Vacuum fields in the Schwinger model

A. V. Smilga*
Institute for Theoretical Physics, Bern University, Sidlerstrasse 5, 3012 Bern, Switzerland
(Received 16 June 1992)

We suggest a simple model of a "vortex-antivortex liquid" for the characteristic Euclidean gauge-field configuration giving the main contribution to the path integral in the two-dimensional Schwinger model. The spectrum of the Dirac operator in this model has a nonzero volume density of eigenvalues $\rho(\lambda)$ at $\lambda=0$ which gives rise to a nonzero fermion condensate $\langle\bar{\psi}\psi\rangle_0$. The model reproduces also the correct qualitative behavior for the field-strength correlator $\langle E(x)E(y)\rangle$ and for the Wilson loop. The implications for QCD are discussed.

PACS number(s): 11.15.Tk, 12.38.Aw

I. INTRODUCTION

It is a well-known experimental fact that the chiral symmetry $SU_L(N_f)\otimes SU_R(N_f)$ of the standard QCD Lagrangian with N_f flavors of massless quarks is broken spontaneously and the fermion condensate $\langle\bar{\psi}_f\psi_f\rangle$ is formed. We are still unable, however, to derive this crucial phenomenological feature of the theory rigorously from first principles.

At present, the most interesting but not yet quite clear question is what the characteristic vacuum field configurations in the path integral are which are responsible for the formation of the quark condensate. This question has been discussed intermittently in the literature during the last 10 years. The streamline of activities takes its origin from the work of Banks and Casher [1], in which the fermion condensate has been related to the mean volume density of the fermion eigenvalues in the Euclidean gauge fields:

$$\langle 0|\bar{q}q|0\rangle=-\pi\langle\rho(\lambda=0)\rangle \tag{1}$$

(the averaging over the Euclidean gauge-field configurations with the weight determined by the corresponding path integral is assumed).

The nonzero level density $\rho(0)$ means that the low eigenvalues of the Dirac operator λ_n exhibit the behavior

$$\lambda_n \propto \frac{\pi n}{V|\langle\bar{q}q\rangle_0|}\ , \tag{2}$$

where V is the volume of the four-dimensional box introduced to provide the infrared regularization of the theory and to make the spectrum discrete.[1]

The behavior (2) is, however, highly unusual. (E.g., the eigenvalues for the free Dirac operator behave quite differently: $\lambda_n^{\text{free}}\propto n/L$, where $L=V^{1/4}$ is the length of the box.) Thus, it is highly desirable to understand physically what characteristic gauge-field configurations are responsible for this behavior. Different models have been discussed in the literature [3–7]. The discussion has necessarily been done on the qualitative level—it is very difficult to analyze quantitatively the spectrum of the Dirac operator in a complicated four-dimensional gauge-field background.

That is why we have chosen to address this question in the very simple two-dimensional Schwinger model, which bears many essential physical features of QCD and where the quantitative analysis *is* possible.

II. SCHWINGER MODEL: PATH-INTEGRAL APPROACH

The Schwinger model is the two-dimensional QED with one massless fermion [8]. The action is

$$S=\int d^2x\left(-\tfrac{1}{4}F_{\mu\nu}^2-i\bar{\psi}\mathbb{D}\psi\right)\ , \tag{3}$$

with $\mathbb{D}=\gamma_\mu(\partial_\mu-igA_\mu)$. The charge g has the dimension of mass. Many features of this model are similar to QCD. Like in QCD, the axial-vector current $j_\mu^5=\bar{\psi}\gamma_\mu\gamma^5\psi$ is anomalous:

$$\partial_\mu j_\mu^5=\frac{1}{2\pi}\varepsilon^{\alpha\beta}F_{\alpha\beta}\ ; \tag{4}$$

the notion of topological charge may be introduced (so that there are instantons, etc.), and the fermion condensate

$$\langle\bar{\psi}\psi\rangle_0=-\frac{e^\gamma}{2\pi^{3/2}}g \tag{5}$$

is formed [9–13] (γ is the Euler constant). Note that the formation of the fermion condensate does not imply here spontaneous symmetry breaking—the U(1) chiral symmetry is already broken by the anomaly. The theory is, in fact, analogous to QCD with only one quark flavor where the formation of a quark condensate is also not associated with spontaneous chiral-symmetry breaking.

Anyway, the nonzero condensate (5) must imply the nonzero level density $\rho(0)$, and correspondingly, the presence of the low-energy fermion modes (2) in the Dirac

*On leave of absence from ITEP, Moscow, Russia.

[1]There is also a question of the possible zero-mode contributions, but it can be shown that, in the region $m|\langle qq\rangle_0|V\gg 1$ (m is a small quark mass), they are not relevant. This and other related questions are discussed in detail in our recent work [2].

operator spectrum. Let us derive it in the manner that will be useful for us in the following analysis.

To this end, let us assume that the Euclidean space time is compactified. (Whether the compact manifold where the theory is now defined is a sphere [11] or a torus [12] is not important for us. The important point is only the assumption that the characteristic size of the manifold L is large $gL \gg 1$. For definiteness and simplicity, we assume that the theory lives on a two-dimensional symmetrical torus with the volume $V=L^2$.) The gauge field $A_\mu(x)$ can be presented as

$$A_\mu(x) = -\varepsilon_{\mu\nu}\partial_\nu\phi(x) + \partial_\mu\chi(x) , \quad (6)$$

where $\chi(x)$ is the irrelevant gauge degree of freedom. We assume also that the topological charge of gauge-field configurations,

$$\nu = \frac{g}{2\pi}\int E(x)d^2x = \frac{g}{2\pi}\int \Delta\phi(x)d^2x , \quad (7)$$

is zero.

