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# Two-dimensional black hole physics

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Two-dimensional black hole solutions of string-motivated two-dimensional (2D) gravity are studied. It is shown that the main results of 4D black hole physics (such as the uniqueness theorems, mass formulas, and thermodynamical analogy) have their counterparts in the 2D case. The existence of static solutions describing black holes with tachyon hairs is proved. It is shown that the entropy of a charged 2D black hole is $2\pi \exp(\Phi_H)$, where $\Phi_H$ is the value of the dilaton field at the black hole horizon.

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## I. INTRODUCTION

Black holes are regions in spacetime with a strong gravitational field from where no information-carrying signals can escape to infinity. The idea of black holes has proved to be highly fruitful. Originally, black holes arose as special solutions of the Einstein equations. Later, it was understood that the black holes may play an important role in the Universe. The theoretical discovery by Hawking of quantum radiation from black holes provided us with understanding that black holes may also play the role of a "Rosetta stone" to relate gravity, quanta, and thermodynamics.

Many properties of black holes are directly connected with the nontrivial causal structure of spacetime in the presence of a black hole. There are also a number of fundamental results (such as the increasing of the black hole surface [1], uniqueness theorems [2–6], mass formulas, and thermodynamic analogy [7]), which for their proof require the Einstein equations.[1] An interesting problem is how far these results can be extended to other versions of the theory of gravity. Recently, this problem has become important for string theory because it was found that black-hole-like solutions may play an important role in it.

The problem of the unification of quantum mechanics with general relativity is almost as old as both these theories. A quite remarkable development in solving this problem was proposed by string theory. This theory not only provides us with manifestly covariant reliable perturbative calculations in second-quantized theory, but also gives hope for treating non-perturbative effects, such as the change of spacetime topology.

The evolution of a classical string is described by a two-dimensional (2D) sheet in a $d$-dimensional target space. If $g$ is an external metric in the target space, then the conditions of self-consistency of quantum string equations require that the metric $g$ obey special relations. These relations, determined by setting the $\beta$ function to zero, can be obtained also by varying some effective action [9]. This effective action contains the curvature. It describes the theory of gravity, which is a generalization of Einstein theory and which (for bosonic strings) in addition to the gravitational field $g$ contains also the dilation field $\Phi$ and tachyon field $T$ (as well as some other fields).

In this construction one can choose the number $d$ of target space dimensions arbitrarily. The case when $d=2$ plays an important role. It is singled out by the property that, because of the absence of the transversal directions, a string can not vibrate. As a consequence, the string-motivated 2D gravity contains only one dynamical degree of freedom, which is connected with the tachyon field. It was shown by Witten [10] and Mandel, Sengupta, and Wadia [11] that in the absence of the tachyon field string-motivated 2D gravity allows black-hole-like solutions. These solutions, their properties, and quantum effects in their presence appeared to be highly important for string theory [10–14]. On the other hand, in the framework of these 2D toy models, the most intriguing problems of black hole physics (such as singularities, loss of quantum coherence, and final state of an evaporating black hole) can be analyzed. Many papers on this subject appeared recently. Without pretending to be complete, we refer only to some of them [15–27].

In this paper[2] we would like to stress another side of the problem. Namely, string-motivated 2D gravity provides us with a nontrivial example of the theory which allows the same nontrivial causal structure of spacetime as black holes in general relativity, while the basic equations of the theory are quire different. The possibility of a rather detailed study of 2D string-motivated gravity allows one to obtain and compare those results of "standard" black hole physics which are based on the equations.

The main aim of this paper is to develop the theory of 2D black holes. It is remarkable that many basic results

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[1]For a general discussion, see [8] and references therein.

[2]This paper is an extended version of the invited talk given at the "Black Holes, White Holes, and Wormholes—A Symposium in Honour of Werner Israel" (May 20–23 1992, Banff).

 

of "standard" 4D black hole physics find their counterpart in the 2D case.

This paper is organized as follows. Our starting point (the 2D effective action and basic equations) is formulated in Sec. II. The uniqueness theorem for 2D black holes in the absence of a tachyon field is proved in Sec. III. The mass formula for 2D black holes is obtained in Sec. IV. Static solutions with the tachyon field are discussed in Secs. V and VI. It is shown that the complete system of 2D equations for the gravitational, dilaton, and tachyon fields can be reduced to one nonlinear second-order equation for the tachyon field ("the master equation"). The existence of regular solutions of the master equation which generate black hole metrics with a nonvanishing tachyon field is proved. The analytic continuation of static 2D black hole solutions to the Euclidean region is discussed in Sec. VII. The reduced Euclidean action approach developed in Refs. [28–31] is used to relate the entropy of a charged 2D black hole with the value of the dilaton field at the horizon. Four laws of 2D black hole physics are formulated and discussed in Sec. VIII.

The sign conventions of Misner, Thorne, and Wheeler (MTW) [32] and the units in which $G=\hbar=c=1$ are used.

## II. BACKGROUND THEORY AND BASIC EQUATIONS

Our starting point is the action

$$S=S[g,\Phi,T,A]$$
$$\equiv\tfrac{1}{2}\int_V d^2x\sqrt{-g}\,e^{\Phi}(R+L)-\int_{\partial V}dx\sqrt{|h|}\,e^{\Phi}K\ , \tag{2.1}$$

where

$$L=(\nabla\Phi)^2-|DT|^2+\mu^2|T|^2+\lambda-\tfrac{1}{2}F_{\mu\nu}F^{\mu\nu}\ , \tag{2.2}$$

$$D_\mu=\nabla_\mu T-ieA_\mu T\ , \tag{2.3}$$

$$F_{\mu\nu}=\nabla_\mu A_\nu-\nabla_\nu A_\mu\ . \tag{2.4}$$

This action is to be considered as describing 2D gravity gravity, $\Phi$, $T$, and $A_\mu$ being the dilaton, tachyon, and electromagnetic field, respectively. The surface term is added in Eq. (2.1) in order to make the variation procedure self-consistent. The quantity $h$ is the induced metric on $\partial V$, and $K$ is its extrinsic curvature.

In the absence of an electromagnetic field, the action Eq. (2.1) coincides with the effective action for closed bosonic strings [9–11,33,34]. We include the electromagnetic field into the action (2.1) because many results of 2D black hole physics can be easily obtained in the general case of charged black holes. On the other hand, the consideration of charged black holes allow us to trace more deeply the analogy between 2D black hole solutions and 4D ones. This action (for uncharged tachyons) naturally arises in the 2D heterotic string theory [35].

The following system of equations is obtained by varying the action (2.1):

$$T_{\mu\nu}=0\ , \tag{2.5}$$

$$\Box\Phi+(\nabla\Phi)^2-\tfrac{1}{2}(R+L)=0\ , \tag{2.6}$$

$$e^{-\Phi}D^\mu(e^{\Phi}D_\mu T)+\mu^2T=0\ , \tag{2.7}$$

$$(e^{\Phi}F^{\mu\nu})_{;\nu}=e^{\Phi}J^\mu\ , \tag{2.8}$$

where

$$T_{\mu\nu}=e^{\Phi}[\Phi_{;\mu\nu}+D_{(\mu}T\bar{D}_{\nu)}\bar{T}+F_{\mu\alpha}F_\nu{}^\alpha-g_{\mu\nu}U]\ , \tag{2.9}$$

$$U=\Box\Phi+\tfrac{1}{2}(\nabla\Phi)^2+\tfrac{1}{2}|DT|^2-\tfrac{1}{2}\mu^2|T|^2-\tfrac{1}{2}\lambda+\tfrac{1}{4}F_{\alpha\beta}F^{\alpha\beta}\ , \tag{2.10}$$

$$J_\mu=-\frac{ie}{2}(\bar{T}D_\mu T-T\bar{D}_\mu\bar{T})\ . \tag{2.11}$$

This system can be simplified. The electromagnetic-field strength being antisymmetric, the tensor can be written in the form

$$F_{\mu\nu}=Fe_{\mu\nu}\ , \tag{2.12}$$

where $e_{\mu\nu}=e_{[\mu\nu]}$ and $e_{01}=(-g)^{1/2}$. The electromagnetic-field equation reads now

$$(e^{\Phi}F)_{,\alpha}=-e_{\alpha\nu}e^{\Phi}J^\nu \tag{2.13}$$

and

$$T_{\mu\nu}=e^{\Phi}[\Phi_{;\mu\nu}+D_{(\mu}T\bar{D}_{\nu)}\bar{T}-\tfrac{1}{2}g_{\mu\nu}(R+2F^2)]\ . \tag{2.14}$$

