Nonlinear noise in cosmology

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This paper derives and analyzes exact, nonlocal Langevin equations appropriate in a cosmological setting to describe the interaction of some collective degree of freedom with a surrounding "environment." Formally, these equations are much more general, involving as they do a more or less arbitrary "system," characterized by some time-dependent potential, which is coupled via a nonlinear, time-dependent interaction to a "bath" of oscillators with time-dependent frequencies. The analysis reveals that, even in a Markov limit, which can often be justified, the time dependences and nonlinearities can induce new and potentially significant effects, such as systematic and stochastic mass renormalizations and state-dependent "memory" functions, aside from the standard "friction" of a heuristic Langevin description. One specific example is discussed in detail, namely, the case of an inflaton field, characterized by a Landau-Ginzburg potential, that is coupled quadratically to a bath of scalar "radiation." The principal conclusion derived from this example is that nonlinearities and time-dependent couplings do *not* preclude the possibility of deriving a fluctuation-dissipation theorem, and do *not* change the form of the late-time steady state solution for the system, but *can* significantly shorten the time scale for the approach towards the steady state.

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I. INTRODUCTION

Over the past two decades or so, a great deal of attention has focused on the problem of understanding statistical field theory in a cosmological context, allowing correctly for the fact that the spacetime is not flat, or even static. This is a problem of interest both from an abstract field-theoretic viewpoint and from a more practical viewpoint which seeks to understand what the Universe was like at very early times. When addressing the latter sorts of issues, one is led oftentimes to implement various models and/or approximations which, although not justified rigorously, seem physically well motivated and even essential, if one wishes to obtain concrete results.

One important ingredient in this sort of modeling is the idea that, at some level, the physical degrees of freedom of the Universe divide into two coupled pieces: a "system" component, the detailed evolution of which is for some reason of particular relevance, and a "bath" component, the detailed evolution of which is somehow irrelevant. This sort of picture has arisen in at least four different settings. One such setting entails an understanding of inflation in terms of an inflaton field evolving in the rest of the Universe, which serves as an external environment or bath [1–3]. Another involves the general notion of "coarse graining" as a physical mechanism in terms of which to extract quantum decoherence, this facilitating a "quantum-to-classical" transition in the early Universe [4]. A third entails a more systematic development of statistical quantum field theory [5], which uses a closed-time-path formalism to derive quantum dissipation and memory loss. And finally, there is the intriguing, but not yet completely understood, program of stochastic inflation originally proposed by Starobinsky [6].

Much work along these lines has been predicated upon the formulation of essentially *ad hoc* Fokker-Planck equations, which effectively introduce a bilinear interaction between the system and the bath. It seems crucial to understand the extent to which these sorts of heuristic models are in fact reasonable, i.e., approximately true in some appropriate limit, and, especially, how relaxing the basic assumption of bilinearity changes the underlying physics. In particular, is the standard sort of modeling legitimate if one allows, as in certain cases one must, for time-dependent couplings and frequencies and incorporates realistic nonlinearities?

These are extremely difficult questions to answer in complete generality. However, as will be seen in this paper, they *can* be examined in certain cases by considering special models of systems coupled to baths comprised of time-dependent harmonic oscillators, where it is possible to derive exact, nonlocal Langevin equations simple enough to understand, both physically and mathematically. These models are nonlinear, time-dependent generalizations of phenomenological models which have proven quite successful in other branches of physics, such as condensed matter physics or quantum optics. And, as in

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those settings, the models *are* well motivated phenomenologically, even if they are not derived *ab initio*.

One of the objectives here is to derive exact generalized Langevin equations for these models, and in an appropriate Markov limit, their Fokker-Planck realizations. One is then poised to understand the sorts of new effects arising in these exact nonlocal equations which are absent both from more heuristic Langevin descriptions and from the exact nonlocal equations that can be derived for the special case of time-independent bilinear couplings. What this entails is an analysis of the interplay between three different sorts of effects: namely, linear and nonlinear noise, which may well have very different natural time scales, and the explicit time dependence of the environment, reflecting the overall expansion of the Universe, which introduces yet another time scale.