Consider now the Euclidean correlator

$$\langle \bar\psi\psi(x)\bar\psi\psi(y)\rangle_\phi = -\mathrm{Tr}\{G_\phi(x,y)G_\phi(y,x)\} \quad (8)$$

in a *particular* external gauge field $\phi(x)$ [the physical correlators do not depend, of course, on the gauge parameter $\chi(x)$]. The great simplification in the Schwinger model compared to four-dimensional theories is that the explicit expression for the Green's function $G_\phi(x,y)$ may be written [11,12]

$$G_\phi(x,y) = \exp\{-g\gamma^5\phi(x)\}S(x-y)\exp\{-g\gamma^5\phi(y)\} , \quad (9)$$

where $S(x-y)$ is the free fermion Green's function [the gauge parameters $\chi(x)$ being set to zero]. Thus we have

$$\langle \bar\psi\psi(x)\bar\psi\psi(y)\rangle_\phi = \frac{1}{2\pi^2(x-y)^2}\cosh\{2g[\phi(x)-\phi(y)]\} . \quad (10)$$

On the other hand, we can write the spectral decomposition for the Green's function:

$$G_\phi(x,y) = \sum_n \frac{\psi_n(x)\psi_n^\dagger(y)}{-i\lambda_n} , \quad (11)$$

where all λ_n are nonzero [this is assured by our assumption that the field configuration (6) has the *zero* topological charge (7)]. Substituting Eq. (11) into Eq. (8), comparing it with Eq. (10), and integrating both sides over d^2xd^2y, we obtain the following *sum rule* for the eigenvalues λ_n in a particular external field $\phi(x)$:

$$\sum_n \frac{1}{\lambda_n^2} = \int \frac{d^2xd^2y}{2\pi^2(x-y)^2}\exp\{2g[\phi(x)-\phi(y)]\} \quad (12)$$

[we substituted $\cosh\{\,\}\to\exp\{\,\}$ as odd powers of $\phi(x)-\phi(y)$ give zero after integration]. Let us average Eq. (12) over the field configurations $\phi(x)$. The proper weight in the path integral is

$$W[\phi] = \exp\left\{-\tfrac{1}{2}\int d^2x[\phi\Delta^2\phi - \mu^2\phi\Delta\phi]\right\} . \quad (13)$$

The first term in the exponential is just the classical action $\propto \int d^2x\, F_{\mu\nu}^2$ and the second term comes from the fermion determinant ($\mu = g/\sqrt{\pi}$ is the photon mass induced thereby). Again, the splendid feature of the Schwinger model is that the fermion determinant can be calculated exactly and the resultant path integral is purely Gaussian. As a result, the average of Eq. (12) over the field configurations $\phi(x)$ with the weight (13) can also be explicitly found:

$$\left\langle \sum_n \frac{1}{\lambda_n^2}\right\rangle = \int \frac{d^2xd^2y}{2\pi^2(x-y)^2}\exp\{4g^2[\mathcal{G}(0)-\mathcal{G}(x-y)]\} , \quad (14)$$

where $\mathcal{G}(x-y)$ is the Green's function of the operator $(\Delta^2-\mu^2\Delta)/2$ on the corresponding manifold.

The behavior of the integral (14) depends crucially on the large distance asymptotics of the Green's function $\mathcal{G}(x-y)$. The presence of the term $\propto \mu^2\phi\Delta\phi$ in the effective action makes it logarithmic:

$$\mathcal{G}(x-y) \propto -\frac{1}{4\pi\mu^2}\ln(x-y)^2 + \text{const} . \quad (15)$$

Thus, the factor $\propto (x-y)^{-2}$ in the integrand in Eq. (14) is, in fact, compensated by the growing factor coming from the exponential. Taking the proper account of all constants, we get

$$\left\langle \sum_n \frac{1}{\lambda_n^2}\right\rangle = \frac{(V\Sigma)^2}{2} , \quad (16)$$

where $-\Sigma$ is the fermion condensate (5). Actually, the sum rule (16) belongs to the series of sum rules for inverse powers of λ_n that may be derived in a very general way without resorting to the explicit expressions for the Green's functions and fermion determinant and employing only Ward identities. A general sum rule has the form

$$\left\langle \sum_{n_1\neq\cdots\neq n_k} \frac{1}{\lambda_{n_1}^2\cdots\lambda_{n_k}^2}\right\rangle = \frac{(V\Sigma)^{2k}}{2^{2k}(k!)^2} . \quad (17)$$

We refer the reader to our paper [2], where essentially the same sum rules have been derived in QCD. [The sum in Eq. (17) extends only on positive eigenvalues λ_n.]

The sum rules (16) and (17) imply obviously that the scale of the lowest eigenvalues is $\propto 1/\Sigma V$. A more careful analysis [2] shows that the characteristic eigenvalues are just proportional to the zeros of the Bessel function $J_0(x)$ and, at $n \gg 1$, the relation (2) holds.

It is instructive to look at two-dimensional QED with $N_f \neq 1$. Consider first the case $N_f = 0$ (the so-called *quenched* Schwinger model). In this case, the factor $\int \mu^2\phi\Delta\phi\, d^2x$ is absent in the effective action and the Green's function $-\mathcal{G}(x-y)$ entering Eq. (14) is that of the operator Δ^2, which grows $\propto (x-y)^2\ln[L^2/(x-y)^2]$ at large distances. This means that the integral in Eq. (14) grows exponentially with the volume and the characteristic eigenvalues λ_n contributing to the sum are exponentially small. And that implies that the volume density $\rho(\lambda=0)$ and the fermion condensate $\langle\bar\psi\psi\rangle_0$ are infinite in the quenched approximation. We return to the discussion of this issue in the next section.

Consider now the case $N_f > 1$. The path integral has exactly the same form as in the Schwinger model but the photon mass μ depends now on the number of flavors:

$$\mu^2(N_f) = \frac{g^2 N_f}{\pi} . \tag{18}$$

The large distance asymptotics of the Green's function is of the same form (15) as in the Schwinger model, but the coefficient at the logarithm is now scaled down by the factor $1/N_f$. As a result, the integral (14) diverges now not as V^2 but rather as $V^{(N_f+1)/N_f}$. The scale of the lowest eigenvalues is not $1/V$ but rather

$$\lambda_{\text{char}}(N_f) \propto V^{-(N_f+1)/2N_f} . \tag{19a}$$

The volume level density $\rho(\lambda=0)$ and the fermion condensate are now zero. It is not difficult to see that

$$\rho(\lambda) \propto \lambda^{(N_f-1)/(N_f+1)} g^{2/(N_f+1)} \tag{19b}$$

at $\lambda \ll g$.

The result that $\langle \bar{\psi}\psi \rangle_0 = 0$ for $N_f \geq 2$ is very natural. A nonzero fermion condensate would break spontaneously the chiral symmetry $SU_L(N_f) \otimes SU_R(N_f)$ of the model. But spontaneous symmetry breaking is just not possible in two dimensions [14].

