Quantum collapse of a self-gravitating shell: Equivalence to Coulomb scattering

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A spherically symmetric thin shell of dust with a fixed rest mass M is considered as a model for gravitational collapse in general relativity. For a special choice of the time variable, the dynamical equations of the shell have the same form as those of a charged relativistic particle moving radially in an external Coulomb potential. The critical charge of the Coulomb potential, $Z=137/2$, corresponds to the rest mass M of the shell attaining the Planck mass value M_P. A boundary condition for wave functions at the singularity is determined by requiring that the Klein-Gordon product and the total energy be conserved. This leads uniquely to the spectrum of the relativistic "scalar hydrogen" obtained long ago by Sommerfeld, if $Z=137(M^2/2M_P^2)$ is substituted for the "central charge." All stationary wave functions are expressed by means of standard special functions. The scattering states are symmetric under time reversal for arbitrarily high energies. In particular, their asymptotic form shows that precisely the same amount of probability and energy comes out as was sent in. This is surprising, because energy and/or information losses down black holes are to be expected. The full solvability and the analogy to the charged particle does not, however, automatically remove some interpretational problems typical for quantum gravity.

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I. INTRODUCTION

In quantum gravity, progress has often been achieved by studying simplified, solvable models (see, e.g., [1,2]). For this reason, fields or strings on lower-dimensional spacetimes (as, e.g., [3,4]), lower-dimensional theories of gravity (as [5]), or finite-dimensional minisuperspace models (as [6,7]) abound in the literature.

In the present paper, we study another system with a finite number of degrees of freedom: a spherically symmetric thin self-gravitating shell (we follow papers [8,9]). Our motivation is twofold. First, as with any solvable minisuperspace model, the dynamics of the shell helps us to clarify certain conceptual problems. Second, some of the spacetimes in which the shell moves contain an asymptotically flat region; thus, unlike in cosmological models, one can investigate the scattering of matter by the gravitational interaction. This scattering is a quantum version of gravitational collapse, and as such it invokes the problem of singularities and of the irreversibility of the collapse due to the formation of black holes [10].

Our shell is made out of incoherent dust. Let us first call attention to the fact that the three-dimensional stress-energy tensor of any spherically symmetric shell is necessarily isotropic and describes thus an ideal fluid. Not to contaminate the dynamics by a physical force, we put the surface tension equal to zero. This reflects the idea of matter particles that interact only through gravitation.

In the present era of strings and membranes (whose action is most naturally chosen to be proportional to the n-volume of the "n-brane") one may feel forced to justify the use of dust. While the action that is proportional to the three-volume is physically sensible for, say, domain walls between the false and true vacua, it does not lead to the familiar form for gravitational collapse that we would like to study. The reason is that, roughly speaking, the rest mass of a three-brane is proportional to the second power of its radius. Hence, the membrane will in general reach its gravitational radius by expanding.

In Ref. [9], the time coordinate for the shell dynamics was chosen to be the proper time along the shell history. The corresponding Hamiltonian operator

$$H_9 = M\cosh\left[-\frac{i}{M}\frac{\partial}{\partial R}\right]-\frac{m^2}{2R}$$

[for the classical version, see Eq. (2)] has been shown to possess a positive self-adjoint extension if the rest mass M of the shell is smaller than about one Planck mass M_P

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($m = M/M_P$). In this case, a unitary quantum theory was constructed. The corresponding boundary condition, on a boundary of the configuration space, not of a spacetime, kept the shell away from the singularity. However, many questions remained open, such as the following: How many self-adjoint extensions exist below the Planck mass? Are there any physically meaningful theories above the Planck mass? Does the shell possess any bound states and what is their spectrum? Are there scattering states and what is the S matrix?, etc. One difficulty was that the Schrödinger equation was not a differential equation—it contained derivatives of all orders.

In the present paper we recast the problem into a much more elegant and natural form by a clever choice of the time variable. The classical system then becomes formally identical to a relativistic charged particle with zero spin which is radially falling in a Coulomb potential. A quantum version of this problem was studied in the 1920's by Sommerfeld [11], and later by Bethe [12]. Both Sommerfeld and Bethe looked for solutions of the Klein-Gordon equation with a Coulomb potential of the form $m^2/2R$:

$$\left[-i\frac{\partial}{\partial T}-\frac{m^2}{2R}\right]^2\Psi+\frac{\partial^2}{\partial R^2}\Psi-M^2\Psi=0 \; .$$

This corresponds to the Wheeler-DeWitt equation for the shell, and it is this method of quantization on which we shall focus our attention in the present paper. Another option is to take the corresponding square-root Hamiltonian

$$H_{13}=\left[-\frac{\partial^2}{\partial R^2}+M^2\right]^{1/2}-\frac{m^2}{2R}$$

[see Eq. (3) of Sec. II] as was done in the electromagnetic case by Herbst [13]. (Thus, altogether there are three nonequivalent quantum treatments of the same system, two of them associated with the Hamiltonians H_9, H_{13}, and the third one with the Wheeler-DeWitt equation above.) Sommerfeld almost threw away all s waves because they did not seem to him to be sufficiently regular at the origin, but, in spite of his misgivings, he wrote down the spectrum anyway. Bethe found two different spectra, but he kept only Sommerfeld's because the matrix elements of the kinetic energy between the states corresponding to the other spectrum seemed to him to diverge. Bethe's argument is, however, inconclusive because the self-adjoint extensions of various operators of the model do not act as the "naive" differential operators which he uses. (This follows from the existence of a so-called "weak zero mode," and will be explained in the next paper, Ref. [14].) As a by-product of the present paper, we find an independent, self-consistent and rather simple argument in favor of Sommerfeld's spectrum.

