Investigating decoherence in a simple system

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I present the results of some simple calculations designed to study the loss of quantum coherence. The relevant physical issues are briefly reviewed, and then a very simple "toy" model is analyzed. Exact solutions are found using numerical techniques. The type of decoherence exhibited by the model can be changed by varying a coupling strength. I study the system from two points of view. One, the Schmidt paths approach, is closely related to the conventional approach of studying decoherence by checking the form of the density matrix. The consistent histories approach is also used, and the relationship between the two approaches is explored.

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I. INTRODUCTION

The "loss of quantum coherence" plays an important role in our understanding of quantum mechanics. Loss of quantum coherence provides a mechanism whereby effects often attributed to "the collapse of the wave function" can arise in a system whose evolution is entirely unitary. This is accomplished by introducing a sufficiently complex "environment" into the calculation [1–8]. (For an introductory treatment see [9].) Such environments are presented in most realistic physical situations. Indeed, one finds that this mechanism is constantly in operation all around us, and decoherence is in large part responsible for the "state" in which we find many commonplace objects.

For example, one could never hope to observe a macroscopic pendulum in an energy eigenstate of its "harmonic-oscillator" Hamiltonian, even if an initial state could be prepared that way [10]. Local interactions of the pendulum with its internal degrees of freedom, the gas in the room, or even just with the cosmic microwave background [11] would ensure its rapid "collapse" into a much more localized state. This would be a consequence of correlations being set up with these other degrees of freedom which destroy the coherence of the initial state. At any given time the wave function of the world would then describe many localized copies of the pendulum, each at different positions, and each correlated with different environment states. The delocalized property of the initial state would just be reflected in these different positions being broadly distributed.

The physics of decoherence often points to the existence of preferred states, whose coherence is not destroyed by interactions with the environment. In the case of the pendulum these resemble the "coherent states," which are localized, and follow classical pendulum trajectories [12]. Following the pioneering work of Zurek [2–4,6], the preferred states are often referred to as the "pointer basis." The word "pointer" is used because of the importance of the pointer basis in understanding measurement devices (many of which have pointers). However (as Zurek and others have noted), the process of decoherence and the ability of special states to survive the decohering effects of the environment is a widespread phenomenon, and is not limited to manmade laboratory equipment. For many objects (such as the macroscopic pendulum) the pointer basis states are highly localized in both position and momentum. This fact can give rise to a classical treatment of these objects, in which both position and momentum are sharply defined.

The role of quantum decoherence in the early Universe is of particular interest. It may provide valuable insights into questions relating to "initial conditions." One can ask which properties of the Universe relate directly to the "initial state" and which are a consequence of dynamics forcing systems into the preferred pointer basis states. For example, in inflationary cosmology the state of the Universe during inflation is a highly symmetric spatially homogeneous one, even though many modes are excited. The fact that we observe an inhomogeneous matter distribution today is attributed to local interactions destroying the coherence of this "initial" state [7] (see also [13–18]). The pointer basis states in this case are not homogeneous. Halliwell [19] has also emphasized the importance of quantum decoherence when studying emerging classical behavior in quantum cosmology. There has recently been interest in the possible role that certain quantum gravity effects ("wormholes") could play as an environment responsible for decoherence. In [20] Coleman shows that the wormhole interactions serve to define a pointer basis. Unlike general states, states which are members of the pointer basis (or "A eigenstates" in Coleman's language) do not lose coherence very rapidly to wormhole interactions.

In familiar examples it is often easy to guess the nature of the pointer basis. For example, pointer basis states for the center-of-mass coordinates of macroscopic objects tend to be localized in space. This can be attributed to

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the locality of interactions with the environment. In addition, these localized states need to have fairly sharply defined momenta in order to remain localized in the course of time. Of course, there are notable exceptions to this rule. Superconducting Josephson junctions, for example, involve highly delocalized states which maintain their coherence [21–23].

In any case, when venturing into unfamiliar territory such as the early Universe, one might want something more than heuristic arguments to work with. A common approach is to guess a pointer basis, and to check the diagonality of the reduced density matrix (in that basis) produced by tracing out over the environment. Care must be taken, however, since *any* density matrix can be diagonalized, but a pointer basis does not always exist. In many cases the decohering effects of the environment will rapidly destroy the coherence of any state. A related approach involves following the time evolution of the eigenstates of the reduced density matrix (which define "Schmidt paths"). To the extent to which the eigenstates retain characteristic properties (such as locality) throughout time, a pointer basis may be said to exist.

This paper includes a study of a simple system from the "Schmidt paths" point of view. The system has a parameter which can be adjusted to change the nature of the decoherence. Using the Schmidt point of view, I show that in some limits a static pointer basis emerges, while in other cases the pointer basis is dynamically evolving. In many cases the system is just noisy, and *no* state can survive the decohering effects of the environment.

The Schmidt paths provide information about instantaneous probabilities assigned to correlations, and allow one to follow the time development of these probabilities and correlations. The "consistent histories" or "decoherence functional" scheme is related to the sum over histories formulation of quantum mechanics. It is used to determine whether one can assign well-defined probabilities to different *histories* for a given system. When the answer is in the affirmative, the histories are said to be "consistent" or "decohering."

In many cases correlations between a system and its environment (or "records") play a crucial role in causing certain histories to be consistent. I show how the Schmidt paths can be useful in identifying which properties of a subsystem have been recorded. In turn this can help determine which histories are consistent. I show how the presence of a pointer basis allows a particularly simple form of consistent histories to be constructed, but the absence of a pointer basis does *not* prevent the construction of consistent histories.

Section II is a review of the basic ideas of decoherence, and Sec. III provides an introduction to the Schmidt paths point of view. In Sec. IV, the toy model is introduced. In Sec. V and VI, the behavior of the toy model is analyzed from the Schmidt point of view. Section VII starts with brief introduction to the consistent histories approach, and discusses the important role of correlations or "records." The toy model is analyzed from the consistent histories point of view in Sec. VIII. Comparisons with previous work are made in Sec. IX. Conclusions are presented in Sec. X. A number of technical issues are addressed in the appendices. Throughout this paper I use units in which $\hbar=1$.

II. CORRELATIONS AND DECOHERENCE

A. The exactly separable limit

Whenever one studies a system quantum mechanically, the wave function of the world, $|\psi\rangle_w$, is usually implicitly assumed to have the direct product form

$$|\psi\rangle_w=|\psi\rangle_s\otimes|\psi\rangle_r\ . \tag{1}$$

Here $|\psi\rangle_s$ is the wave function of the *system* under study, and $|\psi\rangle_r$ is a state for the *rest* of the degrees of freedom of the world, which do not concern the particular calculation at hand. In fact, one usually further assumes that $|\psi\rangle_s$ itself can be written as a direct product:

$$|\psi\rangle_s=|\psi\rangle_1\otimes|\psi\rangle_2\otimes|\psi\rangle_3\otimes\cdots\otimes|\psi\rangle_n\otimes\cdots\ . \tag{2}$$

The subscripts indicate the numerous subsystems into which one divides the "system under consideration." These subsystems may be an incoming particle, a target, a clock, etc.

The product form for a wave function is far from general, and one might wonder why, instead of Eq. (1), one is not forced to consider the more general case:

$$|\psi\rangle_w=\sum_{i,j}\alpha_{i,j}|i\rangle_s\otimes|j\rangle_r\ , \tag{3}$$

where $\{|i\rangle_s\}$ and $\{|j\rangle_r\}$ are bases which span the Hilbert spaces of the "system" and the "rest of the world," respectively. One could just embrace the product form as one of the initial assumptions, but there is another point of view which is much more physically motivated. This second point of view is closely tied with the notion of "quantum decoherence" and is the subject of this paper. I will start the discussion with some formal remarks, and then bring in the additional complications which make the picture more interesting and physical.

In order to exactly preserve the product form of Eq. (1), the total Hamiltonian must generally be of the form

$$H_w=H_s\otimes I_r+I_s\otimes H_r \tag{4}$$

where I represents the identity operator in the labeled subspace.[1]

However, given a separable Hamiltonian, one could go beyond the simple product form of Eq. (1), and instead have

$$|\psi\rangle_w=\sum_i i\,\alpha_i|\psi_i\rangle_s\otimes|\psi_i\rangle_r\ . \tag{5}$$

Each $|\psi_i\rangle_s$ will evolve independently according to H_s. Furthermore, let us assume

[1]Another possibility is that $|\psi\rangle_s\otimes|\psi\rangle_r$ is an eigenstate of the total Hamiltonian. This possibility will be relevant to later discussion.

$$ {}_s\langle\psi_i|\psi_j\rangle_s = {}_r\langle\psi_i|\psi_j\rangle_r = \delta_{i,j} \ . \tag{6} $$

Note that there is only one summation in Eq. (5), so each state $|\psi_i\rangle_s$ is uniquely correlated with its own member of the orthonormal set $|\psi_i\rangle_r$. The special correlations represented here have interesting implications.

In particular, the reduced density matrix for system s takes the form

$$ \rho_s = \sum_{i,j} \alpha_j^* \alpha_i |\psi_i\rangle_s \, {}_s\langle\psi_j| \, {}_1\langle\psi_j|\psi_i\rangle_1 $$
$$ = \sum_i \alpha_i^* \alpha_i |\psi_i\rangle_s \, {}_s\langle\psi_i| \ . \tag{7} $$

The correlations described by Eqs. (5) and (6) result in a ρ_s which is diagonal and which has eigenstates given by $|\psi_i\rangle_s$'s.

The probability for system s to in a state $|x\rangle_s$ is thus given by

$$ \langle x|\rho_s|x\rangle = \sum_i \alpha_i^* \alpha_i |_s\langle x|\psi_i\rangle_s \ , \tag{8} $$

where interference terms proportional to ${}_s\langle x|\psi_i\rangle_s \, {}_s\langle\psi_j|x\rangle_s$ are completely absent.

The separability of the Hamiltonian allows one to think of system s as being in *independently evolving* states $|\psi_i\rangle_s$, each with probability $\alpha_i^* \alpha_i$. The correlations with system r ensure that interference among these states will never be observed. The possibility for this kind of behavior is a very important feature of quantum mechanics. It forms the basis for the realization of Everett's "many worlds" [24,25], in which different outcomes of a quantum measurement subsequently evolve in an independent, noninterfering manner.

B. Interactions and the origin of the correlations

This discussion has been focused on correlations between the system and the *"rest"* subspaces. So far, I have had nothing to say about the origin of these correlations. Because of the exact separability assumed for the Hamiltonian, the correlations are not a result of dynamics, but are just a property of the initial state which is preserved by the dynamics.