However, the scale of the lowest eigenvalues and the spectral density are not the same as one can expect for the free fermions [$\lambda_{\text{char}} \propto 1/L$ and $\rho(\lambda) \propto \lambda$]. They are modified by the nonperturbative effects in the case of several fermions too, though less drastically than in the Schwinger model.

The vacuum average becomes nonzero if we allow for the small nonzero fermion mass—the chiral symmetry is then broken explicitly in the Lagrangian and nothing prevents the formation of the condensate. The latter goes to zero in the limit $m \to 0$ but the dependence is more weak than in the perturbation theory [13]:

$$\langle \bar{\psi}\psi \rangle_m \propto m^{(N_f-1)/(N_f+1)} g^{2/(N_f+1)} . \tag{20}$$

III. WILSON LOOP IN SCHWINGER MODEL

In the following, we shall concentrate not on the general result (16) but rather on the formula (12), which is specific for the Schwinger model and makes it possible to explore the question of the characteristic field configurations. A viable model for the field configurations will be suggested in the next section. But before that, we need to specify further requirements (other than that the fermion condensate is formed) which characteristic fields must satisfy. In particular, characteristic fields should provide for the correct qualitative behavior of the field-strength correlators and the Wilson loop.

In the Schwinger model, the correlator of the field strengths

$$E(x) = \varepsilon_{\mu\nu}\partial_\mu A_\nu(x) = \Delta\phi(x) \tag{21}$$

is known explicitly [11,12]:

$$C(x) = \langle E(x)E(0) \rangle = \delta(x) - \frac{\mu^2}{2\pi} K_0(\mu|x|) , \tag{22}$$

where $|x|$ is assumed to be much smaller than the characteristic size L of the manifold so that boundary effects are not important. The correlator (22) exhibits the finite correlation length (related physically to the screening phenomena) and falls down rapidly at $|x| \gg \mu^{-1}$. The fluctuations of the field are Gaussian so that all higher correlators are factorized into the products of two-point correlators (22).

Consider now the Wilson loop

$$W(C) = \exp\left\{ ig \int_C A_\mu dx_\mu \right\} = \exp\left\{ ig \int_D E(x) d^2x \right\}$$
$$= \exp\left\{ -\frac{g^2}{2} \int_{D\times D} C(x-y) d^2x\, d^2y \right\} \tag{23}$$

(the factorization property of the correlators has been used). Suppose now that the contour C embracing the region D is large compared to the scale μ^{-1} of the theory but small compared to the whole manifold. The correlator $C(x)$ gives zero after integrating over the whole volume:

$$\int_M d^2x\, C(x) = 0 . \tag{24}$$

This means that if x sites well within the region D, the integral over d^2y on the right-hand side (RHS) of Eq. (23) gives zero. If, however, the point x is close to the border C of the region D, the integral does not vanish. It depends only on the distance from the point x to the boundary and the contribution is essential as soon as this distance is of order of the correlation length μ^{-1}. As a result, we get the perimeter law[2]

$$W(C) = \exp\{-g^2 P / 4\mu\} , \tag{25}$$

where P is the length of the contour C. The results (22) and (25) hold for any number of flavors $N_f > 0$.

The property (24) and the perimeter law (25) reflect the screening of an external source by massless fermions. The perimeter law for Wilson loops is also to be expected in QCD (in contrast with the pure Yang-Mills theory where external sources are not screened and confinement leads to the area law).

It is instructive to look also at the Wilson loop in the quenched Schwinger model where the effects due to fermion vacuum polarization are switched off. In that case, the correlator $C(x)$ is just $\delta(x)$ (the fields at different points are totally uncorrelated), and we obtain the area law for the Wilson loop [15].

[2]Note the difference with the result $W(C)=1$ quoted in Ref. [12] where the boundary effects in the integral in Eq. (23) were disregarded. However, the result obtained there for the thermal Polyakov loop, which is a special kind of Wilson loop, is correct and coincides with Eq. (25) in the limit $\beta = T^{-1} \ll L_{\text{spacial}}$.

IV. MODEL FIELD CONFIGURATIONS

We are ready now to discuss various models for the vacuum field configurations in order to select the one that gives reasonable behavior for the Dirac operator spectrum, field strength correlator, and Wilson loop. Our main goal is to get some insight for QCD, and analogies with the corresponding four-dimensional gauge-field configurations will be constantly traced back.

The main ingredient in the four-dimensional models discussed so far in the literature is the instanton solution. It has two distinguishing properties. (i) It is essentially local—the field strength and action density fall down rapidly at large distances, and (ii) it carries a nonzero net topological charge. The closest two-dimensional analogue of the instanton is the vortex field configuration

$$\phi^{\rm vort}(x)=\frac{1}{2g}\ln(x^2+\rho^2)\ . \tag{26}$$

The field strength

$$E^{\rm vort}(x)=\Delta\phi^{\rm vort}(x)=\frac{2\rho^2}{g(x^2+\rho^2)^2} \tag{27}$$

falls down rapidly at large x, and the net two-dimensional topological charge (7) of the vortex is equal to 1.

There is, of course, an important difference between the vortex (26) and the four-dimensional instanton—the former does not realize a minimum of the action. There are also exact solutions to the Euclidean equations of motion with a nonzero topological charge, but in two dimensions, they are essentially nonlocal, being characterized by the constant field strength and spread out along the whole manifold. However, in the Schwinger model, as well as in QCD, fields fluctuate strongly and the weight of these special global field configurations in the path integral is negligible.

Let us discuss this last point a little bit in more detail. If one tries to calculate the path integral in QCD in the framework of quasiclassical approximation, i.e., expand the action at its stationary points (instantons), retain only quadratic terms in the expansion, and integrate them over, one sees immediately that the main contribution to the path integral comes from the instantons of large sizes $\rho\sim\Lambda_{\rm QCD}^{-1}$ where the effective coupling constant $g^2(\rho)$ is large, characteristic fluctuations are large too, and the Gaussian approximation breaks down. In the exactly solvable Schwinger model, the path integral is exactly Gaussian and can be done explicitly. It is, however, still true that the characteristic field fluctuations are large and the weight of the field configurations close to the stationary points of the path integral is small.