Instead of Eq. (2.6), one may use the following relation which follows from it and from the trace equation $T^\mu_\mu=0$:

$$(\nabla\Phi)^2-|DT|^2+R-\mu^2|T|^2-\lambda+3F^2=0\ . \tag{2.15}$$

Before considering a general case, it is worthwhile making the following remark. In the absence of the tachyon and electromagnetic fields, the basic system of 2D gravity equations reads

$$\Phi_{;\mu\nu}-\tfrac{1}{2}g_{\mu\nu}R=0\ , \tag{2.16}$$

$$(\nabla\Phi)^2+R=\lambda\ . \tag{2.17}$$

It is possible to show that the solutions of these equations are identical to the solutions of the theory described by the action

$$\tilde{S}=\tilde{S}[g]=\int_V d^2x\sqrt{-g}\,R\ln|R|\ , \tag{2.18}$$

provided one puts $\Phi=-\ln|R|$. In the latter case, the parameter $\lambda$ arises as the integration constant.

The action (2.18) may be considered as the special limiting case ($\gamma\to 0$) of a nonlinear in curvature two-dimensional gravitational theory,

$$\tilde{S}'=\tilde{S}'[g]=\int_V d^2x\sqrt{-g}\,R^{1+\gamma}\ , \tag{2.19}$$

discussed in Ref. [36].

## III. TACHYON-FREE SOLUTIONS. UNIQUENESS THEOREM

In this section we show that the tachyon field is the only dynamical one in the theory (2.1). In the absence of this field, the solutions are static; i.e., they allow a Killing vector. To prove this we note that for $T=0$ the 2D gravity-field equations are of the form

$$\Phi_{;\mu\nu}=\tfrac{1}{2}g_{\mu\nu}(R+2F^2)\ . \tag{3.1}$$

We assume at first that $\Phi_{;\alpha}\neq 0$. Then Eq. (3.1) implies that the vector field

$$\xi^{\mu}\equiv -e^{\mu\alpha}\Phi_{;\alpha} \tag{3.2}$$

obeys the Killing equation $\xi_{(\mu;\nu)}=0$. Moreover, Eqs. (2.13) and (3.2) show that

$$\xi^{\mu}\Phi_{;\mu}=0\ ,\quad \xi^{\mu}F_{;\mu}=0\ , \tag{3.3}$$

and hence the dilaton and electric fields are constant along the Killing trajectories. This result may be considered as a 2D version of Birkhoff's theorem.[3]

In two-dimensional space, the tensor $\xi_{[\mu}\xi_{\nu;\lambda]}$ vanishes identically, and according to Frobenius's theorem, one has

$$\xi_{\mu}=\beta\eta_{,\mu}\ . \tag{3.4}$$

The scalar functions $\eta$ and $\Phi$ can be chosen as coordinates. One can use Eq. (3.1) to show that $\beta=B$, and the metric in these coordinates takes the form

$$ds^2=-Bd\eta^2+\frac{d\Phi^2}{B}\ . \tag{3.5}$$

The general solution to the electromagnetic-field equation for $T=0$ reads

$$F=Qe^{-\Phi}\ . \tag{3.6}$$

Taking into account that the curvature $R$ for the metric Eq. (3.5) is

$$R=-\frac{d^2B}{d\Phi^2}\ , \tag{3.7}$$

one can find the solution of Eq. (3.1) for $\mu=\nu=0$ in the form

$$B=B_0+B_1e^{-\Phi}+Q^2e^{-2\Phi}\ . \tag{3.8}$$

The other equations are satisfied, provided the integration constant $B_0$ is chosen so that

$$B_0=\lambda\ . \tag{3.9}$$

The obtained solution can be rewritten in a more familiar way by introducing new coordinates

$$t=\alpha\eta\ ,\quad r=\alpha^{-1}\Phi\ ,\quad \alpha\equiv|\lambda|^{1/2}\ . \tag{3.10}$$

In these coordinates,

$$ds^2=-Adt^2+\frac{dr^2}{A}\ , \tag{3.11}$$

$$A=1-\frac{2M}{\alpha}e^{-\alpha r}+\frac{Q^2}{\alpha^2}e^{-2\alpha r}\ , \tag{3.12}$$

$$\Phi=\alpha r\ ,\quad F=Qe^{-\alpha r}\ . \tag{3.13}$$

(Here we put $M=-B_1/2\alpha$.)

For $M\geq Q$ this solution describes a charged (with charge $Q$) two-dimensional black hole, the coordinates $t$ and $r$ being valid out of the horizon defined by the relation

$$r=r_{\pm}\equiv\alpha^{-1}\ln\{\alpha^{-1}[M\pm(M^2-Q^2)^{1/2}]\}\ . \tag{3.14}$$

The global structure of this solution is the same as for the four-dimensional charged black hole. For $Q>M$ the horizons are absent, and instead of a black hole one has a naked singularity. The above-described charged solutions were obtained in Refs. [35,37]. For $Q=0$ the solution (3.11)–(3.13) coincides with the black hole solution in string theory [10,11] (see also, [16,17]).

Now we consider the degenerate case when $\Phi_{;\alpha}=0$. In this case $\Phi=\text{const}$ and it cannot be used as the coordinate. Under the assumption $\Phi=\text{const}$, the complete set of field equations is reduced to the relations

$$F=\alpha\ ,\quad R=-2\alpha^2\ . \tag{3.15}$$

In the absence of an electric field, these relations show that the only possible solution is flat spacetime and this solution exists only if the cosmological term $\lambda=\alpha^2$ in the action Eq. (2.1) vanishes. In the presence of an electric field, there are new nontrival solutions.[4] For these solutions the curvature is constant and the two-dimensional spacetime metric is an anti–de Sitter one. This metric can be written in one of three forms:

$$ds^2=-e^{-2\alpha\xi}d\eta^2+d\xi^2\ , \tag{3.16}$$

$$ds^2=-\sinh^2(\alpha\xi)d\eta^2+d\xi^2\ , \tag{3.17}$$

$$ds^2=-\cosh^2(\alpha\xi)d\eta^2+d\xi^2\ . \tag{3.18}$$

It is worthwhile noting that the metric in the form (3.16) can also be obtained as a limiting case of a metric near the horizon of a charged extreme black hole [Eqs. (3.11) and (3.12)]. For this purpose one must put

$$M=Q\ ,\quad r=\frac{1}{\alpha}\ln\left[\frac{Q}{\alpha}\right]+e^{-\alpha\xi}\ , \tag{3.19}$$

and consider the limit when $\xi\to\infty$. Equation (3.13) shows that in this limit $F=\alpha$ and $\Phi=\ln(Q/\alpha)=\text{const}$. All the degenerate solutions allow three parameter groups of isometries.

## IV. MASS FORMULA. SURFACE GRAVITY

Though in the presence of a tachyon field the generic solution of 2D gravity equations is time dependent, there are also static solutions. We prove the existence of these solutions describing a static black hole with tachyon "hairs" in Sec. VI. In this section we describe the generic properties of static solutions. In particular, we prove the so-called *mass formula*, which relates the asymptotic

[3]The theorem of uniqueness for uncharged black holes was also proved in Ref. [38].

[4]The author is grateful to Robert Mann, who has indicated on this possibility.

properties of the black hole metric with the parameters of the event horizon, and obtain the useful integral representation of the surface gravity of a black hole.