The reason these considerations are interesting, however, is that in the physical works one never has *exact* separability. While there are cases which are separable enough from a practical point of view, there are often ways we can manipulate the situation to expose the additional degrees of freedom (or subspaces), and to exhibit interactions among different $|\psi_i\rangle$'s in a wave function of the form of Eq. (5). (In some interesting cases, including in an example I will present in this paper, the interaction between system and environment will actually dominate over the respective self-Hamiltonians.)

An important example is that of the "quantum measurement." A decaying nucleus can interact with a Geiger counter, setting up correlations with its internal degrees of freedom. Afterwards, the total wave function can be thought of as having two essentially independent terms: one describing the case where the Geiger counter has clicked, and the other where it has not clicked. The correlations will prevent interference between these two terms. In this case, the lack of exact separability allows one to link the creation of the relevant correlations to the interactions between the decay products and the Geiger counter.

There has been a lot of theoretical work linking quantum measurement with the setting up of correlations such as those discussed above. I refer the reader to [1,3,11] for further discussion.

With the abandonment of exact separability, one is able to point to a physical origin for correlations under discussion. However, confusion sometimes arises as to how the correlations are to be discussed. For example, the specially correlated form for the wave function [Eq. (5)] is not generally preserved under time evolution unless the Hamiltonian is exactly separable. In a typical discussion one often sees

$$ |\psi\rangle_w \approx \sum_i \alpha_i |\psi_i\rangle_s \otimes |\psi_i\rangle_r \tag{9} $$

and

$$ {}_r\langle\psi_i|\psi_j\rangle_r \approx {}_s\langle\psi_i|\psi_j\rangle_s \approx \delta_{i,j} \ , \tag{10} $$

which are just Eqs. (5) and (6) with $\approx$ replacing $=$. However, the ambiguity introduced by "$\approx$" can be confusing. One wonders, for example, how close to equality is "good enough" in each of these expressions. This is where the "Schmidt decomposition" can be quite helpful.

III. THE SCHMIDT PATHS

In the approach outlined above, one studies

$$ \rho_s \equiv \mathrm{tr}_r \rho_w \ , \tag{11} $$

where one has traced out the *rest* space in $\rho_w (\equiv |\psi\rangle_{ww}\langle\psi|)$ to produce the reduced density matrix for the *system*. One notes that if $|\psi\rangle_w$ takes on the specially correlated form [Eq. (5)] then

$$ \rho_s = \sum_{ij} |\psi_i\rangle_s \, \alpha_i^* \alpha_j \, {}_s\langle\psi_j| \, {}_r\langle\psi_i|\psi_j\rangle_r \tag{12} $$
$$ = \sum_i |\psi_i\rangle_s \, \alpha_i^* \alpha_i \, {}_s\langle\psi_i| \ , \tag{13} $$

and the density matrix for the system is diagonal.

It is quite common to then reverse the argument, and say that if the density matrix is found to be diagonal in a particular basis, then it is these basis states which are specially correlated with the environment. The "Schmidt decomposition" is derived from the fact that *any* density matrix can be diagonalized.

In fact, it turns out that the "specially correlated" form of the wave function [Eq. (5)] is actually completely general, even when the orthogonality relations [Eq. (6)] are satisfied for both subsystems. This is an old result due to Schmidt [26], and is related to the fact that, in addition to ρ_s, one can construct

$$ \rho_r \equiv \mathrm{tr}_s \rho_w \ . \tag{14} $$

Both ρ_s and ρ_r can be diagonalized, and both density ma-

trices will have identical eigenvalues . (The ρ corresponding to the larger Hilbert space will have additional zero eigenvalues.)

A basis for the whole Hilbert-space can be formed as the direct product space of the eigenstates of the two density matrices. When the state of the world is expanded in this direct product basis one finds

$$|\psi\rangle_w = \sum_i \sqrt{p_i}\,|i\rangle_s^S |i\rangle_r^S , \tag{15}$$

where the $|i\rangle_s^S$ and $|i\rangle_r^S$ denote the eigenstates (or "Schmidt states") corresponding to the nonzero eigenvalues p_i of each density matrix. Phase information can be incorporated into the eigenstates to allow positive real values for the expansion coefficients $\sqrt{p_i}$.

Equation (15) shows that given a $|\psi\rangle_w$, and a particular direct product form for the Hilbert space, the "spatially correlated" from of Eq. (5) may always be exactly obtained using the Schmidt procedure. In general, however, time evolution will not preserve this form, and the Schmidt decomposition must be recalculated at each moment of time. People are often surprised by the Schmidt result. Appendix A gives a brief proof and further discussion designed to give the result more intuitive appeal.

By nature, the Schmidt form of a state gives an exact account of the correlations present between any two subsystems. (Each $|i\rangle_s^S$ is uniquely correlated with its own $|i\rangle_r^S$.) That makes it a natural starting point for a discussion of decoherence. A particular application of interest here is the search for a pointer basis. That means determining if there are any states for the system whose coherence is not continually destroyed by the setting up of correlations with the environment. The Schmidt states will, in general, vary in some crazy way with time, as $|\psi\rangle_w$ evolves, indicating every-changing correlations between the system and the environment. My favorite approach to the search for a pointer basis is to follow the evolution of the Schmidt states (which trace out "Schmidt paths"). When their evolution becomes sufficiently regular, one can say that they represent a pointer basis. This point of view was first proposed by Zeh [1], and has been further discussed in [27–29].

One of the goals of this article is to illustrate these ideas with some simple examples. I turn now to the description of a simple "toy system," with which calculations are easy, yet for which the decohering behavior can be adjusted by varying a parameter. The Schmidt paths point of view will be used to analyze the system. Then the "consistent histories" point of view will be reviewed and applied to the same system.

IV. THE TOY SYSTEM

The system which I study is designed to exhibit decoherence in a primitive form, with as little extra baggage as possible. The world is divided into a two-state subsystem (subsystem number 2) coupled to an "environment" of variable size, n_1 (subsystem number 1). Thus, the Hilbert space of the "world" is $2n_1$ dimensional. The Hamiltonian can be written

$$H = H_1 \otimes I_2 + I_1 \otimes H_2 + H_I , \tag{16}$$

where I_k represents the unit operator in the space of subsystem k. The first two terms represent the self-Hamiltonians of the environment and system, respectively, and the last term gives an interaction between system and environment. I choose the self-Hamiltonian of the two-state system to be

$$H_2 = E_2(|\uparrow\rangle\langle\downarrow| + |\downarrow\rangle\langle\uparrow|) . \tag{17}$$

This causes the spin to rotate, from the point of view of the $\{|\uparrow\rangle, |\downarrow\rangle\}$ basis, with a frequency proportional to E_2. The self-Hamiltonian of system 1 (the "environment") is

$$H_1 = E_1 \times \hat{R} , \tag{18}$$

where $\hat{R}$ is a Hermitian matrix with the real and imaginary parts of each independent matrix element initially chosen randomly in the interval $[-0.5, 0.5)$ (by the computer's random number generator), and held fixed throughout the calculation.

The idea here is to have the environment (subsystem 1) evolve in a way which exhibits no special relationship with the "system" (subsystem 2). The random form of H_1 insures this, without trying to mimic any particular physical environment.

In a similar spirit, the interaction Hamiltonian is

$$H_I = E_I(|\uparrow\rangle\langle\uparrow| \otimes H_1^{\uparrow} + |\downarrow\rangle\langle\downarrow| \otimes H_1^{\downarrow}) . \tag{19}$$

The matrices $H_1^{\uparrow}$ and $H_1^{\downarrow}$ are each different random matrices constructed in the same fashion as $\hat{R}$. The idea of this interaction is to set up different correlations between the system and the environment depending on whether the spin is up or down. If the spin is up, the first term in H_I causes the environment state to be pushed in one direction (in its Hilbert space), while if the spin is down, the second term pushes the environment in another direction. In this sense the interaction can be thought of as providing for a primitive "measurement," with the $\{|\uparrow\rangle, |\downarrow\rangle\}$ basis being the "pointer basis" in which the measurement occurs. However, although the two different random H_1's ensure that the environment is pushed in the two different directions in each of the $|\uparrow\rangle$ versus $|\downarrow\rangle$ cases, there is purposely no attempt to attach any additional interpretation to $H_1^{\uparrow}$ and $H_1^{\downarrow}$. I have made this choice in order to keep the discussion focused on the simplest possible example.

In real physical examples of the loss of quantum coherence, the environment may or may not be as "anonymous" as I have depicted it here. In some cases a real laboratory measurement is being performed, and we would easily identify some subsystem of the environment as a "pointer" or a mark in a lab notebook, which has become correlated with the state of the system under examination. In other cases the role of the environment may be played by a photon scattering off the object in question, and propagating off into "empty" space. In this second situation one would have a fairly clear idea of the state of the photon (at least until it interacted with a dust particle or star). In yet other cases decoherence can be caused by interactions with some sort of thermal bath

(such as Earth's atmosphere). In such a case one only has a very rough idea of the specific states of the environment that become correlated with the system, and the situation begins to resemble the one I have set up here. In each of these cases, an essential ingredient is that the interactions serve to set up correlations between the system and environment in a particular way. It is this feature which I have sought to incorporate here, while keeping specific details to a minimum.

The reader may notice a similarity between this system and others that have appeared in the literature on quantum decoherence. I will save until Sec. IX a discussion of the relationship between this and other work.

V. CALCULATING SCHMIDT PATHS

The calculations I undertake are quite simple. I use a computer program (described in Appendix B) to evolve an initial state unitarily under the action of the Hamiltonian presented in the previous section. The density matrix for system 2 (ρ_2) is easily constructed at any time by tracing over the system 1 subspace (the "environment"). The density matrix is diagonalized, and the Schmidt paths, described by the time history of the eigenvectors, can be followed. This process has been repeated for a variety of initial states, and for different relative strengths of the couplings E_1, E_2, and E_I in the Hamiltonian.

A. Weak coupling

I start the discussion with the following case. The parameters of the Hamiltonian are

$$E_1=1, \quad E_2=1, \quad E_I=0.3 \ . \tag{20}$$

This represents weak coupling, since the self-energies of systems 1 and 2 (E_1 and E_2) are higher than the interaction energy (E_I). The initial state is

$$|\psi(\text{initial})\rangle = |\uparrow\rangle \otimes |\text{random}\rangle_1 \ , \tag{21}$$

where $|\text{random}\rangle_1$ is a state chosen randomly in the environment subspace. The random state is generated by choosing both the real and imaginary parts of its expansion coefficients in the working basis randomly on the interval $[-0.5, 0.5)$ and then normalizing (see Appendix B for further details). Figure 1 displays a variety of information about this example. The dotted curve on the lower plot represents the entropy of the system, in units where the maximum entropy is unity:

$$S \equiv -\text{tr}[\rho_2(\log_2 \rho_2)] \ . \tag{22}$$

Note how the initial entropy is zero. Because the initial state [Eq. (21)] has a simple direct product form, the individual subsystems are in "pure" states, the density matrix has only one nonzero eigenvalue, and the entropy is zero. Naturally enough, the entropy increases with time. The solid line on the lower plot shows the value of the larger eigenvalue of ρ_2. Since there are only two eigenvalues, this curve contains no more information than the entropy (one can be calculated from the other), but it is handy to have both forms of the information available.