This analysis shows that (a) allowing for a nontrivial time dependence necessarily induces qualitatively new effects such as mass (or frequency) "renormalization," even for the special case of bilinear couplings [3]; and, moreover, (b) allowing for nonlinearities in the system-environment coupling induces new effects aside from the usual "friction" term. One discovers, e.g., that nonlinearities give rise to an additional renormalization of the system potential, and that they imply a "memory" kernel which involves the state of the system.

These results might suggest that, in the presence of such nonlinearities, one's naive intuition is completely lost. This, however, is not so: Even allowing for nonlinearities and time-dependent couplings in the interaction between the system and environment, one can, at least for the case of time-independent oscillators, where the bath may still be viewed as being "at equilibrium," still derive a simple fluctuation-dissipation theorem [7]. When the oscillators become time dependent, the fluctuation-dissipation theorem will no longer be exact. However, this theorem *does* remain at least approximately true to the extent that the coupling between the system and environment is dominated by modes of sufficiently short wavelength. As will be shown below, in a cosmological setting this implies that, on scales short compared with the horizon length, it is still possible to speak of an approximate equilibrium and an approach towards that equilibrium.

This paper focuses on obtaining a qualitative understanding of the effects of nonlinear couplings and a time-dependent environment. A subsequent paper will present a concrete calculation, applying the technology of the time-dependent renormalization group to a simple cosmological phase transition.

Section II of this paper focuses on the general problem of couplings between a system and a bath, motivating in particular a rather general time-dependent Hamiltonian which is amenable to a systematic analysis. Section III derives an exact, nonlocal Langevin equation from this Hamiltonian and then discusses its physical implications. All of this is completely general, not restricted in any way to a cosmological context. Section IV then turns to a consideration of one specific cosmological model, deriving Langevin and/or Fokker-Planck equations for some collective degree of freedom, such as the dilaton mode, evolving in a Landau-Ginzburg potential and coupled to scalar "radiation." This equation can provide one with a simple tool in terms of which to model a cosmological phase transition associated either with inflation or the formation of a cosmic string. Section V provides approximate solutions to this equation, which enable one to study the approach towards a (time-dependent) steady state. It is observed that such an approach towards "equilibrium" can be strongly influenced by the nonlinearities in the couplings ("multiplicative noise") and/or the effects of the time-dependent expansion of the Universe.

II. THE SYSTEM-ENVIRONMENT SPLITTING

Two sorts of equations are ubiquitous in nonequilibrium statistical mechanics: namely, (collisional) Boltzmann equations and Fokker-Planck equations. Boltzmann equations are appropriate for transport-type problems, involving strongly interacting particles, whereas Langevin and/or Fokker-Planck equations are useful in the study of phase transitions, Brownian motion, and the like.

These two ubiquitous equations also find a place in cosmology. In the past, kinetic theory in the manner of Boltzmann has been studied extensively, most prominently perhaps in the analysis of nucleosynthesis [8]. More recently, however, phase transitions have come into vogue, especially with regard to the inflationary scenario [9], and it is here that one encounters Fokker-Planck equations.

That the standard methodology (with essentially trivial modifications) can be applied to the very early Universe involves a certain leap of faith; and while on the whole cosmologists appear comfortable with the status quo, doubts have certainly been voiced in the literature [10]. The main objections relate to (1) the assumption of thermal equilibrium, (2) the absence of a clear separation of time scales, (3) the neglect of fluctuations, (4) the validity of heuristic master equations, and, related to this, (5) the lack of a fundamental Liouville description, derived from a Hamiltonian. While it is fair to say that some of these objections have not been stated in a concrete, quantitative way, they *do* lead to a feeling of unease.

In order to address some of these issues concretely, this paper will consider a prototypical Hamiltonian which incorporates more or less realistic nonlinearities and time dependences, and then extract from that Hamiltonian an *exact* Langevin equation for the system variable. This is a Liouville approach [10] in which the fundamental equation is derived systematically without any *ad hoc* assumptions. However, the more difficult problem of justifying a full-blown nonlinear Boltzmann equation will not be treated here.

The exact Langevin equation involves at least three distinct time scales: (1) the Hubble time t_H, (2) the relaxation time t_R on which the system is affected by the surrounding environment, and (3) the time scale t_C set by the decay of the noise autocorrelation function. If the system evolves under the influence of some nontrivial system potential U, there is also a fourth time scale t_S, the time

scale on which the system changes in response to U. The implementation of any approximation entails an assumption regarding the separation of these time scales. In particular, the possibility of an approximate local description, i.e., the existence of a Markov limit, depends critically on the assumption that $t_C \ll t_R$, t_H, and t_S. An obvious point then is that nonlinearities, especially with respect to the system-bath coupling, can play an important role by inducing multiplicative noise which can significantly alter t_R.