We begin by deriving the mass formula. Let us assume that all the field equations except the gravitational one [Eq. (2.5)] are satisfied. The coordinate invariance of the action (2.1) implies that

$$T^{\mu\nu}{}_{;\nu}=0\ , \tag{4.1}$$

where $T^{\mu\nu}$ is given by Eq. (2.9). We assume that the gravitational field is static and allows a timelike Killing vector $\xi^\mu$. The general form of a static metric is

$$ds^2=-\xi^2dt^2+\frac{dx^2}{P}\ , \tag{4.2}$$

where $\xi^2\equiv-\xi_\mu\xi^\mu$. The ambiguity in the choice of a coordinate $x$ can be eliminated by imposing a gauge-fixing condition. The condition $P=\xi^2$ was used in the previous section.

In a static gravitational field, one has

$$(T^{\mu\nu}\xi_\mu)_{;\nu}=0\ . \tag{4.3}$$

In two dimensions Eq. (4.3) implies that there exists a scalar function $m$ such that

$$m_{,\alpha}=-e_{\alpha\mu}T^{\mu\nu}\xi_\nu\ . \tag{4.4}$$

[Equation (4.3) plays the role of the integrability condition for Eq. (4.4).] Straightforward calculations show that the "mass function" $m$ at a point $x$ can be written in the explicit form [37]

$$m(x)=e^{\Phi(x)}w(x)\xi(x)+\varphi(x)Q(x)\ , \tag{4.5}$$

where

$$\xi=|\xi_\alpha\xi^\alpha|^{1/2}\ ,\quad w=|w_\alpha w^\alpha|^{1/2}\ ,\quad w_\alpha=\nabla_\alpha\ln\xi\ , \tag{4.6}$$

$\varphi(x)\equiv A_\alpha(x)\xi^\alpha(x)$ is the value of the electric potential at a given point, and $Q(x)\equiv e^{\Phi(x)}F(x)$ is the electric charge "inside" this point. In what follows we assume that the Killing vector is normalized by the condition $\xi(x=\infty)=1$ and we choose the electric potential to vanish at infinity.

Equation (4.5) is valid for any static gravitational field. In the case when the field equations $T_{\mu\nu}=0$ are satisfied, Eq. (4.4) implies that $m=$const. In other words, Eq. (4.5) gives the first integral of the gravitational-field equations. This first integral can be used to relate the mass of a black hole seen by a distant observer with the parameters of the black hole horizon. The asymptotic value of $e^\Phi w\xi$ at infinity coincides with the Arnowitt-Deser-Misner (ADM) mass $M$ of the black hole and hence $m=M$. On the other hand [see Eq. (A6)], the limiting value of $w\xi$ at the horizon gives the surface gravity of the black hole:

$$\kappa=[w\xi]_H\ . \tag{4.7}$$

Hence we have

$$M=e^{\Phi_H}\kappa+\varphi_HQ\ , \tag{4.8}$$

where $\varphi_H$ is the electric potential and $Q=[e^\Phi F]_H$ is the charge of the black hole. The obtained relation [Eq. (4.8)] is the desired mass formula for 2D black holes.

For the tachyon-free solutions (3.11)–(3.13), the values of $\Phi_H$, $\kappa$, and $\varphi_H$ are

$$\Phi_H=\ln\left|\frac{M+(M^2-Q^2)^{1/2}}{\alpha}\right|\ , \tag{4.9}$$

$$\kappa=\frac{\alpha(M^2-Q^2)^{1/2}}{M+(M^2-Q^2)^{1/2}}\ , \tag{4.10}$$

$$\varphi_H=\frac{Q}{M+(M^2-Q^2)^{1/2}}\ . \tag{4.11}$$

These formulas resemble the analogous formulas for a 4D Reissner-Nordström black hole.

In the presence of the tachyon field, these relations are modified. It is interesting that some conclusions about the properties of the surface gravity of a black hole can be obtained in a quite general form without using explicit solutions. We obtain now a useful representation for the surface gravity $\kappa$. Our starting point is Eq. (A10) of Appendix A:

$$\kappa=\tfrac{1}{2}\int_\Sigma R\xi^\alpha d\Sigma_\alpha\ . \tag{4.12}$$

The integration in the right-hand side is taken over the exterior of the black hole.

In a static case, $\xi^\mu\Phi_{;\mu}=0$ and hence $\xi^\alpha{}_{;\mu}\Phi^{;\mu}=\xi^\mu\Phi^{;\alpha}{}_{;\mu}$. This equation together with the gravitational-field equation (2.5) allows one to show that

$$(\xi^\alpha\Phi^{;\mu}-\xi^\mu\Phi^{;\alpha})_{;\mu}=\Box\Phi\xi^\alpha=(R+2F^2-|DT|^2)\xi^\alpha\ . \tag{4.13}$$

This relation allows us to rewrite $\kappa$ in the form

$$\kappa=\int_\Sigma(\xi^{[\alpha}\Phi^{;\mu]})_{;\mu}d\Sigma_\alpha+\int_\Sigma(-F^2+\tfrac{1}{2}|DT|^2)\xi^\alpha d\Sigma_\alpha\ . \tag{4.14}$$

The first integral in the right-hand side can be taken by using Stokes' theorem:

$$\int_\Sigma(\xi^{[\alpha}\Phi^{;\mu]})_{;\mu}d\Sigma_\alpha=\tfrac{1}{2}e_{\alpha\mu}\xi^{[\alpha}\Phi^{;\mu]}=\left[\frac{\xi}{2w}w^\mu\Phi_{;\mu}\right]_H^\infty\ . \tag{4.15}$$

The quantity in the right-hand side of Eq. (4.15) vanishes at the horizon, while at the infinity it gives

$$\left[\frac{\xi}{2w}w^\mu\Phi_{;\mu}\right]_\infty=\frac{\alpha}{2}\ . \tag{4.16}$$

After simple transformations we finally obtain the desired representation for the surface gravity of a charged 2D black hole:

$$\kappa=\frac{\alpha}{2}+\frac{1}{2}\int_\Sigma[|\nabla T|^2-e^2A_\mu A^\mu|T|^2]\xi^\alpha d\Sigma_\alpha-\int_\Sigma[F^2+A_\mu J^\mu]\xi^\alpha d\Sigma_\alpha\ . \tag{4.17}$$

This relation shows that in the absence of an electromagnetic field the surface gravity of the black hole with tachyon hairs is always higher than $\alpha/2$, the value of the surface gravity for tachyon-free solutions.

## V. STATIC SOLUTIONS WITH TACHYON FIELD. MASTER EQUATION

In this section we consider solutions describing 2D black holes with tachyon hairs. For simplicity, we assume that a black hole has no charge and we put $A_\mu=0$. The tachyon field is also assumed to be noncharged. (We remind the reader that this case is of high interest for string theory.) The following set of equations can be used as a complete system of equations for the case under consideration:

$$\frac{\xi^\mu\xi^\nu}{\xi^2}\Phi_{;\mu\nu}=\frac{1}{2}R \ , \tag{5.1}$$

$$\Box\Phi=-(\nabla T)^2+R \ , \tag{5.2}$$

$$(\nabla\Phi)^2-(\nabla T)^2+R-\mu^2T^2-\alpha^2=0 \ , \tag{5.3}$$

$$e^{-\Phi}[\nabla(e^{\Phi}\nabla T)]+\mu^2 T=0 \ . \tag{5.4}$$

The first two equations are obtained from Eq. (2.5) by applying to it the projection operator $\xi^\mu\xi^\nu/\xi^2$ and by taking the trace of this equation. Equation (5.3) is the constraint equation (2.15), which together with the tachyon-field equation (5.4) guarantees the satisfaction of the dilaton-field equation (2.7).