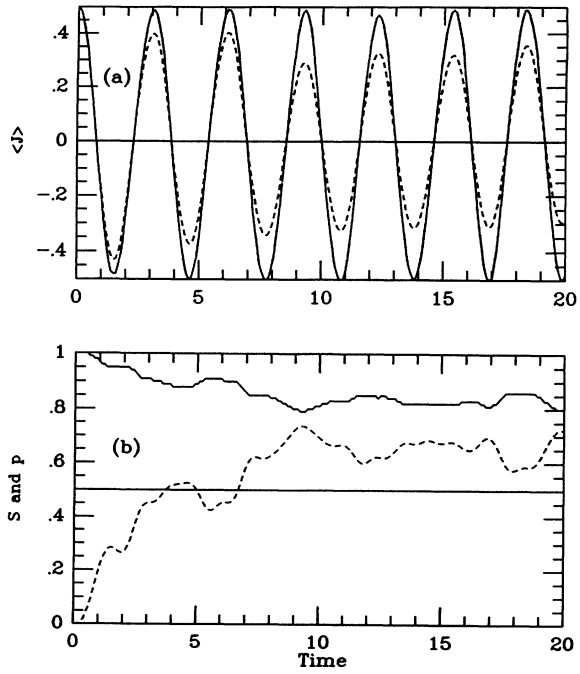

FIG. 1. (a) The solid line is $\langle \hat{J} \rangle$ for an eigenstate of ρ_2. The dashed line is ${}_w\langle \psi | \hat{J} \otimes I_1 | \psi \rangle_w$. (b) The dashed line is the entropy, the solid line is the largest eigenvalue of ρ_2. This figure shows weak coupling ($H_1=H_2=1$, $H_I=0.3$) with a $|\uparrow\rangle_2 \otimes |\text{random}\rangle_1$, initial state.

It is important to remember that the whole "world" remains in a pure state throughout the evolution. It is just relative to the subdivision into a spin system and environment that the density matrix becomes "mixed," and the entropy becomes nonzero.

The dashed line on the top plot in Fig. 1 gives the expectation value of the spin operator $I_1 \otimes \hat{J}$, where

$$\hat{J} \equiv |\uparrow\rangle(+\tfrac{1}{2})\langle\uparrow| + |\downarrow\rangle(-\tfrac{1}{2})\langle\downarrow| \ . \tag{23}$$

This curve starts at $\frac{1}{2}$, as it should, given that the initial state is pure "$|\uparrow\rangle$," and oscillates with the period π, corresponding to the self-Hamiltonian of system 2. However, the amplitude of the oscillation declines with time. This coincides with the increasing entropy, and is indicative of the fact that the spin system is no longer in a pure state. To the extent that the spin is not definitely in a particular state, the quantity

$${}_w\langle \psi | I_1 \otimes \hat{J} | \psi \rangle_w \tag{24}$$

will not clearly exhibit the evolution of a pure spin state.

The solid curve in the top box of Fig. 1 represents the expectation value of the spin in one of the eigenstates of ρ_2 (the one corresponding to the larger of the two eigenvalues). This quantity allows one to follow one of the two "Schmidt paths" associated with the two-state system. This curve also exhibits the oscillatory behavior induced by the self-Hamiltonian. An important feature of the solid curve is that the amplitude remains steady. There is

no sign that the clean oscillatory behavior is degrading with time, as it is with the dotted curve. This fact allows one to regard the oscillating states as a pointer basis. The wave function describes two possible paths for the spin, each oscillating, but with different phases. The decohering effect of the environment does not dramatically effect the evolution of the oscillating states.

B. What does it mean?

One way to ascribe physical meaning to these results is to speak about how other systems interact with the spin system. If one were to adhere strictly to the spirit of this work, one would need to enlarge the Hilbert space so as to include the additional systems. I will not do that here, but instead describe in words what might be expected.

The results in Fig. 1 can be taken to mean the following: If one were to interact at some time with the spin system using an apparatus which couples only to the spin, and not to the environment, then one may treat the spin as being "in two different states" (the Schmidt states), each of which leads to its own independent outcome of the interaction. (Such an interaction could be described along the lines of Sec. II A, where the state "$|x\rangle_2$" would be an eigenstate of the operator representing the measurement apparatus.) The type of states one could expect to find would be oscillating and 180° out of phase. Here "oscillating" means that the states one would actually find would depend on when in the phase of the oscillation one chose to interact. (This discussion assumes the interaction with the third system occurs rapidly compared with the time it takes the eigenvalues of ρ_2 to change.)

The (oscillating) "state" of the spin is determined by the dynamics. The only property which is determined by the state of the "world" is the relative probability of finding the spin with one phase or another. This way in which correlations, rather than just the initial conditions, determine the "state" one finds a system in is of particular interest in the field of cosmology. In cosmology one is driven to contemplate the "initial conditions of the Universe." It is very important, however, that we clarify what properties of our Universe are a reflection of the initial conditions, and which properties are the result of correlations set up by the dynamics, and the nature of the particular measurements we are able to make. (Of course, such effects may also be related in some less direct way to the initial conditions.)

C. Strong coupling

I now vary the problem by increasing the strength of the interaction, relative to the self-Hamiltonians. Figure 2 describes a situation identical to that in Fig. 1, except that now

$$E_1=1,\quad E_2=1,\quad E_I=1\ . \tag{25}$$

Although there are some traces of the periodicity present in the previous case, the clean effect is completely gone. There is no clear pointer basis. Whatever states are specially correlated with the environment at one time, the

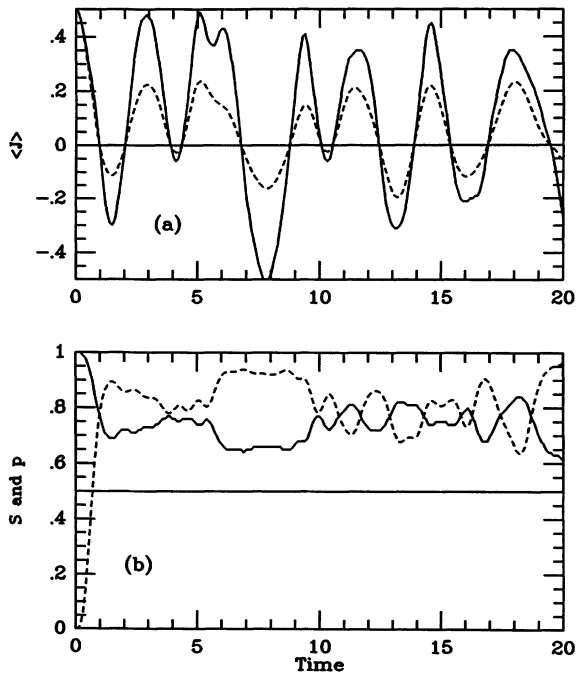

FIG. 2. Medium coupling ($H_1=H_2=1$, $H_I=1$) with a $|\uparrow\rangle_2\otimes|\text{random}\rangle_1$ initial state. Plots as in Fig. 1.

interactions always decohere them. One is left with no chance to view the evolution of the spin system in a simple way. Note that the entropy increases much more rapidly than in Fig. 1, in keeping with the disordered nature of the more strongly coupled case.

For Fig. 3, the interaction strength has been increased

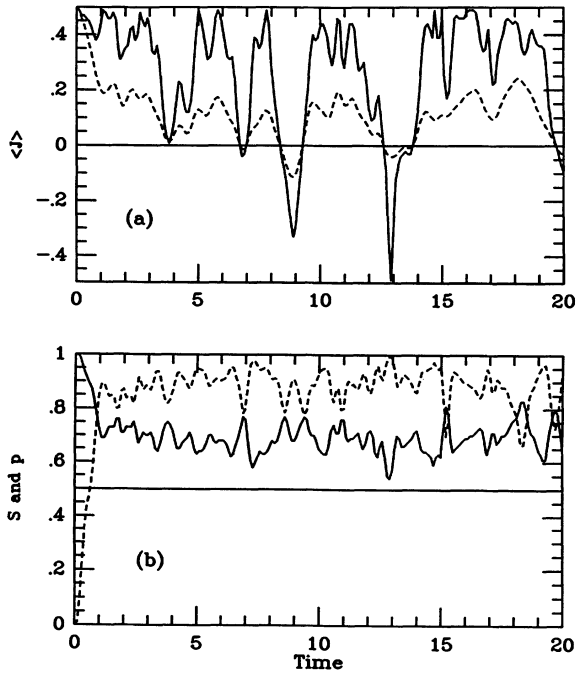

FIG. 3. Stronger coupling ($H_1=H_2=1$, $H_I=3$), with a $|\uparrow\rangle_2\otimes|\text{random}\rangle_1$ initial state. Plots as in Fig. 1.

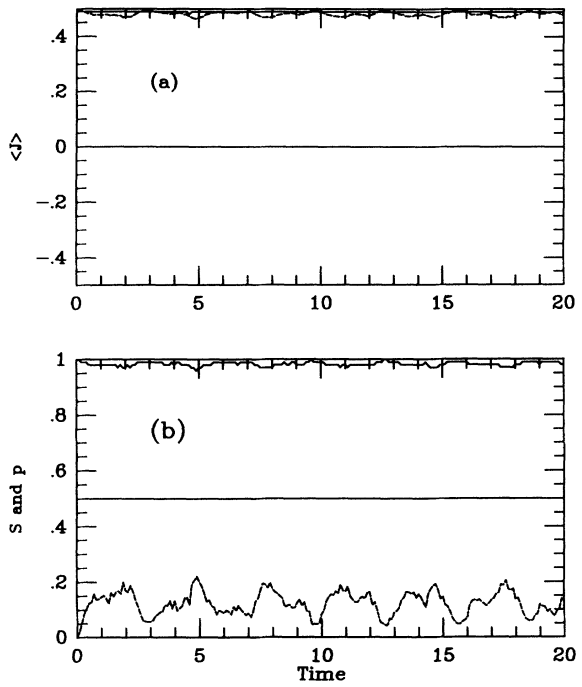

FIG. 4. Strong coupling ($H_1=H_2=1$, $H_I=50$) with a $|\uparrow\rangle_2\otimes|\text{random}\rangle_1$ initial state. Plots as in Fig. 1.

to $E_I=3$. Again, there is no simple behavior on the part of the Schmidt paths, but there is a new feature: The plotted Schmidt path is now more likely to be spin up than spin down.

