Phase evolution of the photon in Kerr spacetime

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In this paper, we explore some aspects of the gravitational lens effects due to a Kerr black hole. Under the eikonal approximation of the Maxwell equations in curved space, the spin function of a photon in the degenerate metric is determined. Furthermore, we present an investigation of the phase factor that a photon acquires in Kerr spacetime. The resulting phase consists of two parts: a real and an imaginary one. The real part has been interpreted as contributing a rotational angle of the plane polarization for linearly polarized light, and the imaginary one results in the light intensity amplification along with the photon's trajectory in the gravitational field. Finally, we provide the so-called "Sagnac factor" related to the phase shift.

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I. INTRODUCTION

With Berry's elegant analysis, a new nonintegrable phase factor was recognized in an adiabatic process of a quantum system [1]. Recently much attention to this phase and its analogies has been paid [2]. In fact, it has been realized that Berry's phase is, in essence, topological and it is related to the holonomy group of a line bundle over the space of parameters [3]. The first experimental suggestion was proposed by Chiao and Wu [4]. In their proposal, a laser beam propagating down a helical single-mode optical fiber will acquire Berry's phase; this phase contributes a rotation angle for a linearly polarized electromagnetic wave. Subsequently, the experiment performed by Tomita and Chiao [5] justified this expected optical activity.

Another phase factor concerned with the dragging effect of an inertial frame, called the Sagnac factor, was noted more than 70 years ago [6]. In general relativity, in a stationary gravitational field, a clock cannot be uniquely synchronized for accelerated observers at different points; thus, as one proceeds along any closed path, the Sagnac factor appears due to the time delay. By definition of synchronization, the phase is [7]

$$\gamma_s = \omega \oint \frac{g_{0i}}{g_{00}} dx^i \ , \tag{1.1}$$

where ω is the frequency of the photon. In the simple case of a uniformly rotating disk, two beams of light propagating along two lines which form a certain contour will interfere with each other since each beam gets a different extra phase. The net phase received by a photon after it follows a loop is proportional to the area of the loop projected onto the plane perpendicular to the direction of the angular velocity. Namely, the phase is $4\omega\mathbf{\Omega}\cdot\mathbf{A}$, where $\mathbf{\Omega}$ is the angular velocity and $\mathbf{A}$ is the surface area vector. The Sagnac factor is, in fact, the coupling effect between the classical angular momentum of the photon and the rotation of the reference frame.

On the other hand, the phase shift of photons in a rotating frame of reference has been discussed by Feng and his colleague [8]. It is shown that, apart from Berry's phase and the Sagnac factor, the coupling of the intrinsic spin of the photon with the rotation gives rise to an extra phase shift

$$\gamma_n = \pm \mathbf{\Omega} \cdot \int \frac{d\mathbf{x}}{1 - \mathbf{g}\cdot\boldsymbol{\kappa}} \ , \tag{1.2}$$

where $g = \{g_{0i}\}$, κ is the unit wave vector, and $\pm$ corresponds to the two helicity states of the photon. Similar to Berry's phase, this phase induces an optical rotation for linearly polarized light. It is reasonable to expect that this term will be significant in a strong rotating gravitational field.

The gravitational lens effect due to a Kerr black hole has been widely discussed in the current literature. The majority of works done so far were mainly concerned with problems such as deflection of light rays, amplification of intensity, and differential phase shift. In this paper, we develop a unified scheme to deal with these problems.

In Sec. II, adopting Hanni's definition [9], we write the Maxwell equations and cast them into a Schrödinger-like equation in three-dimensional curved space and then in the degenerate metric [10].

Although in the particular situation when the typical wavelength is much less than the inhomogeneous or anisotropic length scale of the medium, the wave theory provides a complete description of physical phenomena in nature; the correspondence principle permits us to make a simple analysis in terms of particle interpretation. The traditional approach is based on the eikonal approximation [11], well known as the WKB method or geometrical optics approximation. The procedure of the eikonal approximation may be simply summarized as follows: (1) the zero-order approximation gives a disperson relation which induces a Hamiltonian structure that determines the ray system associated with the wave, and (2) the equation at the first order determines the evolution of amplitude, phase, and polarization of the wave along the classical trajectory of the particle. In the spirit of this method, Sec. III begins with the derivation of the effective Hamiltonian for a photon in the eikonal limit and of its eigenstate. From the dispersion relation we derive the Hamilton-Jacobi equation for calculating the photon's orbit in the degenerate metric. Section IV is devoted to finding the overall phase factor of a photon. In Sec. V, we summarize the paper, make some comments, and provide discussion. Throughout the present paper we use the natural units in which $\hbar=c=1$ and we take the signature $(1,-1,-1,-1)$ for the spacetime metric.