In constructing our quantum mechanics, we cannot avoid some well-known conceptual problems of quantum gravity. First of all, by choosing different time coordinates and quantization methods mentioned above, we obtain quantum theories which are not unitarily equivalent to each other. In particular, the theory of the Hamiltonian operator H_{13} is not unitarily equivalent to that of H_9. This reflects the "multiple choice problem" of time [2]. The nonequivalence of the quantum theory associated with H_{13} from that of the present paper based on the Wheeler-DeWitt equation is due to different factor orderings (see [2]). One should note, however, that these different quantum theories have at least some qualitative features in common, especially the existence of a unitary evolution (generated by a positive Hamiltonian) if the rest mass of the shell is lower than about one Planck mass as well as the symmetry of the scattering states under time reversal.

Another problem is how to construct a Hilbert space from solutions to the Wheeler-DeWitt equation (this is the "Hilbert space problem" of Ref. [2]).

For our particular model, we can use the method described in Ref. [15], which is based on the existence of time symmetry (the conservation of energy), and on the positivity of energy. There is, of course, the question of how general such a method can be: in full (classical) general relativity, there is no such time symmetry [16]. However, at least for the broad class of asymptotically flat spacetimes, there could be such a symmetry [17], albeit only for the asymptotic evolution, and, moreover, the total energy *is* classically well defined, and it *is* positive [18] for all systems in this class.

The plan of the paper is as follows. In Sec. II, we write the basic equations of the classical version of the model as they were given in Refs. [8,9]. Then, we introduce the time coordinate such that the new equation is formally identical with the dynamical equation of a spinless charged relativistic massive particle in a Coulomb potential. In Sec. III, we write down the corresponding Wheeler-DeWitt equation and use the formal analogy to the classical charged scalar field in a static external potential to find two conserved currents: that of charge and that of total energy. We discuss the physical interpretation of these two currents in the quantum theory of the shell. Finally, we study a time-reversal transformation. In Sec. IV, we fix the boundary condition at the singularity by requiring the global conservation of probability and of energy. We find that these requirements can be satisfied only if the rest mass of the shell is strictly smaller than the Planck mass. The boundary condition allows us to complete the Dirac quantization program by constructing a well-defined unitary time evolution with positive Hamiltonian generator on a well-defined Hilbert space. The details of this construction will be given in Ref. [14]. In the present paper, we simply write down the resulting formula for the Hilbert space inner product. This enables us to interpret the positive-energy solutions as quantum wave functions. In Sec. V, using the boundary condition, we rederive the Sommerfeld spectrum and write down the corresponding wave functions in terms of standard special functions. The wave functions are all invariant under our time-reversal operation. Finally, in Sec. VI, we draw those conclusions which are reasonable at the present rudimentary stage of the theory, and list a number of open problems. Throughout the paper, units are chosen such that $\hbar = c = 1$.

II. THE MODEL

In this section, we briefly recapitulate the definition of the model (following Ref. [9]). Then, we perform the promised change of time variable and write down the corresponding Hamiltonian and super-Hamiltonian.

The world sheet Σ of the shell is a three-dimensional hypersurface that separates the spacetime into two parts—the inside and the outside. The inside spacetime is a part of flat, Minkowskian, spacetime; the outside spacetime is a part of the Kruskal spacetime with the value E/M_P^2 of its mass parameter (which has the dimension of length). The two spacetimes must induce the same internal geometry on Σ (for more details, see [19] or [20]). In the coordinates τ, ϑ, and φ adapted to the spherical symmetry of Σ, the internal metric has the form

$$ds^2 = -d\tau^2 + R^2(\tau)(d\vartheta + \sin^2\vartheta\, d\varphi^2) .$$

Here, τ is the proper time along the shell and $R(\tau)$ is the radius of the shell at the time τ. The three-dimensional energy-momentum tensor S_{kl} of the shell has the components

$$S_{kl} = \frac{M}{4\pi R^2}\delta_k^0\delta_l^0 ,$$

where the real constant M is the total rest mass of the shell. In analogy with a single relativistic particle, we will consider M as a parameter which distinguishes different dynamical systems from each other. Hence, unlike E, M is fixed and not a dynamical variable.

Einstein's equations imply the equation of motion for the shell in the form of a first integral:

$$E = M\left[1 + \left[\frac{dR}{d\tau}\right]^2\right]^{1/2} - V(R) , \tag{1}$$

where the potential $V(R)$ is given by

$$V = \frac{m^2}{2R} ,$$

and m is the dimensionless parameter associated with the rest mass of the shell, $m = M/M_P$. Hence, the energy is indeed conserved, and the system is symmetric with respect to time translations $\tau \to \tau + \delta\tau$. We remark that E is not positive for all possible values of R and $\dot{R}$. However, the spacetime geometries corresponding to $E < 0$ do not contain any asymptotically flat region (this is explained in Ref. [9]). Thus, they cannot communicate with infinity, and no perpetua mobile can be constructed; Witten's positivity theorem is not violated.

If one looks for the Hamiltonian H_9 that generates the motion determined by Eq. (1) (evolution in the proper time τ), and such that the value of H_9 is equal to that of E, one obtains [9]

$$H_0 = M\cosh(P_R/M) - V(R) , \tag{2}$$

where P_R is the momentum conjugate to R.

In the present paper, we choose a different time variable. Let T be the Minkowskian time coordinate in the flat spacetime inside the shell. Physically, T is measured by clocks synchronized by light signals with the central clock at $R = 0$ [20]. Then,

$$\left[\frac{dT}{d\tau}\right]^2 - \left[\frac{dR}{d\tau}\right]^2 = 1$$

and

$$\left[\frac{dR}{d\tau}\right]^2 = \left[\frac{dR}{dT}\right]^2\left[1 - \left[\frac{dR}{dT}\right]^2\right]^{-1} .$$

Substituting this into Eq. (1), we obtain

$$E = M\left[1 - \left[\frac{dR}{dT}\right]^2\right]^{-1/2} - V .$$

However, this is the energy of the radial motion of an ordinary relativistic particle with rest mass M in a Coulomb potential $V(R)$. It is easy to verify that the corresponding motion is generated by the super-Hamiltonian

$$h = -(P_T - V)^2 + P_R^2 + M^2$$

on the extended phase space spanned by the variables T, R, and P_T, P_R, where P_T is the momentum conjugate to T. The conserved value of P_T coincides with $-E$. By solving the Hamiltonian constraint $h = 0$ for $-P_T$, we identify the true Hamiltonian

$$H_{13} = \sqrt{P_R^2 + M^2} - V . \tag{3}$$

The corresponding Schrödinger equation is again difficult to solve [the potential $V(R)$ does not commute with the square root, so that the Schrödinger equation is not equivalent to a partial differential equation of second order]. The spectrum of the Hamiltonian H_{13} has been studied in Ref. [13]. In the present paper, we base the quantum theory of the shell on the super-Hamiltonian h, which is quadratic in all momenta.