This feature becomes more pronounced as the interaction strength is increased further. Figure 4 shows the case where $E_I=50$. In this case the two Schmidt paths are very closely pinned to spin up (shown) and spin down (orthogonal to the one shown). The paths are now constant in time, except for small fluctuations. One can again say that there is a pointer basis, but it is a different one, as compared with the weakly coupled oscillating case. The Schmidt paths for the spin are now roughly constant, one $\uparrow$ and the other $\downarrow$.

D. The watchdog effect

It is tempting to view the strongly coupled case as a nice illustration of the "watchdog effect" whereby a frequent measurement of a system in a particular state can prevent it from evolving [30–35]. I argued in Sec. IV that the environment (system 1) plays the role of a primitive measurement apparatus. Increasing the interaction strength decreases the time it takes for correlations to be set up with the environment, and thus increases the frequency with which the "measurements" take place. The spin is measured in the $\{|\uparrow\rangle,|\downarrow\rangle\}$ basis. When the frequency of measurement is increased (by increasing E_I) the spin is prevented from evolving out of these states, as exhibited by the constant Schmidt paths.[2]

[2]The idea that the environment can often provide the "measurements" necessary for the watchdog effect has been discussed in the literature [4,11].

Including the measurement apparatus in the wave function can help demystify the watchdog effect. Something which is accomplished in other treatments by the insertion of projection operators can be achieved by explicitly increasing a coupling strength in this time. However, it is not clear that the system described here is a good one for the purpose. For example, it is not at all clear that the case with medium coupling really describes a good measurement which merely happens to be occurring more slowly than in the strongly coupled case. Figures 2 and 3 suggest a much more confused state of affairs. Further discussion of the degree to which this system is able to describe a good quantum measurement will appear in Sec. IV C.

VI. THE HIGH ENTROPY CASE

So far I have discussed only direct product initial states, which have zero entropy. In the situations where a nice pointer basis appeared the entropy never really got very large and, correspondingly, there was always one eigenvalue of the density matrix which was clearly the largest. I now shift the discussion to higher entropy states.

A. Strong coupling

In the strongly coupled case (Fig. 4), the initial state was pure spin up, and the evolution was one in which the $|\uparrow\rangle$ states remained stable. It is not surprising that the world does not deviated too far from its low entropy initial state. The evolution looks the same even at much later times and the entropy simply does not increase.

However, one can simply start with a high entropy initial state and see how the Schmidt paths behave. One might expect that if the coupling were strong, the same simple evolution would be obtained. Figure 5 shows the

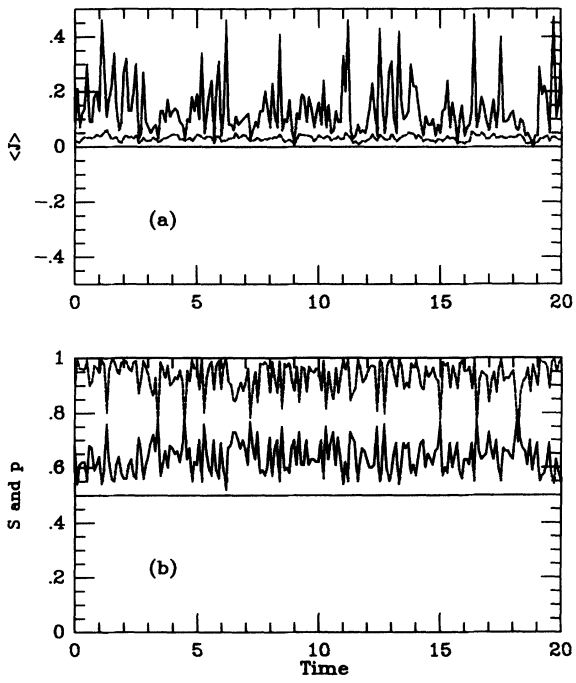

FIG. 5. Strong coupling ($H_1=H_2=1$, $H_I=50$) with a random initial state. Plots as in Fig. 1.

results from a calculation that started with "random initial conditions." Specifically, I chose the real and imaginary parts of each expansion coefficient (in the working basis) randomly on the interval $[-0.5, 0.5)$, and then normalized. Note how the entropy is initially close to maximal, as would be expected. The Schmidt paths clearly do not reproduce the simple constant spin (up or down) evolution of the low entropy case. One does see some long-time features (on time scales ~ 10) which might suggest a tendency toward constant behavior, but that is far from clear.

Perhaps the high entropy case needs a larger coupling to achieve the simple constant spin pointer basis evolution. Figure 6 shows the results of increasing the relative strength of the interaction even more ($E_I/E_2 = E_I/E_1 = 10^7$), and starting with the same initial state. The behavior of the Schmidt paths in this case is not qualitatively different.

What we are seeing here might be related to the definition of the Schmidt paths. Schmidt paths are based on the eigenvectors of the density matrix ρ_2. The high entropy case corresponds to the eigenvalues of ρ_2 being nearly degenerate (note how the larger of the two eigenvalues, given by the solid line in the lower plot, is close to $\frac{1}{2}$). When the eigenvalues of a matrix are exactly degenerate, the eigenvectors are not uniquely specified. It seems reasonable that when the eigenvalues are close to degenerate, the eigenvectors depend in a delicate way on small fluctuations in the elements of ρ_2. This dependence on small fluctuations appears to be washing out the nice pointer basis behavior exhibited by the low entropy case.

While this explanation seems reasonable, the effect should not be universal. For example, one should be able to construct an experiment where the probability that a Geiger counter has ticked is as close to $\frac{1}{2}$ as one might want. There should be no confusion about the existence of two distinct outcomes in that situation. Certainly there has been no observed degradation of Geiger counter performance in that limit. (Since this work was completed, I have made progress in modifying H_I and the initial state to allow simple Schmidt behavior in strongly coupled cases with fairly high entropy [36]. The problem seems to be one of degrees. The higher entropy one wants to cope with, the larger one must make the size of the environment and/or the interaction strength.)

B. Weak coupling

I now turn to the case of high entropy, weakly coupled systems. Figure 7 shows the results of using the same high entropy initial state as above but now with

$$E_1 = 1, \quad E_{2=1}, \quad E_I = 0.3 \; . \tag{26}$$

The story appears to be similar to the strongly coupled case. Although there are vestiges of the oscillating behavior observed in the low entropy states (compare with Fig. 1), the evolution is not as clean, and it is clear that the same type of pointer basis behavior is not present.

Figure 8 shows the same arrangement as in Fig. 1 (weak coupling, with low initial entropy), viewed much later in its evolution. As with the strongly coupled case, the early behavior of the Schmidt paths persists at later times. (Actually there does appear to be some modulation of the amplitude here, but the effect is limited, and it may be attributable to the small size of the environment.) What is interesting is that in this case the entropy is very high, at least as high as the entropy in the case of random

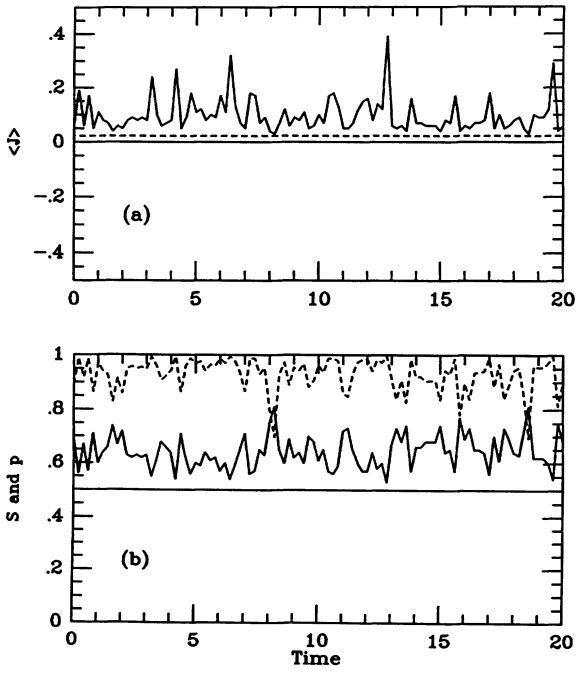

FIG. 6. Ultrastrong coupling ($H_1 = H_2 = 0.001$, $H_I = 10\,000$) with a random initial state. Plots as in Fig. 1.

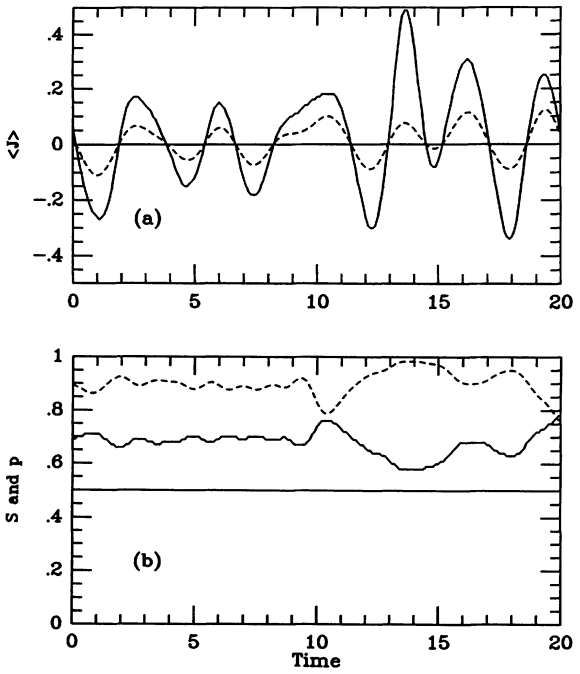

FIG. 7. Weak coupling ($H_1 = H_2 = 1$, $H_I = 0.3$) with a random initial state. Plots as in Fig. 1.