II. THE MAXWELL EQUATIONS IN CURVED SPACE

A. General form in curved space

According to different definitions of electromagnetic vectors, there have been several ways to write the Maxwell equations in curved space. In this paper, we adopt Hanni's definitions in which the spatial metric is defined as

$$\gamma_{ij}=-g_{ij}\ ,\quad \gamma^{ij}=-g^{ij}+g^{0i}g^{0j}/g^{00}\ , \tag{2.1}$$

and the three-dimensional (3D) dragging vector is

$$g^{i}=g^{i0}/g^{00}\ ,\quad g_i=g_{0i}\ . \tag{2.2}$$

Identifying electromagnetic vectors as

$$\begin{aligned} &E_i=F_{i0}\ ,\quad B^i=\epsilon^{ijk}F_{jk}/2\sqrt{\gamma}\ ,\\ &D^i=F^{0i}/\sqrt{g^{00}}\ ,\quad H_i=\sqrt{\gamma}\epsilon_{ijk}F^{jk}/2\sqrt{g^{00}}\ , \end{aligned} \tag{2.3}$$

where γ is the determinant of the 3D metric γ_{ij}, we may write the Maxwell equations in the noncovariant form

$$\begin{aligned} &\nabla\times\mathbf{E}=-(\sqrt{\gamma}\mathbf{B})_{,t}/\sqrt{\gamma}\ ,\quad \nabla\cdot\mathbf{B}=0\ ,\\ &\nabla\times\mathbf{H}=(\sqrt{\gamma}\mathbf{D})_{,t}/\sqrt{\gamma}\ ,\quad \nabla\cdot\mathbf{D}=0\ , \end{aligned} \tag{2.4}$$

with the constitutive equations

$$\begin{aligned} &\mathbf{D}=(g^{00})^{1/2}(\mathbf{E}+\mathbf{g}\times\mathbf{B})\ ,\\ &\mathbf{B}=(g^{00})^{1/2}(\mathbf{H}+\mathbf{D}\times\mathbf{g})\ . \end{aligned} \tag{2.5}$$

Formally, the Maxwell equations (2.4) in curved space may be understood to describe an electromagnetic wave propagating in an inhomogeneous medium while the dielectric tensor of the medium is given by the constitutive equations (2.5) [12].

By defining the wave function of the photon as $|\Psi\rangle=\mathbf{D}+i\mathbf{B}$, Eq. (2.4) is cast into the form of a Schrödinger-like equation,

$$i\frac{\partial}{\partial t}|\Psi\rangle=\nabla\times(|\Psi\rangle/\sqrt{g^{00}}+i\mathbf{g}\times|\Psi\rangle)\ , \tag{2.6a}$$

together with the transverse condition,

$$\nabla\cdot|\Psi\rangle=0\ , \tag{2.6b}$$

where the constitutive equations have been used. It must be noted that all of the above equations were calculated working in the three-dimensional curved space with metric γ_{ij}.

By denoting $g=(g^{00}\gamma)^{-1/2}$ and by making the replacement $|\Psi\rangle\to|\Psi\rangle/\sqrt{\gamma}$, it follows from Eq. (2.6) that

$$\begin{aligned} i\frac{\partial}{\partial t}|\Psi\rangle=&g(\kappa\cdot\mathbf{s})|\Psi\rangle^c+(\mathbf{g}\cdot\kappa)|\Psi\rangle+\nabla g\times|\Psi\rangle^c\\ &+i(|\Psi\rangle\cdot\nabla)\mathbf{g}-i(\nabla\cdot\mathbf{g})|\Psi\rangle\ , \end{aligned} \tag{2.7}$$

where $\kappa=-i\nabla$ is the momentum operator, $s=\{s^i\}$ is the photon's spin operator given by the adjoint representation of SO(3), i.e., $(s^i)^{jk}=-i\epsilon^{ijk}$, and $|\Psi\rangle^c$ is the contravariant three-vector corresponding to $|\Psi\rangle$, namely, $|\Psi\rangle^c_i=\gamma_{ij}|\Psi\rangle^j$. The transverse equation then takes the form $\kappa\cdot|\Psi\rangle=0$.