III. WHEELER-DeWITT EQUATION AND ITS SYMMETRIES

In the present section, we perform the first step in quantizing the shell by the Dirac method. We write down the Wheeler-DeWitt equation, and use its formal identity with the field equation of a spherically symmetric classical charged scalar field in a Coulomb potential to find the conserved currents.

Let us choose the polarization so that the states will be described by complex functions $\Psi(T,R)$ of T and R. Then, the Wheeler-DeWitt equation corresponding to the super-Hamiltonian h reads

$$\left[-i\frac{\partial}{\partial T} - V\right]^2\Psi + \left[\frac{\partial}{\partial R}\right]^2\Psi - M^2\Psi = 0 . \tag{4}$$

In Eq. (4), we chose the simplest factor ordering of the one-dimensional Laplacian: one which is symmetric with respect to the measure dR. As has been explained in Ref. [9], we are free in the choice of the measure because, if we reorder the Laplacian to be symmetric with respect to a new measure, we get a unitarily equivalent theory. In particular, we could have used the measure

$d\mu(R)=4\pi R^2 dR$ associated with the interpretation of R as a radial coordinate in a three-dimensional space.

Equation (4) is formally identical to the *classical* field equation for a spherically symmetric charged scalar field in a Coulomb potential, written in the Coulomb gauge. Equation (4) follows from the action

$$S=\tfrac{1}{2}\int dR\, dT[\eta^{\mu\nu}(\overline{\nabla_\mu\Psi})(\nabla_\nu\psi)-M^2\overline{\Psi}\Psi] , \tag{5}$$

where $\eta^{\mu\nu}$ is the metric tensor of a two-dimensional Minkowskian spacetime with the pseudo Cartesian coordinates T and R, ∇_μ is the U(1)-covariant derivative,

$$\nabla_\mu\Psi=\partial_\mu\Psi+iA_\mu\Psi , \quad A_\mu=(V,0) ,$$

and the overbar denotes complex conjugation. The action (5) is invariant under the gauge transformation

$$A_\mu\to A_\mu+\partial_\mu\Lambda ,$$
$$\Psi\to e^{-i\Lambda}\Psi ,$$

and also, if the potential A_μ is time independent (as it is in our case), under the time shift $T\to T+\Delta T$. The corresponding conserved Noether currents are the charge current J^μ and the energy current E^μ:

$$J^\mu=\frac{\delta S}{\delta A_\mu} ,$$
$$E^\mu=\frac{\partial L}{\partial\Psi_{,\mu}}\dot{\Psi}+\frac{\partial L}{\partial\overline{\Psi}_{,\mu}}\dot{\overline{\Psi}}-\delta^\mu_0 L ,$$

where L is the Lagrange function in the action (5). These can be generalized into conserved bilinear currents constructed from any two solutions Φ and Ψ of the Wheeler-DeWitt equation (4):

$$J^0(\Phi,\Psi)=-\frac{i}{2}\left[\overline{\Phi}\frac{\partial}{\partial T}\Psi-\Psi\frac{\partial}{\partial T}\overline{\Phi}\right]+V\overline{\Phi}\Psi ,$$
$$J^1(\Phi,\Psi)=\frac{i}{2}\left[\overline{\Phi}\frac{\partial}{\partial R}\Psi-\Psi\frac{\partial}{\partial R}\overline{\Phi}\right] ,$$
$$E^0=\tfrac{1}{2}(\dot{\overline{\Phi}}\dot{\Psi}+\overline{\Phi}'\Psi'+M^2\overline{\Phi}\Psi-V^2\overline{\Phi}\Psi) ,$$
$$E^1=-\tfrac{1}{2}(\overline{\Phi}'\dot{\Psi}+\dot{\overline{\Phi}}\Psi') .$$

Indeed, the Noether currents which one immediately obtains from the action are only a special case of the bilinear currents, corresponding to $\Psi=\Phi$. The conservation of the bilinear currents follows immediately from that of Noether's—one just has to replace Ψ by $\Psi+c\Phi$, and use the arbitrariness of the complex number c. We call the conserved bilinear forms the "charge form" $q(\Phi,\Psi)$ and the "energy form" $e(\Phi,\Psi)$:

$$q(\Phi,\Psi)=\int_0^\infty dR\, J^0(\Phi,\Psi) ,$$
$$e(\Phi,\Psi)=\int_0^\infty dR\, E^0(\Phi,\Psi) .$$

The names "charge" and "energy" form originate in the analogy with the classical charged field. But what is the meaning of these forms for the Wheeler-DeWitt equation (4) that describes a quantum system which, physically, is *not* charged? The charge form is, of course, nothing else but the well-known Klein-Gordon product. (This will be used in Ref. [14] for the construction, along the lines of Ref. [15], of the physical inner product, which will define the final Hilbert space of the system—see also Sec. IV.)

The energy operator $-\mathbf{P}_T$ commutes with the super-Hamiltonian operator of Eq. (4): there are "common eigenstates" [15]. This is due to the time-translation symmetry of the equation and leads to its separability. The separating ansatz is

$$\Psi_E(T,R)=e^{-iET}\psi(R) . \tag{6}$$

Then, Eq. (4) is equivalent to the "radial" equation

$$\psi''+\left[(E^2-M^2)+\frac{m^2E}{R}+\frac{m^4}{4R^2}\right]\psi=0 . \tag{7}$$

The name "energy form" for e is justified, since we can now show that

$$e(\Psi_{E'}\Psi_E)=Eq(\Psi_{E'}\Psi_E)+\text{boundary terms} \tag{8}$$

and

$$q(\Psi_{E'},\Psi_E)=\text{boundary terms} \tag{9}$$

for all $E\neq E'$. Thus, if the "boundary terms" in Eqs. (8) and (9) vanish, then the states Ψ_E diagonalize both the q and the e forms and the diagonal elements of the e form are equal just to the corresponding values of the energy, if the states are q normalized.