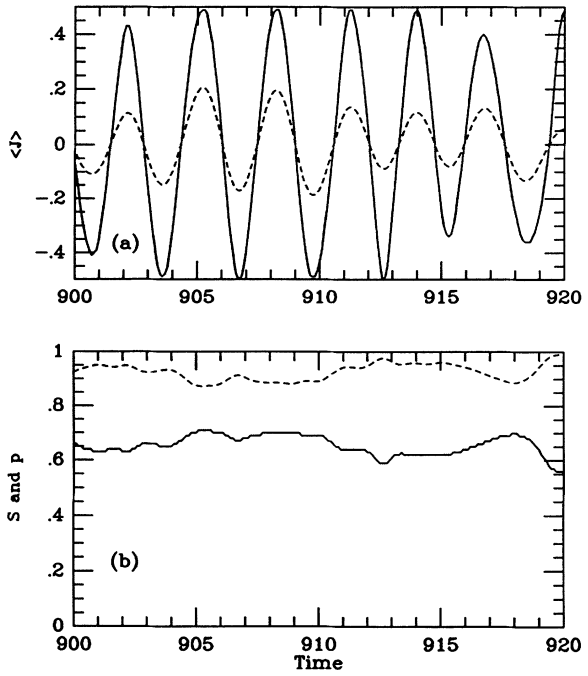

FIG. 8. Weak coupling ($H_1=H_2=1$, $H_I=0.3$) with a $|\uparrow\rangle_2\otimes|\text{random}\rangle_1$ initial state, viewed at late times.

initial conditions (Fig. 7). This shows that for weak coupling the system *is* able to exhibit a simple pointer basis in a situation where density matrix eigenvalues are nearly degenerate. Thus, the evolution of the Schmidt paths need not be dominated by the fluctuations in this limit.

It is intriguing that Figs. 7 and 8 both show (pure) states with high entropy from the point of view of the spin-environment subdivision. Although the entropy is similar, the states have very different behavior. One was chosen randomly, and the other was reached by evolving a particular low entropy initial state for a long time. A more thorough investigation of the differences between these two states is currently underway.

C. Quantum measurement

It would be nice to use these calculations to illustrate the "collapse" of the wave function associated with a quantum measurement. In principle, one would do this by setting up an initial state of the form

$$|\psi(\text{initial})\rangle_w=\left[\frac{|\uparrow\rangle+|\downarrow\rangle}{\sqrt{2}}\right]\otimes|\text{something}\rangle_1 \tag{27}$$

and tracing the evolution of the Schmidt paths. Initially one Schmidt state would be $(|\uparrow\rangle+|\downarrow\rangle)/\sqrt{2}$. In the strongly coupled case, as time progressed, one could simply watch the evolution of this Schmidt path and watch the state "collapse" into a $|\uparrow\rangle$ or $|\downarrow\rangle$ state. At the same time, of course, the other Schmidt path would have its probability increasing from zero, and would become peaked around the other alternative.

The catch is, at least when the initial states of the environment are the random ones I chose here, that the final state is identical to the state of the high entropy, strong-coupling case already discussed. (One can see that the two paths should end up with roughly equal probabilities, and that means high entropy.) Thus, the outcome is not two nice $|\uparrow\rangle$ and $|\downarrow\rangle$ Schmidt paths. This might not be too surprising since a random initial state for the environment makes it a very noisy measurement apparatus. I am presently investigating the possibility of properly "preparing" the environment in a suitable initial state so as to make it work like a good measurement apparatus in this example. It is possible, however, that my toy system is too simplistic to play that role. (Since this work was completed I have successfully accomplished the goals outlined in this subsection using a suitable initial state and a somewhat larger environment [36].)

VII. CONSISTENT HISTORIES

A. Review of the consistent histories approach

It is tempting to view the Schmidt paths as a set of different paths or histories followed by the spin. Starting with the work of Griffiths [37], a number of authors [38–44] have developed specific ideas as to when different histories of a quantum system may be regarded independently. This "consistent histories" or "decoherence functional" point of view is rooted in the sums over histories formulation of quantum mechanics.

By way of illustration, consider an initial state $|\psi_0\rangle$ evolved to a time t (the Schrödinger picture will be used here). One can insert sets of projections at will throughout the time evolution, resulting in

$$|\psi_0(t)\rangle=e^{-iH(t)(t-t_n)}\left\{\sum_k\hat{P}_k\right\}e^{-iH(t_n-t_{n-1})}\left\{\sum_l\hat{P}_l\right\}\cdots \times\left\{\sum_m\hat{P}_m\right\}e^{-iH_w(t_2-t_1)}\left\{\sum_j\hat{P}_j\right\}e^{-iH_w(t_1-t_0)}|\psi_0\rangle\ . \tag{28}$$

Here I am assuming

$$\sum_m\hat{P}_m=I,\quad \hat{P}_m\hat{P}_n=\delta_{mn}\hat{P}_m \tag{29}$$

for each set of projections inserted (the sets corresponding to different times need not be the same). Equation (28) is thus just an identity, regardless of when and how often the sets of projections are inserted. For the system discussed in this work, an interesting set of projections is

$$\hat{P}_\uparrow\equiv|\uparrow\rangle\langle\uparrow|\otimes I_1,\quad \hat{P}_\downarrow\equiv|\downarrow\rangle\langle\uparrow|\otimes I_1\ , \tag{30}$$

which clearly obey $\hat{P}_\uparrow+\hat{P}_\downarrow=I$.

It is useful to define a "path projected state" corresponding to a particular choice of projection at each of the chosen times. For example, the path projected state which results from uniformly choosing the $\uparrow$ projection is

$$|[\uparrow\cdots\uparrow\uparrow]\rangle\equiv\hat{P}_\uparrow e^{-iH_w(t_n-t_{n-1})}\cdots \times\hat{P}_\uparrow e^{-iH_w(t_2-t_1)}\hat{P}_\uparrow e^{-iH_w(t_2-t_1)}|\psi\rangle\ , \tag{31}$$

If the $\downarrow$ projection is chosen at t_2 one has instead

$$|[\uparrow\cdots\uparrow\downarrow\uparrow]\rangle \equiv \hat{P}_{\uparrow} e^{-iH_w(t_n - t_{n-1})} \cdots \hat{P}_{\uparrow} e^{-iH_w(t_3 - t_2)} \hat{P}_{\downarrow} e^{-iH_w(t_2 - t_1)} \hat{P}_{\uparrow} e^{-iH_w(t_1 - t_0)} |\psi\rangle \ . \tag{32}$$

By collecting all the $\sum$'s at the front of the right side of Eq. (28), one can write $|\psi_0(t)\rangle$ as the sum of many path projected states. The requirements given by Eq. (29) ensure that paths are "exhaustive" and "mutually exclusive." When constructing the path-integral formulation of quantum mechanics a similar procedure is followed, and the time between projection is taken arbitrarily small. For present purposes the time between projections can remain finite, representing a coarse graining in time. Different choices of the t_i's, when projections are inserted, correspond to different temporal coarse grainings.

One can attempt to use the squared magnitude of the path projected states as the "probability" assigned to the corresponding path. However, in general, this quantity will not behave like a probability. For example, consider the case where just two projections are performed. A probability should obey the sum rule

$$\langle[\uparrow\cdot]|[\uparrow\cdot]\rangle = \langle[\uparrow\uparrow]|[\uparrow\uparrow]\rangle + \langle[\uparrow\downarrow]|[\uparrow\downarrow]\rangle \ , \tag{33}$$

where the centered dot on the left side represents the absence of any projection at the first time. This is just saying that the probability of the path which ignores the state at the first time should be equal to the sum of the probabilities of the paths for which all possible projections are made at the first time (in this example there are only two). Using the identity

$$|[\uparrow\cdot]\rangle = |[\uparrow\uparrow]\rangle + |[\uparrow\downarrow]\rangle \ , \tag{34}$$

one can see that the general case is

$$\begin{aligned}\langle[\uparrow\cdot]|[\uparrow\cdot]\rangle &= \langle[\uparrow\uparrow]|[\uparrow\uparrow]\rangle + \langle[\uparrow\downarrow]|[\uparrow\downarrow]\rangle \\ &\quad + \langle[\uparrow\uparrow]|[\uparrow\downarrow]\rangle + \langle[\uparrow\downarrow]|[\uparrow\uparrow]\rangle \\ &= \langle[\uparrow\uparrow]|[\uparrow\uparrow]\rangle + 2\,\mathrm{Re}(\langle[\uparrow\downarrow]|[\downarrow\downarrow]\rangle) \\ &\quad + \langle[\uparrow\downarrow]|[\uparrow\downarrow]\rangle \ .\end{aligned} \tag{35}$$

Only if the middle term happens to be negligible will the sum rule [Eq. (33)] be obeyed. If, for a given set of projections and projection times, all the appropriate sum rules are obeyed, then the corresponding set of paths or histories are said to be "consistent" or "decohering." Since their probabilities add in the normal way, consistent histories may be regarded as "independent" or "noninterfering." On the other hand, if the sum rules are not obeyed, it is not reasonable to regard the amplitude squared of the path projected states as a probability.

Gell-Mann and Hartle [41] call the inner products of path projected states the elements of the "decoherence functional." If the two paths are different, the inner product gives an "off-diagonal" element of the decoherence functional. The inner product of a state with itself gives an "on-diagonal" decoherence functional element. As illustrated by Eq. (33), the probability sum rules are relations among the on-diagonal elements of the decoherence functional. For each probability sum rule there is a corresponding expression which involves the addition of off-diagonal decoherence functional elements and which is always correct [an example of this is Eq. (35)]. For the sum rules to be valid, the off-diagonal decoherence functional elements should be small. [In fact, the net effect of *all* the off-diagonal terms in the generalization of Eq. (35) must be small. If many projections are made, individual off-diagonal elements will typically be much smaller than unity, but the generalization of Eq. (35) will contain *many* off-diagonal elements, so the net effect can still be large.]

A particular choice of projection times and sets of projectors to use at each time [each set obeying Eq. (29)] completely specifies a (possibly coarse-grained) set of histories. One says that the *set* of histories is consistent if all the relevant sum rules are obeyed.

B. Further discussion

The result of acting with $\hat{P}_{\uparrow}$ on a state $|\psi\rangle$ is certain to be of the form

$$\hat{P}_{\uparrow}|\psi\rangle = |\uparrow\rangle \otimes |\phi\rangle_1 \equiv \alpha_{\phi} |\uparrow\rangle \otimes |\tilde{\phi}\rangle_1 \tag{36}$$

for some state of the environment $|\phi\rangle_1$. The second equality uses $\alpha_{\phi} \equiv ({}_1\langle\phi|\phi\rangle_1)^{1/2}$ to extract the normalization. For all paths which end in $\uparrow$, the path projected states look like Eq. (36).

Now consider the case [as in Eq. (34)] where there are only two projection times. This results in four path projected states. Two of these will have the form given by Eq. (36). For the sum rules to be valid, the inner product of these two path projected states, which takes the form

$$\alpha_1^* \alpha_2 \langle\phi_1|\phi_2\rangle \ , \tag{37}$$

must be small. There are two things that can make this expression small: the norm (α) of one of the states can be small, or the overlap between the environment states (the $|\phi\rangle$'s) can be small. Should the spin subsystem undergo purely unitary evolution, the first case may always be achieved for suitably chosen paths. One simply defines one of the earlier projections as the final projection ($\hat{P}_{\uparrow}$) evolved back to t_1 with the (unitary) time evolution operator. Operating by the orthogonal projection at t_1 and then by $\hat{P}_{\uparrow}$ at t_2 will give a path projected state with zero amplitude. The inner product between this path projected state and any other will give zero, and the sum rule will be obeyed. In fact, this procedure works for *any* final projection, so infinitely many consistent histories can be constructed for a subsystem which evolves unitarily.

