PHYSICAL REVIEW D VOLUME 46, NUMBER 12 15 DECEMBER 1992

# QED vacuum polarization on a momentum lattice

H. Kröger, R. Lafrance, and L. Marleau
*Département de Physique, Université Laval, Québec, Québec, Canada G1K 7P4*
(Received 26 May 1992)

We study the effect of a momentum ($k$) lattice as a regulator of quantum field theory. An an example, we compute the vacuum polarization in noncompact (linearized) QED from $k$-lattice perturbation theory to one-loop order and study the continuum limit. The amplitude has a finite part plus logarithmically, linearly, and quadratically divergent terms. The amplitude violates gauge invariance (Ward identity) and Lorentz (Euclidean) invariance and is nonlocal. For example, the linear term $\sim \Lambda|k|$ is nonlocal. Renormalization requires nonlocal counterterms, which is not inconsistent because the original action on the $k$ lattice already has a nonlocality. We explicitly give the counterterms, which render the amplitude Lorentz and gauge invariant to recover the standard result.

PACS number(s): 11.15.Ha, 12.20.Ds

## I. INTRODUCTION

Since Wilson's [1] proposal to use a space-time lattice as a regulator of a field theory, it has been noticed that some symmetries are conserved while others are not. For example, gauge symmetry is conserved by Wilson's construction by expressing the action in terms of link variables instead of the gauge fields (compact or periodic formulation of the action). However, there is a problem to reconcile the conservation of chiral symmetry with the absence of fermion doubling as has been noticed by Karsten and Smit [2], as well as by Nielsen and Ninomiya [3]. The Nielsen-Ninomiya no-go theorem is based on the assumption that the action is local. Over the years a number of proposals have been made to overcome those problems. An early attempt was made by Drell, Weinstein, and Yankielowicz [4] who suggested expressing a derivative in the action as in continuum momentum space (SLAC derivative). This introduces a nonlocality of the action on a space-time lattice. A crucial test for a chiral fermion lattice proposal is the axial anomaly (Adler-Bell-Jackiw anomaly [5,6]). The Drell-Weinstein-Yankielowicz proposal conserves chiral symmetry, and produces no fermion doubling, but yields a vanishing axial anomaly, as has been shown in a nonperturbative argument by Ninomiya and Tan [7]. The QED vacuum polarization in the Drell-Weinstein-Yankielowicz formulation has been computed in Ref. [8].

A number of other proposals have been made to deal with the chiral fermion problem. Rebbi's chiral fermion proposal [9] yields fermion spectrum doubling for the vacuum polarization and the anomaly in two dimensions, as has been shown by Bodwin and Kovács [10]. Bodwin and Kovács [11] have considered the Schwinger model to compare the lattice fermion proposals by Wilson, by Kogut and Susskind, and by Drell, Weinstein, and Yankielowicz by computing the anomaly, the mass gap, and the chiral order parameter $\langle \bar{\psi}\psi \rangle$. The proposals by Aoki [12] and by Funakubo and Kashiwa [13], which are local formulations, have been shown by Bodwin and Kovács [14] to require nonlocal counterterms for renormalization in the continuum limit. In the proposal by Smit [15] fermion doubling is present, but the fermion doublers are decoupled by giving them a large mass. Recently, a proposal has been made by Bodwin and Kovács [18] introducing auxiliary Dirac fields. Recent reviews are given in Refs. [16,17].

The above results show the failure of lattice theory to yield gauge invariance, chiral invariance, absence of fermion doubling, correct continuum limit of quantum expectation values in weak-coupling perturbation theory, in particular for vacuum polarization and the axial anomaly. This holds for a local lattice formulation by the Nielsen-Ninomiya no-go theorem [3], as well as for the nonlocal Drell-Weinstein-Yankielowicz formulation based on a nonperturbative argument [7].

As an alternative to a space-time lattice, the use of a momentum ($k$) lattice has been investigated recently. Kuti, Lin, and Shen [19,20] have applied it to the scalar $\phi^4$ model (in one and four components) to estimate an upper limit of the Higgs mass. Bérubé *et al.* [21] have investigated the critical behavior of the scalar $\phi^4$ model on a small momentum lattice and compared it with Lüscher and Weisz's [22] renormalization-group solutions of high-temperature-expansion data. Brockmann and Frank [23] have used a momentum lattice to compute the equation of state of nuclear matter from quantum hadrodynamics.

The potential advantages of a momentum lattice can be summarized as follows: (a) Observables determined by the low-momentum behavior of a theory (renormalized mass, wave function renormalization, renormalized coupling constant, etc.) may eventually be determined more easily on a $k$ lattice. (b) Critical slowing down can be opposed by Fourier acceleration, which is related to modifying the zero-momentum behavior. This can be implemented on a $k$ lattice in a straightforward manner. The results of Ref. [21] show that critical behavior can be extracted on a quite small $k$ lattice ($3^4$). (c) There is no fermion doubling.

The implications of using a momentum lattice for gauge or chiral theories has been investigated by Bérubé *et al.* in Refs. [24–26]. In particular, the action has been treated in the noncompact (linearized) formulation, i.e., by writing the action in terms of the gauge and chiral fields, instead of using the link variables. The question arises: What happens to the symmetry group? Is the classical action invariant? Are quantum expectation values in the continuum limit invariant? In the naive $k$-lattice formulation all answers are negative. However, a one-to-one correspondence between a symmetry group of local gauge transformations on a space-time lattice and a corresponding symmetry group on a $k$ lattice can be established (then a gauge transformation is given by a circulant and unitary matrix on the $k$ lattice). This is also possible for chiral transformations. The classical action is invariant in a weak form, i.e., under infinitesimal transformations with compact support (which means that the exponential exponent is a nonvanishing matrix only in a finite domain which is smaller than the size of the lattice). To see what happens to quantum expectation values, the axial anomaly has been computed in Ref. [26]. The standard Ward identities and axial anomaly are found, however, only if the amplitude is renormalized by using nonlocal counterterms. The appearance of nonlocal counterterms is not surprising, because the original action contains a nonlocality in the form of a jump discontinuity in the kinetic term.

In Ref. [26] the Ward identities have been computed from the triangle diagram, but not the vertex amplitude itself. Here we want to study another example and compute the full amplitude. We consider QED and compute the vacuum polarization from $k$-lattice perturbation theory to one-loop order. We give explicitly all nonlocalities and compute the counterterms to find the standard Lorentz and gauge-invariant result. Recently, Bodwin, Kovács, and Sloan [28] have studied noncompact (linearized) QED on a space-time lattice and studied renormalization. They also find counterterms not present in the compact theory or in the continuum theory with a gauge-invariant regulator. Also in their study of the Yukawa model in light-cone quantization, Burkardt and Langnau [27] have found the necessity to introduce noncovariant counterterms in order to restore Lorentz invariance.