To derive Eqs. (8) and (9), we substitute Eq. (6) for the wave functions in the expression for the charge and energy forms and currents:

$$q(\Psi_{E'}\Psi_E)=e^{i(E'-E)T}\int_0^\infty dR\left[\frac{E+E'}{2}+V\right]\overline{\psi}_{E'}\psi_E , \tag{10}$$

$$e(\Psi_{E'},\Psi_E)=\tfrac{1}{2}e^{i(E'-E)T}\times\int_0^\infty dR[\overline{\psi}'_{E'}\psi'_E+(M^2-V^2+E'E)\overline{\psi}_{E'}\psi_E] , \tag{11}$$

$$J^1(\Psi_{E'},\Psi_E)=-(i/2)e^{i(E'-E)T}(\overline{\psi}'_{E'}\psi_E-\overline{\psi}_{E'}\psi'_E) , \tag{12}$$

$$E^1(\Psi_{E'},\Psi_E)=(i/2)e^{i(E'-E)T}(E'\overline{\psi}'_{E'}\psi_E-E\overline{\psi}_{E'}\psi'_E) . \tag{13}$$

Then, from Eq. (7),

$$\overline{\psi}'_{E'}\psi'_E+(M^2-V^2+E'E)\overline{\psi}_{E'}\psi_E=(\overline{\psi}_{E'}\psi'_E)'+E(E+E'+2V)\overline{\psi}_{E'}\psi_E .$$

This, together with Eqs. (10) and (11), yields Eq. (8).

To derive Eq. (9), we multiply Eq. (7) with $\psi=\psi_E$ by $\overline{\psi}_{E'}$ then interchange E with E', complex conjugate, and subtract the result from the original expression. We obtain

$$(E-E')\left[\frac{E+E'}{2}+V\right]\overline{\psi}_{E'}\psi_E=\tfrac{1}{2}(\overline{\psi}'_{E'}\psi_E-\overline{\psi}_{E'}\psi'_E)' .$$

Integration of this equation over $(0,\infty)$ gives immediately Eq. (9).

Strictly speaking, one should integrate over a large spherical box $(0,R_0)$ and the boundary terms in Eqs. (8) and (9) will arise both from the origin $R=0$ and from the wall $R=R_0$ of the box. However, we assume the usual infinite box limit and shall only be concerned with boundary terms arising from the origin.

Similarly, for any two solutions Φ and Ψ of the Wheeler-DeWitt equation, the charge and energy forms will be independent of T, if some boundary terms vanish. A detailed study of boundary conditions is postponed to the next section.

Another important symmetry of Eq. (4) is the time-reversal transformation $\mathbf{T}$:

$$\mathbf{T}\Psi(T,R)=\bar{\Psi}(-T,R) .$$

The behavior of the currents under the time reversal is given by

$$J^0(\mathbf{T}\Phi,\mathbf{T}\Psi)=J^0(\Psi,\Phi) ,$$

$$J^1(\mathbf{T}\Phi,\mathbf{T}\Psi)=-J^1(\Psi,\Phi) ,$$

$$E^0(\mathbf{T}\Phi,\mathbf{T}\Psi)=E^0(\Psi,\Phi) ,$$

$$E^1(\mathbf{T}\Phi,\mathbf{T}\Psi)=-E^1(\Psi,\Phi) .$$

The solutions of form (6) transform in the following way:

$$\mathbf{T}(e^{-iET}\psi)=e^{-iET}\bar{\psi} .$$

Thus, $\mathbf{T}$ preserves the energy; in particular, it leaves invariant the spaces which are spanned only by the positive- or only by the negative-energy states. The complex conjugate of a solution to the radial equation (7) is again a solution and, as usual, it corresponds to the time reversal of the original solution.

IV. BOUNDARY CONDITION AT THE SINGULARITY

In this section, we will study possible choices of boundary conditions at the singularity $R=0$ which make the boundary terms in Eqs. (8) and (9) vanish. We will find that the global conservation of the energy and probability determines such a condition uniquely. We briefly mention how this boundary condition leads to a well-defined Hilbert space inner product on the corresponding positive-energy solutions.

From the point of view of the ordinary differential equation (7), the point $R=0$ is a regular singular point. We set

$$\psi(R)=R^{\lambda}\varphi(R) , \tag{14}$$

where $\varphi(0)$ is nonzero and regular, and obtain the characteristic equation for λ,

$$\lambda(\lambda-1)+\frac{m^4}{4}=0 ,$$

with the roots

$$\lambda_{\pm}=\tfrac{1}{2}\pm\tfrac{1}{2}\sqrt{1-m^4} .$$

Hence, the leading terms at the origin will have the form

$$\psi_E(T,R)\simeq u(E)(MR)^{\lambda_+}+v(E)(MR)^{\lambda_-} , \tag{15}$$

where $u(E)$ and $v(E)$ are some constants.

These constants can be fixed by a boundary condition at the origin. It seems natural to require that the charge current and the energy current through the boundary $R=0$ both vanish. This condition is necessary and sufficient for the total energy and charge to be conserved.

In fact, the boundary "point" $R=0$ of the configuration space can mean two quite different boundaries in the classical spacetime that has the shell as its classical source—namely, it can mean either a past or future singularity. This is similar to the situation at the configuration space boundary $R=\infty$, which also does not distinguish the two timelike infinities, i^+ and i^-, of the spacetime from each other. Thus, the condition of vanishing currents through $R=0$ can lead to a correlation between the incoming current at the future Schwarzschild singularity with the outgoing current at the past one. Let us stress that as long as one insists that the configuration space of a constrained system is the domain on which the wave function is defined, one is not free to distinguish the past and the future spacetime singularities, and one is *forced* to impose the boundary condition on the configuration space only.

