Nonrelativistic field-theoretic scale anomaly

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We construct a nonrelativistic scalar field theory with a quartic self-interaction in 2+1 dimensions as an infinite mass limit of a relativistic theory, and calculate the two-particle scattering amplitude and two-particle bound-state energy. We show that the results are the same as for quantum mechanics of two scalar particles interacting via a δ-function potential. Renormalization of the theory reveals an anomalous breaking of scale symmetry. The renormalization group structure is presented and the anomaly is expressed in terms of the trace of the energy-momentum tensor.

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I. INTRODUCTION

Two- and three-dimensional δ-function potentials have been studied as models for applying renormalization theory to nonrelativistic (NR) quantum mechanics [1]. The bare δ-function potentials do not give an interaction between particles. This has been used to argue that $\lambda\phi^4$ theory, at least in the nonrelativistic limit, is trivial. However, a renormalization procedure produces a point interaction with a nonzero scattering amplitude and a single bound state. More recently, planar point interactions have arisen in the N-anyon system. A system of N particles interacting with a Chern-Simons gauge field, and interacting among themselves via δ-function potentials, has been shown to obey fractional statistics. In addition, classical soliton solutions have been constructed for this system [2].

For two particles in two dimensions interacting via a δ-function potential of strength α_0, the quantum mechanical scattering amplitude to second order in the Born approximation contains a UV-divergent integral. Upon regularization, it becomes

$$T(p,\alpha_0,\Lambda)=\alpha_0\left[1-\frac{\alpha_0 m}{4\pi}\left[\ln\frac{\Lambda}{p}+\frac{i\pi}{2}\right]+O(\alpha_0^2)\right], \tag{1.1}$$

where p is the relative momentum and Λ is the momentum cutoff. T diverges in the limit $\Lambda\to\infty$, unless a redefinition of the coupling constant is made:

$$\alpha_0^{-1}=\alpha^{-1}-\frac{m}{4\pi}\ln\frac{\Lambda}{\mu} \tag{1.2}$$

where α is independent of the cutoff, and μ is an arbitrary scale. The amplitude is then reexpressed as

$$T(p,\alpha,\mu)=\alpha\left[1-\frac{\alpha m}{4\pi}\left[\ln\frac{\mu}{p}+\frac{i\pi}{2}\right]+O(\alpha^2)\right]. \tag{1.3a}$$

The perturbation expansion can now be summed to all orders to give

$$T(p,\alpha,\mu)=\alpha\left[1+\frac{\alpha m}{4\pi}\left[\ln\frac{\mu}{p}+\frac{i\pi}{2}\right]\right]^{-1}. \tag{1.3b}$$

In addition a bound state is found, with energy

$$E_B=-\frac{\mu^2}{2m}e^{8\pi/m\alpha} \tag{1.4}$$

The scattering amplitude can be rewritten

$$T(p,E_B)=\frac{8\pi}{m}\left[\ln\frac{2mE_B}{p^2}\right]^{-1} \tag{1.5}$$

which shows that it depends on only one dimensionful parameter (other than the mass) E_B and not on α and μ separately. The branch of the logarithm is chosen real for a negative real p^2. From (1.2) we see that a finite bound state energy requires α_0 to be negative, implying that the attractive point interaction leads to nontrivial physics, whereas the repulsive one does not.

This nontrivial point interaction can also be viewed as a self-adjoint extension of a free Hamiltonian on a space with one point removed [3]. The self-adjoint extension parameter then has the interpretation of the renormalized coupling constant α.

The δ-function potential in quantum mechanics is the formal NR limit of the relativistic $\lambda\phi^4$ theory [4]. In 2+1 dimensions, the relativistic $\lambda\phi^4$ interaction is super-renormalizable. The loop corrections to the scattering amplitude are finite, and therefore no coupling constant renormalization is necessary. In the NR limit, the amplitude becomes

$$A(p)=4m^2\alpha_0\left[1-\frac{\alpha_0 m}{4\pi}\left[\ln\frac{2m}{p}+\frac{i\pi}{2}\right]+O(\alpha_0^2)\right] \tag{1.6}$$

in the center-of-mass (c.m.) frame. The relativistic coupling constant has been written as $4m^2\alpha_0$. Since $m \gg p$ in the NR limit, the logarithmic divergence of the quantum mechanical amplitude (1.1) is also apparent in (1.6), so apart from the prefactor $4m^2$ the two amplitudes are the same. The prefactor is compensated for by kinematic factors, which are different for relativistic and NR physics, so that the resulting cross sections are the same.