The motivation of this work is based on the following. (a) The general interest in a noncompact (linearized) formulation of field theory on a lattice. For example, for U(1) gauge theory, the compact and the noncompact formulation give a different phase structure. (b) Further exploration of the $k$-lattice regularization, which has been shown to be a viable tool in numerical simulations. (c) Finally, there is a more speculative point. The above results giving quantum amplitudes which require nonlocal counterterms come from applying weak-coupling perturbation theory (loops giving rise to infinities). There are reasons to speculate that nonlocal counterterms would not be needed by doing perturbation theory in a way which gives finite amplitudes. Before investigating this further, it is necessary in contrast to study in detail the nonlocal structure in weak-coupling perturbation theory.

## II. MOMENTUM LATTICE REGULARIZATION

We introduce a regular, hypercubic $M^D$ momentum ($k$) lattice, symmetric with respect to the origin $k=0$. It has a lattice spacing (momentum resolution) $\Delta k$ and a high momentum cutoff $\Lambda$, related by $\Lambda=\Delta k M/2$ if $M$ is even and $\Lambda=\Delta k(M-1)/2$ if $M$ is odd. The correspondence to a space-time lattice with lattice spacing $a$ is given by $\Lambda=\pi/a$. We write the Euclidean action as follows

$$S=\sum_{k_I}\left[\frac{\Delta k}{2\pi}\right]^4\tilde{\bar{\psi}}(k_I)[\gamma_\mu(-i)(\tilde{k}_I)_\mu+m]\tilde{\psi}(k_I)+ie\sum_{k_I,k_J,k_M}\left[\frac{\Delta k}{2\pi}\right]^{12}(2\pi)^4\tilde{\delta}(k_I-k_J-k_M)\tilde{\bar{\psi}}(k_I)\gamma_\mu\tilde{A}_\mu(k_J)\tilde{\psi}(k_M)$$
$$+\tfrac{1}{4}\sum_{k_I}\left[\frac{\Delta k}{2\pi}\right]^4\tilde{F}_{\mu\nu}(-k_I)\tilde{F}_{\mu\nu}(k_I)+\frac{\lambda}{2}\sum_{k_I}\left[\frac{\Delta k}{2\pi}\right]^4(-i)(-\tilde{k}_I)_\mu\tilde{A}_\mu(-k_I)(-i)(\tilde{k}_I)_\nu\tilde{A}_\nu(k_I)\ , \tag{1}$$

where

$$\tilde{F}_{\mu\nu}=-i(\tilde{k}_I)_\mu\tilde{A}_\nu-(-i)(\tilde{k}_I)_\nu\tilde{A}_\mu(k_I)\ . \tag{2}$$

Here $\tilde{f}(k_I)$ stands for a periodic, lattice function (with a period $M\,\Delta k$ in each dimension), i.e., we assume that all functions (fields, $\delta$ function etc.) defined on the $k$ lattice are periodic. This periodic structure was introduced in Refs. [24,25] to give the gauge transformations a group structure. [Note that for a periodic function $\tilde{v}(k_I)$, the matrix defined by $\tilde{V}(k_I,k_J)=\tilde{v}(k_I-k_J)$ is a circulant matrix, and circulant unitary matrices form a group.] Euclidean Green's functions are expressed on the $k$ lattice by

$$\langle\cdots\rangle=\frac{\int\prod_{k_I}d\tilde{\psi}(k_I)d\tilde{\bar{\psi}}(k_I)\prod_{k_J}d\tilde{A}_\mu(k_J)\cdots\exp(-S[\tilde{\bar{\psi}},\tilde{\psi},\tilde{A}_\mu])}{\int\prod_{k_I}d\tilde{\psi}(k_I)d\tilde{\bar{\psi}}(k_I)\prod_{k_J}d\tilde{A}_\mu(k_J)\exp(-S[\tilde{\bar{\psi}},\tilde{\psi},\tilde{A}_\mu])}\ . \tag{3}$$

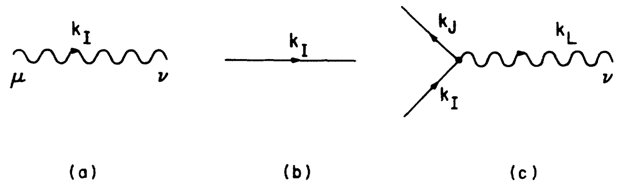


FIG. 1. (a) Photon line, (b) fermion line, and (c) vertex in QED on $k$ lattice.

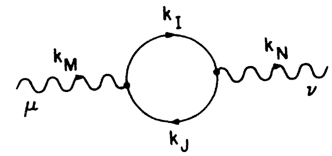


FIG. 2. One-loop vacuum-polarization diagram in QED on $k$ lattice.

The Euclidean Feynman rules (in the Feynman gauge, $\lambda=1$) are: An internal photon line [Fig. 1(a)] is given by $\delta_{\mu\nu}/(\tilde{k}_I)^2$. An internal fermion line [Fig. 1(b)] corresponds to $1/[\gamma_\mu(-i)(\tilde{k}_I)_\mu+m]^{-1}$, and the vertex [Fig. 1(c)] is $(-i)e\gamma_\mu(2\pi)^4\tilde{\delta}(k_I-k_J-k_L)$. According to Quinn and Weinstein [29] the vertex employed in the SLAC derivative formulation reads,

$$\Gamma_\mu(k,q)=e\gamma_\mu\frac{D_\mu(k+q)-D_\mu(k)}{\sin(q_\mu^2 a/2)}\ , \tag{4}$$

where $D_\mu(k)$ denotes the SLAC derivative. One should note that this vertex differs from ours.

Let us consider now vacuum polarization. The standard result for the vertex amplitude for vacuum polarization computed to one-loop order in the continuum limit by Pauli-Villars regularization is given by Ref. [30]. It reads in Euclidean form

$$I_{\mu\nu}=\frac{-2\alpha}{\pi}[\delta_{\mu\nu}k^2-k_\mu k_\nu]\times\int_0^1 dx\,x(1-x)\ln\left[\frac{M_{\rm PV}^2}{m^2+x(1-x)k^2}\right]\ , \tag{5}$$

where $M_{\rm PV}$ is the Pauli-Villars mass. This amplitude satisfies the Ward identity

$$k_\mu I_{\mu\nu}(k)=0\ , \tag{6}$$

which expresses gauge invariance, and it is Lorentz (Euclidean) covariant.