B. The Maxwell equation in the degenerate metric

For the purpose of the present paper, it is convenient to use the degenerate metric

$$g_{\mu\nu}=\eta_{\mu\nu}-2ml_\mu l_\nu\ ,\quad \eta^{\mu\nu}l_\mu l_\nu=0\ , \tag{2.8}$$

where m is an arbitrary constant. For Kerr spacetime, we choose m as the mass of the central compact object. Hereafter we set $R_s=2m=1$.

Let us briefly review some useful properties of the degenerate metric. For a detailed discussion see Ref. [10]. It is easy to see that the matrix $g^{\mu\nu}$ is given by

$$g^{\mu\nu}=\eta^{\mu\nu}+l^\mu l^\nu\ , \tag{2.9}$$

where the contravariant four-vector l^μ corresponding to l_μ is

$$l^\mu=g^{\mu\nu}l_\nu=\eta^{\mu\nu}l_\nu\ . \tag{2.10}$$

In the stationary case, one may introduce a three-vector λ_j via the equation

$$l_\mu=l_0(1,\lambda)\ , \tag{2.11}$$

where λ is a flat-space unit vector, i.e., $\lambda^2=1$. From Einstein's equation it follows that

$$\partial_i \lambda_j = \alpha(\delta_{ij} - \lambda_i \lambda_j) - \beta \epsilon_{ijk} \lambda_k \ , \tag{2.12}$$

where two parameters α, β have been introduced. From (2.12), we have

$$\nabla \cdot \lambda = 2\alpha \ , \tag{2.13}$$

$$\nabla \times \lambda = -2\beta \lambda \ . \tag{2.14}$$

Defining a complex function $\Gamma = \alpha + i\beta$, we may show that Einstein's equation reduces to a simple partial differential equation for Γ,

$$\nabla \Gamma = -\Gamma^2 + i(\nabla \Gamma \times \lambda) \ , \tag{2.15}$$

or, equivalently, in the form of Laplace and eikonal equations,

$$\nabla^2 \Gamma = 0 \ , \quad (\nabla \chi)^2 = 1 \ , \quad \chi = \frac{1}{\Gamma} \ . \tag{2.16}$$

Meanwhile we have

$$l_0^2 = \mathrm{Re}(\Gamma) = \alpha \ . \tag{2.17}$$

Using the above algebraic properties of the degenerate metric, we have $\mathbf{g} = \Omega \lambda$, $\Omega = g - 1$, $g = 1/(1+\alpha)$, and $|\Psi\rangle^c = |\Psi\rangle + \alpha \lambda (\lambda \cdot |\Psi\rangle)$. Finally, Eq. (2.7) can be written in the explicit form

$$i \frac{\partial}{\partial t} |\Psi\rangle = g(\kappa \cdot \mathbf{s}) |\Psi\rangle + \mathbf{g} \cdot \kappa |\Psi\rangle + i\alpha g(\kappa \times \lambda)(\lambda \cdot |\Psi\rangle) + H_s |\Psi\rangle \ , \tag{2.18}$$

where

$$\begin{aligned} H_s |\Psi\rangle = {} & \nabla g \times |\Psi\rangle + g \nabla \beta (\lambda \cdot |\Psi\rangle) + i(|\Psi\rangle \cdot \nabla g)\lambda \\ & - i(\lambda \cdot \nabla g)|\Psi\rangle + \Omega \Gamma^* (\lambda \times |\Psi\rangle) \\ & - i\Omega \Gamma^* |\Psi\rangle - i\Omega \Gamma \lambda (\lambda \cdot |\Psi\rangle) \ . \end{aligned} \tag{2.19}$$

The Kerr geometry may be described by one of the solutions of Eq. (2.16):

$$\Gamma = [x^2 + y^2 + (z - ia)^2]^{-1/2} \ . \tag{2.20}$$

The scalar functions α, β and the 3D unit vector λ are obtained by straightforward algebraic calculations:

$$\alpha = \frac{\rho}{\rho^2 + \sigma^2} \ , \quad \beta = -\frac{\sigma}{\rho^2 + \sigma^2} \ , \tag{2.21}$$

where

$$\begin{aligned} \rho^2 &= \frac{r^2 - a^2}{2} + \left[\frac{(r^2 - a^2)^2}{4} + a^2 z^2 \right]^{1/2} \ , \\ \sigma &= -\frac{az}{\rho} \ , \end{aligned} \tag{2.22}$$

and λ may be written in the quite simple form

$$\lambda = \frac{\rho}{\rho^2 + a^2} \left[\mathbf{r} + \frac{a^2 z}{\rho^2} \mathbf{e}_k + \frac{a}{\rho} (\mathbf{r} \times \mathbf{e}_k) \right] \ , \tag{2.23}$$

where $\mathbf{e}_k$ is the unit three-vector along the z axis (0,0,1).