We shall study the (2+1)-dimensional point interaction from a NR field-theoretic (second-quantized) approach. In Sec. II we derive the NR Lagrangian density, calculate the two-particle scattering amplitude, renormalize the theory, and find the bound-state energy. In Sec. III the issues of scale invariance in the NR theory and its anomalous breaking are discussed through renormalization-group techniques.

II. NONRELATIVISTIC FIELD THEORY

The Lagrangian density of the real relativistic scalar field theory can be written as

$$\mathcal{L}=\frac{1}{2}\partial_\mu\chi\partial^\mu\chi-\frac{m^2}{2}\chi^2-\frac{\lambda_0}{4!}\chi^4-\frac{\delta m^2}{2}\chi^2\ , \tag{2.1}$$

where m is the physical mass and δm^2 is the mass counterterm, determined in an expansion in powers of λ_0. We shall discuss the mass counterterm in the context of the NR field theory, where we will show that it actually vanishes.

Before taking the limit $m\to\infty$, we must rewrite (2.1) in terms of NR fields, so we substitute

$$\chi=\frac{1}{\sqrt{2m}}(e^{-imt}\phi+e^{imt}\phi^*)\ . \tag{2.2}$$

Terms in $\mathcal{L}$ which oscillate with a frequency $\propto m$ will not contribute to $\int d^3x\,\mathcal{L}$ in the NR limit, so we drop them. The resulting Lagrangian density is given by

$$\mathcal{L}=\phi^*\left[i\partial_t+\frac{\nabla^2}{2m}\right]\phi-\frac{\alpha_0}{4}(\phi^*\phi)^2\ , \tag{2.3}$$

where $\alpha_0=\lambda_0/4m^2$. The mass counterterm has been dropped. For convenience we eliminate the mass from the action by the transformation

$$\begin{aligned}&\mathcal{L}\to m\mathcal{L}\ ,\\ &t\to t/m\ ,\\ &\mathbf{x}\to\mathbf{x}\ .\end{aligned} \tag{2.4}$$

The mass dimension of time is now -2. The action is unchanged, and the Lagrangian density becomes

$$\mathcal{L}=\phi^*\left[i\partial_t+\frac{1}{2}\nabla^2\right]\phi-\frac{v_0}{4}(\phi^*\phi)^2\ , \tag{2.5}$$

where $v_0\equiv m\alpha_0$. The free fields can be written as Fourier integrals

$$\begin{aligned}\phi(\mathbf{x},t)&=\int\frac{d^2k}{(2\pi)^2}\alpha(\mathbf{k})e^{-i(\omega_k t-\mathbf{k}\cdot\mathbf{x})}\\ \phi^*(\mathbf{x},t)&=\int\frac{d^2k}{(2\pi)^2}\alpha^*(\mathbf{k})e^{i(\omega_k t-\mathbf{k}\cdot\mathbf{x})}\end{aligned} \tag{2.6}$$

where $\omega_k=\frac{1}{2}\mathbf{k}^2$.

Quantization. The theory is quantized by promoting the fields to operators ($\phi^*\to\phi^\dagger$), and imposing the commutation relations

$$\begin{aligned}&[\phi(\mathbf{x},t),\phi(\mathbf{y},t)]=[\phi^\dagger(\mathbf{x},t),\phi^\dagger(\mathbf{y},t)]=0\ ,\\ &[\phi(\mathbf{x},t),\phi^\dagger(\mathbf{y},t)]=\delta(\mathbf{x}-\mathbf{y})\ .\end{aligned} \tag{2.7}$$

When we posit the existence of a vacuum state $|0\rangle$, such that $\alpha(\mathbf{k})|0\rangle=0$, the particle picture follows. $\alpha(\mathbf{k})$ annihilates a particle of momentum $\mathbf{k}$, and $\alpha^\dagger(\mathbf{k})$ creates one. Therefore, ϕ destroys particles and $\phi^\dagger$ creates particles. As a consequence of the U(1) symmetry of (2.5) the number of particles is conserved, as it should be in NR quantum mechanics. [We could equally as well have started with a *complex* relativistic field theory instead of a real one, and taken its NR limit. The resulting NR theory would contain antiparticles, and would possess a U(1)×U(1) symmetry, implying *separate* conservation of particle number and antiparticle number. We could then choose to work in the zero-antiparticle sector by convention, and the same Lagrangian density would result.]