Fluctuations of fields imply also the fluctuations of the topological charge. At some space-time points, the topological charge density is positive, while at others it is negative. The field in the region with a positive topological charge density may be associated with an instanton (vortex) and the field in the region with a negative topological charge density with an anti-instanton (antivortex). We thus arrive at the picture of an instanton–anti-instanton medium in QCD [4–7] and at the picture of a vortex-antivortex medium in the Schwinger model. Let us explore now whether or not this picture is reasonable.

We consider first the model of the instanton-anti-instanton "crystal" discussed in detail by Diakonov and Petrov [4(b)]. In this model, the field presents a regular "polar crystal" of instantons and anti-instantons as depicted in Fig. 1. The analogous two-dimensional field configuration in the Schwinger model is given by the sum

$$\phi^{\rm crystal}(x)=\sum_{\mathbf{n}}\phi^{\rm vort}(\mathbf{x}-\mathbf{n}a)(-1)^{n_1+n_2}\ , \tag{28}$$

where $\mathbf{n}$ is a two-dimensional integer vector and $\phi^{\rm vort}(\mathbf{x})$ is given in Eq. (26). For the notion of the separate vortices and antivortices to make sense, their characteristic size ρ should be at least several times less than the distance a between adjacent crystal sites. We assume also that the ratio $L/2a$ is an integer and the sum in Eq. (28) extends up to $|n_1^{\rm max}|=|n_2^{\rm max}|=L/2a$ (L is the size of the torus).

Unfortunately, the model (28) does not provide the correct behavior (2) for the fermion eigenvalues and is unable to generate the fermion condensate (1). That follows just from the fact that the sum in Eq. (28) *is* convergent and the resultant $\phi^{\rm crystal}(x)$ presents a finite periodic function.

Thus the factor $\exp\{2g[\phi(x)-\phi(y)]\}$ in Eq. (12) is bounded from above and the integral (12) diverges only as the first power of the volume V [*not* as V^2 as is required to obtain the behavior (2) for the eigenvalues λ_n].

The model (28) exhibits also the unpleasant property of long-range correlations that contradict both physical intuition and the exact result (22) for the field-strength correlator. It was noted earlier that a realistic model for the field configuration should implement stochastic properties so that long-range correlations are absent [15,4–7]. To implement stochasticity, consider the following model for characteristic field configurations:

$$\phi(\mathbf{x})=\sum_{\mathbf{n}}\eta_{\mathbf{n}}\phi^{\rm vort}(\mathbf{x}-\mathbf{n}a)\ , \tag{29}$$

where $\eta_{\mathbf{n}}$ are not fixed as in Eq. (28), but are stochastic variables.

Equation (29) may be thought of as a linear transformation of variables in the functional integral. As the original field correlators were Gaussian [higher correlators were expressed via the pair correlator $\langle\phi(x)\phi(0)\rangle$], one can expect that the same is true for $\eta_{\mathbf{n}}$ and the whole physics depends only on the pair correlator $\langle\eta_{\mathbf{n}}\eta_{\mathbf{m}}\rangle$.

Of course, it is not *quite* true. The transformation (29) is not a change of variables in the exact sense as it diminishes greatly the number of degrees of freedom—we

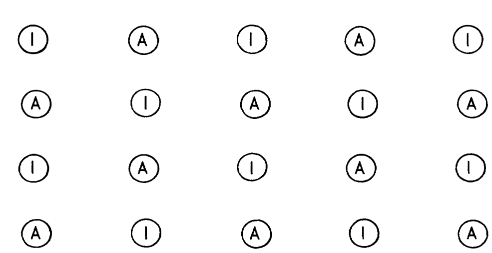

FIG. 1. Instanton–anti-instanton crystal.

have now a discrete set of variables $\{\eta_n\}$ instead of a continuous set $\{\phi(\mathbf{x})\}$. Thus we cannot expect to get in this way any information on the field correlators $\langle\phi(x)\phi(a)\rangle$, $\langle E(x)E(0)\rangle$, etc., at short distances $x \lesssim a$. However, if we are interested only in the *global* characteristics such as the fermion condensate and Wilson loop average for large contours, the ansatz (29) with Gaussian η_n and a proper assumption on $\langle\eta_n\eta_m\rangle$ should be equivalent to original field theory. Let us check whether or not it is true. The simplest assumption on the pair correlator is

$$\langle\eta_n\eta_m\rangle = C^2\delta_{nm} . \tag{30}$$

The model (29) and (30) imitates well, in fact, the features of the *quenched* Schwinger model. The space time is decomposed here in domains surrounding the sites $\mathbf{x}_n = \mathbf{n}a$ and the fluxes $(g/2\pi)\int\Delta\phi(x)d^2x$ in the different domains are totally uncorrelated. In this model, the Wilson loop (23) exhibits the area law behavior characteristic of the quenched Schwinger model (that follows just from the simple fact that the mean square of the flux in the large region D grows linearly with the area $\mathcal{A}_D$ [15]).

Let us look now at the Dirac operator spectrum in the model (29) and (30). To this end, we use the exact result (12).[3] Thus we substitute the field (29) on the RHS of Eq. (12) and perform the averaging over all realizations $\{\eta_n\}$ bearing in mind the Gaussian nature of η_n [all higher correlators are expressed via elementary pair correlator (30)]. We obtain

$$\left\langle \exp\{2g[\phi(x)-\phi(y)]\}\right\rangle = \exp\left\{\frac{C^2}{2}\sum_n \ln^2\frac{(\mathbf{x}-\mathbf{n}a)^2+\rho^2}{(\mathbf{y}-\mathbf{n}a)^2+\rho^2}\right\} . \tag{31}$$

The sum in the exponent diverges logarithmically at large $|\mathbf{n}|a \gg |\mathbf{x}|,|\mathbf{y}|$. The cutoff is provided by the finite size of the torus L. Thus we get an estimate

$$\left\langle \exp\{2g[\phi(x)-\phi(y)]\}\right\rangle \propto \exp\left\{\frac{\pi C^2(x-y)^2}{a^2}\ln\frac{\kappa L^2}{(x-y)^2}\right\} , \tag{32}$$

where κ is a geometric numerical coefficient. The dependence $\propto (x-y)^2\ln[L^2/(x-y)^2]$ in the exponent is extremely reasonable and is nothing other than the Green's function of the double Laplacian operator Δ^2, which appears when calculating the average (32) exactly by the "quenched" path integral (see Sec. II). Substituting the estimate (32) in the integral (12), we see that the integral grows exponentially with the volume. And that conforms exactly with what we have observed from the exact analysis in Sec. II for the quenched Schwinger model and means that the low characteristic eigenvalues λ_n are exponentially small and the fermion condensate is infinite.