Let us calculate the corresponding leading terms in J^1 using Eqs. (12), (13), and (15):

$$J^1(\Psi_{E'},\Psi_E)\simeq(\bar{\lambda}_+-\lambda_+)\bar{u}(E')u(E)(MR)^{\bar{\lambda}_++\lambda_+-1}+(\bar{\lambda}_--\lambda_-)\bar{v}(E')v(E)(MR)^{\bar{\lambda}_-+\lambda_--1}$$
$$+(\bar{\lambda}_+-\lambda_-)\bar{u}(E')v(E)(MR)^{\lambda_-+\bar{\lambda}_+-1}+(\bar{\lambda}_--\lambda_+)\bar{v}(E')u(E)(MR)^{\lambda_++\bar{\lambda}_--1} ,$$

and

$$E^1(\Psi_{E'},\Psi_E)\simeq(E\bar{\lambda}_+-E'\lambda_+)\bar{u}(E')u(E)(MR)^{\bar{\lambda}_++\lambda_+-1}+(E\bar{\lambda}_--E'\lambda_-)\bar{v}(E')v(E)(MR)^{\bar{\lambda}_-+\lambda_--1}$$
$$+(E\bar{\lambda}_+-E'\lambda_-)\bar{u}(E')v(E)(MR)^{\lambda_-+\bar{\lambda}_+-1}+(E\bar{\lambda}_--E'\lambda_+)\bar{v}(E')u(E)(MR)^{\lambda_++\bar{\lambda}_--1} .$$

The following three cases should be treated separately.

(1) $m<1$. Then $\lambda_+\neq\lambda_-$, both roots being real. It follows that

$$J^1(\Psi_{E'},\Psi_E)\simeq(\lambda_+-\lambda_-)[\bar u(E')v(E)-\bar v(E')u(E)] ,$$

$$E^1(\Psi_{E'},\Psi_E)\simeq\lambda_+(E-E')\bar u(E')u(E)(MR)^{2\lambda_+-1}+\lambda_-(E-E')\bar v(E')v(E)(MR)^{2\lambda_--1}+(E\lambda_+-E'\lambda_-)\bar u(E')v(E)+(E\lambda_--E'\lambda_+)\bar v(E')u(E) .$$

As $2\lambda_--1<0$ and $2\lambda_+-1>0$, $\bar v(E')v(E)$ has to vanish for all $E'\neq E$. It follows that $v(E)=0$ for all E, and Eq. (15) implies

$$\psi_E(R)\simeq u(E)(MR)^{\lambda_+} . \tag{16}$$

This is Sommerfeld's boundary condition.

(2) $m=1$. Then $\lambda_+=\lambda_-=\frac{1}{2}$, $J^1|_{R=0}$ vanishes, and

$$E^1\simeq(E-E')[\bar u(E')+\bar v(E')][u(E)+v(E)] .$$

This implies immediately that $\psi=0$, and there is no nontrivial solution to Eq. (7) for which the energy and charge are simultaneously conserved.

(3) $m>1$. Then $\bar\lambda_\pm=\lambda_\mp=\frac{1}{2}\pm i\nu$, and we obtain

$$J^1(\Psi_{E'},\Psi_E)\simeq(\lambda_+-\lambda_-)[\bar u(E')u(E)-\bar v(E')v(E)] .$$

Hence,

$$v(E)/u(E)=e^{2i\alpha} , \tag{17}$$

where α is independent of E. Similarly,

$$E^1(\Psi_{E'},\Psi_E)\simeq\bar u(E')u(E)\left\{[(E-E')-2i\nu(E+E')]+[(E-E')+2i\nu(E+E')]\frac{\bar v(E')}{\bar u(E')}\frac{v(E)}{u(E)}+[(E-E')-2i\nu(E-E')]\frac{v(E)}{u(E)}(MR)^{-2i\nu}+[(E-E')+2i\nu(E+E')]\frac{\bar v(E')}{\bar u(E')}(MR)^{2i\nu}\right\} .$$

By using (17), we obtain easily

$$E^1(\Psi_{E'},\Psi_E)\simeq\bar u(E')u(E)[1+m^2\cos(\arctan 2\nu)+2\alpha+2\nu\ln(MR)] .$$

There is no α for which this expression vanishes at $R=0$, and so there is no boundary condition for solutions of Eq. (7) which guarantees the conservation of both the charge and the energy.

The boundary condition (16) together with the radial equation (7) determines uniquely a definite set $\{\Psi_E(T,R)\}$ of solutions. One can then complete the Dirac quantization program along the lines proposed in Ref. [15] by constructing the quantum Hilbert space $\mathcal{H}$ from wave packets made from the positive-energy solutions in $\{\Psi_E(T,R)\}$. The Hilbert space inner product is then defined by

$$(\Psi_E,\Psi_{E'})_{\mathcal{H}}=q(\Psi_E,\Psi_{E'}) .$$

One should show that such wave packets are complete in the sense that any function of R (satisfying suitable falloff conditions) may be expressed as the restriction of such a wave packet to, say, the $T=0$ surface. Further, the same construction then represents dynamics by a unitary time evolution with a positive quantum Hamiltonian generator. The details of this construction will be given in Ref. [14] where it will be necessary to adopt a somewhat different formalism based from the start on wave packets.