The free NR propagator is given by

$$\begin{aligned}D(\mathbf{x},t)&=\int\frac{d^2k\,d\omega}{(2\pi)^3}\frac{1}{\omega-\frac{1}{2}k^2+i\epsilon}e^{-i(\omega t-\mathbf{k}\cdot\mathbf{x})}\\ &=-\frac{\theta(t)}{2\pi t}\exp\left[\frac{ix^2}{2t}\right]\ .\end{aligned} \tag{2.8}$$

The interaction is handled through the usual Dyson perturbation expansion of the scattering matrix. There is an ambiguity in the ordering of the fields in the interaction, but the physics is independent of the choice of ordering. We choose to normal order the interaction, so it has the form $\phi^\dagger\phi^\dagger\phi\phi$. This is what allows us to drop the mass counterterm.

The only nontrivial Green's functions in this theory are the $(2n)$-point functions given by

$$\begin{aligned}&G^{(2n)}(x_1,\ldots,x_n,y_1,\ldots,y_n)\\ &\quad=\langle 0|T\phi(x_1)\cdots\phi(x_n)\phi^\dagger(y_1)\cdots\phi^\dagger(y_n)|0\rangle\ ,\end{aligned} \tag{2.9}$$

where $x_i\equiv(\mathbf{x}_i,t_i)$. We will be interested in the one-particle-irreducible (1PI) parts of these Green's functions in momentum space, defined in the same way as in relativistic field theories.

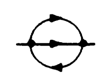
Exact propagator. The bare 1PI two-point function is given by

$$\Gamma_B^{(2)}(\mathbf{p},\omega)=\omega-\frac{1}{2}\mathbf{p}^2-\Sigma(\mathbf{p},\omega)-\Pi(\mathbf{p},\omega) \tag{2.10}$$

where, diagrammatically,

$-i\Sigma(\mathbf{p},\omega)=$  $+$  $+\cdots$

and

$-i\Pi(\mathbf{p},\omega)=$  $+\cdots$.

$\Sigma(\mathbf{p},\omega)$ vanishes to all orders due to normal ordering of the interaction. It is easiest to show that $\Pi(\mathbf{p},\omega)$ also vanishes by noticing that the lowest-order diagram, in coordinate space, contains the factor $\theta(y^0-z^0)\theta(z^0-y^0)$.

This is seen by the opposing arrows in one of the loops. This feature persists to all orders; hence, $\Pi(\mathbf{p},\omega)$ vanishes exactly. (Note that the second- and higher-order diagrams in $\Sigma(\mathbf{p},\omega)$ vanish for this reason as well.) The renormalization condition

$$\Gamma_R^{(2)}(\mathbf{p},\omega)=\omega-\tfrac{1}{2}\mathbf{p}^2 \tag{2.11}$$

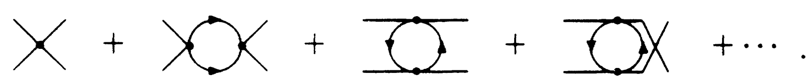
is then satisfied if $\delta m^2=0$, so that neither mass renormalization nor field renormalization are required. The absence of field renormalization allows us to drop the B and R subscripts from now on.

The resulting Lagrangian density

$$\mathcal{L}=\phi^\dagger\left[i\partial_t+\frac{1}{2}\nabla^2\right]\phi-\frac{v_0}{4}\phi^\dagger\phi^\dagger\phi\phi \tag{2.12}$$

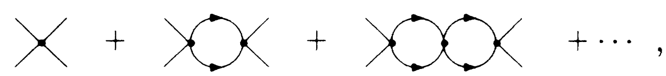
contains no dimensionful parameters, so the theory is classically scale invariant. The relativistic theory (2.1) is not scale invariant, but its limit as $m\to\infty$ is. The quantum theory, however, requires further examination.

The NR scattering amplitude. The bare 1PI four-point function is given diagrammatically by

$-i\Gamma^{(4)}=$ 

The third and fourth terms vanish because they contain opposing arrows on their loops. Because of the form of the NR propagator (2.8), the direction of the arrows corresponds to increasing time (in coordinate space), so these diagrams represent unphysical contributions. In a *relativistic* theory, these diagrams would correspond to pair creation and annihilation, and would have a nonvanishing contribution. The vanishing of such diagrams in the NR theory can also be seen as a consequence of analyticity properties in momentum space.