Since the spin in the toy system does not generally evolve unitarily, paths which are consistent because the $|\phi\rangle$'s have little overlap are of particular interest. If $\langle\phi_1|\phi_2\rangle$ is small, one can say that the environment contains a "record" of the path, since the state of the envi-

ronment is different depending on which path was followed. This clearly has to do with correlations between the spin and environment being established at one time and being preserved, in some sense, as time goes on. Gell-Mann and Hartle use the term "medium decoherence" when it is correlations with the environment or "records" which suppress the inner products between path projected states and allow the sum rules to be obeyed [45].

I should remark that by requiring the off-diagonal decoherence functional elements be small, I am exceeding the less restrictive requirement that only the real parts need be small, which is all that one needs to obey the sum rules. Gell-Mann and Hartle also do this [41], and they remark that there are no known physical examples where the real parts are small but the imaginary parts are not.

VIII. EVALUATING HISTORIES FOR THE TOY SYSTEM

I consider the following generalization of Eq. (35):

$$\langle[1\cdot]|[1\cdot]\rangle=\langle[11]|[11]\rangle+2\,\mathrm{Re}(\langle[11]|[12]\rangle)+\langle[12]|[12]\rangle\ , \tag{38}$$

$$\langle[2\cdot]|[2\cdot]\rangle=\langle[21]|[21]\rangle+2\,\mathrm{Re}(\langle[21]|[22]\rangle)+\langle[22]|[22]\rangle\ , \tag{39}$$

where 1 and 2 refer to projections which are generalizations of Eq. (30):

$$P_i\equiv|i\rangle_{22}\langle i|\otimes I_1\ . \tag{40}$$

The basis $\{|1\rangle,|2\rangle\}$ which defines the projections will be called the "projection basis" (PB).

A. Results

Figure 9 represents the various terms of Eqs. (38) and (39). The heavy curve gives the left-hand side and the dashed curve gives the sum of the first and last terms (the on-diagonal decoherence functional elements) from the right-hand side. To the extent that the sum rule is obeyed, the dashed curve and the heavy curve should coincide. The first term from the right-hand side is given by the light solid curve. When the light solid curve is close to the heavy curve or close to zero, the probability of the coarse-grained path comes predominantly from one or the other of the fine-grained paths. The top and bottom panels represent Eqs. (38) and (39), respectively. The first projection occurs at $t=10$. The x axis gives the time elapsed until the second projection is made.

The initial conditions and coupling strengths used for Fig. 9 are identical to those of Fig. 1 (weak coupling, spin initially up). The projection basis is $\{|\uparrow\rangle,|\downarrow\rangle\}$. The fact that the dashed and heavy curves are quite different indicates that the sum rules are not obeyed. Figure 10 represents the same system as Fig. 9 but with the projection basis being the Schmidt states calculated when no projections were done. (Note that the projection basis depends on time now, as shown in Fig. 1). The heavy and

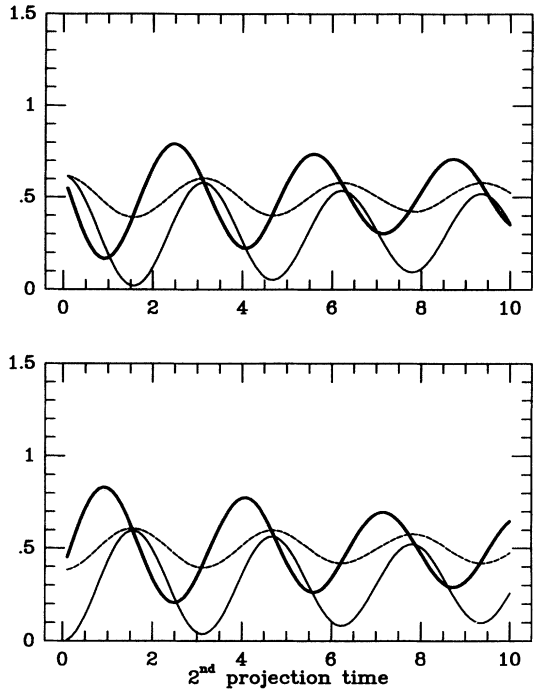

FIG. 9. Figures 9–14 show values of the decoherence functional elements in Eq. (38) (upper panel) and Eq. (39) (lower panel). The heavy curve gives the left-hand side and the dashed curve gives the sum of the two on-diagonal terms of the right-hand side. The heavy and dashed curves coincide when the sum rules are obeyed. The light solid curve gives the first term on the right-hand side. For this figure the projection basis states are $\{|\uparrow\rangle,|\downarrow\rangle\}$ and the coupling is weak (as in Fig. 1).

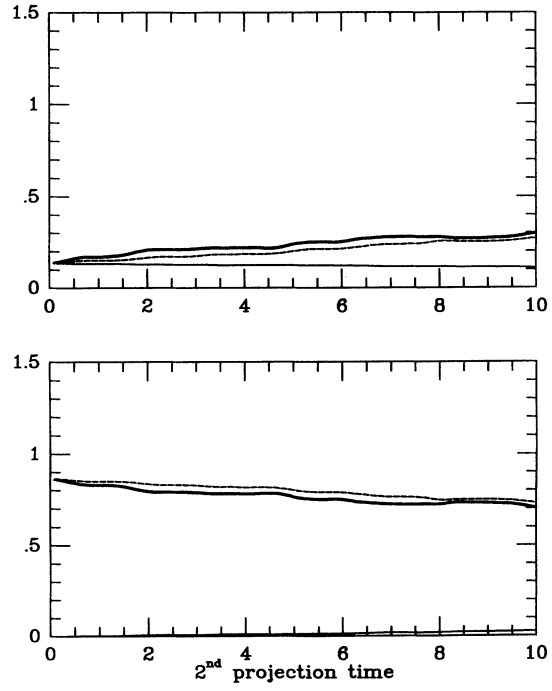

FIG. 10. Checking the sum rules (as in Fig. 9). The projection basis states are the eigenstates of the unprojected ρ_2 and the coupling is weak (as in Fig. 1).

dashed curves are much closer, indicating greater validity of the sum rules.

This result is not surprising. In the first case the projections were made with no particular regard to what records had been made in the environment. The Schmidt states are precisely the states which have been "recorded" by the environment. In the second case the Schmidt states were used as the PB, and much better validity of the sum rules was achieved.

I should note that for the sum rules to be valid, records must not only be made, but also *preserved.* This tends to work out naturally for this system, since once one is considering two orthogonal environment states, the random environment evolution is pretty good about keeping them orthogonal, especially when the size of the environment subspace is large. (But see Sec. VIII B for further comments on this point.)

Figures 11 and 12 correspond to Figs. 9 and 10, but with the coupling set to the "medium" value ($E_I=1$, as depicted in Fig. 2). Again, when projections were made on the Schmidt states the sum rules are more closely obeyed, due to records being present in the environment.

The strongly coupled case is a little different. For one, the $\{|\uparrow\rangle,|\downarrow\rangle\}$ basis *is* the Schmidt basis, and the initial $|\uparrow\rangle$ state is so stable that there is very little loss of quantum coherence. Still, the (very nearly) direct product form of the state describes important correlations between the spin and environment and one can choose a "bad" projection basis by ignoring these correlations. Figure 13 depicts the strongly coupled case ($E_I=50$) where the first PB was $|\pm\rangle[\equiv(|\uparrow\rangle\pm|\downarrow\rangle)/\sqrt{2})]$ and the second was $\{|\uparrow\rangle,|\downarrow\rangle\}$. Here no regard is paid to the

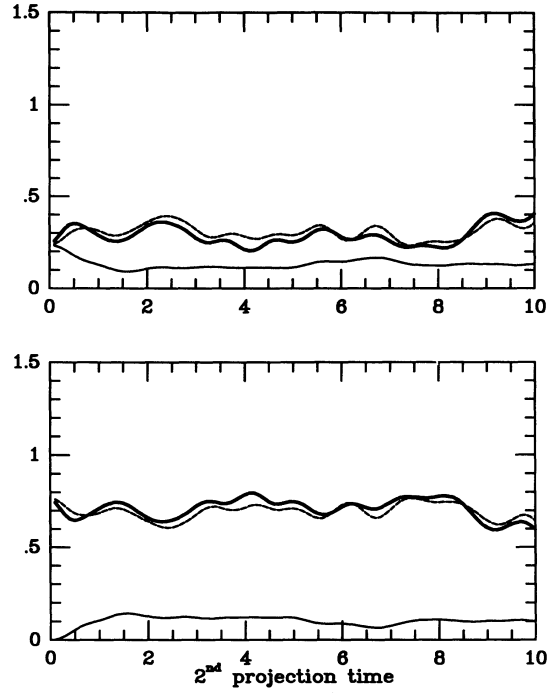

FIG. 12. Checking the sum rules (as in Fig. 9). The projection basis states are the eigenstates of the unprojected ρ_2 and the coupling is medium (as in Fig. 2).

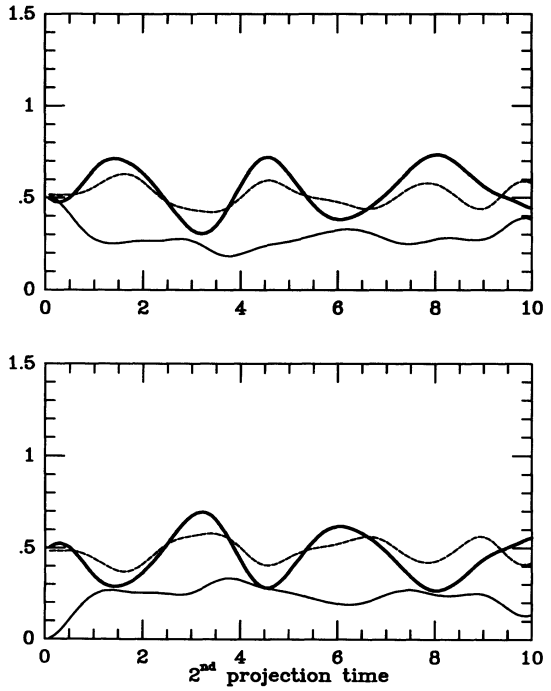

FIG. 11. Checking the sum rules (as in Fig. 9). The projection basis states are $\{|\uparrow\rangle,|\downarrow\rangle\}$ and the coupling is medium (as in Fig. 2).