III. HELICITY STATE OF THE PHOTON IN THE DEGENERATE METRIC

A. Geometrical optics approximation

Under the geometrical optics approximation, the wave function $\Psi(\mathbf{x}, t)$ of photons characterized by a typical wave number k and a frequency ω, which are large enough in comparison to the spatial and temporal rates of variation of the propagating medium, can be regarded locally as a plane wave. For a Kerr black hole, the length and time scale of the variation of the gravitational field are, respectively, the radius of curvature, i.e., the Schwarzschild radius R_s, and the spin-rotation frequency Ω_s of the black hole. Introducing a small dimensionless parameter ϵ,

$$kR_s \sim \frac{\omega}{\Omega_s} \equiv \epsilon^{-1} \ , \tag{3.1}$$

we can expand $\Psi(\mathbf{x}, t)$ in the powers of ϵ,

$$|\Psi\rangle = \sum_{n=0} \epsilon^n |\Psi_n\rangle \exp\left[\frac{i\Phi}{\epsilon} \right] \ , \tag{3.2}$$

and then define the local wave vector and frequency as

$$\begin{aligned} \mathbf{k}(\mathbf{x}, t) &= \nabla \Phi(\mathbf{x}, t) \ , \\ \omega(\mathbf{x}, t) &= -\partial_t \Phi(\mathbf{x}, t) \ . \end{aligned} \tag{3.3}$$

Substituting the expansion in (3.2) into the Maxwell equation (2.18) and keeping the leading term only, we have

$$H_0 |\Psi_0\rangle = \omega |\Psi_0\rangle \ . \tag{3.4}$$

Here H_0 is the matrix operator $H_0 = \{H_0^{ij}\}$,

$$H_0^{ij} = gk^l s_l^{ij} + \mathbf{g} \cdot \mathbf{k} \delta^{ij} + i\alpha g (\mathbf{k} \times \lambda)^i \lambda^j \ , \tag{3.5}$$

where the definitions (3.3) have been used. The above equation is the eigenstate equation for a photon in the degenerate metric. In the case of flat space, $\alpha = 0$, $\mathbf{g} = 0$, Eq. (3.5) reduces to the familiar form

$$(\mathbf{k} \cdot \mathbf{s}) |\Psi_0\rangle = \omega |\Psi_0\rangle \ ,$$

and we find two eigenstates, the positive helicity state with the energy eigenvalue $\omega = |\mathbf{k}|$ and the negative one with $\omega = -|\mathbf{k}|$. These two helicity states correspond to the right and left polarization states of the photon, respectively. However, the positivity of energy allows us to have only the positive state. For the negative-energy state one may redefine the wave function as $|\Psi\rangle = \mathbf{D} - i\mathbf{B}$ and obtain the equation $\mathbf{k} \cdot \mathbf{s} |\Psi_0\rangle = -\omega |\Psi_0\rangle$. Accordingly, the opposite helicity states are mutually complex conjugates. The zero helicity state is eliminated by the transverse condition.

Let the unit vectors $\{\mathbf{e}_i\}$, $i = 1, 2, 3$ form an orthogonal tetrad with $\mathbf{e}_3$ along the direction of the wave vector $\mathbf{k}$. Denoting the helicity states $\mathbf{e}_\pm$ as those states which satisfy the helicity eigenstate equation

$$(\mathbf{e}_3 \cdot \mathbf{s}) \mathbf{e}_\pm = \pm e_\pm \ , \tag{3.6}$$

we may express $\mathbf{e}_\pm$ in terms of the real vector basis $\{\mathbf{e}_i\}$,

$$\mathbf{e}_\pm = \mp\frac{1}{\sqrt{2}}(\mathbf{e}_1 \pm i\mathbf{e}_2) \ . \tag{3.7}$$

Because of the transverse condition, $\mathbf{e}_\pm$ span a Hilbert space for the photon's states. Thus, the wave function of the photon may be written as a linear superposition of the two helicity states $\mathbf{e}_\pm$,

$$|\Psi_0\rangle = \xi\mathbf{e}_+ + \eta\mathbf{e}_- \ . \tag{3.8}$$

Here we introduce some useful notations. For an arbitrary 3D vector $\mathbf{v}$, we make component expansions in terms of bases $\{\mathbf{e}_i\}$ and $\{\mathbf{e}_\pm, \mathbf{e}_3\}$, respectively. Define $v_i = \mathbf{v}\cdot\mathbf{e}_i$ and $v_\pm = \mp\mathbf{v}\cdot\mathbf{e}_\pm$. We define

$$|v_\perp|^2 = v_+ v_- = \tfrac{1}{2}[\mathbf{v}\cdot\mathbf{v} - (\mathbf{v}\cdot\mathbf{e}_3)(\mathbf{v}\cdot\mathbf{e}_3)]$$

and introduce an angle variable γ_v defined by $v_\pm = |v_\perp|\exp(\pm\gamma_v/2)$.