If the rest mass of the shell is larger than the Planck mass, then the wave function changes its behavior: it becomes oscillatory near the origin. We will see in Ref. [14] that there is then no unitary quantum theory of the shell, at least within the framework we have employed. The change of the behavior of the wave functions if the rest mass crosses the Planck border is analogous to what happens with the wave function of the relativistic electron (Dirac equation) or relativistic charged spin-0 particle (Klein-Gordon equation) in a Coulomb field of a central charge larger than $Z=1/(2\alpha)$ (α is the fine-structure constant, $\alpha\simeq\frac{1}{137}$). One can find two explanations of what happens in the literature (these are not contradictory to each other): Landau and Lifshitz [21] speak about a "fall of the electron into the center." This catastrophic behavior of the wave function is associated in Ref. [21] with the well-known nonexistence of relativistic quantum mechanics for strong potentials. This is emphasized even more strongly by Bjorken and Drell [22], where the oscillatory behavior of the wave function is classified as a Klein-paradox effect. In the real world, there is no such effect. Quantum field theory must be applied to the system so that pairs will be created or the vacuum will be polarized by the strong potential, and the potential will decay or will be screened. This seems to indicate that in quantum gravity, one has to "second quantize" the shell. However, such a procedure is not viable if one wants to perform it in full formal analogy to the second quantization of a charged particle. Indeed, the map corresponding to charge conjugation sends the wave functions satisfying Eq. (4) into functions obeying an analogous equation with the reversed potential $-V$. The potential for the scalar field of charge q in an external Coulomb potential of charge Q is proportional to the product qQ. The sign change of the potential can thus be naturally interpreted: the antiparticle has the charge $-q$. However, the shell potential contains the factor $m^2/2$ in place of qQ. There is no conceivable shell for which this quantity could be negative. Considerations like these assign quite a new role to the Planck mass in the quantum theory of collapsing objects. This must be studied carefully, and one should not restrict oneself from the beginning just to the solutions suggested by the Coulomb analogy. In principle, one could also try to abandon the positivity of energy. (The operators corresponding to the Hamiltonians H_9 and H_{13} written down in the Introduction will have nonpositive self-adjoint extensions above the Planck mass.) In fact, the classical theory has negative-energy states which cannot communicate with infinity, so that no paradoxes can be based on their existence. One should check if this property can be carried over into a quantum theory with a nonpositive Hamiltonian.

V. SOLUTIONS TO THE RADIAL EQUATION

In this section, we will describe the solutions to the radial equation (7) satisfying the boundary condition (16) for each $M<M_P$. We shall restrict our attention to "bound states" and "scattering states," where by bound states we mean solutions which decay exponentially at infinity, and by scattering states we mean solutions which are oscillatory at infinity. For the physical interpretation of the quantum theory of the shell, it is important to restrict oneself to the positive-energy solutions. Then, the states will be "bound" or "scattering" in the sense of the quantum theory on the Hilbert space $\mathcal{H}$ mentioned in the previous section. The bound states and their spectrum were found long ago by Sommerfeld [11]. We rederive these results and calculate the wave functions of the scattering states, using the accumulated knowledge of special functions, as given, e.g., in Ref. [23]. As far as we know, the results on scattering states are published here for the first time.

A. Scattering states

Let us assume that $E^2 \geq M^2$. If $E^2>M^2$, we can define a dimensionless variable ρ by

$$\rho=\sqrt{E^2-M^2}R \ .$$

Equation (7) then takes the form

$$\psi''+\left[1-2\eta\rho^{-1}+\frac{m^4}{4}\rho^{-2}\right]\psi=0 \ , \tag{18}$$

where

$$\eta=\frac{m^2E}{2\sqrt{|E^2-M^2|}} \ , \tag{19}$$

and the primes denote derivatives with respect to ρ. Equation (18) is called the Coulomb wave equation (see Ref. [23], p. 538). It has precisely one solution satisfying condition (16): the Coulomb wave function $F_{\lambda_+-1}(-\eta,\rho)$ with index λ_+-1. Here η can be either positive or negative, depending on the sign of E. The Coulomb wave functions are well defined for either $-\infty<E<-M$ or $M<E<\infty$; these intervals define the continuum part of the spectrum. For the shell, only the positive-energy solutions are relevant. The negative ones play a role in the corresponding electrodynamics problem. The Coulomb wave function, both of whose arguments are real and whose index is real, is a real function. Thus, the scattering states are invariant under the time reversal of Sec. III.

The Coulomb wave functions have been tabulated only for non-negative integral values of the index; in the nonrelativistic theory, this is just the angular momentum number l. (As already observed by Sommerfeld, solutions of the relativistic theory are given by the same special functions, but with rescaled arguments and shifted indices.) Our index, λ_+-1, lies, however, between $-\frac{1}{2}$ and 0. Thus, to study the behavior of the solution near the Schwarzschild radius, one should put the problem on a computer. Fortunately, the asymptotic form of the solution can be obtained from that of the Kummer function $M(a,b,x)$. This is related to the Coulomb wave function by

$$F_{\lambda-1}(\eta,\rho)=C_{\lambda-1}(\eta)\rho^{\lambda}e^{-i\rho}M(\lambda-i\eta,2\lambda,2i\rho) \ ,$$

where $C_{\lambda-1}(\eta)$ is a real constant (see Ref. [23], p. 538). Formula 13.5.1 on page 508 of Ref. [23] yields

$$F_{\lambda-1}(\eta,\rho)=C_{\lambda-1}(\eta)\left[\frac{2^{-\lambda+i\eta}e^{(\pi/2)(\eta+i\lambda)}e^{-i(\rho-\eta\ln\rho)}}{\Gamma(\lambda+i\eta)}\right][1+O_1(\rho^{-1})]$$
$$+C_{\lambda-1}(\eta)\left[\frac{2^{-\lambda-i\eta}e^{(\pi/2)(\eta-i\lambda)}e^{i(\rho-\eta\ln\rho)}}{\Gamma(\lambda-i\eta)}\right][1+O_2(\rho^{-1})] \ .$$

This is a particular combination of an ingoing and an outgoing wave: the shell bounces at the singularity with a well-defined phase shift. The asymptotic behavior is spoiled by logarithmic terms because the Coulomb potential is long range. To define the scattering matrix, one typically introduces "the distorted free Hamiltonian" (see, e.g., Refs. [24,25]). Then, the scattering matrix will be uniquely determined by the above asymptotic expression. We will not go into details here.