The specific objective here is to motivate a phenomenological Hamiltonian for the system plus environment, and to analyze it rigorously to extract the physical effects contained therein. The system is taken to be some "collective coordinate" evolving under the influence of a "heat bath." Particular interest focuses on phase transitions, but the development here is in no way restricted to such a setting. Indeed, it should be stressed that this general approach has been used successfully in many other areas of physics, such as condensed matter physics, nonlinear optics, nuclear physics, etc.

Given an arbitrary composite Hamiltonian, one wishes to introduce a splitting into a "system" piece, a "bath" piece, and an interaction term. If this split is to be useful, it must be true that in some sense the system is "small" compared with the environment. What this means is that, as far as the system is concerned, the full Hamiltonian

$$H_T = H_S + H_B + H_I \tag{1}$$

is well approximated by

$$H = H_S + \delta H_B + \delta H_I \ , \tag{2}$$

where δH_B is the Hamiltonian for a collection of harmonic oscillators and δH_I an interaction Hamiltonian linear in the oscillator variables q_A [11,12]. The heat bath may well be one in which the oscillators are "fundamental" (e.g., the modes of some free field, as in blackbody photons), but this is by no means necessary: Assume that, in the absence of any coupling with the system, the environment is characterized by some fixed, possibly time-dependent, solution. Now allow for a weak coupling with the system, weak in the sense that each path mode is only changed marginally. Then identify the q_A's as perturbed variables, i.e., degrees of freedom defined relative to the fixed solution (e.g., phonons). This has two implications: (1) the environment can be visualized as a collection of oscillators with (possibly time-dependent) frequencies, so that δH_B is quadratic in bath variables q_A, and (2) because the interaction of the environment with the system is assumed to be weak, in the sense that the individual bath modes are not altered significantly, δH_I must be linear in the q_A's. Note that one does *not* want to assume that the system is only weakly altered, so the interaction δH_I is *not* necessarily linear in the system variable x. In principle one can proceed without imposing any restrictions on the form of the system Hamiltonian H_S.

Given the above set of assumptions, one can write that

$$H_S = \tfrac{1}{2}v^2 + V_{\rm ren}(x,t) \ , \tag{3}$$

$$\delta H_B = \frac{1}{2}\sum_A [p_A^2 + \Omega_A^2(t) q_A^2] \ , \tag{4}$$

and, in terms of relatively arbitrary functions Γ_A,

$$\delta H_I = -\sum_A \Omega_A^2(t)\Gamma_A(x,t) q_A \ . \tag{5}$$

It is, however, convenient to rewrite H in the manifestly positive form

$$H = \tfrac{1}{2}v^2 + U(x,t) + \frac{1}{2}\sum_A \{p_A^2 + \Omega_A^2(t)[q_A - \Gamma_A(x,t)]^2\} \ , \tag{6}$$

where, in terms of the "renormalized" potential $V_{\rm ren}$,

$$U(x,t) = V_{\rm ren} - \frac{1}{2}\sum_A \Omega_A^2(t)\Gamma_A^2(x,t) \ . \tag{7}$$

Couplings of a system to some environment can induce finite and stochastic renormalizations in the system potential, although this is not always so [12]. In this paper, the words "system potential" will always refer explicitly to the renormalized system potential. A physical restriction on the form of the couplings Γ_A arises from the requirement that the renormalized potential not change the qualitative form of the bare potential. Thus, e.g., if the bare potential is a polynomial of order n, the renormalization should induce no terms of order higher than n. This condition also ensures stability of the system towards the destabilizing effects of multiplicative noise, since it implies that the stochastic forcing terms in the potential must be a polynomial of order $\leq n$.

Finally, as emphasized, e.g., by Caldeira and Leggett [12], it should be stressed that this is more than simply a toy model. This form of the Hamiltonian generally provides a correct description for *any* system which is only weakly coupled to its surroundings. To facilitate a concrete calculation, this Hamiltonian need only be supplemented by two inputs, namely, the spectral distribution of the environmental modes and the form of the coupling to the system. For a general physical problem, these may either be extracted from experimental data or derived from theoretical considerations.