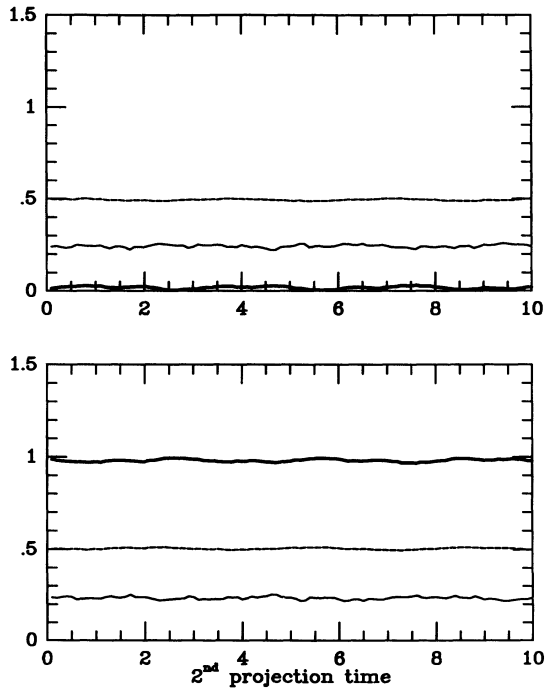

FIG. 13. Checking the sum rules (as in Fig. 9). The projection basis states are $|\pm\rangle$ at t_1 and $\{|\uparrow\rangle,|\downarrow\rangle\}$ at t_2. The coupling is strong (as in Fig. 4).

correlations and the price one pays is the lack of validity of the sum rules. Figure 14 shows the strongly coupled case with PB=$\{|\uparrow\rangle,|\downarrow\rangle\}$ throughout. As expected the sum rules are obeyed quite well.

In Sec. VII B I made the point that valid sum rules could arise either because the unitary subsystem evolution was being followed, or because suitable records were being made in the environment. These two "different" circumstances are actually unified by the use of the Schmidt basis as the PB. Should the spin evolution be unitary, the spin will continue to be in a pure state, and its density matrix will continue to have but one nonzero eigenvalue. The eigenstate (or "Schmidt state") corresponding to that eigenvalue will be *the* state of the spin. Thus, the Schmidt basis naturally follows unitary evolution, when it is present. However, as noted in Sec. VII B, when the evolution is truly unitary a multitude of sets of consistent histories can be constructed. Only one of these sets is chosen by projecting on the Schmidt basis. It is the one which maximizes the extent to which the probability is peaked, rather than spread out among many paths.

If the evolution is only slightly nonunitary, one eigenvalue of the density matrix will remain close to unity. It is easy to show that the choice of Schmidt of the PB always maximizes the probability of the highest probability path. Thus, there is a sense in which using Schmidt to select the "best recorded" paths also selects the "most unitary" paths.

B. Adding more projection times

We have seen how the presence of good records in the environment have allowed consistent histories to be found for systems with radically different behaviors. It might seem particularly surprising that any sense could be made of the medium coupled case, whose evolution appears noisy and random. One might ask is there no advantage to be had, from the consistent histories point of view, when a nice pointer basis is present?

The answer to this question becomes apparent when one contemplates adding additional projection times. It is a simple matter to evolve each path projected state, and consider its Schmidt decomposition. That will tell you what additional records have been made in the environment since the last projection. However, in general, the Schmidt basis for each path will vary greatly from path to path.

Figure 15 illustrates this point. What is plotted is the norm square of the overlap of Schmidt states with $|\uparrow\rangle$. Each panel has two curves, representing the two path projected states which one has following the first projection time ($t_1=10$). The x axis is the time elapsed since t_1. In each case the Schmidt state corresponding to the largest probability (or largest eigenvalue of ρ_2) is chosen. The top panel corresponds to weak coupling, and the lower panel corresponds to medium coupling.

Figure 15 shows that for the weakly coupled case the Schmidt basis is the same for the two path projected states. (The "most probable" Schmidt states are actually

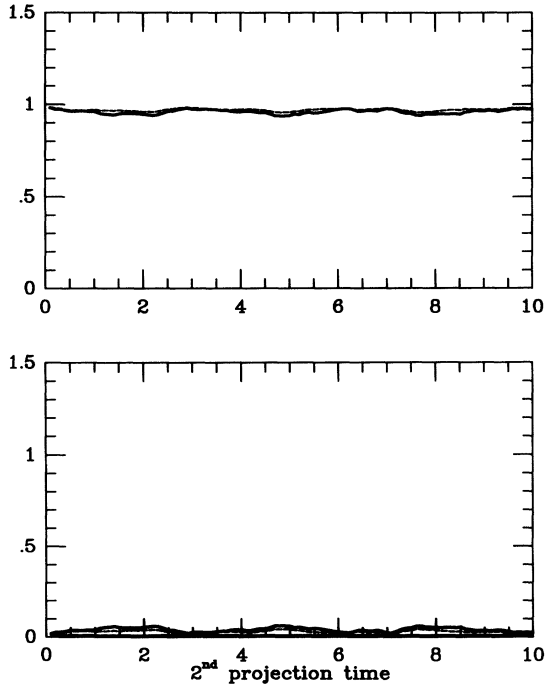

FIG. 14. Checking the sum rules (as in Fig. 9). The projection basis states are $\{|\uparrow\rangle,|\downarrow\rangle\}$ and the coupling is strong (as in Fig. 4).

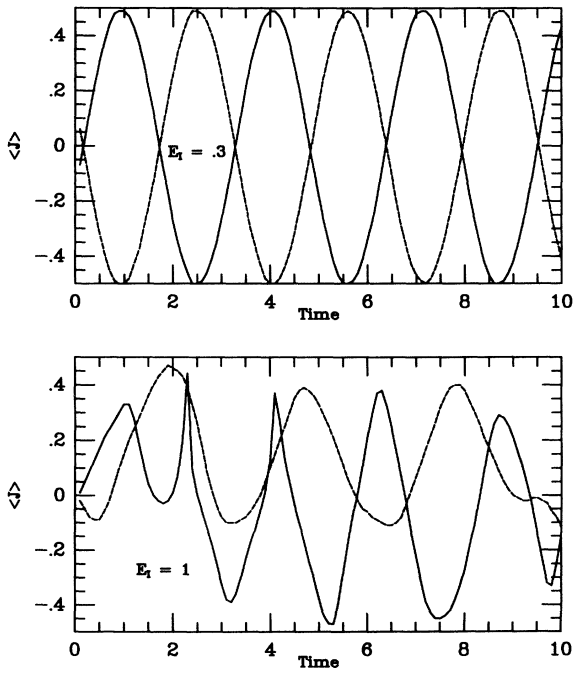

FIG. 15. Comparing the Schmidt basis for the two path projected states. For weak coupling (upper panel) the two bases are essentially the same, while for medium coupling the bases have no simple relationship.

orthogonal.) This will allow one to use the same projection basis on *both* paths at the next projection time, without compromising the goal of projecting on the "best recorded" basis. The universality of the Schmidt basis across the different paths is a consequence of the presence of a preferred "pointer basis" for the weakly coupled system. In contrast, the medium coupled case (lower panel) has no particular connection between the Schmidt states on the different paths (and no particular pointer basis). To project on the best recorded basis, one must choose a different projection basis on each path.

Gell-Mann and Hartle [41] stress that one should not expect the best projections to always be onto the same basis on different paths. I agree with them. This example is simply an illustration of how the presence of a pointer basis can be reflected in the nature of the consistent histories.

I should note that as more projection times are added, one puts more strain on the simple system discussed here. I have been counting on the largeness of the environment space, combined with the randomness of the evolution within that space, to ensure that environment states are orthogonal (to a good approximation) whenever this is necessary for a "good record" to be present (just as in [3]). The size of the environment space in the toy model is only 12, so there are only so many mutually orthogonal "records" one can have. To emphasize this point, I note that the "Schmidt paths" depicted in Figs. 1–4 are resolved to a time scale of $\Delta t = 0.1$ over a period of 20 units of time. If the same resolution were used for consistent histories, one would require 2^{200} different paths, which is much more than one could hope to record in the small environment used here. This serves to illustrate not only the limitations of the toy model, but also the lack of a simple one-to-one correspondence between Schmidt paths (of which there are only two in this example) and consistent histories.

On the other hand, the Schmidt decomposition appears to be a helpful way to analyze path projected states. It can point one toward useful projections to choose when constructing the decoherence functional. Furthermore, it is likely that the Schmidt paths and the consistent histories can be brought closer together in the case where the environment is better resolved. It might be the case that the environment space can be decomposed into separate subspaces, so that the location of each record can be identified (as a spot on a magnetic tape, for example). The Schmidt decomposition may be applied repeatedly to separate out each subspace. The resulting decomposition of the wave function will have many more factors in each term (corresponding to many more subsystems) and, consequently, many more terms (resulting in more Schmidt paths). If all the relevant records are resolved in this way, the Schmidt paths should closely resemble the particular consistent histories for which those records are relevant.

C. Reducing the time between projections

So far I have not discussed the role of the coarse graining in time which is brought about by the finite time between projections. If one can rely on the Schmidt decomposition to determine which measurements have actually been made, and choose the projections accordingly, what is to prevent one from letting the time between projections go to zero? Should not this prescription give perfectly consistent histories?

The answer to this question is "no." The reason the prescription fails lies in the fact that nowhere have I managed to get *perfect* accordance with the probability sum rules. None of the Figs. 9–14 have the dashed line exactly on top of the heavy line. This imperfection will have an increasing impact on the results as the time between projections is decreased.

Consider for a moment Fig. 10. When only a small amount of time has elapsed since the first projection, two of the paths have very low probability. This can be deduced from the fact that the light solid curves are close to either the heavy curve or zero on the left side of Fig. 10. The reason for this is that there has not been enough time for much branching to occur. As a result the sum rule violations are large compared to the probabilities assigned to some of the paths. In general, when the projections are made sufficiently frequently, the probabilities assigned to most of the paths will be smaller than the sum rule violations. For example, in Fig. 10 (top panel) the light solid curve eventually deviates significantly from the heavy curve, indicating that the second path eventually acquires a large probability. If many projections had been made in the same time period, this probability would have been divided among many paths, and in most cases the path probability would be smaller than the sum rule violation.

Which paths one requires to have probabilities larger than the sum rule violation depends on the physical problem one is addressing. The point of this subsection is that the coarse graining in time is way to control this effect.