This contradicts the statement of Ref. [16] that in the quenched Schwinger model, the volume density of the characteristic eigenvalues and the fermion condensate are finite. This statement followed from a variational estimate

$$\rho(\lambda=0) > Kg \tag{33}$$

(K is a numerical constant) and the guess that the true $\rho(\lambda=0)$ is not far from this estimate. This last guess is just not true. It is possible to present a more accurate variational estimate that shows that the low eigenvalues are actually exponentially small and $\rho(\lambda)$ diverges at $\lambda=0$. This is done in the Appendix.

The fact that $\langle\bar\psi\psi\rangle_0$ is infinite in the quenched Schwinger model has a natural interpretation in the dual (as compared to that adopted in the present paper) approach. So far, we related the appearance of the condensate to the small nonzero eigenvalues (2). In the massless Schwinger model (as well as in the massless QCD with $N_f=1$), the condensate may also be related to the *zero* fermion modes in the field configurations with a topological charge $\nu=1$ [11,12]. In this language, the condensate is determined by the path integral

$$\langle\bar\psi\psi\rangle_0 = Z^{-1}\lim_{m\to 0}\int_{\nu=\pm1}\Pi dA_\mu \det(-i\not{D}+m)G_A(x,x) \times\exp\left\{-\tfrac{1}{4}\int F_{\mu\nu}^2 d^2x\right\} , \tag{34}$$

where $G_A(x,x)$ is given by Eq. (11) with $-i\lambda_n$ being substituted by $-i\lambda_n+m$. The finite result for $\langle\bar\psi\psi\rangle_0$ in the normal Schwinger model appears after the cancellation of the large factor $\propto 1/m$ in the zero-mode contribution in the Green's function and the small factor $\propto m$ in the determinant. If the determinant is absent (Schwinger model is quenched), the factor $\propto 1/m$ has nothing to cancel with and leads to the infinite condensate in the limit $m\to 0$.

The relationship between representations (1) and (34) is discussed in detail in [2]. Here, we mention only that the Banks-Casher approach is physically relevant as soon as $mgV \gg 1$ while the instanton approach (34) is relevant in the opposite limit $mgV \ll 1$. The result (5) for the condensate does not depend, however, on the value of the parameter mgV as long as $m \ll g$ and $V \gg g^{-2}$.

We have seen that the crystal model (28) is not realistic while the model (29) and (30) describes the quenched Schwinger model. Can one suggest a model describing the features of the normal (unquenched) Schwinger model? The answer is yes.

Consider the configuration (29) where, again, η_n are Gaussian stochastic variables but another form for the

---

[3]Strictly speaking, the spectral decomposition (11) on which the derivation of the sum rule (12) was based implied the absence of zero eigenvalues, i.e., the zero net topological charge (7). In our model, that can be implemented by imposing the global constraint $\sum_n\eta_n=0$, which amounts to averaging with the factor $\exp\{i\alpha\sum_n\eta_n\}$ and integrating the result over α afterwards. This constraint does not change, however, the local properties of the theory and is not crucial. It is also possible to derive sum rules for *nonzero* λ_n in the sectors with a nonzero topological charge [2]. Their characteristic scale is also $\propto 1/\Sigma V$.

pair correlator $\langle \eta_n \eta_m \rangle$ is adopted:

$$\langle \eta_n \eta_n \rangle = C^2 , \quad \langle \eta_n \eta_{n+e} \rangle = -\frac{C^2}{4} , \tag{35}$$

where $\mathbf{e}$ is the integer vector of unit length, and all other $\langle \eta_n \eta_m \rangle$ are zero. As we assumed that the vortices are well separated, η_n has the meaning of the flux (7) in the domain surrounding the site $\mathbf{x}_n = \mathbf{n}a$. The correlator (35) of the fluxes enjoys the property

$$\sum_m \langle \eta_n \eta_m \rangle = 0 , \tag{36}$$

which models the property (24) of the exact field-strength correlator (22). In fact, the additional negative correlators of the nearest neighbors in Eq. (35) imitate the second term in the correlator (22).

Let us now determine the vacuum average of the Wilson loop (23). It is convenient to present it in the form

$$W(C) = \exp\{-2\pi^2 \langle \Phi^2 \rangle_D\} , \tag{37}$$

where the mean-square flux on the region D is just

$$\langle \Phi^2 \rangle_D = \sum_{n,m \in D} \langle \eta_n \eta_m \rangle . \tag{38}$$

Most terms in the sum (38) cancel out due to the property (36). The terms that survive come from $\mathbf{x}_n = \mathbf{n}a$ lying near the border of the region such that some of its closest neighbors $\mathbf{x}_m$ find themselves outside the contour C. It is not difficult to show that if the region is large and the contour is smooth, the universal result

$$\langle \Phi^2 \rangle_D = C^2 \frac{P}{4a} \tag{39}$$

is valid (P is the perimeter of the contour). Substituting it in Eq. (37) and comparing it with the exact results (25), we see that the perimeter law for the Wilson loop is reproduced and the correct coefficient may be obtained if we require that the parameters of the model satisfy the relation

$$C^2 = \frac{g^2 a}{2\pi^2 \mu} = \frac{ga}{2\pi^{3/2}\sqrt{N_f}} . \tag{40}$$

Consider now the Green's function

$$\langle \phi(x)\phi(y) \rangle = \frac{1}{4g^2} \sum_{nm} \langle \eta_n \eta_m \rangle \ln[(\mathbf{x}-\mathbf{n}a)^2+\rho^2] \times \ln[(\mathbf{y}-\mathbf{n}a)^2+\rho^2] . \tag{41}$$