III. TIME-DEPENDENT LANGEVIN EQUATIONS WITH MULTIPLICATIVE NOISE

The equations of motion generated from the Hamiltonian (6) clearly take the form

$$\begin{aligned} \dot{x} &= v \ , \\ \dot{v} &= -\frac{\partial}{\partial x}U(x,t) + \sum_A \Omega_A^2(t)[q_A - \Gamma_A(x,t)]\frac{\partial}{\partial x}\Gamma_A(x,t) \ , \\ \dot{q}_A &= p_A \ , \\ \dot{p}_A &= -\Omega_A^2[q_A - \Gamma_A(x,t)] \ , \end{aligned} \tag{8}$$

where an overdot denotes a time derivative $\partial/\partial t$.

The fact that the equation for $\dot{p}_A$ is linear in q_A implies

that one can immediately write down a formal solution for $q_A(t)$ in terms of $\Gamma_A(x,t)$ at retarded times $s<t$. Indeed, let $S_A(t)$ and $C_A(t)$ denote two linearly independent solutions to the homogeneous oscillator equation

$$\ddot{\Xi}_A+\Omega_A^2\Xi_A(t)=0 , \tag{9}$$

chosen without loss of generality to satisfy the initial conditions $C_A(0)=\dot{S}_A(0)=1$ and $S_A(0)=\dot{C}_A(0)=0$ at some time $t=0$. One then concludes exactly that

$$\begin{aligned} q_A(t)=&q_A(0)C_A(t)+p_A(0)S_A(t) \\ &+\int_0^t ds\ \Omega_A^2(s)\Gamma_A(x(s),s) \\ &\times[S_A(t)C_A(s)-S_A(s)C_A(t)] . \end{aligned} \tag{10}$$

The integrand in (10) vanishes in the coincidence limit $s\to t$. This, however, may be remedied by replacing S_A and C_A by $-\ddot{S}_A/\Omega_A^2$ and $-\ddot{C}_A/\Omega_A^2$ and then integrating by parts. The net result is a formal solution

$$\begin{aligned} q_A(t)=&\Gamma_A(x(t),x) \\ &+[q_A(0)-\Gamma_A(x,0)]C_A(t)+p_A(0)S_A(t) \\ &+\int_0^t ds\ W_A(s,t)\frac{\partial}{\partial s}\Gamma_A(x(s),s) , \end{aligned} \tag{11}$$

where the Wronskian

$$W_A(s,t)\equiv\dot{S}_A(s)C_A(t)-\dot{C}_A(s)S_A(t) . \tag{12}$$

Note that, for the special case of time-independent frequencies, $C_A(t)=\cos\Omega_A t$ and $S_A(t)=\Omega_A^{-1}\sin\Omega_A t$.

By inserting (11) into the equation for $\dot{v}$ and grouping terms suggestively, one then recovers an exact, nonlocal equation of the form

$$\begin{aligned} \dot{v}=&-\frac{\partial U}{\partial x}-\int_0^t ds\ \sum_A \Omega_A^2(t)W_A(s,t)\frac{\partial}{\partial x}\Gamma_A(x(t),t)\frac{\partial}{\partial s} \\ &\times\Gamma_A(x(s),s)+F_s(t) , \end{aligned} \tag{13}$$

where

$$\begin{aligned} F_s(t)=&\sum_A\Omega_A^2(t)\left[\frac{\partial}{\partial x}\Gamma_A(x(t),t)\right] \\ &\times\{[q_A(0)-\Gamma_A(x(0),0)]C_A t \\ &+p_A(0)S_A(t)\} . \end{aligned} \tag{14}$$