Inserting (3.8) into the eigenstate equation (3.4) and taking the inner product of both sides by $\mathbf{e}_\pm$, we have

$$\begin{aligned} (gk + \alpha gk|\lambda_\perp|^2 - \omega_g)\xi &= \alpha gk\lambda_-^2\eta \ , \\ (gk + \alpha gk|\lambda_\perp|^2 + \omega_g)\eta &= \alpha gk\lambda_+^2\xi \ , \end{aligned} \tag{3.9}$$

where $k = |\mathbf{k}|$, $\omega_g = \omega - \mathbf{g}\cdot\mathbf{k}$. From Eq. (3.9), we have the dispersion relation

$$(gk + \alpha gk|\lambda_\perp|^2)^2 - \omega_g^2 = \alpha^2 g^2 k^2 |\lambda_\perp|^4$$

or, in another form,

$$\omega^2 - k^2 + \alpha(\omega + \lambda\cdot\mathbf{k})^2 = 0 \ . \tag{3.10}$$

By defining a covariant four-vector $k_\mu = -\partial_\mu\Phi = (\omega, -\mathbf{k})$, it is easy to check that the above equation is the null wave-vector equation

$$g^{\mu\nu}k_\mu k_\nu = 0 \tag{3.11}$$

from which we get the geodesic equation $\nabla_{\mathbf{k}}\mathbf{k} = 0$.

We may solve Eq. (3.9) for ξ, η:

$$\begin{aligned} \xi &= \cos\frac{\theta}{2}e^{-i\gamma_\lambda/2} \ , \\ \eta &= \sin\frac{\theta}{2}e^{i\gamma_\lambda/2} \ , \end{aligned} \tag{3.12}$$

where θ is given by

$$\tan\frac{\theta}{2} = \frac{gk + \alpha gk|\lambda_\perp|^2 - \omega_g}{\alpha gk|\lambda_\perp|^2} \tag{3.13a}$$

or, in an equivalent form,

$$\tan\frac{\theta}{2} = \frac{\alpha gk|\lambda_\perp|^2}{gk + \alpha gk|\lambda_\perp|^2 + \omega_g} \ . \tag{3.13b}$$

B. Photon's orbit in Kerr spacetime

In the usual eikonal approximation scheme, the dispersion relation generates a Jacobi-Hamilton structure, which, in turn, is used to derive the particle trajectory in phase space. Here we do not use this approach but an alternative equivalent method. By introducing a super-Hamiltonian [13]

$$H = \tfrac{1}{2}g^{\mu\nu}k_\mu k_\nu \ , \tag{3.14}$$

the photon geodesic is determined by the Hamilton-Jacobi equations

$$\begin{aligned} \dot{x}^\mu &= \frac{\partial H}{\partial k_\mu} \ , \\ \dot{k}_\mu &= -\frac{\partial H}{\partial x^\mu} \ , \end{aligned} \tag{3.15}$$

where the overdots stand for differentiation with respect to an affine parameter. Note that, meanwhile, we also have an on-shell condition $H = 0$ for the photon.

From Eq. (3.15), we find the equations of motion for the photon

$$\dot{\mathbf{x}} = \mathbf{k} - \alpha\lambda(\omega + \lambda\cdot\mathbf{k}) \ , \tag{3.16a}$$

$$\begin{aligned} \dot{\mathbf{k}} = &-\tfrac{1}{2}\nabla\alpha(\omega + \lambda\cdot\mathbf{k})^2 \\ &+ \alpha(\omega + \lambda\cdot\mathbf{k})\{\alpha[\mathbf{k} - \lambda(\lambda\cdot\mathbf{k})] + \beta\mathbf{k}\times\lambda\} \ , \end{aligned} \tag{3.16b}$$

and

$$\omega = \text{const} \ . \tag{3.16c}$$

Since the super-Hamiltonian H is the constant of motion, the constraint condition $H = 0$ can be automatically satisfied as long as we properly select the initial momentum of the photon to satisfy $H = 0$. If the photon were set off at infinity where spacetime is asymptotically flat, we would simply have $\omega = k$ there.