We write down the static metric in the form

$$ds^2=-e^{2U}dt^2+\frac{dx^2}{P} \ , \tag{5.5}$$

where

$$U=\tfrac{1}{2}\ln|\xi^2| \ . \tag{5.6}$$

The Killing vector is normalized by the condition $|\xi^2(x=\infty)|=1$, so that we have $U(x=\infty)=0$. Denote by $w$ the acceleration of a Killing observer:

$$w=|w_\mu w^\mu|^{1/2} \ , \quad w_\mu=U_{,\mu} \ . \tag{5.7}$$

For the metric (5.5), one has

$$w=\frac{P^{1/2}}{\xi}\frac{d\xi}{dx}\equiv U'P^{1/2} \ . \tag{5.8}$$

[Here and later $(\ )'\equiv d/dx$.] We eliminate the ambiguity in the choice of the coordinate $x$ by imposing the condition

$$x=-\ln\frac{w}{\alpha} \ . \tag{5.9}$$

For this choice we have

$$P=\frac{\alpha^2}{U'^2}e^{-2x} \tag{5.10}$$

and

$$ds^2=-e^{2U}dt^2+\alpha^{-2}e^{2x}U'^2dx^2 \ . \tag{5.11}$$

The curvature $R$ for this metric is

$$R\equiv-2\Box U=-2\alpha^2\frac{e^{-2x}}{U'}(U'-1) \ . \tag{5.12}$$

Equation (5.1) can be rewritten as

$$w^\mu\Phi_{;\mu}=\tfrac{1}{2}R \ , \tag{5.13}$$

which in the chosen coordinates reads

$$\Phi'=1-U' \ . \tag{5.14}$$

The solution of this equation is

$$\Phi=x-U+\ln\frac{M}{\alpha} \ . \tag{5.15}$$

[The integration constant is denoted by $\ln(M/\alpha)$.] Equation (5.2) yields the relation

$$1-T'^2-U'^2+\frac{U''}{U'}=0 \ , \tag{5.16}$$

while Eq. (5.3) gives

$$U'^2=\frac{1-T'^2}{1+e^{2x}(1+\beta^2T^2)} \ . \tag{5.17}$$

Here $\beta\equiv\mu/\alpha$. Finally, Eq. (5.4) reads

$$\frac{T''}{U'^2}-\frac{U''}{U'^3}T'+\beta^2e^{2x}T=0 \ . \tag{5.18}$$

By combining Eq. (5.17) with Eq. (5.18), it is possible to obtain the following equation which contains only the tachyon field $T$:

$$(e^{-2x}+1+\beta^2T^2)T''+(1-T'^2)(1+\beta^2T^2)T' +\beta^2(1-T'^2)T=0 \ . \tag{5.19}$$

It is easy to verify that Eq. (5.16) is identically satisfied.

To summarize, we proved that the general static solution of the 2D gravity equations (5.1)–(5.4) can be written in the form (5.11) and (5.15), where

$$U(x)=-\int_x^\infty\left[\frac{1-T'^2}{1+e^{2x}(1+\beta^2T^2)}\right]^{1/2}dx \ , \tag{5.20}$$

and $T$ is a solution of Eq. (5.19). We shall refer to Eq. (5.19), the solutions of which generate the solutions of the complete system of equations (5.1)–(5.4), as the *master equation*.

## VI. TACHYON HAIRS

For a static black hole, the event horizon coincides with the Killing horizon and hence it is given by the equation $U=-\infty$. This infinite value of $U$ cannot be reached at any finite value of the coordinate $x$ because according to Eq. (5.17) the value of $|U'|$ remains finite ($|U'|\le 1$). Hence the horizon (only if it exists) is located at $x=-\infty$. (At this point the acceleration $w$ of the Killing observer is infinite.) We assume that our solution describes a black hole and hence possesses the horizon. By using the mass formula (4.8), we get the following expression for the surface gravity:

$$\kappa=Me^{-\Phi_H}=\alpha\lim_{x\to-\infty}e^{U-x} \ . \tag{6.1}$$

It means that, near the horizon,

$$U\sim x+\ln\frac{\kappa}{\alpha} \ , \quad \lim_{x\to-\infty}U'=1 \ . \tag{6.2}$$

Equation (5.17) allows us to conclude that a static solution has a regular horizon if and only if the tachyon field obeys the boundary conditions

$$\lim_{x\to-\infty} T'=0\ , \quad \lim_{x\to-\infty} e^{x}T=0\ . \tag{6.3}$$

We show now that the boundary conditions (6.3) single out a regular, decreasing-at-spatial-infinity solution of the master equation (5.19), which is unambiguously specified by its value $T_H=T(-\infty)$. By using Eqs. (5.20), (5.11), and (5.15), this regular solution allows one to get the metric and dilaton field. The solution obtained by this procedure is regular and describes a static black hole with tachyon hairs.

We begin the proof of the above statement by considering the asymptotic behavior of solutions of the master equation (5.19) near $x=-\infty$. For this purpose it is convenient to introduce a new coordinate $\rho=e^{x}$. The master equation in these coordinates takes the form

$$(1+\rho^2 Z)\frac{d^2T}{d\rho^2}+\left[\frac{1}{\rho}+\rho Z+\rho WZ\right]\frac{dT}{d\rho}+\beta^2 WT=0\ , \tag{6.4}$$

where $Z=1+\beta^2T^2$ and $W=1-\rho^2(dT/d\rho)^2$. The boundary conditions (6.3) imply that near the horizon ($\rho=0$) the master equation (6.4) has the asymptotic form

$$\frac{d^2T}{d\rho^2}+\frac{1}{\rho}\frac{dT}{d\rho}+\beta^2T=0\ , \tag{6.5}$$

and a solution obeying the boundary conditions (6.3) allows the following expansion near $\rho=0$:

$$T=T_H\left[1+\frac{\beta^2}{2}\rho^2+\cdots\right]\ . \tag{6.6}$$

This regular solution is defined by its value $T_H$ at the horizon.

As the next step, we show that for any solution of the master equation regular at the horizon one has $|T'|<1$. In order to show this, we rewrite Eq. (5.18) as the integral equation

$$\frac{T'}{U'}=-\int_{-\infty}^{x}\beta^2U'e^{2x}T\,dx\ , \tag{6.7}$$

and use Eq. (5.17) to present it in the form

$$\frac{T'}{(1-T'^2)^{1/2}}=-[1+e^{2x}(1+\beta^2T^2)]^{1/2}\int_{-\infty}^{x}\beta^2U'e^{2x}T\,dx\ . \tag{6.8}$$

The convergence of the integral in the right-hand side of Eq. (6.8) at the lower limit is guaranteed by the boundary conditions (6.2) and (6.3). At the horizon, $|T'|$ vanishes. We assume that $|T'|$ reaches the value 1 at some finite value $x=x_0$. Equation (6.8) implies that it may happen if either, near this point, $T$ is infinitely growing or the integral in the right-hand side of Eq. (6.8) is divergent at this point. In the latter case, $T$ also must be divergent near $x_0$ because, according to Eq. (5.17), $|U'|\le 1$. But infinite growth of $T$ at $x_0$ means that $T'$ is also infinite at this point and hence $|T'|$ reaches the value 1 at some point $x_1<x_0$. The obtained contradiction with our initial assumption shows that $|T'|$ cannot reach the value 1 at any finite point $x<\infty$.

In Appendix B it is shown that *any* solution of the master equation, for which $|T'|<1$, remains finite and is decreasing at infinity provided $\beta<1$. This result together with the above proved statements means that a solution of the master equation (5.19) finite at the horizon is globally regular and vanishing at infinity. The metric (5.11) and dilaton field (5.15) corresponding to this solution are also evidently regular.

## VII. EUCLIDEAN 2D BLACK HOLE

The study of the analytic continuation of 2D black hole spacetime and its Euclidean section is important for understanding quantum aspects of black hole physics and for the development of a thermodynamical analogy. The metric of the Euclidean black hole can be obtained from Eq. (4.2) by putting $\tau=it$:

$$ds_E^2\equiv g^{E}_{\mu\nu}dx^{\mu}dx^{\nu}=\xi_E^2d\tau^2+\frac{dx^2}{P}\ . \tag{7.1}$$

Denote by $l$ a proper distance from the Euclidean horizon $\xi_E^2=0$ ($dl=dx/P^{1/2}$). Then the metric Eq. (7.1) near the horizon has the form

$$ds_E^2\approx\kappa^2l^2d\tau^2+dl^2\ , \tag{7.2}$$

where

$$\kappa=\lim_{l\to 0}(w\xi)=\left[P^{1/2}\frac{d\xi}{dx}\right]_{l=0} \tag{7.3}$$

is the surface gravity of a black hole. The Euclidean black hole is defined as a regular manifold with the metric (7.1). The regularity of the metric at $l=0$ implies that $\tau$ is periodic with a period $2\pi/\kappa$ [$\tau\in(0,2\pi/\kappa)$].