On the $k$ lattice, the vacuum polarization diagram, given by Fig. 2, corresponds to the amplitude

$$(-1)\left[\frac{1}{\tilde{k}_M{}^2}\right]^2(2\pi)^4\tilde{\delta}(k_M-k_N)\sum_{k_I,k_J}\left[\frac{\Delta k}{2\pi}\right]^8(2\pi)^4\tilde{\delta}(k_M-k_I+k_J)\times\mathrm{Tr}\left[(-i)e\gamma_\mu\frac{1}{\gamma_\rho(-i)(\tilde{k}_I)_\rho+m}(-i)e\gamma_\nu\frac{1}{\gamma_\sigma(-i)(\tilde{k}_J)_\sigma+m}\right]\ . \tag{7}$$

We go over to the continuum limit in two steps: (a) $\Delta k\to 0$, but cutoff $\Lambda=$const, (b) $\Lambda\to\infty$. Performing the first step, carrying out the trace and amputating the external lines yields

$$I_{\mu\nu}(k)=-4e^2\int_{-\Lambda}^{+\Lambda}\frac{d^4p}{(2\pi)^4}\int_{-\Lambda}^{+\Lambda}\frac{d^4q}{(2\pi)^4}(2\pi)^4\tilde{\delta}(k+p-q)\frac{\tilde{p}_\mu\tilde{q}_\nu+\tilde{q}_\mu\tilde{p}_\nu-\delta_{\mu\nu}(\tilde{p}\cdot\tilde{q}+m^2)}{[\tilde{p}^2+m^2][\tilde{q}^2+m^2]}\ . \tag{8}$$

All functions of the integrand are periodic with period $2\Lambda$ and hence so is the amplitude

$$I_{\mu\nu}(k)\equiv\tilde{I}_{\mu\nu}(k)\ . \tag{9}$$

The periodic $\delta$ function can be written as

$$\begin{aligned}\tilde{\delta}(k+p-q)=&\,\delta(k+p-q)+\delta(k+p-q+2\Lambda)\\&+\delta(k+p-q-2\Lambda)+\delta(k+p-q+4\Lambda)\\&+\delta(k+p-q+4\Lambda)+\cdots\ .\end{aligned} \tag{10}$$

Let us consider $k_\alpha\in[-\Lambda,+\Lambda]$, $\alpha=1,\ldots,4$. Then terms with shifts $\pm4\Lambda,\pm6\Lambda,\ldots$ do not contribute to the integral. We have to consider only the shifts 0, $\pm2\Lambda$.

### A. Contribution from shift zero

First, let us consider the contribution from shift zero,

$$I_{\mu\nu}(k)=-\frac{4e^2}{(2\pi)^4}\int_{\mathrm{Box}(k,\Lambda)}d^4p\,f_{\mu\nu}(k,p)\ ,$$

$$f_{\mu\nu}(k,p)=\frac{2p_\mu p_\nu+p_\mu k_\nu+k_\mu p_\nu-\delta_{\mu\nu}(p^2+p\cdot k+m^2)}{[p^2+m^2][(p+k)^2+m^2]}\ , \tag{11}$$

$$\mathrm{Box}(k,\Lambda)=\{p\,|\,|p_\alpha|\le\Lambda,|(p+k)_\alpha|\le\Lambda,\ \ \alpha=1,\ldots,4\}\ .$$

In order to exhibit the singularity structure we expand

$$\frac{1}{(p+k)^2+m^2}=\frac{1}{p^2+m^2}\sum_{N=0}^{\infty}\left[\frac{-k^2-2k\cdot p}{p^2+m^2}\right]^N . \tag{12}$$

The series converges as a function of $k$ for $|k|<m/2$. We split the right-hand side (RHS) into terms with $N=0,1,2$ and $N\geq 3$,

$$\frac{1}{(p+k)^2+m^2}=\frac{1}{p^2+m^2}\sum_{N=0}^{2}\left[\frac{-k^2-2k\cdot p}{p^2+m^2}\right]^N +\left[\frac{1}{(p+k)^2+m^2}\right]^{[N=0,1,2\,\text{subtr}]} , \tag{13}$$

where

$$\left[\frac{1}{(p+k)^2+m^2}\right]^{[N=0,1,2\,\text{subtr}]} =\frac{[-k^2-2k\cdot p]^3}{[p^2+m^2]^3[(p+k)^2+m^2]}\xrightarrow[p\to\infty]{}\frac{1}{p^5} . \tag{14}$$

Let us define $I_{\mu\nu}^{[N=0,1,2\,\text{subtr}]}$ and $f_{\mu\nu}^{[N=0,1,2\,\text{subtr}]}$ by Eq. (11), when replacing $[(p+k)^2+m^2]^{-1}$ by $[((p+k)^2+m^2)^{-1}]^{[N=0,1,2\,\text{subtr}]}$. This yields $I_{\mu\nu}^{[N=0,1,2\,\text{subtr}]}(k,\Lambda)$ to be finite and convergent with $\Lambda\to\infty$.

We will not compute $I_{\mu\nu}^{[N=0,1,2\,\text{subtr}]}$ explicitly, because it gives the same finite contribution as computed from standard continuum perturbation theory with Pauli-Villars regularization. We will show, however, that it satisfies the Ward identity. Let $N\geq 3$ and let us consider the contribution for a given $N$ [see Eq. (12)].

$$I_{\mu\nu}^N(k,\Lambda)=-\frac{4e^2}{(2\pi^4)}\int_{\text{Box}(k,\Lambda)}d^4p\frac{[2p_\mu p_\nu+p_\mu k_\nu+k_\mu p_\nu-\delta_{\mu\nu}(p^2+p\cdot k+m^2)][-k^2-2k\cdot p]^N}{[p^2+m^2]^{N+2}} . \tag{15}$$

Note that the integral is convergent, which allows us to replace for large $\Lambda$ the integration domain $\text{Box}(k,\Lambda)$ by a symmetric domain. Then one obtains

$$\begin{aligned}k_\mu I_{\mu\nu}^N(k,\Lambda)&=-\frac{4e^2}{(2\pi)^4}(-1)^N\int_{-\Lambda}^{+\Lambda}d^4p\left[\frac{[k^2+2k\cdot p]^{N+1}}{[p^2+m^2]^{N+2}}p_\nu-\frac{[k^2+2k\cdot p]^N}{[p^2+m^2]^{N+1}}k_\nu\right]\\&=-\frac{4e^2}{(2\pi)^4}\frac{(-1)^{N+1}}{2(N+1)}\int_{-\Lambda}^{+\Lambda}d^3p\frac{[k^2+2\mathbf{k}\cdot\mathbf{p}+2k_\nu p_\nu]^{N+1}}{[\mathbf{p}^2+p_\nu^2+m^2]^{N+1}}\Bigg|_{p_\nu=-\Lambda}^{p_\nu=+\Lambda}\xrightarrow[\Lambda\to\infty]{}0 .\end{aligned} \tag{16}$$

Here we have used partial integration and made essential use of $N\geq 3$. For $N\leq 2$ nonzero surface terms occur. This result implies

$$k_\mu I_{\mu\nu}^{[N=0,1,2\,\text{subtr}]}(k)=0 , \tag{17}$$

i.e., the correct Ward identity.