Next, let us set $E=\pm M$. Equation (7) then becomes

$$\psi''+[\pm m^2R^{-1}+m^4(2R)^{-2}]\psi=0 \ .$$

We introduce a new variable ρ by

$$\rho=\sqrt{\pm 4m^2MR} \ ,$$

and a new function $w(\rho)$ by $\psi=\rho w(\rho)$. Then, w satisfies the Bessel equation of order $\kappa=2\lambda-1$. The solution which is regular at the origin is

$$\psi=\rho I_\kappa(\rho) \ .$$

For large ρ, the leading term in the positive-energy solutions is proportional to $\rho^{-1/2}\cos(\rho-\pi\kappa/2-\pi/4)$, whereas the negative-energy solution increases exponentially.

B. Bound states

Let us assume that $E^2<M^2$, define

$$\rho=2\sqrt{M^2-E^2}R ,$$

and set

$$\psi=e^{-\rho/2}\rho^{\lambda}w(\rho) .$$

Equation (7) then implies an equation for w:

$$\rho w''+(2\lambda-\rho)w'-(\lambda-\eta)w=0 . \quad (20)$$

This is known as the Kummer equation (or confluent hypergeometric equation). The only solution to this equation with a behavior at infinity that is appropriate for a bound state is

$$w(\rho)=U(\lambda_+-\eta,2\lambda_+,\rho)$$

where $U(a,b,\rho)$ is Kummer's function defined in Ref. [23], p. 504. Then, asymptotically, $\psi\simeq e^{-\rho/2}$. The behavior of $\psi(R)$ at the origin is determined by formulas 13.1.2 and 13.1.3 of Ref. [23]:

$$\psi(R)\simeq\frac{\Gamma(-2\nu)}{\Gamma(\lambda_--\eta)}2^{\lambda_+}(1-E^2M^{-2})^{\lambda_+/2}(MR)^{\lambda_+} + \frac{\Gamma(2\nu)}{\Gamma(\lambda_+-\eta)}2^{\lambda_-}(1-E^2M^{-2})^{\lambda_-/2}(MR)^{\lambda_-} , \quad (21)$$

where ν is given by

$$\nu=\tfrac{1}{2}\sqrt{1-m^4} .$$

Condition (16) applied to the wave function (21) implies that E must satisfy the condition

$$\Gamma(\lambda_+-\eta)=\pm\infty \quad (22)$$

because Γ does not vanish on the real axis. Since the poles of Γ are at nonpositive integers, all solutions to Eq. (22) are given by

$$\eta=\lambda+n , \quad (23)$$

where n is a non-negative integer. Then,

$$w(\rho)=\frac{n!\Gamma(2\lambda)}{\Gamma(2\lambda+n)}L_n^{(2\lambda-1)}(\rho) ,$$

where $L_n^{(\alpha)}(x)$ is a generalized Laguerre polynomial (see Ref. [23], p. 775). By substituting expression (23) for η into Eq. (19), we obtain the energy spectrum of the bound states:

$$E_n=M\frac{2(\lambda+n)}{\sqrt{m^4+4(\lambda+n)^2}} .$$

In all these formulas, one should set $\lambda=\lambda_+$. The lowest positive energy is easily calculated to be

$$E_0=M\lambda^{1/2} , \quad (24)$$

with the corresponding (unnormalized) wave function

$$\psi_0=e^{-\rho/2}\rho^{\lambda} . \quad (25)$$

What is the meaning of a wave function such as the ground-state wave function (25)? We know that in the Klein-Gordon theory, the wave functions are *not* probability amplitudes for the particle to have a given coordinate. To calculate such probabilities, one should first define a position operator, find its eigenstates, and expand the wave functions in these eigenstates. For a free particle, this was done by Newton and Wigner (see, e.g., Ref. [26]; a generalization for a field in an external potential was proposed in Ref. [27]). In spite of all this caution, let us find the maximum of the wave function (25) and tentatively interpret the corresponding value of the radius R as giving order-of-magnitude information about the approximate position of the shell. Let the maximum of the wave function (25) be achieved for the value $\rho_{\max}$ of ρ. Equation (25) determines

$$\rho_{\max}=2\lambda ,$$

or

$$R_{\max}=2\left[\frac{\lambda^{1/2}}{m}\right]^3 M_P^{-1} .$$

The Schwarzschild radius of the spacetime classically generated by the shell with energy E is $R_S(E)=2E/M_P^2$. In particular, if the shell has the ground-state energy,

$$R_S(E_0)=2m\lambda^{1/2}M_P^{-1} .$$

The ratio of the two radii $R_{\max}$ and R_s is a decreasing function of m satisfying

$$\infty>R_{\max}/R_s>\tfrac{1}{2}$$

for all $m\in[0,1]$. Thus, the bound states seem to lie well outside the Schwarzschild radius only if m is small with respect to 1.

VI. CONCLUSIONS

In the last section, we have found the system $\{\Psi_E(T,R),E>0\}$ of wave functions which satisfy the Klein-Gordon constraint (4) under the boundary condition (16). This, as stated in Sec. IV, is the main ingredient in constructing the unitary quantum mechanics with positive energy on the Hilbert space $\mathcal{H}$.

One of the most interesting results (to anticipate the completion of the program in Ref. [14]) is that there *is* a quantum theory of the self-gravitating shell (at least for small rest masses) compatible with the requirements of unitarity, of positivity of energy and of energy conservation, and that this theory seems to be *unique*. The uniqueness question is, however, both important and subtle. One must check carefully whether or not there are any technical assumptions on which our present results are based for which there is no good physical reason. For example, the form of the charge and energy functionals is fixed only for wave functions which are zero in some neighborhoods of the singularity $R=0$ [that is, for functions from $\mathscr{C}_0^\infty(\mathbb{R}_+)$]. One can speculate that the extensions of these forms to C^∞ functions on $[0,\infty)$ can differ from our "naive" forms by some boundary terms, like $\bar{f}(0)f'(0)$, etc. Self-consistency of such proposals can be studied only within a more elaborate mathematical framework. We thus postpone further discussion of this question to Ref. [14].