It is a sign-alternating series and most terms have the tendency to cancel out due to the property (36). The main contribution comes from the regions $\mathbf{n}a \sim \mathbf{m}a \sim \mathbf{x}$ and $\mathbf{n}a \sim \mathbf{m}a \sim \mathbf{y}$. Consider the second region. We shall see that the series converges rapidly at $|\mathbf{n}a - \mathbf{y}| \gg a$ and we may substitute $\mathbf{n}a \to \mathbf{y}$ in the first logarithm if $|\mathbf{x}-\mathbf{y}| \gg a$. We have

$$\langle \phi(x)\phi(y) \rangle_{\mathbf{n}a \sim \mathbf{y}} \simeq \frac{C^2}{4g^2} \ln[(\mathbf{x}-\mathbf{y})^2+\rho^2] \sum_n \left\{ \ln[(\mathbf{y}-\mathbf{n}a)^2+\rho^2] - \tfrac{1}{4}\sum_{i=1}^{4} \ln[(\mathbf{y}-\mathbf{n}a-\mathbf{e}_i a)^2+\rho^2] \right\} \tag{42}$$

with $\mathbf{e}_i^2 = 1$. Expanding the logarithms in the internal sum over the parameter $a/|\mathbf{y}-\mathbf{n}a|$, one can be convinced, indeed, that the sum over $\mathbf{n}$ converges rapidly when $|\mathbf{y}-\mathbf{n}a|$ is large. Note that a sum

$$S(\mathbf{y}-\mathbf{e}_i a) = \sum_n \ln[(\mathbf{y}-\mathbf{e}_i a - \mathbf{n}a)^2+\rho^2] \tag{43}$$

coincides formally with $S(\mathbf{y})$ if changing the summation variable $\mathbf{n} \to \mathbf{n} - \mathbf{e}_i$. Thus seemingly, the sum in the RHS of Eq. (42) is just zero. However, this is not true as the individual sums (43) involve the quadratic divergence at large $\mathbf{n}$ and the change of variables is not legitimate. To calculate the sum (42), we regularize it at large $\mathbf{n}$ and write

$$R^{\text{reg}}(\mathbf{y}) = S^{\text{reg}}(\mathbf{y}) - \tfrac{1}{4}\sum_{i=1}^{4} S^{\text{reg}}(\mathbf{y}-\mathbf{e}_i a) = \sum_{n \in D} \left\{ \ln[(\mathbf{y}-\mathbf{n}a)^2+\rho^2] - \tfrac{1}{4}\sum_{i=1}^{4} \ln[(\mathbf{y}-\mathbf{n}a-\mathbf{e}_i a)^2+\rho^2] \right\} , \tag{44}$$

where the region D embraces the point $\mathbf{y}$ and has a scale l much larger than a but much less than $|\mathbf{x}-\mathbf{y}|$ (see Fig. 2). Now the cancellation in the sum (44) is not exact and the terms with $\mathbf{n}a$ lying near the border C of the region D survive. If $l \gg a$, the sum (44) may be approximated by the contour integral

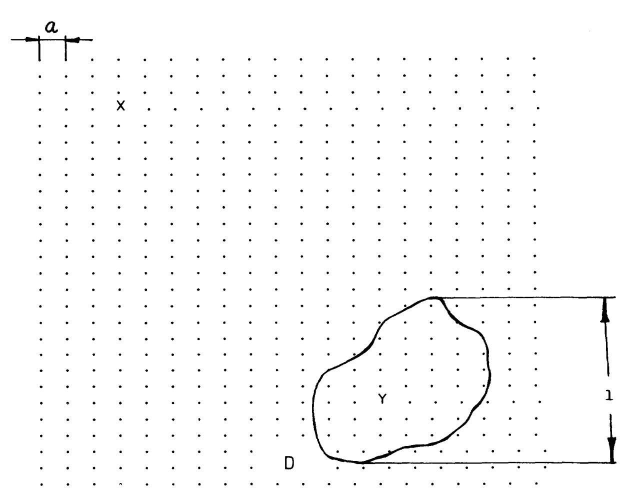

FIG. 2. Relevant region of $\mathbf{n}a$ for the sum (42).

$$R^{\rm reg}(\mathbf{y}) \simeq -\tfrac{1}{4}\int_C (\mathbf{e}_\parallel d\mathbf{y}_C)\{\partial \ln[(\mathbf{y}-\mathbf{y}_C)^2+\rho^2]\mathbf{e}_\perp\}$$
$$= -\tfrac{1}{4}\int_D d^2Y_D \Delta \ln[(\mathbf{y}-\mathbf{y}_D)^2+\rho^2] \simeq -\pi \quad (45)$$

($\mathbf{e}_\parallel$ and $\mathbf{e}_\perp$ are unit vectors tangent and normal to the contour C, respectively). Substituting it in Eq. (42) and adding the similar contribution from the region $\mathbf{n}a \sim \mathbf{m}a \sim \mathbf{x}$ in the sum in Eq. (41), we finally get

$$\langle \phi(x)\phi(y)\rangle \simeq -\frac{\pi C^2}{2g^2}\ln(\mathbf{x}-\mathbf{y})^2 \quad (46)$$

in the region $L \gg |\mathbf{x}-\mathbf{y}| \gg a$. It coincides with the exact Green's function (15) and provides for the correct behavior of $\langle \exp\{2g[\phi(x)-\phi(y)]\}\rangle$ and $\langle \sum_n 1/\lambda_n^2\rangle$ if the parameter C^2 is chosen as

$$C^2 = \frac{g^2}{2\pi^2\mu^2} = \frac{1}{2\pi N_f} . \quad (47)$$

Combining it with the relation (40) obtained earlier, we may fix also the parameter a:

$$a = \frac{1}{g}\sqrt{\pi/N_f} , \quad (48)$$

which coincides exactly with the correlation length μ^{-1} of the model.

Thus we see that the model (29) and (35) provides, indeed, the correct behavior of the Dirac operator spectrum and of the Wilson loop if the parameters C^2 and a are chosen as in Eqs. (47) and (48).

V. LESSONS FOR QCD

The whole analysis performed so far was based on the exact expression (9) for the fermion Green's function in the arbitrary external field, which may be derived in the Schwinger model. No similar formula exists in QCD and we are unable at the moment to make rigorous statements about the Dirac operator spectrum in this or that model for characteristic four-dimensional field configurations. We may suggest, however, some guesses based on the results of our analysis for the Schwinger model.

