8}e^{-x/8}), e^{-x/8}(1+\frac{1}{8}e^{-x/8})]$ and so does not contribute to Eq. (6.22). Therefore

$$\left[\int_{|\cos\phi_0| \le 1} \frac{ds}{s} \sin\phi_0 \cos\phi_0 \right] \Bigg|_{x=-\infty}^{x=\infty} = \frac{-\pi}{2} \tag{6.26}$$

and $\int_{-\infty}^{\infty} a_0(\tau) d\tau = -2\mu$. Substituting this into Eq. (6.20), we find

$$m_{\text{final}} = \frac{3\mu}{2} + 4\int_{-\infty}^{\infty} a_2(\tau) d\tau + o(1) , \tag{6.27}$$

where $o(1)$ denotes a term tending to zero as $\gamma \to \infty$.

VII. SUMMARY

We have studied in this paper the axisymmetric collision of two black holes at the speed of light, with a view to understanding the more physically realistic collision of two black holes at large but finite γ. Following earlier work of Curtis [10] and Chapman [11], the curved region IV of the space-time depicted in Fig. 1, resulting from this collision of two impulsive plane-fronted waves, has been treated by means of perturbation theory. A large Lorentz boost applied to the incoming states (each with energy μ) yields two null particles with energies λ, ν, where $\lambda \ll \nu$. The metric of the curved region IV can be found as a perturbation of flat space-time, in powers of the small parameter λ/ν, by solving a sequence of characteristic initial-value problems with initial data given just to the future of the strong shock with energy ν. The perturbation theory is expected to be singular: It should give a good description of the parts of the space-time near the forward and backward directions in which the incoming shocks have been delayed and deflected by small angles during the interaction at large distances from the symmetry axis. But it will give a less and less accurate description of the geometry as one examines regions further into the center of the space-time, where formation of a single final Schwarzschild black hole should take place, with associated further emission of gravitational radiation.

In Sec. IV the metric was calculated to first order in λ/ν. On boosting back to the center-of-mass frame this yielded the contribution $a_0(\hat{\tau})$ (shown in Fig. 4) to the series conjectured for the news function in Eq. (1.3). This agrees with the form found in Ref. [7] for the gravitational radiation at angles $\hat{\theta}$ fairly close to the axis, obeying $\gamma^{-1} \ll \hat{\theta} \ll 1$, in the finite-$\gamma$ collision. The form of $a_0(\hat{\tau})$ is such that 25% of the initial energy 2μ would be emitted in gravitational waves, if the radiation were isotropic. The calculation was continued to second order in Sec. V, leading to an integral expression for the next coefficient $a_2(\hat{\tau})$ in the angular expansion (1.3) of the news function near the axis. Further motivation for the computation of $a_2(\hat{\tau})$ was provided in Sec. VI, which showed that if *all* the gravitational radiation near the axis in the finite-γ space-times is accurately described (in a certain precise sense) by Eq. (1.1), and if Eqs. (1.1) and (1.2) match smoothly at angles obeying $\gamma^{-1} \ll \hat{\theta} \ll 1$, then the mass of the (assumed) final static black hole can be found from a knowledge only of $a_0(\hat{\tau})$ and $a_2(\hat{\tau})$, up to corrections which tend to zero as $\gamma \to \infty$. Since Penrose [8] has found an apparent horizon for the speed-of-light collision on the union of the two incoming shocks, the collision space-time is thus providing an interesting test of the assumption of cosmic censorship, which gives a lower bound of $\sqrt{2}\mu$ for the final mass.

In the following paper II we show how the perturbative field equations can be reduced to equations in only two independent variables, because of a conformal symmetry at each order of perturbation theory. This yields an alternative integral expression for $a_2(\hat{\tau})$ which has allowed us to calculate this quantity numerically. The results are presented in the concluding paper III, where it is found that the mass-loss formula of Sec. VI makes the unphysical prediction that the mass of the assumed final static black hole is approximately twice the initial energy of the colliding waves. The most likely explanation for this apparently surprising result is that there is some other gravitational radiation present in the space-time. A "second burst" of radiation produced deep inside the space-time will be delayed by an amount proportional to $\ln\hat{\theta}$ relative to the "first burst" described by Eq. (1.3), which is in part produced at very large radii. As discussed further in paper III, this will have the consequence that the expansion (1.3) is not valid uniformly with respect to retarded time, so that the assumptions made in Sec. VI fail to hold. Nevertheless, knowledge of $a_2(\hat{\tau})$ together with $a_0(\hat{\tau})$ does give some further information about the angular distribution of radiation, and allows a rough estimate of the emitted energy following the conventional formula of Bondi, van der Burg, and Metzner [4].