The four-point function can now be expanded in loops,

$-i\Gamma^{(4)}=$ 

because only insertions of the one-loop s diagram give nonvanishing diagrams. Apart from an energy- and momentum-conserving δ function, the four-point function to one-loop order is given by

$$-i\Gamma_1^{(4)}(\mathbf{p}_i,\omega_i,v_0,\Lambda)=-iv_0+\frac{iv_0^2}{8\pi}\left[\ln\left[\frac{4\Lambda^2}{(\mathbf{p}_1+\mathbf{p}_2)^2-4(\omega_1+\omega_2)}\right]\right],\quad i=1,\ldots,4 \tag{2.13}$$

where Λ is an ultraviolet cutoff. We redefine the coupling constant:

$$\begin{aligned} v_0&=v+\delta v\ ,\\ \delta v&=\frac{1}{4\pi}v^2\ln\frac{\Lambda}{\mu}+O(v^3)\ . \end{aligned} \tag{2.14}$$

This allows us to rewrite the four-point function in a cutoff-independent form

$$-i\Gamma_1^{(4)}(\mathbf{p}_i,\omega_i,v,\mu)=-iv+\frac{iv^2}{8\pi}\left[\ln\left[\frac{4\mu^2}{(\mathbf{p}_1+\mathbf{p}_2)^2-4(\omega_1+\omega_2)}\right]\right]. \tag{2.15}$$

The renormalized scattering amplitude is gotten by taking the momenta on shell and going into the c.m. frame:

$$A(p,v,\mu)=v\left[1-\frac{v}{4\pi}\left[\ln\frac{\mu}{p}+\frac{i\pi}{2}\right]+O(v^2)\right]. \tag{2.16}$$

This agrees with (1.3a), with $m=1$. Because of the vanishing of the t- and u-type diagrams, the exact 1PI four-point function can be expanded in powers of the one-loop term. The series is summed to give

$$-i\Gamma^{(4)}(\mathbf{p}_i,\omega_i,v,\mu)=-iv\left[1+\frac{v}{8\pi}\ln\left[\frac{4\mu^2}{(\mathbf{p}_1+\mathbf{p}_2)^2-4(\omega_1+\omega_2)}\right]\right]^{-1}. \tag{2.17}$$

The exact scattering amplitude is then

$$A(p,v,\mu)=v\left[1+\frac{v}{4\pi}\left[\ln\frac{\mu}{p}+\frac{i\pi}{2}\right]\right]^{-1} \tag{2.18}$$

which agrees with (1.3b). Relation (2.14) can now be made exact:

$$v_0=v\left[1-\frac{v}{4\pi}\ln\frac{\Lambda}{\mu}\right]^{-1}. \tag{2.19}$$

The bound-state energy is given by the position of the pole in (2.18):

$$E_B=-\tfrac{1}{2}\mu^2e^{8\pi/v} \tag{2.20}$$

which agrees with (1.4) with $m=1$. Like Eq. (1.5), Eqs. (2.17)–(2.19) can also be rewritten in terms of only the bound-state energy. In particular, the amplitude is given by

$$A(p,E_B)=8\pi\left[\ln\frac{2E_B}{p^2}\right]^{-1}, \tag{2.21}$$

where the branch of the logarithm is chosen as in (1.5). This shows that the only physical parameter is E_B, and it defines the scale of the physics. The scale dependence in (2.21) contradicts the apparent scale invariance of the theory, and is a sure sign of a *scale anomaly*. In other words, the classical theory possesses a scale symmetry, but quantization and renormalization, which involve an introduction of an arbitrary scale, necessarily break this symmetry.

III. RGE AND THE SCALE ANOMALY

NR scale invariance. Scale transformations in the NR theory take the form

$$\mathbf{x}\to e^{\alpha}\mathbf{x}\ ,\qquad t\to e^{2\alpha}t\ . \tag{3.1}$$

Under an infinitesimal scale transformation, the fields change by

$$\delta\phi=[1+\mathbf{x}\cdot\nabla+2t\partial_t]\phi\ . \tag{3.2}$$

The charge density ρ and current density $\mathbf{J}$ associated with this transformation satisfy the relation

$$\partial_t\rho+\nabla\cdot\mathbf{J}=2\theta_{00}-\sum_{i=1}^{2}\theta_{ii}\ , \tag{3.3}$$

where θ is the energy and momentum tensor [2,5].