Now let us consider the terms with $N=0,1,2$. These will give infinities in the amplitude and nonzero contributions to the Ward identity. Let us see this in some detail. Let us assume $k_\alpha>0$, $\alpha=1,\dots,4$. Then

$$\text{Box}(k,\Lambda)=\{p|p_\alpha:-\Lambda\to+\Lambda-k_\alpha,\ \alpha=1,\dots,4\} . \tag{18}$$

This domain can be decomposed into the following disjoint domains

$$\text{Box}(k,\Lambda)=D^0(\Lambda)-D^1(k,\Lambda)+D^2(k,\Lambda)-D^3(k,\Lambda) , \tag{19}$$

where

$$\begin{aligned}D^0(\Lambda)&=\{p|p_\alpha:-\Lambda\to+\Lambda,\ \alpha=1,\dots,4\} ,\\D^1(k,\Lambda)&=\bigcup_{\alpha=1}^{4}\{p|p_\beta:-\Lambda\to+\Lambda,\ \beta=1,\dots,4,\ \beta\neq\alpha,\ p_\alpha:\Lambda-k_\alpha\to\Lambda\} ,\\D^2(k,\Lambda)&=\bigcup_{\alpha<\beta=1}^{4}\{p|p_\gamma:-\Lambda\to+\Lambda,\ \gamma=1,\dots,4,\ \gamma\neq\alpha,\beta,\ p_\alpha:\Lambda-k_\alpha\to\Lambda,\ p_\beta:\Lambda-k_\beta\to\Lambda\} ,\\D^3(k,\Lambda)&=\bigcup_{\alpha<\beta<\gamma=1}^{4}\{p|p_\delta:-\Lambda\to+\Lambda,\ \delta=1,\dots,4,\ \delta\neq\alpha,\beta,\gamma,\ p_\alpha:\Lambda-k_\alpha\to\Lambda ,\\&\qquad p_\beta:\Lambda-k_\beta\to\Lambda,\ p_\gamma:\Lambda-k_\gamma\to\Lambda\} .\end{aligned} \tag{20}$$

This decomposition is useful to disentangle contributions to the amplitude with different degrees of divergence.

Let us start to consider the contribution from domain $D^0$,

$$I_{\mu\nu}(k)=-\frac{4e^2}{(2\pi)^4}\int_{-\Lambda}^{+\Lambda}d^4p\, f_{\mu\nu}(k,p) . \tag{21}$$

This expression is quadratically divergent by power counting. One obtains for the terms $N=0,1,2$ [see Eq. (12)],

$$I_{\mu\nu}^{[D^0,N=0]}(k) = -\frac{4e^2}{(2\pi)^4}\delta_{\mu\nu}[-2\Lambda^2 S^{(1)} + 2m^2 S^{(2)}] + O(\Lambda^{-1}) , \quad (22)$$

where $S^{(n)}$ is given in the Appendix.

$$I_{\mu\nu}^{[D^0,N=1]}(k) = -\frac{4e^2}{(2\pi)^4}\{k_\mu k_\nu[-I^{(2)}(\Lambda) + 2S^{(2)}] + \delta_{\mu\nu}k^2 I^{(2)}(\Lambda)\} + O(\Lambda^{-1}) , \quad (23)$$

where $I^{(n)}(\Lambda)$ is given in the Appendix.

$$I_{\mu\nu}^{[D^0,N=2]}(k) = -\frac{4e^2}{(2\pi)^4}\left[k_\mu k_\nu\left[\frac{2}{3}I^{(2)}(\Lambda) - \frac{28}{9}S^{(2)} + \frac{16}{9}S^{(3)}\right] + \delta_{\mu\nu}k^2\left[-\frac{2}{3}I^{(2)}(\Lambda) + \frac{4}{9}S^{(2)} + \frac{8}{9}S^{(3)}\right]\right.$$
$$+ \delta_{\mu\nu}k_\mu k_\mu\left[\frac{8}{3}S^{(2)} - \frac{16}{3}S^{(3)}\right] + k_\mu k_\nu k^2\frac{1}{m^2}\left[\frac{8}{3}S^{(2)}\right]$$
$$\left. + \delta_{\mu\nu}(k^2)^2\frac{1}{m^2}\left[\frac{-8}{3}S^{(2)}\right]\right] + O(\Lambda^{-1}) . \quad (24)$$

Next, we compute the contribution from domain $D^{(1)}(k,\Lambda)$ [see Eq. (20)]. One obtains

$$I_{\mu\nu}^{[D^1,N=0]}(k) = -\frac{4e^2}{(2\pi)^4}\left[k_\mu k_\nu[2S^{(2)}] + \delta_{\mu\nu}k^2\left[-\frac{2}{3}S^{(2)} - \frac{4}{3}S^{(3)}\right] + \delta_{\mu\nu}k_\mu k_\mu\left[-\frac{10}{3}S^{(2)} + \frac{16}{3}S^{(3)}\right]\right.$$
$$\left. + \Lambda\sum_{\alpha=1}^{4}k_\alpha\left[-\frac{1}{3}S^{(1)} - \frac{2}{3}S^{(2)}\right] + \Lambda k_\mu\left[-\frac{2}{3}S^{(1)} + \frac{8}{3}S^{(2)}\right]\right] + O(\Lambda^{-1}) ,$$
$$I_{\mu\nu}^{[D^1,N=1]}(k) = -\frac{4e^2}{(2\pi)^4}\left[k_\mu k_\nu\left[-\frac{8}{3}S^{(2)} + \frac{8}{3}S^{(3)}\right] + \delta_{\mu\nu}k^2\left[\frac{2}{3}S^{(2)} + \frac{4}{3}S^{(3)}\right]\right.$$
$$\left. + \delta_{\mu\nu}k_\mu k_\mu\left[\frac{10}{3}S^{(2)} - \frac{22}{3}S^{(3)}\right]\right] + O(\Lambda^{-1}) . \quad (25)$$

Note that due to Eq. (19), these terms enter into the total amplitude with an additional minus sign.

Finally, from domain $D^2(k,\Lambda)$ [see Eq. (20)] one obtains

$$I_{\mu\nu}^{[D^2,N=0]}(k) = -\frac{4e^2}{(2\pi)^4}\left[k_\mu k_\nu[2T^{(2)}] + \delta_{\mu\nu}k^2[-T^{(2)}] + \delta_{\mu\nu}k_\mu k_\mu[T^{(1)} - 2T^{(2)}] + \delta_{\mu\nu}k_\mu\sum_{\alpha=1}^{4}k_\alpha[-T^{(1)}]\right. \quad (26)$$
$$\left. + \delta_{\mu\nu}\sum_{\alpha,\beta=1}^{4}k_\alpha k_\beta\left[\frac{1}{2}T^{(2)}\right]\right] + O(\Lambda^{-1}) ,$$

where $T^{(n)}$ is given in the Appendix. The contributions from $D^{(1)}$, $N=2$ and from $D^{(2)}$, $N=1,2$ are $O(\Lambda^{-1})$.