Various thermodynamical quantities for a system containing a black hole can be obtained by differentiation of the Massieu function:

$$\Omega(\beta,\Phi_B,\varphi_B)=-\frac{1}{\beta}\ln Z_{\rm GC}\ , \tag{7.4}$$

where $Z_{\rm GC}$ is the grand-canonical partition function. It is assumed that a black hole is inside a cavity with a boundary $B$, and $\beta$, $\Phi_B$, and $\varphi_B$ are the inverse temperature, the value of the dilaton field, and the chemical potential at the boundary. The chemical potential for a charged black hole is conjugate to the electric charge, and hence it is an electrostatic potential energy per unit charge. The electric charge $Q$ and the value of the dilaton field at the horizon, $\Phi_H$, are not fixed and enter the Massieu function as parameters. In Refs. [28–31] there was proposed a method for obtaining an approximation to $\ln Z_{\rm GC}$ based upon finding the stationary points of the Euclidean action [39] from which the constraints have been eliminated. We use this approach in order to show that the entropy of a charged 2D black hole is

$$S_H=2\pi e^{\Phi_H}\ . \tag{7.5}$$

We also show that for an uncharged black hole the entropy coincides with the value of the action evaluated for the Euclidean black hole solution after subtraction of the vacuum contribution from it.[5]

Following [31], we rewrite the metric Eq. (7.1) in the form

$$ds^2 = B d\eta^2 + \frac{dx^2}{P} , \quad (7.6)$$

where $\eta$ is periodic with a period $2\pi$ and $x \in [0,1]$. The boundary surrounding a black hole is located at $x=1$. Denote

$$B(1) = B_B , \quad \Phi(1) = \Phi_B , \quad \Phi(0) = \Phi_H . \quad (7.7)$$

Then the inverse temperature $\beta$ measured at the boundary is

$$\beta = \int_0^{2\pi} B_B^{1/2} d\eta = 2\pi B_B^{1/2} . \quad (7.8)$$

The regularity condition of the metric and dilaton field at the origin give

$$[B^{1/2}(P^{1/2})']_{x=0} = 1 , \quad (7.9)$$

$$[P\Phi'^2]_{x=0} = [(\nabla\Phi)^2]_{x=0} = 0 . \quad (7.10)$$

Here $(\ )' = \partial/\partial x$.

The vector potential for the electromagnetic field can be chosen in the form

$$A_\mu dx^\mu = -i\varphi(x) d\eta , \quad (7.11)$$

where $\varphi(x)$ is a real function obeying the boundary conditions

$$\varphi(0) = 0 , \quad \varphi(1) = \frac{\beta\varphi_B}{2\pi} . \quad (7.12)$$

The reduced action is obtained from the Euclidean action

$$S_E = S_E[g, \Phi, A]$$
$$\equiv -\tfrac{1}{2}\int_V d^2x \sqrt{g}\, e^{\Phi}[R + (\nabla\Phi)^2 + \alpha^2 - \tfrac{1}{2}F_{\mu\nu}F^{\mu\nu}]$$
$$- \int_{\partial V} dx \sqrt{|h|}\, e^{\Phi} K , \quad (7.13)$$

by substituting the solution of the constraints and boundary conditions into it. The electromagnetic-field constraint at $\eta = \text{const}$,

$$(e^{\Phi} F^{\eta\mu})_{;\mu} = 0 , \quad (7.14)$$

gives

$$\left[ \left[ \frac{P}{B} \right]^{1/2} e^{\Phi} \varphi' \right]' = 0 . \quad (7.15)$$

This equation has a solution

$$\varphi' = Q \left[ \frac{B}{P} \right]^{1/2} e^{-\Phi} . \quad (7.16)$$

[5]Another approach for the calculation of the entropy of a black hole was proposed in Ref. [38]. For a neutral black hole, the result of [38] coincides with Eq. (7.5).

The gravitational constraint $T_{\eta x} = 0$ is identically satisfied, while $T_{\eta\eta} = 0$ gives

$$[(P\Phi'^2 - \alpha^2)e^{\Phi} - Q^2 e^{-\Phi}]' = 0 . \quad (7.17)$$

The solution of this constraint is

$$W \equiv P\Phi'^2 = \alpha^2 - De^{-\Phi} + Q^2 e^{-2\Phi} . \quad (7.18)$$

Equation (7.9) can be used to get the following expression for the constant $D$ connected with the mass of a black hole $M$:

$$D \equiv 2\alpha M = \alpha^2 e^{\Phi_H} + Q^2 e^{-\Phi_H} . \quad (7.19)$$

Denote by $I$ the reduced action, which is obtained from the Euclidean action by substituting the constraint solutions and using the boundary conditions. The calculations give

$$I \equiv I(\beta, \Phi_B, \varphi_B; \Phi_H, Q)$$
$$= -\beta e^{\Phi_B}\sqrt{W_B} - 2\pi e^{\Phi_H} - \beta Q \varphi_B , \quad (7.20)$$

where

$$W_B = (1 - e^{\Phi_H - \Phi_B})(\alpha^2 - Q^2 e^{-\Phi_B - \Phi_H}) . \quad (7.21)$$

The stationary points of the reduced action with respect to the parameters $\Phi_H$ and $Q$ are defined by a differentiation of $I$ with respect to these parameters with $\beta$, $\Phi_H$, and $\varphi_H$ fixed. By using these relations, we get

$$\varphi_B = Q \frac{e^{-\Phi_H} - e^{-\Phi_B}}{\sqrt{W_B}} , \quad (7.22)$$

$$\beta = \frac{4\pi\sqrt{W_B}\, e^{\Phi_H}}{\alpha^2 e^{\Phi_H} - Q^2 e^{-\Phi_H}} = \frac{\sqrt{W_B}}{\alpha}\beta_H , \quad (7.23)$$

where

$$\beta_H = \frac{1}{T_H} = \frac{2\pi}{\kappa} , \quad (7.24)$$

with $\kappa$ given by Eq. (4.10). The entropy of the black hole is defined by the relation

$$S_H = \left[ \beta \frac{\partial I}{\partial \beta} \right]_{\Phi_B, \varphi_B} - I . \quad (7.25)$$

Finally we get

$$S_H = 2\pi e^{\Phi_H} . \quad (7.26)$$

In the 4D Einstein theory of gravity, the entropy of an uncharged black hole coincides with the value of the action calculated for the Euclidean black hole solution after subtracting from the latter the vacuum contribution. The same is true for a 2D black hole. One can demonstrate it by evaluating the value of $I$ given by Eq. (7.20) for the black hole metric. Another more direct way is to use the dilaton equation (2.6) and to transform the Euclidean action (7.13) to the form

$$S_E=-\int_V d^2x\sqrt{g_E}e^{\Phi}[\Box_E\Phi+(\nabla_E\Phi)^2]+\int_{\partial V}dx\sqrt{h_E}e^{\Phi}K_E=-\int_{\partial V}dx\sqrt{h_E}e^{\Phi}(n^{;\mu}\Phi_{;\mu}-K_E)\ . \tag{7.27}$$

The subscript $E$ in these relations indicates that the corresponding quantities are to be calculated in the metric (7.1) of the Euclidean black hole.