As it was shown by Mashhoon, the photon's trajectory is independent of its polarization state. That is, the spin-spin coupling between the photon and the rotation of the black hole will not make the photon's trajectory deviate from its null geodesic in the regime of geometric optics. Actually, we may consider the classical force $f^a = Dp^a/Ds$ due to the coupling of spin to spacetime curvature, which is written as [14]

$$f_a = \tfrac{1}{2}\epsilon^{cdef}R^{ab}_{cd}S_e v_b v_f \ , \tag{3.17}$$

where R^{ab}_{cd} is the Riemann curvature tensor, v_a is the four-velocity of the particle, and S_a is the spin vector. For a massless particle, its spin orientation must be parallel or antiparallel to its velocity; that is, $S^a = \alpha_s v^a$. It follows from Eq. (3.17) that $f^a = 0$, which means that the world lines of the massless particle are geodesic. Finally, from the condition $S^a v_a = 0$, we have $v^a v_a = 0$ (if S^a is nonzero), namely, the world lines of a massless spinning particle are null geodesics.

IV. PHASE FACTOR OF THE PHOTON IN KERR SPACETIME

Under the geometrical optics approximation, the wave function of a photon depends only on its propagating path length. Let C be the integral curve of $\mathbf{k}$ and let the path length s parametrize this curve. Introducing a phase factor term φ, the wave function is written in the form

$$|\Psi\rangle = \varphi(s)|\Psi_0\rangle \ . \tag{4.1}$$

Using $\nabla = \mathbf{n}_k(\partial/\partial s)$, where $\mathbf{n}_k = \nabla s$ is the unit wave vector, we may check that the above form of the wave function satisfies the transverse condition. Substituting (4.1) into Eq. (2.18), we obtain

$$ig\mathbf{n}_k\cdot\mathbf{s}\frac{\partial}{\partial s}(|\Psi_0\rangle\varphi) + i\mathbf{g}\cdot\mathbf{n}_k\frac{\partial}{\partial s}(|\Psi_0\rangle\varphi) - \alpha g(\mathbf{n}_k\times\lambda)\left[\lambda\cdot\frac{\partial}{\partial s}(|\Psi_0\rangle\varphi)B\right] = H_s|\Psi_0\rangle\varphi \ . \tag{4.2}$$

Using the zero-order Hamiltonian (2.3), this can be rewritten as

$$iH_0\frac{\partial}{\partial s}|\Psi_0\rangle\varphi = kH_s|\Psi_0\rangle\varphi \ . \tag{4.3}$$

Taking the inner product of both sides of the equation with $\langle\Psi_0|$, we have

$$\frac{d\varphi}{ds} = -\frac{\langle\Psi_0|H_0|(d/ds)\Psi_0\rangle}{\langle\Psi_0|H_0|\Psi_0\rangle}\varphi - ik\frac{\langle\Psi_0|H_s|\Psi_0\rangle}{\langle\Psi_0|H_0|\Psi_0\rangle}\varphi \ . \tag{4.4}$$

By letting $\varphi = \exp(i\gamma_p^+)$ and integrating, we finally find

$$\gamma_p^+ = i\int_0^s ds\frac{\langle\Psi_0|H_0|(d/ds)\Psi_0\rangle}{\langle\Psi_0|H_0|\Psi_0\rangle} - \int_0^s ds\frac{\langle\Psi_0|H_s|\Psi_0\rangle}{\mathbf{g}\cdot\mathbf{n}_k + g\sqrt{1+2\alpha|\lambda_\perp|^2}} \ . \tag{4.5}$$

The first term on the right-hand side is clearly Berry's phase. In this paper, we will not discuss this phase in detail. Berry's phase of quantum systems in the presence of gravitational and inertial fields has been extensively discussed by Cai and Papini [15]. In their approach, Berry's phase in a covariant form is given by using the proper-time method. The second term is the phase factor arising from the higher-order corrections of the Maxwell equation under the geometrical optics approximation. It will be shown that this term corresponds to the gravitational lensing and the spin-rotation coupling effect.