The spectrum obtained in the foregoing section contains bound states with discrete energy levels between $E_0 \geq (\frac{1}{2})^{-1/2}M$ and M. Such bound states do not exist in the classical theory. They are a direct consequence of our boundary condition in the *configuration* space of the shell which implies a particular correlation between the wave function describing the shell coming out of the "initial" singularity and the wave function describing the shell going into the "final" singularity in the associated classical *spacetime*. One possible description of this result may be that the uncertainty principle of quantum mechanics leads to an effective repulsive force which prevents the shell from being squeezed into a small volume. However, one should be careful about this interpretation: One would intuitively expect such forces to succeed only if they keep the shell in the bound state hovering well outside its Schwarzschild radius, which we were not yet able to show.

The quantum scattering states have energies in the interval (M,∞). They are quite definite linear combinations of ingoing and outgoing waves. There are no losses of probability and energy. This means, in particular, that there are no losses down the black holes. This is a surprising result. Indeed, the unitary quantum mechanics exists for all values of the incoming energy of the shell (only the rest mass must be smaller than the Planck mass). The Schwarzschild radius of a shell with arbitrarily large energy is, however, itself arbitrarily large. It is quite stunning that no part of such an energetic shell would go down into the hole. One can trace this unintuitive result back to the condition at the boundary $R=0$ of the configuration space that leads to superposition of two wave functions, one of which describes the collapse down a black hole, and the other one the time reversal of such a process, that is, an emergence of the shell from a white hole. (One has to keep in mind that the arena for quantum dynamics of the shell is its configuration space rather than any spacetime. Therefore, it is misleading to draw a Penrose-Kruskal diagram and to search for a "smeared trajectory" of the quantum shell therein.) As a result of our boundary condition, the quantum theory seems to acquire more time-reversal symmetry than the original classical one had because the scattering states are invariant under the time-reversal transformation; this is not the case for the classical scattering solutions, which describe either a shell collapsing from past infinity to the future singularity or a time reversal of this process.

We have been unable to construct a reasonable quantum theory if the rest mass of the shell is not smaller than the Planck mass. The behavior is analogous to that of a relativistic particle moving in a supercritical potential.

The next question which naturally arises is the following: What are the physical predictions of our "unique and existing" theory? This is quite a difficult question because we do not know how the wave functions are to be interpreted. So far, we have only one observable: the energy. The energy operator commutes with the super-Hamiltonian, and so it *is* an observable even according to the traditionally stringent (and quite likely misguided [28]) criteria. It is not difficult to turn it into a self-adjoint operator (this is again postponed to Ref. [14]). However, even for a system which has only one degree of freedom (as our system has), there are a lot of physically interesting questions, other than the question of what is the conserved energy of the system, which one would like to ask and answer.

The most interesting among these concerns the position of the shell. Consider questions such as does the shell in a scattering state cross the horizon? To spell out what this question means, we need to have a position (radius) operator. As already mentioned, multiplication by R is *not* such an operator, and our wave function is *not* a probability amplitude for the corresponding value of R. For a free theory, one can use the Newton-Wigner construction, but for our system with a potential term, there are problems. An observable must commute with the projector onto the positive part of the spectrum, and it must be symmetric with respect to the charge form. Clearly, the momentum P_R, the radius R, or, say, the variable RP_R (which may be more suitable for quantization on the half-axis [29]) are not such quantities. An approximate solution to this problem, as given in Ref. [27], may be useful.

Suppose next that we could calculate the position operator representation of our stationary states. It turns out that even then it will not be simple to give the above question about the shell crossing the horizon a precise meaning. We could try to use the energy operator to get information about the spacetime geometry. Indeed, in the classical version of the theory, the total energy is related to the Schwarzschild mass parameter. Thus, if we know the energy of the shell, we seem to know everything about the geometry *outside* the shell. However, the wave functions of the stationary scattering states will be smeared over all values of the radius and, hence, one not only does not know where the shell is, but there is no region which could be called "outside the shell." If we try to localize the shell by forming wave packets, to know better where the shell is, and to be sure that there is an approximately "shell-free" region including infinity, we would have to smear the energy, and consequently lose information about the size of the Schwarzschild radius. We clearly need more specific observables: some suitable operators giving direct information about the spacetime geometry. However, the spacetime geometry is described by quantities which play the role of "dependent variables" in the canonical theory (analogous to the Coulomb field in electrodynamics). Thus, in order to obtain any information about geometry, we need relations determining the geometry in terms of the true degrees of freedom. Such relations are not yet at our disposal because we have circumvented the minisuperspace reduction by guessing the super-Hamiltonian of the reduced system directly from the equations of the motion. We intend to perform such a reduction procedure and to calculate the geometrical observables in another paper.

One might think that one could construct the scattering (i.e., asymptotic) observables more easily. However, even here, there are unexpected problems. First, the construction is complicated by the long-range potential. For example, the "free" Hamiltonian is a nonlocal, time-dependent operator (see Refs. [24,25]). Second, the variables R and T are not suitable for the asymptotic theory: in the classical version of the model, they coincide with the Minkowskian coordinates of the shell as measured by an observer that is situated *inside* the shell. An observer far *outside* the shell will observe the Schwarzschild coordinates t and r. Across the shell $R(\tau)=r(\tau)$, but $t(\tau)$ depends on the dynamical state of the shell through the relation [20]

$$\dot{T}^2-\dot{R}^2=[1-2E/(RM_P^2)]\dot{t}^2-[1-2E/(RM_P^2)]^{-1}\dot{R}^2 .$$

Thus, one can define a function of the variables T, R, P_T, and P (that is, a function on the phase space of the system), which will coincide with $\dot{t}$ if the equations of motion are satisfied. By integration, differences of t along a classical orbit can be calculated from $\dot{t}$, but nothing more. This is understandable because if a piece $(T(\tau),R(\tau))$, $\tau\in(a,b)$, of a classical orbit can be matched with a piece $(t(\tau),r(\tau))$, $\tau\in(a',b')$, of hypersurface in the Schwarzschild spacetime, then it can also be matched with $(t(\tau)+\Delta t,r(\tau))$, $\tau\in(a',b')$, where Δt is any constant. It is not clear to what extent this is only a technical difficulty rather than a manifestation of a deeper problem.

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