It should be stressed that the Euclidean action is divergent at large distances already for flat 2D spacetime where

$$ds_E^2=d\tau^2+dl^2\ ,\quad \Phi=\alpha l\ . \tag{7.28}$$

On the other hand, the difference between the action $S_E$ and its flat-space limit $S_E^0$ is finite. Namely, this difference $S_H=S_E-S_E^0$, which describes the change of the Euclidean action due to the presence of a black hole, has physical meaning and it is directly related to the black hole entropy. The following procedure allows one to define $S_H$. Denote by $V(\phi)$ the spacetime region where the value of the dilaton field is less than or equal to a given value $\Phi$, and let $S_E(\Phi)$ be the corresponding value of the euclidean action Eq. (7.27) calculated for $V=V(\Phi)$. Then the black hole's contribution to the Euclidean action can be defined as

$$S_H=\lim_{\Phi\to\infty}[S_E(\Phi)-S_E^0(\Phi)]\ . \tag{7.29}$$

Here $S_E^0(\Phi)$ is the value of the Euclidean action for a flat metric (7.28) with the same periodicity in the imaginary time coordinate $\tau$ as for the black hole solution. By using Eq. (5.8), the extrinsic curvature $K_E$ of the line $x=\text{const}$ in the metric Eq. (7.1) can be written

$$K_E\equiv-\frac{1}{2}\frac{P^{1/2}}{\xi_E^2}\frac{d(\xi_E^2)}{dx}=-w\ . \tag{7.30}$$

The quantities $n^{;\mu}\Phi_{;\mu}$ and $K_E$ are regular at the Euclidean horizon. For a neutral (noncharged) black hole, the contribution to $S_E$ due to the horizon vanishes and we have

$$S_E(\Phi)=-\frac{2\pi}{\kappa}(\xi_E we^{\Phi}+\xi_E e^{\Phi}n^{\mu}\Phi_{;\mu})\ . \tag{7.31}$$

The asymptotic form of $S_E(\Phi)$ for large values of $\Phi$ is

$$S_E(\Phi)=\frac{2\pi}{\kappa}(M-\alpha e^{\Phi})+O(e^{-\Phi})\ , \tag{7.32}$$

where $M$ is the mass of a black hole. By subtracting from this expression its flat space limit

$$S_E^0(\Phi)=-(2\pi/\kappa)\alpha e^{\Phi} \tag{7.33}$$

and taking the limit $\Phi\to\infty$, we get

$$S_H=\frac{2\pi}{\kappa}M\ . \tag{7.34}$$

This expression evidently coincides with Eq. (7.26).

It is interesting to note that the Euclidean action of a 4D black hole is proportional to its surface. In the 2D case, the "surface" of a black hole is null dimensional (a point) and there is no sense in speaking about its area. Nevertheless, it might be useful to think about the value of the dilaton field at the horizon as of some quantity playing the role of the logarithm of an effective area of the black hole in 2D black hole physics.

## VIII. FOUR LAWS OF 2D BLACK HOLE PHYSICS

In this section we make some remarks concerning the thermodynamical analogy in 2D black hole physics. We begin by considering general (not necessarily static) solutions describing black holes and prove some results on the behavior of the event horizon.

We assume that a spacetime described by a solution of Eqs. (2.5)–(2.11) is asymptotically flat. As usual, one may define the event horizon as a line $\dot{J}^-(\mathcal{J}^+)$. The standard arguments used for the proof of the Penrose theorem show that this line at its regular points is to be null. Any null line in two dimensions is geodesic. Denote by $V$ an affine parameter along the horizon, and let $l^{\mu}\partial_{\mu}\equiv\partial_V$ be its tangent vector.

The results of the previous section allow us to conclude that the entropy of a black hole is to be defined as

$$S_H=2\pi e^{\Phi_H}\ . \tag{8.1}$$

The general equation which describes the evolution of $\Phi_H$ can be obtained by using the gravitational-field equation (2.5) and the expression (2.14) for $T_{\mu\nu}$. These equations imply the following relation, which must be satisfied at the horizon:

$$\frac{d^2\Phi_H}{dV^2}\equiv[l^{\mu}(l^{\nu}\Phi_{;\nu})_{;\mu}]_H=-|l^{\mu}D_{\mu}T|_H^2\equiv-\Pi(V)\ . \tag{8.2}$$

The subscript $H$ in these relations indicates that the corresponding quantities are calculated at the horizon. The solution of Eq. (8.2) under the assumption that the black hole becomes static at a late time can be written as

$$\Phi_H(V)=\Phi_H(\infty)-\int_V^{\infty}dV'\int_{V'}^{\infty}dV''\Pi(V'')\ , \tag{8.3}$$

where $\Phi_H(\infty)$ is the future asymptotic value of the dilaton field at the horizon. For our choice of the action, the local "energy flux" of the tachyon field through the horizon is non-negative $[\Pi(V)\geq 0]$ and Eq. (8.3) implies that $\Phi_H(V)$ (and hence the entropy of the black hole $S_H$) is a monotonically nondecreasing function of time. This result resembles the Hawking theorem for 4D black holes.

Equation (8.3) shows us also that for any moment of time before the moment when the black hole parameters reach their asymptotic values the value of $d\Phi_H/dV$ is strictly negative provided there was a flux of the tachyon field through the horizon. This conclusion may look a bit surprising. We show now that this behavior is natural and is to be expected.

We illustrate it by considering two examples. We assume at first that the spacetime region containing the nontrivial time evolution is located between two moments of the advanced time parameters $V_1$ and $V_2$. We assume that before $V_1$ and after $V_2$ the spacetime is described by static solutions. Figure 1 represents the Penrose diagram

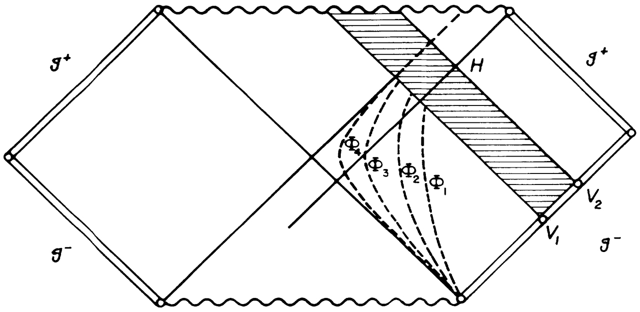


FIG. 1. Penrose diagram for a spacetime with an external 2D black hole. The lines of constant values $\Phi_1 > \Phi_2 > \Phi_3 > \Phi_4$ of the dilaton field are shown.

for spacetime for the case when before $V_1$ there was a static eternal black hole. The null ray $H$ representing the event horizon being monitored back in the advanced time $V$ after passing the time-dependent region is propagating in a static spacetime. For $\Pi \geq 0$ the event horizon $H$ passes in the region $V < V_2$ outside the apparent horizon, and for decreasing values of $V$ it crosses the lines of smaller and smaller values of the dilaton field (see Fig. 1). By using the explicit form of the solution (3.11)–(3.13), it is easy to prove that in the absence of the tachyon field the value of the dilaton field $\Phi$ can be taken as an affine parameter along null geodesics and hence in this region $\Phi(V) = \Phi_1 - \dot{\Phi}_1 V$, where $\Phi_1$ and $\dot{\Phi}_1 \geq 0$ are constants. This result is in complete agreement with the negative value of $d\Phi_H/dV \leq 0$. If the initial eternal black hole possesses tachyon hairs, then even in a static spacetime there may be fluxes of this field through the horizon. In this case $d\Phi_H/dV$ is also negative, but its value might be dependent on time.

Another possibility is shown at Fig. 2, which represents the Penrose diagram for the spacetime of a 2D black hole which arises as a result of tachyonic matter collapse.[6] In the flat region lying before the beginning of the collapse, the dilaton field $\Phi$ can be chosen as the affine parameter, and being monitored back in time, this parameter along the horizon is monotonically decreasing until its asymptotic value $-\infty$.

After the discussion of the general behavior of the event horizons, we are able to formulate the four laws of 2D black hole physics. The analogue of the zeroth law of thermodynamics is quite evident. For a static equilibrium configuration, the black hole temperature $\theta = \kappa/2\pi$ is constant at the horizon.