### B. Contribution from shift $\pm 2\Lambda$ in one dimension

We continue to make the assumption $k_\alpha > 0$, $\alpha = 1,\ldots,4$. Because we take $k_\alpha \in [-\Lambda, +\Lambda]$, Eq. (10) shows that nonzero contributions come only from $\delta(k_\alpha + p_\alpha - q_\alpha - 2\Lambda)$, i.e., only the shift $-2\Lambda$ plays a role. Such a shift $-2\Lambda$ can occur in one, two, three, or four dimensions. In this section we consider the contribution from shift $-2\Lambda$ in one dimension, say dimension $\alpha$. According to Eq. (10), one has to sum over $\alpha$. This gives the following contribution [see Eq. (11)],

$$I_{\mu\nu}^{[\text{one-dim shift } -2\Lambda]}(k) = -\frac{4e^2}{(2\pi)^4}\sum_{\alpha=1}^{4}\int_{\text{Box}(\bar{k}^{(\alpha)},\Lambda)} d^4p\, f_{\mu\nu}(\bar{k}^{(\alpha)}, p) , \quad (27)$$

where $\bar{k}_\beta^{(\alpha)} = k_\beta - 2\Lambda\delta_{\alpha\beta}$. The integration domain is

$$\text{Box}(\bar{k},\Lambda) = \bigcup_{\alpha=1}^{4}\{p \mid p_\beta: -\Lambda \to +\Lambda - k_\beta,\ \beta = 1,\ldots,4,\ \beta \neq \alpha,\ p_\alpha: +\Lambda - k_\alpha \to +\Lambda\} . \quad (28)$$

This domain can be decomposed into the following disjoint domains

$$\text{Box}(\bar{k},\Lambda) = D^1(k,\Lambda) - D^2(k,\Lambda) + D^3(k,\Lambda) . \quad (29)$$

Then we compute the contribution from $D^1$

$$I_{\mu\nu}^{[\text{one-dim shift } -2\Lambda, D^1, N=0]}(k)=I_{\mu\nu}^{[D^1,N=0]}(k)-\frac{4e^2}{(2\pi)^4}\delta_{\mu\nu}\left[k^2\left[\frac{5}{3}S^{(2)}-4S^{(3)}\right]+\Lambda\sum_{\alpha=1}^{4}k_\alpha[+2S^{(2)}]+\Lambda k_\mu[-4S^{(2)}]+k_\mu k_\mu\left[-\frac{1}{6}S^{(2)}-\frac{5}{3}S^{(3)}\right]\right]+O(\Lambda^{-1}),$$

$$I_{\mu\nu}^{[\text{one-dim shift } -2\Lambda, D^1, N=1]}(k)=I_{\mu\nu}^{[D^1,N=1]}(k)-\frac{4e^2}{(2\pi)^4}\left[k_\mu k_\nu\left[\frac{8}{3}S^{(2)}-\frac{8}{3}S^{(3)}\right]+\delta_{\mu\nu}k^2[3S^{(3)}]+\delta_{\mu\nu}k_\mu k_\mu[-2S^{(2)}+10S^{(3)}]\right]+O(\Lambda^{-1}). \quad (30)$$

The contribution corresponding to $D^2$ is given by

$$I_{\mu\nu}^{[\text{one-dim shift } -2\Lambda, D^2, N=0]}(k)=I_{\mu\nu}^{[D^2,N=0]}(k)-\frac{4e^2}{(2\pi)^4}\left[k_\mu k_\nu(-2T^{(2)})+\delta_{\mu\nu}k^2(+T^{(2)})+\delta_{\mu\nu}k_\mu k_\mu(-\tfrac{1}{2}T^{(2)})+\delta_{\mu\nu}\sum_{\alpha,\beta=1}^{4}k_\alpha k_\beta(-T^{(2)})+\delta_{\mu\nu}k_\mu\sum_{\alpha=1}^{4}k_\alpha(+\tfrac{5}{2}T^{(2)})\right]+O(\Lambda^{-1}). \quad (31)$$

Note, this term contributes to the total amplitude with a minus sign.

### C. Contribution from shift $\pm 2\Lambda$ in two dimensions

In this section we consider the contribution from shift $-2\Lambda$ in two dimensions, say dimensions $\alpha$ and $\beta$, $\alpha\neq\beta$. Summing over $\alpha,\beta$ gives the following contribution [see Eq. (11)],

$$I_{\mu\nu}^{[\text{two-dim shift } -2\Lambda]}(k)=-\frac{4e^2}{(2\pi)^4}\sum_{\alpha<\beta=1}^{4}\int_{\mathrm{Box}(\bar{\bar{k}}^{(\alpha,\beta)},\Lambda)}d^4p\, f_{\mu\nu}(\bar{\bar{k}}^{(\alpha,\beta)},p), \quad (32)$$

where $\bar{\bar{k}}_\gamma^{(\alpha,\beta)}=k_\gamma-2\Lambda\delta_{\alpha\gamma}-2\Lambda\delta_{\beta\gamma}$. The integration domain is

$$\mathrm{Box}(\bar{\bar{k}},\Lambda)=\bigcup_{\alpha<\beta=1}^{4}\{p|p_\gamma:-\Lambda\to+\Lambda-k_\gamma,\ \gamma=1,\ldots,4,\ \gamma\neq\alpha,\beta,\ p_\alpha:+\Lambda-k_\alpha\to+\Lambda\ \ p_\beta:+\Lambda-k_\beta\to+\Lambda\}. \quad (33)$$

This domain can be decomposed into the following disjoint domains

$$\mathrm{Box}(\bar{\bar{k}},\Lambda)=D^2(k,\Lambda)-D^3(k,\Lambda). \quad (34)$$

Then we compute the contribution from $D^2$

$$I_{\mu\nu}^{[\text{two-dim shift } -2\Lambda, D^2, N=0]}(k)=I_{\mu\nu}^{[D^2,N=0]}(k)-\frac{4e^2}{(2\pi)^4}\left[k_\mu k_\nu(-4T^{(2)})+\delta_{\mu\nu}k^2(-2T^{(2)})-\delta_{\mu\nu}k_\mu k_\mu(-8T^{(2)})+\delta_{\mu\nu}k_\mu\sum_{\alpha=1}^{4}k_\alpha(-4T^{(2)})+\delta_{\mu\nu}\sum_{\alpha,\beta=1}^{4}k_\alpha k_\beta[2T^{(2)}]\right]+O(\Lambda^{-1}). \quad (35)$$

We close this section with the remark that contributions with shift $-2\Lambda$ in three or four dimensions are $O(\Lambda^{-1})$.

### D. Symmetry of the amplitude

In Sec. II, we have seen that the vacuum-polarization amplitude $I_{\mu\nu}(k)$ computed from the $k$-lattice regularization, given by Eq. (8), is periodic. Moreover, as Eqs. (8) and (10) show, $I_{\mu\nu}(k)$ is invariant under reflection of $k$. More precisely, it is invariant under the transformation

$$\text{(a)}\quad \begin{Bmatrix} k_\mu\to -k_\mu \\ k_\nu\to -k_\nu \end{Bmatrix},\ \text{simultaneously for } \mu \text{ and } \nu. \quad (36)$$

and under the transformation

$$\text{(b)}\quad k_\alpha\to -k_\alpha,\ \text{independently for each } \alpha\notin\{\mu,\nu\}. \quad (37)$$