First of all, it seems that the model of an instanton–anti-instanton liquid, in its gross features, describes the physics adequately. It is not really relevant whether the main ingredients of the model are instantons or something else—what is relevant is that field fluctuates and topological charge density fluctuates too. What *is* also relevant is that the instanton–anti-instanton medium presents a *liquid* exhibiting short-range but not long-range correlations.

Based on our results for the Schwinger model, we guess that in the field configurations presenting regular instanton–anti-instanton crystal with long-range correlations, the level density $\rho(\lambda=0)$ is, in fact, zero and the fermion condensate is not generated. It is remarkable that this guess has actually been confirmed in the analysis of Ref. [6]. Shuryak and Verbaarschot aceepted the ideology of Ref. [4] that the nonzero density of eigenvalues at $\lambda=0$ may arise due to collectivization of individual instanton and anti-instanton fermion zero modes, but after diagonalizing the corresponding overlap matrix, they observed that the "conductivity zones" happen to lie apart from the region $\lambda=0$ so that $\rho(\lambda=0)=0$. Perhaps this "experimental fact" has a deep physical reason and, in *any* regular gauge-field background with long-range correlation, the fermion condensate does not appear. At least, this is definitely true in the Schwinger model—in the periodic external field $\phi(x)$, the large x behavior of the fermion Green's function (9) is qualitatively the same as for free fermions and $\langle \bar\psi\psi\rangle_{\phi_{\rm reg}}=0$.

The stochastic gauge-field configurations were analyzed in detail by Simonov [7]. The stochasticity was, however, implemented in a way that describes adequately the physics in the pure Yang-Mills theory (or, if one chooses, in quenched QCD) with the characteristic area law for the Wilson loop. Our guess is that in the *gas* models of this type (both short-range and long-range correlations are absent), the level density and the fermion condensate are actually infinite for the same reasons as they are in the quenched Schwinger model [see the discussion before and after Eq. (34)—it can be applied without change to QCD].

In the correct model for the relevant gluon field configurations, topological charge densities should fluctuate stochastically but in such a way that the perimeter law for the Wilson loop average is satisfied and the topological susceptibility

$$\chi = \frac{1}{V^{(4)}}\left[\frac{g^2}{16\pi^2}\int d^4x\, {\rm Tr}(G_{\mu\nu}\tilde G_{\mu\nu})\right]^2 , \quad (49)$$

which is the direct four-dimensional analogue of the quantity $\langle \Phi_A^2\rangle/A$ in the Schwinger model, is zero. (Ward identities imply that χ should turn to zero in the massless quark limit.) Alternatively, one may consider the quantity

$$\mathfrak{W}[S^{(3)}] = \exp\left\{ i\int_{S^{(3)}} d^3x\, K_\mu n_\mu \right\}$$
$$= \exp\left\{ \frac{ig^2}{16\pi^2}\int_{D^{(4)}} d^4x\, {\rm Tr}\{G_{\mu\nu}\tilde G_{\mu\nu}\}\right\} , \quad (50)$$

where

$$K_\mu = \frac{ig^2}{16\pi^2}\varepsilon_{\mu\nu\alpha\beta}{\rm Tr}\left\{ A_\nu G_{\alpha\beta} + \frac{2ig}{3}A_\nu A_\alpha A_\beta \right\} \quad (51)$$

and n_μ is the normal vector on the three-dimensional ($3D$) surface $S^{(3)}$, which is the border of the $4D$ region $D^{(4)}$. $\mathfrak{W}[S^{(3)}]$ is the direct $4D$ analogue of the Wilson loop $W[C]$ of Schwinger model. If the region $D^{(4)}$ is large, $\mathfrak{W}[S^{(3)}]$ must enjoy the four-volume law

$$\mathfrak{W}[S^{(3)}] \propto \exp\left\{ -c_{\rm YM}\Lambda_{\rm YM}^4 V_D^{(4)}\right\} \quad (52a)$$

in pure Yang-Mills theory, and the three-surface law

$$\mathfrak{W}[S^{(3)}] \propto \exp\left\{ -c_{\rm QCD}\Lambda_{\rm QCD}^3 \mathcal{A}_S^{(3)}\right\} \quad (52b)$$

in QCD.

The coefficient $2c_{YM}\Lambda_{YM}^4$ differs from the topological susceptibility (49) as, in contrast with the Schwinger model, field fluctuations are not Gaussian here. The lattice measurements of this coefficient would be interesting as the relative difference

$$\eta = 1 + \frac{2}{V_D^{(4)}\chi_{YM}} \ln \left\{ \mathfrak{W}[S^{(3)}] \right\} \tag{53}$$

may serve as an integral measure of the "non-Gaussian nature" of the characteristic vacuum field fluctuations.

Speaking of QCD, the three-surface law (52b) for $\mathfrak{W}[S^{(3)}]$ implying the property $\chi=0$, as well as the perimeter law for the ordinary Wilson loop, may be satisfied if the topological charges in *adjacent* regions correlate and their correlator is negative. Thus if we rely on our experience with the Schwinger model, the instanton–anti-instanton *liquid* model that satisfies this requirement (in the technique of Ref. [6], the correlations appear due to instanton–anti-instanton interaction via their zero modes) should provide, indeed, for a nonzero finite value of $\rho(\lambda=0)$ and the fermion condensate.

A very interesting question is what the mechanism is for providing the melting down of the fermion condensate at high temperature and restoration of chiral symmetry in QCD with several light quark flavors. We cannot, however, say much new about it now. The analogies with the Schwinger model do not help here.

The guesses about the Dirac operator spectrum in the different models for the four-dimensional field configurations can, in principle, be checked numerically. The first results for the Dirac operator spectrum in some models (unfortunately, not yet in QCD) have already appeared [17]. But to perform the calculations in a *particular* external field is a much more easy task than to do the path integration.

ACKNOWLEDGMENTS

I am indebted to E. V. Shuryak, A. Wipf, and especially to H. Leutwyler for very useful discussions. This work was supported in part by Schweizerischer Nationalfonds.

APPENDIX: VARIATIONAL ESTIMATE FOR SPECTRUM LEVEL DENSITY

Elaborating the variational arguments of Ref. [16], we show here that the level density $\rho(\lambda=0)$ and the fermion condensate are actually infinite in the quenched Schwinger model.