For the formulation of the first law, we assume that the system incorporating a black hole switches from one static state to another. The mass formula (4.8) shows that its mass changes by

$$\delta M = \delta(e^{\Phi_H})\kappa + e^{\Phi_h}\delta\kappa \ . \tag{8.4}$$

[6]The behavior of the event horizon for the processes of 2D black hole formation and evaporation in the $1/N$ expansion of a two-dimensional model was analyzed in [15]. See also Refs. [19,20,22,25,26].

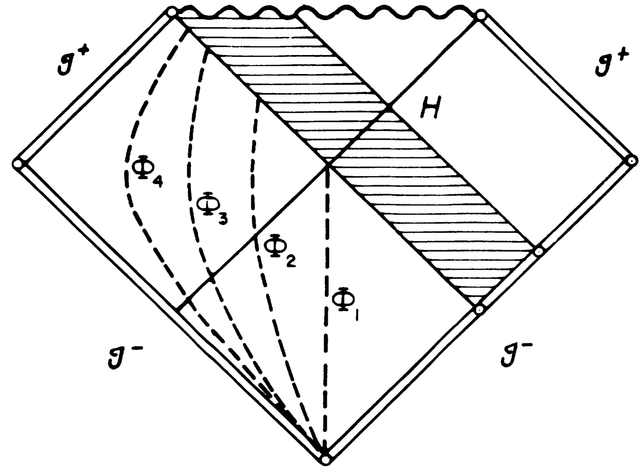


FIG. 2. Penrose diagram for a spacetime in the case when a 2D black hole arises as a result of the gravitational collapse of tachyon matter. The lines of constant values $\Phi_1 > \Phi_2 > \Phi_3 > \Phi_4$ of the dilaton field are shown.

(For simplicity we consider the uncharged black hole.) The variation of $\kappa$ can be found by using Eq. (4.17):

$$\delta\kappa = \tfrac{1}{2}\delta\int_\Sigma (\nabla T)^2 \xi^\alpha d\Sigma_\alpha \ . \tag{8.5}$$

Using the definitions of the entropy $S_H = 2\pi e^{\Phi_H}$ and the temperature $\theta = \kappa/2\pi$ of a black hole, we get

$$\delta M = \theta\delta S_H + \tfrac{1}{2}e^{\Phi_H}\delta\int_\Sigma (\nabla T)^2 \xi^\alpha d\Sigma_\alpha \ . \tag{8.6}$$

The last equation ("differential mass formula"), relating the change of the energy of a black hole to the change of its entropy, plays the role of the first law in the thermodynamical analogy. The second term on the right-hand side of Eq. (8.6) describes the change of the contribution of matter (the tachyon field) outside the black hole to the total energy.

In the beginning of this section, it was shown that $S_H$ is a nondecreasing function of time,

$$\delta S_H \geq 0 \ , \tag{8.7}$$

provided $|l^\mu D_\mu T|_H^2 \geq 0$. The latter inequality ("weak energy condition") is valid for a classical tachyon field, but it can be violated as a result of quantum effects (e.g, during the process of quantum evaporation of a black hole). Nevertheless, one may expect that the generalized entropy (the entropy $S_H$ of a black hole plus the entropy $S_r$ of the quantum radiation) is to be a nondecreasing function of time:

$$\delta S_H + \delta S_r \geq 0 \ . \tag{8.8}$$

This inequality is the second law.

The third law, i.e., the statement that the temperature of a black hole cannot be reduced to zero by a finite number of operations, is evidently valid for uncharged black holes because, for them, $\theta \geq \alpha/4\pi$. For charged tachyon-free black holes, Eq. (4.10) shows that any attempt to reach $\theta = \kappa/2\pi = 0$ must destroy the black hole in the same manner as happens in the 4D case. It means that the same arguments as in the 4D case [40] can be applied to 2D black holes. It would be interesting to obtain a rigorous general proof of the third law for 2D black holes.

In conclusion, we make some remarks concerning tachyon hairs. The positive integral $J[T] = \int_\Sigma (\nabla T)^2 \xi^\alpha d\Sigma_\alpha$ standing in the right-hand side of Eq. (8.6) vanishes only for tachyon-free solutions. It may be used to characterize the deflection of a solution describing a black hole with tachyon hairs from a tachyon-free one. The classical radiation of the tachyon field carries away energy, and hence it decreases the total mass of a black hole. On the other hand, the second law (8.7) implies that the entropy of the black hole cannot decrease, and hence for this process $\delta M - \theta\delta S_H \le 0$. The first law allows one to conclude that $\delta J[T] < 0$. In other words, by radiating the tachyon field, the black hole becomes closer to its "no-hair state." It means that one may expect classical instability of tachyon hairs unless there exists some conservation law preventing this process.

## ACKNOWLEDGMENTS

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## APPENDIX A: GEOMETRY OF A STATIC 2D SPACETIME

This appendix contains some useful results concerning the geometric properties of static two-dimensional spacetimes.

Let $\xi^\mu$ be a timelike Killing vector

$$\xi_{(\mu;\nu)} = 0 \ . \tag{A1}$$

Then

$$\xi_{\alpha;\beta;\gamma} = R_{\alpha\beta\gamma\delta}\xi^\delta \ . \tag{A2}$$

In the 2D spacetime, $\xi_{[\mu}\xi_{\nu;\lambda]} = 0$. By using this identity, it is easy to prove that

$$\xi_{\mu;\nu} = \xi_\mu w_\nu - \xi_\nu w_\mu = -\xi w e_{\mu\nu} \ , \tag{A3}$$

where $\xi = |\xi_\alpha \xi^\alpha|^{1/2}$, $w = |w_\alpha w^\alpha|^{1/2}$, and $w_\alpha = \nabla_\alpha \ln\xi$.

The Killing horizon (which in a static spacetime coincides with the event horizon) is defined by the equation

$$\xi^2 = 0 \ . \tag{A4}$$

Since $\xi^2$ is constant at the horizon, the vector $(\xi^2)_{;\mu}$ is normal to it and hence

$$-\tfrac{1}{2}(\xi^2)^{;\nu} \equiv \xi^\mu \xi^\nu{}_{;\mu} = \kappa \xi^\nu \ , \tag{A5}$$

where $\kappa$ is the surface gravity of a black hole. It is easy to show that

$$\kappa = [\xi w]_H \ , \tag{A6}$$

where the subscript $H$ indicates that the limit is taken for a point at the horizon.

The Eq. (A2) implies

$$\xi^{\alpha;\beta}{}_{;\beta} = \tfrac{1}{2} R \xi^\alpha \ . \tag{A7}$$

By using Stokes' theorem, it is easy to show that

$$\tfrac{1}{2}\int_\Sigma R\xi^\alpha d\Sigma_\alpha = \int_{\partial\Sigma} \xi^{\alpha;\beta} d\sigma_{\alpha\beta} \equiv \tfrac{1}{2}[e_{\alpha\beta}\xi^{\alpha;\beta}]_{\partial\Sigma} \ . \tag{A8}$$

In the case of a static 2D spacetime with a black hole, $\partial\Sigma$ consists of two points: infinity and the horizon. The value of $e_{\alpha\beta}\xi^{\alpha;\beta}$ vanishes at infinity, while at the horizon it is

$$\tfrac{1}{2}[e_{\alpha\beta}\xi^{\alpha;\beta}]_H \equiv \kappa \ . \tag{A9}$$

By combining these formulas, we get the following useful expression for the surface gravity:

$$\kappa = \tfrac{1}{2}\int_\Sigma R\xi^\alpha d\Sigma_\alpha \ , \tag{A10}$$

where the integration is taken over the part of the Cauchy surface lying outside the black hole.