In the preceeding sections we have computed the amplitude under the assumption $k_\alpha>0$, $\alpha=1,\ldots,4$. We found the following structure of the amplitude

$$I_{\mu\nu}(k)=Ak_\mu k_\nu+B\delta_{\mu\nu}k^2+C\delta_{\mu\nu}k_\mu k_\mu+D\delta_{\mu\nu}k_\mu\sum_{\alpha=1}^{4}k_\alpha+E\delta_{\mu\nu}\sum_{\alpha,\beta=1}^{4}k_\alpha k_\beta+F\delta_{\mu\nu}\Lambda k_\mu+G\delta_{\mu\nu}\Lambda\sum_{\alpha=1}^{4}k_\alpha+H(\Lambda)\delta_{\mu\nu}+I_{\mu\nu}^{[\text{regular}]}(k)+O(\Lambda^{-1})\ , \tag{38}$$

where

$$I_{\mu\nu}^{[\text{regular}]}(k)=[k_\mu k_\nu-\delta_{\mu\nu}k^2]f(k^2)\ . \tag{39}$$

Reflection symmetry dictates that the general amplitude, valid for arbitrary sign of $k_\alpha$, reads,

$$I_{\mu\nu}(k)=Ak_\mu k_\nu+B\delta_{\mu\nu}k^2+C\delta_{\mu\nu}k_\mu k_\mu+D\delta_{\mu\nu}|k_\mu|\sum_{\alpha=1}^{4}|k_\alpha|+E\delta_{\mu\nu}\sum_{\alpha,\beta=1}^{4}|k_\alpha||k_\beta|+F\delta_{\mu\nu}\Lambda|k_\mu|+G\delta_{\mu\nu}\Lambda\sum_{\alpha=1}^{4}|k_\alpha|+H(\Lambda)\delta_{\mu\nu}+I_{\mu\nu}^{[\text{regular}]}(k)+O(\Lambda^{-1})\ . \tag{40}$$

The corresponding changes have to be made in Eqs. (22–(26), (30), (31), and (35). The term $|k|$ corresponds to a nonlocality of the amplitude.

## III. COMPARISON WITH CONTINUUM PERTURBATION THEORY

Thus the result for the vacuum-polarization amplitude obtained in second-order perturbation theory in the $k$-lattice regularization is given by Eqs. (22–26), (30), (31), and (35). The standard expression from second-order continuum perturbation theory, using Pauli-Villars regularization is given by Eq. (5). We can check if the lattice result is consistent with the Pauli-Villars result. The Pauli-Villars result is obtained by taking the continuum integral and subtracting the same amplitude with the mass $m$ replaced by a large mass $M_{\text{PV}}$.

$$I_{\mu\nu}^{\text{PV}}(k)=-\frac{4e^2}{(2\pi)^4}\int_{-\infty}^{+\infty}d^4p\,[f_{\mu\nu}(k,p,m)-f_{\mu\nu}(k,p,M_{\text{PV}})]\ . \tag{41}$$

This expression is finite. We are free to rewrite $f_{\mu\nu}$, given by Eq. (11), using the decomposition of Eq. (13). Thus

$$I_{\mu\nu}^{\text{PV}}(k)=-\frac{4e^2}{(2\pi)^4}\int_{-\infty}^{+\infty}d^4p\left[\sum_{N=0}^{2}[f_{\mu\nu}^{[N]}(k,p,m)-f_{\mu\nu}^{[N]}(k,p,M_{\text{PV}})]+f_{\mu\nu}^{[N=0,1,2\ \text{subtr}]}(k,p,m)+O(M_{\text{PV}}^{-1})\right.\ . \tag{42}$$

The integral over $f_{\mu\nu}^{[N=0,1,2\ \text{subtr}]}(k,p,M_{\text{PV}})$ is finite and vanishes for a large Pauli-Villars mass of $O(M_{\text{PV}}^{-1})$. Now we look at the amplitude obtained on the $k$ lattice. We consider the contributions from shift zero and the symmetric integration domain with respect to cutoff $\Lambda$, i.e., the domain $D^0$. For $N=0,1,2$ these contributions are given by Eqs. (22)–(24). If we subtract the amplitude with $m$ replaced by $M_{\text{PV}}$, we obtain (using $\alpha=e^2/4\pi$)

$$\sum_{N=0}^{2}I_{\mu\nu}^{[D^0,N]}(k,m)-I_{\mu\nu}^{[D^0,N]}(k,M_{\text{PV}})=\frac{\alpha}{2\pi}\delta_{\mu\nu}[M_{\text{PV}}^2-m^2]+\frac{\alpha}{3\pi}[k_\mu k_\nu-\delta_{\mu\nu}k^2]\left[\ln(M_{\text{PV}}^2/m^2)-2\frac{k^2}{m^2}\right]+O(M_{\text{PV}}^{-1})\ . \tag{43}$$

On the other hand, the continuum result with Pauli-Villars regularization, given by Eq. (5), yields after expansion in $k$ up to second order,

$$I_{\mu\nu}^{\text{PV}}(k)=\frac{\alpha}{3\pi}[k_\mu k_\nu-\delta_{\mu\nu}k^2][\ln(M_{\text{PV}}^2/m^2)+O(k^2)]\ . \tag{44}$$

After we impose the Ward identity as a renormalization condition, the constant term vanishes on the RHS of Eq. (43) and up to second order in $k$, both results agree. Note that we have used the expansion (12) in Eq. (43) and the Taylor expansion in Eq. (44). Both are of course equivalent, but there is not a one-to-one correspondence between a given order $N$ in Eq. (12) and a given order of $k$ in the Taylor expansion. Thus the term of order $k^4$ of the amplitude is only partially given by Eq. (43), there is another contribution of that order coming from $I^{[D^0,\,N=0,1,2\ \text{subtr}]}$. On the other hand, the second term on the RHS of Eq. (43) is obtained by evaluating the term

$$\sum_{N=0}^{2}[f_{\mu\nu}^{[N]}(k,p,m)-f_{\mu\nu}^{[N]}(k,p,M_{\text{PV}})]$$

in expression (42). From this we conclude that the amplitude obtained by $k$-lattice regularization, when taking the symmetric domain $D^0$, and when making an additional Pauli-Villars subtraction, agrees with the continuum result obtained with Pauli-Villars regularization (apart from the quadratic constant term, which vanishes when imposing the Ward identity).

## IV. LAGRANGIAN COUNTERTERMS

Let us come back to the amplitude obtained from the $k$-lattice regularization, i.e., without Pauli-Villars subtraction, given by Eqs. (22)–(26), (30), (31), and (35). We split it into two parts, the first part giving the standard result of the amplitude and a second part corresponding to those terms which will be made to vanish by introducing suitable counterterms.