In the spirit of the discussion in Sec. II, suppose now that the interaction between system and bath entails a polynomial coupling

$$\Gamma_A(x,t)=\sum_{n=1}^{N}\frac{1}{n}\gamma_A^{(n)}(t)x^n , \tag{15}$$

where the functions $\gamma_A^{(n)}$ are arbitrary real functions of time. Equation (13) then takes the form

$$\dot{v}=-\frac{\partial U}{\partial x}-\int_0^t ds[K(t,s)v(s)+M(t,s)x(s)]+F_s(t) , \tag{16}$$

where, in terms of the quantities

$$A_A(s,t)=\left[\sum_m \gamma_A^{(m)}(s)x^{m-1}(s)\right]\left[\sum_n \gamma_A^{(n)}(t)x^{n-1}(t)\right] \tag{17}$$

and

$$B_A(s,t)=\frac{1}{n}\frac{\partial}{\partial s}A_A(t,s) , \tag{18}$$

the "memory kernels" $K(t,s)$ and $M(t,s)$ are

$$K(t,s)=\sum_A\Omega_A^2(t)A_A(s,t)W_A(s,t) , \tag{19}$$

$$M(t,s)=\sum_A\Omega_A^2(t)B_A(s,t)W_A(s,t) . \tag{20}$$

The force F_s now reduces to

$$\begin{aligned} F_s(t)=&\sum_A\Omega_A^2(t)\sum_m\gamma_A^{(m)}(t)x^{m-1} \\ &\times\{[q_A(0)-\Gamma_A(x,0)]C_A(t) \\ &+p_A(0)S_A(t)\} . \end{aligned} \tag{21}$$

Equation (16) is considerably more complicated than an ordinary Langevin equation. However, these additional complications need not preclude entirely the possibility of a simple physical interpretation or the proof of a fluctuation-dissipation theorem. Provided that the oscillator frequencies Ω_A are not time dependent, one can still prove a fluctuation-dissipation theorem, even if Γ_A is a nonlinear function of x [7] and/or explicitly time dependent. Indeed, consider an ensemble of initial conditions for which the first moments vanish identically, i.e.,

$$\langle Q_A(0)\rangle=\langle p_A(0)\rangle\equiv 0 , \tag{22}$$

with $Q_A(0)\equiv q_A(0)-\Gamma_A(x(0),0)$, and where the second moments are initially thermal, so that

$$\begin{aligned} \langle p_A(0)p_B(0)\rangle&=\Omega_A(0)\Omega_B(0)\langle Q_A(0)Q_B(0)\rangle \\ &=k_BT\delta_{AB} , \end{aligned} \tag{23}$$

where the angular brackets denote an initial ensemble average. One then computes exactly that

$$\langle F_s(t)\rangle=0 \tag{24}$$

and

$$\langle F_s(t_1)F_s(t_2)\rangle=k_BTK(t_1,t_2) , \tag{25}$$

thereby identifying $F_s(t)$ as a noise and providing a generalized fluctuation-dissipation theorem linking the noise autocorrelator with the "viscosity kernel" $K(t,s)$. [Strictly speaking $K(t,s)$ need give rise to a true viscosity only in the case of an Ohmic environment, a point which will be discussed later.] Equation (16) can now be viewed as a nonlinear, nonlocal Langevin equation.

That a fluctuation-dissipation theorem can hold even in these more complicated settings is a formal consequence of the fact that the nonlinearities and time dependences in (16) enter into the memory kernel $K(t_1,t_2)$ and the autocorrelator $\langle F_s(t_1)F_s(t_2)\rangle$ in exactly the same way. Physically, this result can be understood as follows: If the basic picture is valid, the total energy is dominated by the constant energy of the heat bath (recall that one is

now assuming that the bath frequencies are time independent), so that energy is approximately conserved, even if the system Hamiltonian H_S is time dependent. Since fluctuations induce a monotonic increase in the system energy, there must be some source of dissipation if that energy is to remain bounded. However, if that dissipation is too strong and dominates the fluctuations, the system energy will vanish at late times, which is clearly unphysical for a system coupled to a finite-temperature heat bath. The fluctuations and the dissipation must clearly balance if the system is to have a finite but nonzero energy at late times.

It should be observed that the Langevin equation (16) reduces to a well-known form in an appropriate limit. If one neglects all nonlinearities in the coupling between system and bath, assuming that $\Gamma_A \propto x$, one immediately recovers a special model considered previously [3]. And, moreover, if one assumes further that Ω_A and Γ_A are independent of time, one is reduced to the well-known independent oscillator model [13]. It is thus possible to address systematically the question of how the incorporation of nonlinearities and/or time dependences leads to systematic changes in the Langevin equation derived for that original model.