The Lagrangian density of the NR (2+1)-dimensional theory contains no dimensional parameters, so it transforms infinitesimally by

$$\delta\mathcal{L}=(4+\mathbf{x}\cdot\nabla+2t\partial_t)\mathcal{L}=\nabla\cdot(\mathbf{x}\mathcal{L})+2\partial_t(t\mathcal{L})\ . \tag{3.4}$$

Therefore the action remains invariant, and the dilatation charge and current satisfy the continuity equation $\partial_t\rho+\nabla\cdot\mathbf{J}=0$. In terms of the energy-momentum tensor, we get[1]

$$\sum_{i=1}^{2}\theta_{ii}=2\theta_{00}\ . \tag{3.5}$$

So the spatial trace of the energy momentum tensor minus twice its time-time component (the Hamiltonian density), is a measure of scale invariance breaking in the NR field theory [5].

Ward identity. The Ward identity related to NR scale invariance is derived by considering the vacuum expectation values

$$G_\rho^{(n)}(y,x_1,\ldots,x_n)=\langle 0|T\rho(y)\phi(x_1)\cdots\phi^\dagger(x_n)|0\rangle\ , \tag{3.6a}$$

$$G_{J_i}^{(n)}(y,x_1,\ldots,x_n)=\langle 0|TJ_i(y)\phi(x_1)\cdots\phi^\dagger(x_n)|0\rangle\ , \tag{3.6b}$$

$$G_\theta^{(n)}(y,x_1,\ldots,x_n)=\left\langle 0\left|T\left[2\theta_{00}(y)-\sum\theta_{ii}(y)\right]\phi(x_1)\cdots\phi^\dagger(x_n)\right|0\right\rangle\ , \tag{3.6c}$$

where n is an even integer. Differentiating (3.6a) with respect to y_0 and (3.6b) with respect to $\mathbf{y}$ and comparing the sum with (3.6c), we obtain a relation between $G_\theta^{(n)}$ and the n-point function, given in momentum space by

$$\left[n-4(n-1)-\mathbf{p}_i\cdot\frac{\partial}{\partial\mathbf{p}_i}-2\omega_i\frac{\partial}{\partial\omega_i}\right]G^{(n)}(\mathbf{p}_i,\omega_i)=-iG_\theta^{(n)}(0,\mathbf{p}_i,\omega_i) \tag{3.7}$$

where $i=1,\ldots,n$. Performing a scale transformation on the momenta and energies gives

$$\left[\frac{\partial}{\partial\alpha}-3n+4\right]G^{(n)}(e^{-\alpha}\mathbf{p}_i,e^{-2\alpha}\omega_i)=-iG_\theta^{(n)}(0,e^{-\alpha}\mathbf{p}_i,e^{-2\alpha}\omega_i)\ . \tag{3.8}$$

Scale invariance implies that (3.5) holds, so the right-hand side of (3.8) vanishes.

For 1PI functions equation (3.8) becomes

$$\left[\frac{\partial}{\partial\alpha}+(4-n)\right]\Gamma^{(n)}(e^{-\alpha}\mathbf{p}_i,e^{-2\alpha}\omega_i)=-i\Gamma_\theta^{(n)}(0,e^{-\alpha}\mathbf{p}_i,e^{-2\alpha}\omega_i) \tag{3.9}$$

and, in a scale invariant theory,

$$\left[\frac{\partial}{\partial\alpha}+(4-n)\right]\Gamma^{(n)}(e^{-\alpha}\mathbf{p}_i,e^{-2\alpha}\omega_i)=0\ . \tag{3.10}$$

For the four-point 1PI function this becomes simply

$$\frac{\partial}{\partial\alpha}\Gamma^{(4)}(e^{-\alpha}\mathbf{p}_i,e^{-2\alpha}\omega_i)=0\ . \tag{3.11}$$

We see that scale invariance of the classical action implies a scaling relation for the 1PI functions, and in particular it implies that the four-point function is scale invariant. To see whether scale invariance still holds in the quantum theory, however, we must examine the renormalization-group equation.