$$I_{\mu\nu}^{\text{latt reg}}(k)=I_{\mu\nu}^{S}(k)+I_{\mu\nu}^{R}(k)\ ,$$

$$I_{\mu\nu}^{S}(k)=-\frac{4e^2}{(2\pi)^4}[k_\mu k_\nu-\delta_{\mu\nu}k^2]\left[-\frac{1}{3}I^{(2)}(\Lambda)+\frac{k^2}{m^2}\frac{8}{3}S^{(2)}\right]+I_{\mu\nu}^{[N=0,1,2\text{ subtr}]}(k)\ ,$$

$$I_{\mu\nu}^{R}(k)=-\frac{4e^2}{(2\pi)^4}\left[\delta_{\mu\nu}(-2\Lambda^2S^{(1)}+2m^2S^{(2)})+k_\mu k_\nu\left[2S^{(2)}-\frac{28}{9}S^{(2)}+\frac{16}{9}S^{(3)}\right]+\delta_{\mu\nu}k^2\left[\frac{4}{9}S^{(2)}+\frac{8}{9}S^{(3)}\right]\right.$$
$$\left.+\delta_{\mu\nu}k_\mu k_\mu\left[\frac{8}{9}S^{(2)}-\frac{16}{3}S^{(3)}\right]\right]\ , \tag{45}$$
$$-I_{\mu\nu}^{[D^1,N=0]}(k)-I^{[D^1,N=1]}(k)+I_{\mu\nu}^{[D^2,N=0]}(k)+I_{\mu\nu}^{[\text{one-dim shift }-2\Lambda,D^1,\ N=0]}(k)+I_{\mu\nu}^{[\text{one-dim shift }-2\Lambda,D^1,N=1]}(k)$$
$$-I_{\mu\nu}^{[\text{one-dim shift }-2\Lambda,D^2,\ N=0]}(k)+I_{\mu\nu}^{[\text{two-dim shift }-2\Lambda,D^2,N=0]}(k)+O(\Lambda^{-1})\ ,$$

where $I_{\mu\nu}^{S}$ can be written as [see the Appendix for $I^{(2)}(\Lambda)$]

$$I_{\mu\nu}^{S}(k)=\frac{\alpha}{3\pi}[k_\mu k_\nu-\delta_{\mu\nu}k^2]\left[\ln\left[\frac{\Lambda^2}{m^2}\right]+\frac{C^{(2)}}{\pi^2}-1-2\frac{k^2}{m^2}\right]+I_{\mu\nu}^{[N=0,1,2\text{ subtr}]}(k)\ ,$$
$$=\frac{\alpha}{3\pi}[k_\mu k_\nu-\delta_{\mu\nu}k^2]\left[\ln\left[\frac{\Lambda^2}{m^2}\right]+\frac{C^{(2)}}{\pi^2}-1-6\int_0^1dx\,x(1-x)\ln\left[1+x(1-x)\frac{k^2}{m^2}\right]\right]\ . \tag{46}$$

This is the standard result, the term $\ln(\Lambda^2/m^2)+C^{(2)}/\pi^2-1$ renormalizes multiplicatively the coupling constant. In order to obtain $I_{\mu\nu}^{S}$ as a renormalized lattice amplitude, we have to introduce counterterms, such that $I_{\mu\nu}^{R}$ vanishes. Those Lagrangian counterterms have the following form: (a) Corresponding to the term $\delta_{\mu\nu}\Lambda^2$ in the amplitude, the action counterterm is

$$S_{\text{counter}}^{(a)}=\sum_{k_I}\left[\frac{\Delta k}{2\pi}\right]^4\Lambda^2\tilde{A}_\alpha(-k_I)\tilde{A}_\alpha(k_I)\ . \tag{47}$$

The same type of counterterm is needed to cancel $\delta_{\mu\nu}m^2$. (b) Corresponding to the term $k_\mu k_\nu$ in the amplitude, the action counterterm is

$$S_{\text{counter}}^{(b)}=\sum_{k_I}\left[\frac{\Delta k}{2\pi}\right]^4(-i)(-\tilde{k}_I)_\alpha\tilde{A}_\alpha(-k_I)(-i)\times(\tilde{k}_I)_\beta\tilde{A}_\beta(k_I)\ . \tag{48}$$

(c) Corresponding to $\delta_{\mu\nu}k^2$ the counterterm is

$$S_{\text{counter}}^{(c)}=\sum_{k_I}\left[\frac{\Delta k}{2\pi}\right]^4\tilde{k}_I{}^2\tilde{A}_\alpha(-k_I)\tilde{A}_\alpha(k_I)\ . \tag{49}$$

(d) Corresponding to $\delta_{\mu\nu}k_\mu k_\mu$ the counterterm is

$$S_{\text{counter}}^{(d)}=\sum_{k_I}\left[\frac{\Delta k}{2\pi}\right]^4(-i)(-\tilde{k}_I)_\alpha\tilde{A}_\alpha(-k_I)(-i)\times(\tilde{k}_I)_\alpha\tilde{A}_\alpha(k_I)\ . \tag{50}$$

(e) Corresponding to $\delta_{\mu\nu}|k_\mu|\sum_{\alpha=1}^4|k_\alpha|$, occurring in $I^{[\text{one-dim shift }-2\Lambda,D^2,\ N=0]}$, the counterterm is

$$S_{\text{counter}}^{(e)}=\sum_{k_I}\left[\frac{\Delta k}{2\pi}\right]^4(-i)|\tilde{k}_{I_\alpha}|\tilde{A}_\alpha(-k_I)\times\sum_{\beta=1}^4(-i)|\tilde{k}_{I_\beta}|\tilde{A}_\alpha(k_I)\ . \tag{51}$$

(f) Corresponding to $\delta_{\mu\nu}\sum_{\alpha,\beta=1}^4|k_\alpha||k_\beta|$, occurring in $I^{[\text{one-dim shift }-2\Lambda,D^2,\ N=0]}$, the counterterm is

$$S_{\text{counter}}^{(f)}=\sum_{k_I}\left[\frac{\Delta k}{2\pi}\right]^4\sum_{\alpha,\beta=1}^4(-i)|\tilde{k}_{I_\alpha}|(-i)|\tilde{k}_{I_\beta}|\times\tilde{A}_\gamma(-k_I)\tilde{A}_\gamma(k_I)\ . \tag{52}$$

(g) Corresponding to $\delta_{\mu\nu}\Lambda|k_\mu|$, occurring in $I^{[D^1,N=0]}$, the counterterm is

$$S_{\text{counter}}^{(g)}=\sum_{k_I}\left[\frac{\Delta k}{2\pi}\right]^4\Lambda(-i)|\tilde{k}_{I_\alpha}|\tilde{A}_\alpha(-k_I)\tilde{A}_\alpha(k_I)\ . \tag{53}$$

(h) Corresponding to $\delta_{\mu\nu}\Lambda\sum_{\alpha=1}^4|k_\alpha|$, occurring in $I^{[D^1,N=0]}$ the counterterm is

$$S_{\text{counter}}^{(h)}=\sum_{k_I}\left[\frac{\Delta k}{2\pi}\right]^4\Lambda\sum_{\alpha=1}^4(-i)|\tilde{k}_{I_\alpha}|\times\tilde{A}_\beta(-k_I)\tilde{A}_\beta(k_I)\ . \tag{54}$$

## V. SUMMARY

We have computed the QED vacuum-polarization diagram to one-loop order using a momentum ($k$) lattice as regulator. The result can be expressed as the standard result plus a number of terms, which are nonlocal and violate Lorentz (Euclidean) invariance and gauge invariance. In order to get rid of these unphysical terms, one can renormalize by introducing counterterms in the action, which are also nonlocal and violate Lorentz invariance and gauge invariance. The nonlocality is consistent, because the original Lagrangian already contains a nonlocality in the kinetic term. The counterterms are in terms of momenta of the order 1, $k$, $k^2$. The original Lagrangian has been constructed in terms of periodic functions $\tilde{f}(k_I)$, in order to conserve the group structure of the group of local gauge transformations. If we would have abandoned this constraint, the amplitude would not have the terms coming from shift $\pm 2\Lambda$ in one and two dimensions. As Eqs. (30), (31), and (35) show, the shift terms partially replace terms and have the same structure as terms which occur anyway. The shift terms do neither deteriorate nor improve the result with respect to conservation of Lorentz symmetry and gauge symmetry.