When one neglects both the nonlinearities and the time dependences, the memory kernel $M(s,t)$ vanishes identically and, moreover, the remaining memory kernel $K(s,t)$ contains no explicit x dependence. The stochastic force F_s involves x only linearly, through the propagation of an initial condition. It thus follows that one recovers a relatively simple equation involving a (nonlocal) friction $\propto v(s)$ and purely additive noise. To the extent that the function $K(t,s)$ is sufficiently sharply peaked about the coincidence limit $s \rightarrow t$ (what precisely this entails will be discussed below), one can then approximate $v(s)$ by its value at time t, in which case one obtains a Markovian equation involving a friction $\propto v(t)$.

When one allows for a nontrivial time dependence in the oscillator frequencies or the coupling between the system and bath, but as yet no nonlinearities, an additional nonvanishing kernel $M(s,t)$ appears. In a Markov approximation, this leads to a new term in the Langevin equation proportional to $x(t)$ which corresponds to a time-dependent change in the system potential. In a field-theoretic context, this would be interpreted as a mass renormalization. In this case, F_s still gives rise to additive noise, but that noise acquires an explicit time dependence.

When instead one incorporates nonlinearities but no time dependences, the memory kernel $M(s,t)$ still vanishes, but the other kernel $K(s,t)$ becomes significantly more complicated, involving not simply an autocorrelator for the mode functions $C_A \propto \cos \Omega_A t$, but a correlator of $\partial \Gamma_A / \partial x$ with itself. In other words, the nonlinearity implies that the evolution of x exactly involves an x-dependent memory. This is hardly surprising. Indeed, as one might have anticipated, e.g., by analogy with the theory of the Brownian motion, the integral is nothing other than the autocorrelation function for the forces associated with the interaction of the system with each of the bath modes. In this case, F_s also acquires an additional x dependence implying that the noise will depend not only on the initial conditions $x(0)$, $q_A(0)$, and $p_A(0)$, but upon the x-dependent state of the system as well. In other words, the noise is multiplicative.

It should be observed that the memory kernel $M(t,s)$ does not enter into the fluctuation-dissipation theorem for a time-independent bath. Its only effect is to induce a systematic renormalization of the system potential, which will of course affect the *form* of the late time solution.

One further point should be stressed. The fluctuation-dissipation theorem of Eq. (25) refers explicitly only to the total force F_s and the total memory kernel $K(t,s)$. However, it is easy to see that analogous theorems also hold separately for each term $\propto \gamma_A^{(m)}(s)\gamma_A^{(n)}(t)$. In this sense, the fluctuation-dissipation theorem is truly microscopic.

Equation (16) is an exact, nonlocal equation. Only to the extent that this equation can be approximated as Markovian can one derive from it a Fokker-Planck equation. Such a Markov limit implies (1) that the memory kernels $K(t,s)$ and $M(t,s)$ may be approximated as essentially local in time, and (2) that the autocorrelation function for F_s falls off rapidly as $|t-s|$ increases. It is clear by inspection that, if $K(t,s)$ is essentially local, so is $M(t,s)$. Further, the fluctuation-dissipation theorem guarantees that, if $K(t,s)$ is local, the noise autocorrelation will be as well. In considering the validity of a Markov approximation, it thus suffices to consider $K(t,s)$.

The Markov limit can oftentimes be justified approximately for smooth spectral distributions, given an appropriate separation of time scales. Let t_c denote a characteristic time scale on which the memory kernels $K(t,s)$ and $M(t,s)$ decay, and let $t_{\rm sys}$ denote a characteristic time scale on which the system velocity v and/or position x change significantly, either in response to the environment (t_R) or to the system potential $U(t_S)$. [Provided that the mode functions are all oscillatory, $K(t,s)$ will decay much faster than $|t-s|$ at late times, so that one can clearly identify a time scale t_C.] In a time-dependent setting, the Markov limit then follows when $t_C \ll t_{\rm sys}$. If the oscillators are time dependent, there is another relevant time scale t_H, the characteristic time on which the frequencies change. To the extent that t_H is much larger than both $t_{\rm sys}$ and t_C, one anticipates that the time dependence may be viewed as a perturbation, and that the Markov limit should still obtain. If $t_C \ll t_{\rm sys}$, but is *not* short compared with t_H, the time dependence can no longer be viewed as a perturbation, but it may still be true that a Markov approximation can be justified. This should, e.g., be the case if all modes still oscillate and/or the coupling of the system to the longest wavelength modes is relatively weak. If, however, t_H is not much larger than t_c, one expects that a local Fokker-Planck description will be inappropriate. This is, for example, true for the specific example discussed in Secs. IV and V.