## APPENDIX B: ASYMPTOTIC BEHAVIOR OF SOLUTIONS OF THE MASTER EQUATION

In this appendix we prove the results concerning the behavior of the solutions of the master equation (5.19) in the asymptotic region $x \to \infty$. In this region, $e^{-2x} \ll 1$, and the master equation takes the form

$$(1+\beta^2T^2)T'' + (1-T'^2)(1+\beta^2T^2)T' + \beta^2(1-T'^2)T = 0 \ . \tag{B1}$$

By introducing new variables $T_1 = T$ and $T_2 = T'$, this second-order equation can be rewritten as the following system of first-order equations:

$$T'_a = Z_a(T_b) \ , \quad a,b = 1,2 \ , \tag{B2}$$

where

$$Z_1 = T_2 \ , \quad Z_2 = -(1-T_2^2)[T_2 - F(T_1)] \ , \tag{B3}$$

and

$$F(T_1) = -\frac{\beta^2 T_1}{1+\beta^2 T_1^2} \ . \tag{B4}$$

Figure 3 represents the phase plane for this system. There are three lines $(\gamma, \gamma_\pm)$ in this plane where $Z_2$ vanishes. They are described by the equations $T_2 = -1$ $(\gamma_-)$, $T_2 = +1$ $(\gamma_+)$, and $T_2 = F(T_1)$ $(\gamma)$. We denote by $R_-$ $(R_+)$ the region lying between $\gamma$ and $\gamma_+$ (between $\gamma$ and $\gamma_-$). The curve defined by the equation $T_2 = F(T_1)$ has a maximum at the point $B = (-1/\beta, \beta/2)$ and a minimum at the point $A = (1/\beta, -\beta/2)$. It passes through the origin $O$ of the phase plane, and it has the $T_1$ axis as its asymptotes for $|T_1| \to \infty$. $Z_1$ vanishes at the $T_1$ axis, and the integral lines of the vector field $Z_a$ cross the $T_1$ axis perpendicularly everywhere outside the origin

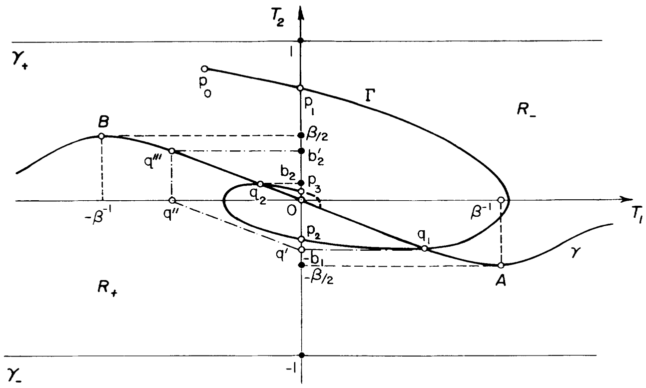


FIG. 3. Phase plane for the system (B2) and (B3).

$O$. The system (B2) and (B3) (and hence the phase plain) is evidently invariant under the transformation

$$T_1 \rightarrow -T_1 \ , \quad T_2 \rightarrow -T_2 \ . \tag{B5}$$

It was shown in Sec. VI that any solution of the master equation regular at the horizon obeys the condition $T'^2<1$. That is why for our purpose it is sufficient to restrict ourselves to considering solutions lying in the strip $|T_2|<1$ of the phase plane. We show that for $\beta<1$ any integral curve of the vector field $Z_a$ with the initial point lying in the strip $|T_2|<1$ reaches the origin $O$ of the phase plane. In order to prove this, we assume that there exists an integral curve $\Gamma$ which does not reach $O$. Let us assume that this curve passes through some point $p_0$ of the strip with the coordinates $T_a^{(0)}$ and monitor its behavior.

Because of the symmetry (B5), without loss of generality we can assume that $T_2^{(0)}>0$. If $T_1^{(0)}<0$, then the solution can cross neither the $T_1$ axis nor $\gamma_+$ at the point where $T_1<0$, and hence it intersects the $T_2$ axis. By the assumption it cannot pass through $O$, and hence it crosses the $T_2$ axis at some point $p_1$ with the coordinates $(0,T_2^{(1)})$ $(0<T_2^{(1)}<1)$. The value of $dT_2/dT_1$ at this point is $-[1-(T_2^{(1)})^2]$. It is easy to verify that in the region $T_1>0, 0<T_2\le T_2^{(1)}$ one has

$$\frac{dT_2}{dT_1} \le -[1-(T_2^{(1)})^2] \ . \tag{B6}$$

It means that $\Gamma$ crosses the $T_1$ axis and enters the region lying between the $T_1$ axis and $\gamma$. It leaves this region crossing $\gamma$ at some point $q_1$ lying between the points $A$ and $O$. We denote the coordinates of the point $q_1$ by $(a_1,-b_1)$. After entering the $R_+$ region (where $Z_2>0$), the curve $\Gamma$ cannot cross $\gamma$ again at the points where $T_1>0$. By our assumption, it also cannot reach $O$. Thus it crosses the $T_2$ axis at some point $p_2$. For symmetry reasons our previous considering can be repeated to show that the curve $\Gamma$ after crossing the $T_1$ axis necessarily crosses $\gamma$ at some point $q_2$ lying between $B$ and $O$. Denote by $(-a_2,b_2)$ the coordinates of this point.

The next step in the proof is to show that the transformation defined by the relation $b_2=f(b_1)$ is a contraction. To show this we consider the line $q_1q'q''q'''$ shown in Fig. 3. Its part $q_1q'$ is a horizontal line. The part $q''q'''$ is vertical, and the direction of the part $q'q''$ coincides with the direction of the vector field $Z_a$ at the point $q'$. Simple arguments show that the curve $\Gamma$ after passing through the point $q_1$ cannot cross the line $q_1q'q''q'''$. It means that $b_2<b_2'$, where $(-a_2',b_2')$ are the coordinates of a point $q'''$. It is easy to find that

$$b_2' = \frac{\beta^2 b_1(1-b_1^2)}{(1-b_1^2)^2+\beta^2 b_1^2} \ . \tag{B7}$$

Denote $y=1-b_1^2$. Then

$$\frac{b_2'}{b_1} \equiv W(y) = \frac{\beta^2 y}{y^2+\beta^2(1-y)} \ . \tag{B8}$$

The value of $b_1$ is less then $\beta/2$. For $\beta<1$, the value of $y$ is positive. For positive $y$ the function $W$ is positive and it reaches its maximum value $W_{\max}=\beta/(2-\beta)$ at $y=\beta$. Thus we proved that, for $\beta<1$,

$$\frac{b_2}{b_1} < \frac{b_2'}{b_1} < \frac{\beta}{2-\beta} < 1 \ . \tag{B9}$$

By our assumption, the curve $\Gamma$ cannot enter $O$. Hence, after passing through $q_2$, it will cross the curve $\gamma$ again and again at the sequence of points which we denote $q_n$. Let $((-1)^{n-1}a_n,(-1)^n b_n)$ be the coordinates of $q_n$. Then one evidently has

$$\frac{b_{n+1}}{b_n} < \frac{\beta}{2-\beta} < 1 \ . \tag{B10}$$

It means that $b_n \rightarrow 0$ as $n\rightarrow\infty$, and the sequence of points $q_n$ has the origin $O$ as its limit.

The behavior of the integral line in the vicinity of $O$ can be described in an explicit form. The system (B2) and (B3) in this vicinity can be linearized, and it has the asymptotic form

$$Z_1=T_2 \ , \quad Z_2=-T_2+\beta^2 T_1 \ . \tag{B11}$$

The corresponding second-order equation

$$T''+T'+\beta^2 T=0 \tag{B12}$$

has the solutions

$$T=C_+e^{-\nu_+x}+C_-e^{-\nu_-x} \ , \tag{B13}$$

where

$$\nu_\pm=\tfrac{1}{2}\pm(\tfrac{1}{4}-\beta^2)^{1/2} \ . \tag{B14}$$

By using this asymptotic solution, one can verify that our original assumption is wrong and the curve $\Gamma$ either enters $O$ after crossing the curve $\gamma$ a finite number of times (it happens for $\beta<\frac{1}{2}$) or the curve $\Gamma$ has $O$ as its limiting point (for $\frac{1}{2}<\beta<1$). In both cases any solution of the master equation regular at the horizon remains regular and it is decreasing at infinity.

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