## ACKNOWLEDGMENTS

One of the authors (H.K.) would like to thank G. P. Lepage and J. Smit for helpful discussions. The authors are grateful for support by NSERC of Canada. H.K. and L.M. would like to acknowledge support by FCAR of Québec.

## APPENDIX

In this section we will give some integrals, which occurred in the preceeding sections. We define

$$I^{(n)}(\Lambda)=\int_{-\Lambda}^{+\Lambda} d^4p \frac{1}{[p^2+m^2]^n} ,$$
$$S^{(n)}=\int_{-1}^{+1} d^3s \frac{1}{[s^2+1]^n} , \tag{A1}$$
$$T^{(n)}=\int_{-1}^{+1} d^2t \frac{1}{[t^2+2]^n} .$$

The integrals $T^{(n)}$ and $S^{(n)}$ are finite. One has

$$T^{(1)}=1.5369\ldots ,$$

$T^{(1)}$ can be expressed in terms of the Lobachevskiy function.

$$T^{(2)}=\frac{\pi}{3\sqrt{3}}, \quad S^{(1)}=4.2868\ldots .$$

$S^{(1)}$ is related to $T^{(1)}$ by

$$S^{(1)}=-\frac{\pi^2}{2}+6T^{(1)}, \quad S^{(2)}=\frac{\pi^2}{4} ,$$
$$S^{(3)}=\frac{\pi^2}{16}+\frac{\pi}{\sqrt{12}} . \tag{A2}$$

The integral $I^{(n)}$ is finite for $n \geq 3$, it is logarithmically divergent for $n=2$, and quadratically divergent for $n=1$. The following relations hold

$$I^{(1)}(\Lambda)=4\Lambda^2 S^{(1)}-m^2 I^{(2)}(\Lambda)-4m^2 S^{(2)}+O(\Lambda^{-1}) ,$$
$$I^{(2)}(\Lambda)=2\pi^2 \ln(\Lambda/m)+C^{(2)}-\pi^2+O(\Lambda^{-1}) , \tag{A3}$$

where

$$C^{(2)}=\int_{-1}^{+1} d^4s \frac{1}{(s^2)^2}-\int_{|s|\leq 1} d^4s \frac{1}{(s^2)^2} \tag{A4}$$

is finite.

[1] K. G. Wilson, Phys. Rev. D **10**, 2445 (1974).
[2] L. H. Karsten and J. Smit, Nucl. Phys. **B183**, 103 (1981).
[3] H. B. Nielsen and M. Ninomiya, Nucl. Phys. **B185**, 20 (1981).
[4] S. M. Drell, M. Weinstein, and S. Yankielowicz, Phys. Rev. D **14**, 487 (1976); **14**, 1627 (1976).
[5] S. Adler, Phys. Rev. **177**, 2426 (1969).
[6] J. Bell and R. Jackiw, Nuovo Cimento **A60**, 47 (1969).
[7] M. Ninomiya and C. I. Tan, Phys. Rev. Lett. **53**, 8611 (1984).
[8] L. H. Karsten and J. Smit, Phys. Lett. **85B**, 100 (1979).
[9] C. Rebbi, Phys. Lett. B **186**, 200 (1987).
[10] G. T. Bodwin and E. V. Kovács, Phys. Lett. B **193**, 283 (1987).
[11] G. T. Bodwin and E. V. Kovács, Phys. Rev. D **35**, 3198 (1987).
[12] S. Aoki, Phys. Rev. Lett. **60**, 2109 (1988).
[13] K. Funakubo and T. Kashiwa, Phys. Rev. Lett. **60**, 2113 (1988).
[14] G. T. Bodwin and E. V. Kovács, in *Lattice '88*, Proceedings of the International Symposium, Batavia, Illinois, 1988, edited by A. S. Kronfeld and P. B. Mackenzie [Nucl. Phys. B (Proc. Suppl.) **9**, 589 (1989)].
[15] J. Smit, Acta Phys. Pol. **B17**, 531 (1986); Nucl. Phys. **B175**, 307 (1980); in *Field Theory on the Lattice*, Proceedings of the International Symposium, Seillac, France, 1987, edited by A. Billoire, R. Lacaze, A. Morel, O. Napoly, and J. Zinn-Justin [Nucl. Phys. B (Proc. Suppl.) **4**, 451 (1988)]; in *Lattice '88*, [14], p. 579.
[16] J. Smit, in *Lattice '89*, Proceedings of the International Symposium, Capri, Italy, 1989, edited by R. Petronzio *et al.* [Nucl. Phys. B (Proc. Suppl.) **17**, 3 (1990)].
[17] M. F. L. Goltermann, in *Lattice '90*, Proceedings of the International Symposium, Tallahassee, Florida, 1990, edited by U. M. Heller, A. D. Kennedy, and S. Sanielevici [Nucl. Phys. B (Proc. Suppl.) **20**, 528 (1991)].
[18] G. T. Bodwin and E. V. Kovács, in *Lattice '90* [17], p. 546.
[19] J. Kuti and Y. Shen, Phys. Rev. Lett. **60**, 85 (1988).
[20] J. Kuti, L. Lin, and Y. Shen, Phys. Rev. Lett. **61**, 678 (1988).
[21] D. Bérubé, H. Kröger, R. Lafrance, and S. Lantagne, Phys. Rev. D **44**, 2604 (1991).
[22] M. Lüscher and P. Weisz, Nucl. Phys. **B290** [FS20], 25 (1987).
[23] R. Brockmann and J. Frank, Phys. Rev. Lett. **68**, 1830 (1992).
[24] D. Bérubé and H. Kröger, Phys. Lett. B **238**, 348 (1990).
[25] D. Bérubé, H. Kröger, R. Lafrance, and L. Marleau, Phys.

Rev. D **43**, 1385 (1991).
[26] D. Bérubé, H. Kröger, R. Lafrance, and L. Marleau, Phys. Rev. D **45**, 1332 (1992).
[27] M. Burkardt and A. Langnau, Phys. Rev. D **44**, 3857 (1991).
[28] G. T. Bodwin, E. V. Kovács, and J. Sloan, Phys. Rev. D **44**, 3924 (1991).
[29] H. R. Quinn and M. Weinstein, Phys. Rev. D **34**, 2440 (1986); Phys. Rev. Lett. **57**, 2617 (1986).
[30] J. D. Bjorken and S. D. Drell, *Relativistic Quantum Mechanics* (McGraw-Hill, New York, 1964).