Now we turn to calculating the matrix element $\langle\Psi_0|H_s|\Psi_0\rangle$. At first, we note some useful formulas. For any 3D arbitrary vector $\mathbf{v}$, it can be proved that

$$\langle\Psi_0|\mathbf{v}\times|\Psi_0\rangle = -i\langle\Psi_0|\mathbf{v}\cdot\mathbf{s}|\Psi_0\rangle = -i\mathbf{v}\cdot\mathbf{n}_k\cos\theta \tag{4.6}$$

and

$$\mathbf{v}\cdot\Psi_0 = |v_\perp|\left[\left[-\cos\frac{\theta}{2}+\sin\frac{\theta}{2}\right]\cos\left[\frac{\gamma_v-\gamma_\lambda}{2}\right] - i\left[\sin\frac{\theta}{2}+\cos\frac{\theta}{2}\right]\sin\left[\frac{\gamma_v-\gamma_\lambda}{2}\right]\right] , \tag{4.7}$$

where

$$\cos\left[\frac{\gamma_v-\gamma_\lambda}{2}\right] = \frac{\mathbf{v}\cdot\lambda-(\mathbf{v}\cdot\mathbf{n}_k)(\lambda\cdot\mathbf{n}_k)}{2|v_\perp||\lambda_\perp|} ,$$
$$\sin\left[\frac{\gamma_v-\gamma_\lambda}{2}\right] = \frac{(\lambda\times\mathbf{v})\cdot\mathbf{n}_k}{2|v_\perp||\lambda_\perp|} \ . \tag{4.8}$$

After using the above formulas, we may get the explicit form of the matrix element

$$\langle\Psi_0|H_s|\Psi_0\rangle = \Xi_s + i\Xi_a \ , \tag{4.9}$$

where

$$\Xi_s = -\Omega\beta[\sin\theta+(\lambda\cdot\mathbf{n}_k)\cos\theta] + \tfrac{1}{2}\cos\theta(\nabla g\times\lambda)\cdot\mathbf{n}_k + \tfrac{1}{2}g(1-\sin\theta)[\nabla\beta\cdot\lambda-(\lambda\cdot\mathbf{n}_k)(\nabla\beta\cdot\mathbf{n}_k)] \ , \tag{4.10a}$$

$$\Xi_a = -\Omega\alpha[2-\sin\theta+(\lambda\cdot\mathbf{n}_k)\cos\theta] - \lambda\cdot\nabla g - (\nabla g\cdot\mathbf{n}_k)\cos\theta + \tfrac{1}{2}g\cos\theta(\nabla\beta\times\lambda)\cdot\mathbf{n}_k + \tfrac{1}{2}(1-\sin\theta)[\nabla g\cdot\lambda-(\lambda\cdot\mathbf{n}_k)(\nabla g\cdot\mathbf{n}_k)] \ . \tag{4.10b}$$

Defining

$$\gamma_s = \int_0^s ds\frac{\Xi_s}{\mathbf{g}\cdot\mathbf{n}_k + g\sqrt{1+2\alpha|\lambda_\perp|^2}} , \tag{4.11a}$$

$$\gamma_a = \int_0^s ds\frac{\Xi_a}{\mathbf{g}\cdot\mathbf{n}_k + g\sqrt{1+2\alpha|\lambda_\perp|^2}} , \tag{4.11b}$$

we can write the phase factor as

$$\gamma_p^+ = -\gamma_s - i\gamma_a \ . \tag{4.12}$$

The phase factor of the photon then consists of two parts: a real one and an imaginary one. Evidently, the imaginary part causes an inherent amplification of the light intensity along the photon's path of propagation. This is the gravitational lens effect of the photon. The real phase contributes a rotation angle for a linearly polarized light. From Eq. (4.10), we note that the real phase factor depends on the scalar function β. Recall that β describes the rotation of a black hole, thus, this real phase emerges from the coupling of the rotation of a black hole with the photon's spin.

For the negative helicity state of the photon, we may get a phase factor by using the definition $|\Psi\rangle = \mathbf{D} - i\mathbf{B}$ and proceed in the same manner as we did for the positive helicity state. Alternatively, since the opposite helicity states are mutually complex conjugates and γ_p^+ is invariant under the replacement $\mathbf{k}\to-\mathbf{k}$ and $\theta\to\pi-\theta$, we may simply deduce the phase factor of the negative helicity state from the above result to be

$$\gamma_p^- = \gamma_s - i\gamma_a \ . \tag{4.13}$$

The imaginary part is then invariant and the real one has an opposite sign with respect to the positive helicity state. Suppose that a beam of linearly polarized light, emitted by a steady radiation source at infinity, goes through a gravitational field produced by a Kerr black hole. Since the linear polarization state is the superposition of the two helicity states, at any time, the spin function of the photon will be of the form $\mathbf{e}_1\cos\gamma_s - \mathbf{e}_2\sin\gamma_s$ (assuming the initial light was polarized along the direction $\mathbf{e}_1$). Consequently, the polarization plane of the light rotates by an angle γ_s. Meanwhile, the amplitude of the wave has grown by the exponential factor γ_a.