Consider a typical gauge-field configuration $E^{\text{char}}(x)$ in the quenched Schwinger model. Let D be a large region of size R, $g^{-1} \ll R \ll L$. A characteristic flux (7) on this region is of the order of gR (which corresponds to the area law for the Wilson loop). Consider now another field configuration $\tilde{E}(x)$ such that

$$\tilde{E}(x) = E^{\text{char}}(x)\ , \quad x \in D\ ,$$
$$\tilde{E}(x) = 0\ , \quad x \notin D\ . \tag{A1}$$

The field configuration (A1) has a finite large flux $\sim gR$ concentrated in the finite region D. According to the index theorem, it supports $\sim gR$ different fermionic zero modes. The corresponding eigenfunctions involve the exponential $\exp\{-g\phi(x)\}$ $[\Delta\phi(x)=E(x)]$ multiplied by a polynomial factor. One can always choose a particular solution $\psi_0(x)$ for the Dirac equation $\mathbb{D}\psi_0(x)=0$ for which this polynomial factor is absent.

The crucial observation is that characteristic fluctuations Δ^{char} of the function

$$g\phi(x) \sim g\int_D E(y) \ln|x-y|^2 y \tag{A2}$$

on the region D are of the order of the characteristic flux Φ_D. E.g., for the uniform field E, $g\phi(x)=gEx^2/2$ and

$$g\Delta_D^{\text{char}}\phi(x) \sim gER^2 \sim \Phi_D\ . \tag{A3}$$

It follows both from Eq. (A2) and from the exact result

$$\langle [\phi(x)-\phi(y)][\phi(x)-\phi(y)] \rangle \propto (x-y)^2 \tag{A4}$$

for the Green's function in the quenched Schwinger model [cf. Eq. (32) and discussion thereafter] that the estimate (A3) holds also for the field configuration of interest, Eq. (A1), up to unessential logarithmic factors.

Under the general choice of the region D, the function $\phi(x)$ achieves its minimal value at the point x_0 well inside the region D. Then the normalized zero-mode function

$$\psi_0(x) \propto \exp\{-g[\phi(x)-\phi(x_0)]\} \tag{A5}$$

is suppressed at the border C of the region D as

$$\psi_0(x \in C) \sim \exp(-\Phi_D) \sim \exp(-AgR)\ . \tag{A6}$$

The function (A5) is the zero mode of the Dirac equation with the gauge-field configuration (A1) but not for the original configuration $E^{\text{char}}(x)$. We may use it, however, to obtain a variational estimate for the lowest characteristic eigenvalues of the Hamiltonian $\mathcal{H}=\mathbb{D}^2$ associated with the wave functions concentrated on a region of size R:

$$\varepsilon_0(R) \sim \lambda_0^2(R) \sim \int \psi_0^*(x)\mathbb{D}^2\psi_0(x)d^2x\ , \tag{A7}$$

where the integral extends on the whole manifold. The integrand is nonzero, however, only outside the region D where it is suppressed according to Eq. (A6). Thus we get the estimate

$$\lambda_0(R) < g\exp(-AgR)\ . \tag{A8}$$

That means that the total number $N(\varepsilon)$ of eigenstates with the eigenvalues $\lambda_n < \varepsilon$ is

$$N(\varepsilon) > \frac{L^2}{R^2(\varepsilon)} \sim \frac{g^2L^2}{\ln^2(g/\varepsilon)} \tag{A9}$$

and the level density

$$\rho(\lambda=0) = \frac{1}{L^2}\lim_{\varepsilon\to 0}\frac{N(\varepsilon)}{\varepsilon} \tag{A10}$$

is infinite. The main reason for that is the exponential suppression (A6) for the quasi-zero-mode functions (A5).

In the unquenched Schwinger model, a suppression is also there but is much more weak,

$$g\Delta_D^{\rm char}\phi(x)|_{\rm unquenched} \propto \ln R \ , \tag{A11}$$

the wave function at the border is suppressed only as a power, and

$$\lambda_0(R) \sim g^{-1/N_f} R^{-(N_f+1)/N_f} \tag{A12}$$

[see Eq. (19)].

[1] T. Banks and A. Casher, Nucl. Phys. **B169**, 103 (1980).
[2] H. Leutwyler and A. V. Smilga, this issue, Phys. Rev. D **46**, 5607 (1992).
[3] E. Floratos and J. Stern, Phys. Lett. **119B**, 419 (1982).
[4] D. I. Diakonov and V. Yu. Petrov, (a) Phys. Lett. **147B**, 351 (1984); (b) Nucl. Phys. **B245**, 259 (1984).
[5] E. V. Shuryak, *The QCD Vacuum, Hadrons and the Superdense Matter* (World Scientific, Singapore, 1988).
[6] E. V. Shuryak and J. J. M. Verbaarschot, Nucl. Phys. **B341**, 1 (1990).
[7] Yu. A. Simonov, Phys. Rev. D **43**, 3534 (1991); Pis'ma Zh. Eksp. Teor. Fiz. **53**, 10 (1991) [JETP Lett. **53**, 9 (1991)].
[8] J. Schwinger, Phys. Rev. **128**, 2425 (1962).
[9] E. Marinari, G. Parisi, and C. Rebbi, Nucl. Phys. **B190**, 734 (1981).
[10] J. E. Hetrick and Y. Hosotani, Phys. Rev. D **38**, 2621 (1988).
[11] C. Jayewardena, Helv. Phys. Acta **61**, 636 (1988).
[12] I. Sach and A. Wipf, Helv. Phys. Acta **65**, 652 (1992).
[13] A. V. Smilga, Phys. Lett. B **278**, 371 (1992).
[14] S. Coleman, Commun. Math. Phys. **31**, 259 (1973).
[15] H. B. Nielsen and P. Olesen, Niels Bohr Institute Report No. NBI-HE-79-45, 1979 (unpublished); P. Olesen, Nucl. Phys. **B200** [FS4], 381 (1982).
[16] A. Casher and H. Neuberger, Phys. Lett. **139B**, 67 (1984).
[17] S. Hands, J. Kogut, and E. Dagotto, Nucl. Phys. **B333**, 551 (1990); M. Cockeler *et al.*, *ibid.* **B334**, 527 (1990); I. Barbour *et al.*, *ibid.* **B368**, 390 (1992).