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[1] For a recent comprehensive review which covers the subject of gravitational wave generation, see K. S. Thorne, in *300 Years of Gravitation,* edited by S. W. Hawking and W. Israel (Cambridge University Press, New York, 1987).

[2] L. L. Smarr, Ann. N. Y. Acad. Sci. **302**, 569 (1977); in *Sources of Gravitational Radiation,* edited by L. L. Smarr (Cambridge University Press, New York, 1979).

[3] P. C. Aichelburg and R. U. Sexl, Gen. Relativ. Gravit. **2**, 503 (1971).

[4] H. Bondi, M. G. J. van der Burg, and A. W. K. Metzner, Proc. R. Soc. London **A269**, 21 (1962).

[5] The speed-of-light collision space-time has also been stud-

ied numerically by J. M. Stewart (private communication).

[6] The collision of two impulsive plane-fronted waves studied here is not to be confused with the problem of colliding *plane* waves, which yields a space-time containing two commuting Killing fields. Many colliding plane-wave space-times have been found analytically, and there is a substantial literature on the subject. See, for example, J. B. Griffiths, *Colliding Plane Waves in General Relativity* (Oxford University Press, New York, 1991), and references therein.

[7] P. D. D'Eath, Phys. Rev. D **18**, 990 (1978).

[8] R. Penrose, presented at the Cambridge University Seminar, Cambridge, England, 1974 (unpublished).

[9] S. W. Hawking and G. F. R. Ellis, *The Large Scale Structure of Space-Time* (Cambridge University Press, New York, 1973).

[10] G. E. Curtis, Ph.D. thesis, Oxford University, 1975; and Gen. Relativ. Gravit. **9**, 987 (1978); **9**, 999 (1978).

[11] G. I. Chapman (unpublished).

[12] P. D. D'Eath and P. N. Payne, following paper, Phys. Rev. D **46**, 675 (1992).

[13] P. D. D'Eath and P. N. Payne, paper III, Phys. Rev. D **46**, 694 (1992).

[14] L. L. Smarr, Phys. Rev. D **15**, 2069 (1977).

[15] P. D. D'Eath, in *Sources of Gravitational Radiation*, edited by L. L. Smarr (Cambridge University Press, New York, 1979).

[16] P. N. Payne, Phys. Rev. D **28**, 1894 (1983).

[17] V. Fock, *The Theory of Space, Time and Gravitation* (Pergamon, Oxford, 1964).

[18] Discontinuous coordinate transformations analogous to that in Eq. (2.3) have also been used by T. Dray and G. 't Hooft, Nucl. Phys. **B253**, 173 (1985), to find the gravitational shock wave due to a massless particle moving at the speed of light along the horizon of a Schwarzschild black hole. Further, G. 't Hooft, Phys. Lett. B **198**, 61 (1987), has shown how quantum scattering amplitudes at very high energies are determined principally by gravitational effects, computed using the discontinuous transformation (2.3). See also T. Schucker, Class. Quantum Grav. **7**, 503 (1990).

[19] R. W. Corkill, Ph.D. thesis, Cambridge University, 1983; J. M. Stewart (private communication).

[20] R. Penrose, *Techniques of Differential Topology in Relativity* (SIAM, Philadelphia, 1972).

[21] H. Friedrich and J. M. Stewart, Proc. R. Soc. London **A385**, 345 (1983).

[22] J. M. Stewart and H. Friedrich, Proc. R. Soc. London **A384**, 427 (1982); R. W. Corkill and J. M. Stewart, *ibid.* **A386**, 373 (1983).

[23] R. Penrose, Gen. Relativ. Gravit. **12**, 225 (1980); R. Penrose and W. Rindler, *Spinors and Space-Time* (Cambridge University Press, New York, 1984), Vol. 1.

[24] F. G. Friedlander, *The Wave Equation on a Curved Space-time* (Cambridge University Press, New York, 1975).