IX. COMPARISON WITH OTHER WORK

Other authors have also studied two-state systems coupled to an environment. In particular, such systems appear in some of the pioneering papers on the loss of quantum coherence [3,11]. The main way the calculations presented here differ from previous work is that here exact (to machine precision) solutions are provided for a much more complex system then most of those which were solved exactly before. The work also represents the first time a side-by-side comparison of the consistent histories and the Schmidt paths has been made. (For more calculations using consistent histories, see [46].) In most previous work, the cases which were solved exactly were very simple. Some cases involved a Hamiltonian which was (or became) exactly separable. In other cases the pointer basis states were exact eigenstates of the total Hamiltonian. These simplifications made it easy for correlations to be discussed, since they were exactly preserved as time evolved. (Another complex example which is solved exactly is given by Unruh and Zurek [7].)

More complex examples have been studied, but various approximations were used. One thing the work here shows clearly is how there are many ways the Schmidt

states can deviate from a simple pointer basis. In some cases the deviations are large, while in other cases the deviations amount to small, stable fluctuations. (Presumably the latter fluctuations are present at some level in *any* realistic example.) I believe that the previous work was sufficiently simplified as to not distinguish among these different possibilities. For example, the *time scale* of these deviations alone is not enough to distinguish between the two cases.

The model studied here is very close in spirit to Zurek's model in [3], but it is less idealized. Although plots appear in [3] of off-diagonal density matrix elements which are small, but not precisely zero, the plots are for the case where the on-diagonal elements are degenerate. In this case the off-diagonal elements would be equally small in *any* basis, and the role of the interactions in choosing a pointer basis is less apparent.

I should stress, however, that I am not calling into question or even adding to the major advances made by Zurek [3] and by Joos and Zeh [11] in identifying the mechanisms which cause the loss of coherence, and their important role in quantum physics. What I am doing in this paper is trying out different ways of viewing these mechanisms in action, on a system with a slightly greater degree of complexity.

X. CONCLUSIONS

I have investigated the decohering properties of a simple toy system from a number of points of view. One goal was to determine the extent to which the interactions with the environment defined a "pointer basis" whose quantum coherence was not destroyed by the interactions.

In the "Schmidt paths" point of view, the system was studied by following the evolution of the eigenstates of the reduced density matrix. Depending on the coupled strength, there was either (1) a constant pointer basis (strong coupling), (2) a simply oscillating pointer basis (weak coupling), or (3) noisy behavior, with no pointer basis (medium coupling).

I also utilized the "consistent histories" approach to study the toy system. Correlations between a system and its environment (or "records") play an important role in causing certain histories to be consistent. The Schmidt decomposition provides an exact account of the records or correlations which are present, and I showed how it can be used to good advantage in constructing consistent histories. The presence of a pointer basis allowed a particularly simple form of consistent histories to be constructed, but the absence of a pointer basis did *not* prevent the construction of consistent histories. I suggested that when the records are better resolved than they are in this particular toy system, the Schmidt paths and certain consistent histories should very closely resemble one another.

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APPENDIX A: THE SCHMIDT DECOMPOSITION

1. Proof

Here is a brief proof that the Schmidt decomposition may always be performed: Consider a state $|\psi\rangle$ in a vector space which we choose to regard as a direct product space. Let $\{|i\rangle_1\}$ and $\{|i\rangle_2\}$ each be some orthonormal basis in the corresponding subspace. There always exist α_{ij}'s such that

$$|\psi\rangle=\sum_{ij}\alpha_{ij}|i\rangle_1|j\rangle_2 \ . \tag{A1}$$

Furthermore, one can define

$$|\tilde{i}\rangle_2\equiv\sum_j\alpha_{ij}|j\rangle_2 \ , \tag{A2}$$

so that one can always write

$$|\psi\rangle=\sum_i|i\rangle_1|\tilde{i}\rangle_2 \ . \tag{A3}$$

In general, the $|\tilde{i}\rangle_2$'s will not be orthogonal or normalized.

Now consider the special case where the $\{|i\rangle_1\}$'s are the (normalized) eigenstates of $\rho_1[\equiv\mathrm{tr}_1(|\psi\rangle\langle\psi|)=\sum_{ijk}\alpha^*_{ij}\alpha_{kj}|i\rangle_1{}_1\langle k|]$, call them $\{|i\rangle_1^S\}$. In this case the $\{|\tilde{i}\rangle_2\}$ *must* be orthogonal because we must have

$$\begin{aligned}{}_1^S\langle i|\rho_1|j\rangle_1^S&={}_1^S\langle i|\left[\sum_{k,l}|k\rangle_1^S{}_2\langle\tilde{k}|\tilde{l}\rangle_2{}_1^S\langle l|\right]|j\rangle_1^S\\&={}_2\langle\tilde{i}|\tilde{j}\rangle_2\propto\delta_{ij} \ .\end{aligned} \tag{A4}$$

One can then see that the $|\tilde{i}\rangle_s$'s must be eigenstates of ρ_2:

$$\rho_2\equiv\mathrm{tr}_1(|\psi\rangle\langle\psi|)=\sum_i|\tilde{i}\rangle_2{}_2\langle\tilde{i}| \ . \tag{A5}$$

Finally, one notes that the nonzero eigenvalues of both ρ_1 and ρ_2 are both given by $p_i={}_2\langle\tilde{i}|\tilde{i}\rangle_2$, and one can construct the normalized states

$$|i\rangle_2^S\equiv(p_i)^{-1/2}|\tilde{i}\rangle_2 \ . \tag{A6}$$

Equation (A3) then becomes

$$|\psi\rangle=\sum_i\sqrt{p_i}|i\rangle_1^S|i\rangle_2^S \ , \tag{A7}$$

which is the quoted result.

2. Remarks

Here is a remark which often helps people develop some intuition about the Schmidt decomposition: If one

is given a particular vector in a vector space, and is allowed complete freedom to choose a basis, one can always choose a basis in which the expansion of the particular vector has but one term. One simply chooses the first basis vector proportional to the state in question. To get a complete basis, one then constructs an orthonormal set around that first basis vector (using the Gram-Schmidt orthogonalization procedure). If one does not have *complete* freedom to choose a basis, but is allowed to choose any basis within two predetermined subspaces, then it should not be surprising that, in general, one cannot get down to a single term in the expansion. However, one should be able to reduce the number of terms, since there is some remaining flexibility, and that is what the Schmidt form does. Note that the number of terms in Eq. (A7) is equal to the *minimum* of the two subspace sizes, rather than the *product* of the two sizes which would arise in a typical expansion.

APPENDIX B: COMPUTATIONAL METHODS

The computational methods employed in this work are straightforward. The total Hilbert space has a size of $2n_1$, where n_1 is the size of the environment subspace. I start by considering an orthonormal set $\{|i\rangle_w | i=1,2n_1\}$ (the "working basis"), which spans the whole space. This set may be viewed as a direct product of two sets of vectors, $\{|\uparrow\rangle_2,|\downarrow\rangle_2\}$ and $\{|j\rangle_1 | j=1,n_1\}$, each of which spans one of the two subspaces. The direct product form for each $|i\rangle_w$ can be realized by writing

$$|i\rangle_w = \begin{cases} |\uparrow\rangle \otimes |(i+1)/2\rangle_1, & i=\text{odd} , \\ |\downarrow\rangle \otimes |(i)/2\rangle_1, & i=\text{even} . \end{cases} \tag{B1}$$

In this way a working basis is defined in each of the subsystems as well. Any state of the system can be represented by a set of $2n_1$ complex numbers, α_i, normalized to $\sum_i \alpha_i^* \alpha_i = 1$, giving the expansion coefficients of the state in the working basis. These numbers can be equivalently labeled $\alpha_{\uparrow,(i+1)/2}$ or $\alpha_{\downarrow,(i)/2}$ according to Eq. (B1). Likewise, any operator can be represented by a $2n_1 \times 2n_1$ array of numbers giving all the matrix elements of the operator in the working basis.

In order to do a calculation, first the expansion coefficients of the initial state in the working basis are calculated. Then the array corresponding to the total Hamiltonian is constructed, and diagonalized [using a packaged subroutine from the International Mathematical and Scientific Library (IMSL)]. One then has a spectrum of eigenvalues $\{E_i\}$ and a unitary operator $\hat{U}$ for transforming in and out of the eigenbasis. Using $\hat{U}$, I calculate the expansion coefficients of the state in the eigenbasis of the Hamiltonian. Then time evolution is reduced to evaluating a new phase $e^{-iE_i t}$ for each of the energy eigenbasis expansion coefficients.

At any time $\hat{U}^\dagger$ can be used to return to the expansion coefficients in the working basis. In this basis, it is easy to construct the matrix elements of ρ_2, or any other quantity of interest. For example,

$$\langle\uparrow|\rho_2|\downarrow\rangle = \sum_{j=1}^{n_1} \alpha_{\uparrow,j}^* \alpha_{\downarrow,j} . \tag{B2}$$

With all the elements of ρ_2 in hand, one can then diagonalize it, and examine the eigenvalues and eigenvectors leading to the Schmidt paths. Likewise, the projections needed to construct the path projected states are easy to perform.

APPENDIX C: OTHER NUMERICAL ISSUES

1. The size of the environment

All the cases discussed here had $n_1 = 12$. I have studied the system for a variety of different n_1's. The value 12 was chosen because it was large enough for the environment to play the desired role, but not much larger, so the computations could run as rapidly as possible.

2. Tests of the code

All calculations were performed with double precision complex numbers on a Vax computer. In general, there should not be a problem in accurately evaluating the time-dependent phases of the energy eigenstates for the time ranges considered. In any case, one does not expect numerical errors to build up in time in this sort of calculation. The state of the world at each time is calculated directly by shifting the phases of the initial state as expanded in the energy eigenstates. There is no dependence on the state at intermediate times.

Perhaps one could be concerned that the correct physics depends on the realization of precise relationships among the energy eigenstates. One example of this is the $E_I = 0$ case, where the simple evolution of system 2 depends on the relationship $\lambda_i - \lambda_j = E_2$ holding among pairs of energy eigenvalues of the "world." I have checked this case, and found the expected sinusoidal evolution to be followed to high precision. This test was performed with a large environment ($n_1 = 50$) so that half of the 100 energy eigenstates have an appreciable overlap with the initial ($|\uparrow\rangle \otimes |\text{random}\rangle$) state. The resulting simple evolution indicates that the special relationships among all these states are being correctly accounted for. In this test the phase of oscillations and direct product form of the state were preserved to $t = 1000$ and beyond.

Another confirmation of the code came from independently calculating all the terms separately and checking that Eqs. (38) and (39) do indeed apply.

Each calculation uses a particular random number seed to generate the parts of the Hamiltonian and of the initial state designated as "random." The seed was changed from time to time, and it does not appear that any results reported here represent atypical realizations. This issue was not investigated systematically, however.

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