Finally, we consider the dynamical phase factor Φ. From the definition in Eq. (3.3), we have

$$\frac{d\Phi}{ds}=k \ . \tag{4.14}$$

Integrating the above equation gives

$$\Phi=-\omega t+\omega\int_0^s \frac{ds}{\mathbf{g}\cdot\mathbf{n}_k+g\sqrt{1+2\alpha|\lambda_\perp|^2}} \ . \tag{4.15}$$

The phase shift relative to the photon's dynamical phase in flat space is given by

$$\Delta\Phi=\omega\int_0^s \frac{ds}{\mathbf{g}\cdot\mathbf{n}_k+g\sqrt{1+2\alpha|\lambda_\perp|^2}}-\omega s \ , \tag{4.16}$$

which is the Sagnac factor of a photon in the Kerr spacetime.

V. DISCUSSION

As discussed in the previous sections, the angular momentum of a Kerr black hole gives rise to a rotation of the plane of polarization of a linearly polarized light. In fact, according to a rotating observer within the framework of quantum mechanics, this effect could be interpreted as the coupling of the intrinsic spin of the photon with the rotation of a black hole.

The basic consideration has been given by Mashhoon [16]. The Hamiltonian associated with this effect is simply given as

$$\delta H_s=\gamma\mathbf{\Omega}\cdot\mathbf{S} \ , \tag{5.1}$$

where the relativistic factor γ has been introduced to indicate the strength of "interaction" as determined by the rotating observer. As a direct application, he considered the neutron interferometry experiment in a rotating frame of reference where the phase shifts of neutrons both from the spin-rotation coupling and the Sagnac factor give the interference pattern. A similar result of optical activity in a rotating frame of reference was given in Ref. [8] [see Eq. (1.2)]. In the case of a photon propagating through the gravitational field produced by a Kerr black hole, the interaction term describing the coupling between the photon's spin and the rotation of the black hole is included in Eq. (2.19).

Although the expression for γ_s is apparently complex, we may consider a special case to show that this formula is, in fact, consistent with the simple version of the Hamiltonian (5.1) associated with the spin-rotation coupling. For instance, we may consider a photon passing through the gravitational field produced by a Kerr black hole. If the wave vector of the photon is perpendicular to the angular momentum vector of the black hole, the straightforward computation from the geodesic equation of Eqs. (4.12) and (4.13) gives $\gamma_s=0$. This result can simply be interpreted using Eq. (5.1), that is obviously $\mathbf{\Omega}\cdot\mathbf{S}=0$.

In this paper, we have developed a unified scheme to deal with the gravitational lens effect due to a Kerr black hole. Usually, the procedure devoted to the same problem is based on the application of the Walker-Penrose constant [17] to the parallel transport along a geodesic in a rotating gravitational field. By this method, the propagation of the parallel transported vector of polarization is determined. As it has been discussed by Connors *et al.* [18], there are two factors affecting the change of the direction of the polarization vector. One is due to the deflection of the light ray, the other is the additional rotation around the propagation vector caused by the angular momentum. It is noted that the rotation of the polarization plane induced by the angular momentum of a central gravitational source is analogous to the well-known Faraday effect of an electromagnetic wave propagating through a magnetized plasma and thus may be referred to as the "gravitational Faraday rotation."

Attention is called to the difference between the Faraday effect and the spin-rotation coupling. In the latter case, the rotation angle is independent of the frequency of the light. Thus, the gravitational Faraday rotation can be observed only by comparison between two images produced by the gravitational lensing. However, even in this case the complete information about the initial polarization state of the photon is required. The rotation angle of the plane of polarization in the particular case of a linear polarized light ray propagating in a weak gravitational field has been calculated in Ref. [19]. Though it is technically difficult to detect directly this gravitational Faraday rotation from the observational views, this effect is of great interest from a theoretical point of view and also for possible applications to the astrophysics of compact objects. For example, in the context of x-ray astrophysics, the rotation of the plane of polarization induced by a rotating object has been used to investigate the total polarization degree of x-ray radiation emitted from an accretion disk surrounding a Kerr black hole [18,20].

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