Renormalization-group equation (RGE). Equation (2.19) gave us the relation between the bare and renormalized couplings. The β function obtained from this relation is given by

$$\beta(v)=\mu\frac{\partial v}{\partial\mu}=\frac{v^2}{4\pi}\ . \tag{3.12}$$

Like equation (2.19) this equation is exact. The RGE for 1PI functions is

$$\left[\mu\frac{\partial}{\partial\mu}+\beta(v)\frac{\partial}{\partial v}\right]\Gamma^{(n)}(\mathbf{p}_i,\omega_i,v,\mu)=0\ . \tag{3.13}$$

The anomalous dimension term is absent from this equation because, as was argued in part 2, field renormalization is absent from the NR theory. Consequently, the

[1] In relativistic field theories the equivalent statement is $\theta^{\mu}{}_{\mu}=0$.

Green's functions retain their canonical dimensions, and the solution to (3.13) has the form

$$\Gamma^{(n)}(\mathbf{p}_i,\omega_i,v,\mu)=Q^{4-n}\mathcal{G}\left[v'\left[v,\ln\frac{\mu}{Q}\right]\right], \quad (3.14)$$

where Q is the appropriate dynamical variable, e.g.,

$$\begin{aligned} &Q^2=\tfrac{1}{2}p^2-\omega \quad \text{for } n=2\ , \\ &Q^2=\tfrac{1}{2}(\mathbf{p}_1+\mathbf{p}_2)^2-\omega_1-\omega_2 \quad \text{for } n=4\ , \end{aligned} \quad (3.15)$$

and v' is the running coupling constant given by

$$v'(v,\ln\mu/Q)=v\left[1+\frac{v}{4\pi}\ln\frac{\mu}{Q}\right]^{-1}. \quad (3.16)$$

For $n=4$, the 1PI function (2.17) does indeed have the form dictated by (3.14)–(3.16), with $\mathcal{G}(v')=v'$. Rescaling the momenta and energies in (3.14) and then differentiating with respect to α and μ gives the identity

$$\left[-\frac{\partial}{\partial\alpha}+\mu\frac{\partial}{\partial\mu}-4+n\right]\Gamma^{(n)}(e^{-\alpha}\mathbf{p}_i,e^{-2\alpha}\omega_i,v,\mu)=0\ . \quad (3.17)$$

Inserting this into (3.13) gives

$$\left[\frac{\partial}{\partial\alpha}+\beta(v)\frac{\partial}{\partial v}+4-n\right]\Gamma^{(n)}(e^{-\alpha}\mathbf{p}_i,e^{-2\alpha}\omega_i,v,\mu)=0\ . \quad (3.18)$$

Finally, for the four-point function we get

$$\left[\frac{\partial}{\partial\alpha}+\beta(v)\frac{\partial}{\partial v}\right]\Gamma^{(4)}(e^{-\alpha}\mathbf{p}_i,e^{-2\alpha}\omega_i,v,\mu)=0\ . \quad (3.19)$$

Comparing this with Eq. (3.11), we see that scale invariance is broken by the $\beta(v)$ term. Going back to (3.9), we can express the violation of scale invariance in the quantum theory as a statement on the 1PI four-point function with the $2\theta_{00}-\sum\theta_{ii}$ insertion,

$$i\Gamma_\theta^{(4)}(0,e^{-\alpha}\mathbf{p}_i,e^{-2\alpha}\omega_i,v,\mu) =\beta(v)\frac{\partial}{\partial v}\Gamma^{(4)}(e^{-\alpha}\mathbf{p}_i,e^{-2\alpha}\omega_i,v,\mu)\ , \quad (3.20)$$

or equivalently as an operator equation reflecting an anomalous trace of the energy and momentum tensor:

$$\sum_{i=1}^{2}\theta_{ii}=2\theta_{00}+\frac{v^2}{16\pi}\phi^\dagger\phi^\dagger\phi\phi\ . \quad (3.21)$$

IV. CONCLUSION

The (2+1)-dimensional NR field theory with quartic self-interaction is exactly solvable, at least up to the two-particle scattering amplitude and bound state. The theory is classically scale invariant, but acquires an anomaly upon quantization of the fields and renormalization of the scattering amplitude. The expression for the anomaly is exact, in contrast with *relativistic* scale anomalies, which can only be expressed to finite order in perturbation theory. Another NR scalar field theory with the above properties is the $\lambda(\phi^\dagger\phi)^3$ theory in 1+1 dimensions.

When the NR scalar theory is coupled to a Chern-Simons gauge field it is suggested that the degree of divergence of the theory is reduced, and renormalization is no longer required [6].

It is yet to be determined whether the field theoretic approach to NR quantum mechanics will yield useful results for more than two particles. I leave these calculations and the inclusion of Chern-Simons gauge field interactions for future work.

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