For the special case of "ohmic" environments, the nonlocal term in the exact Langevin equation involving $K(t,s)$ reduces to the usual linear viscosity seen in heuristic *ad hoc* Langevin descriptions. These environments are

characterized by time-independent frequencies and have a spectral distribution $g(\Omega)\propto\Omega^2$, with an upper cutoff at some $\Omega_{\max}$, and all the oscillators are assumed to couple bilinearly to the system with an equal strength [12,7]. If the coupling is not bilinear, one still recovers a viscosity that is linear in v, but this viscosity will be dependent explicitly on x. If the spectral distribution $g(\Omega)$ differs only slightly from $\propto\Omega^2$, the difference may be treated perturbatively to extract calculable modifications in the form of the Markovian equation. However, for spectral distributions which are very different, e.g., "supra-ohmic" distributions $\propto\Omega^4$, the local Langevin equation can be higher order in time derivatives and need not contain a simple viscosity term $\propto v(t)$. One concrete example thereof is provided by an electron interacting with its self-electromagnetic field [14]. However, for the cosmological example considered in Secs. IV and V, the spectral distribution will be nearly "ohmic," so that the local Langevin description will contain an ordinary viscosity and thus admit a Fokker-Planck realization.

IV. A COSMOLOGICAL EXAMPLE

The objective of this section is to formulate a nonlocal Langevin equation in terms of which to describe the evolution of some system variable, evolving in a Landau-Ginzburg potential $V_{\rm ren}$ and coupled to a bath of scalar blackbody "radiation." The aim of this Langevin equation is to provide a quasirealistic model for a cosmological phase transition. The entire analysis will be classical, allowing for thermal fluctuations but *not* for quantum fluctuations. This may prove appropriate either in the context of some versions of inflation or, alternatively, in the formation of cosmic strings or baryogenesis. Attention here focuses on formulating the problem and discussing its physical potentialities. Quantitative details will be provided in a subsequent paper.

In what follows the bath will be idealized as a collection of oscillators, characterized by time-dependent frequencies appropriate for a scalar field with a general ξR curvature coupling. The case $\xi=\frac{1}{6}$ corresponds to conformal coupling, whereas $\xi=0$ yields minimal coupling. Consistent with the discussion in Sec. II, the interaction Hamiltonian will be taken as linear in the oscillator variables q_A, but it can involve an arbitrary quadratic dependence on the system variable x, with both linear and nonlinear pieces. By allowing for both linear and nonlinear couplings, and varying the relative strengths of these two different contributions, one will be able to compare the effects of additive and multiplicative noise. The analysis will be effected in the conformal frame, in terms of a conformal time coordinate η satisfying $d\eta=a^{-1}dt$, where a denotes the scale factor. It will, moreover, be assumed that the spatial curvature of the $t=$const slices vanishes, so that one is considering a $k=0$ Friedman cosmology.

Given these assumptions, one is led directly to a Hamiltonian of the form motivated in Sec. II, namely,

$$H=\frac{1}{2}v^2+U(x,\eta)+\frac{1}{2}\sum_A\{p_A^2+\Omega_A^2(\eta)[q_A-\Gamma_A(x,\eta)]\}^2\ , \tag{26}$$

where, in terms of constants λ, θ, and σ,

$$U(x,\eta)=\tfrac{1}{4}\lambda(\eta)x^4+\tfrac{1}{3}\theta(\eta)x^3+\tfrac{1}{2}\sigma(\eta)x^2\ , \tag{27}$$

and the coupling

$$\Gamma_A(x,\eta)=\gamma_A^{(1)}(\eta)x+\gamma_A^{(2)}(\eta)x^2\ . \tag{28}$$

Recall from Eq. (7) that U and the Landau-Ginzburg potential $V_{\rm ren}$ are connected by a term involving the couplings Γ_A. It follows that, if the $\gamma_A^{(1)}$'s and $\gamma_A^{(2)}$'s are both nonvanishing, this coupling will in general induce a cubic term in U, although the coefficient θ of that term *could* vanish. The coefficients λ and σ in U are also "dressed" quantities, differing from the bare quantities in $V_{\rm ren}$ because of the terms involving the Γ_A's. At this stage, the explicit time dependence of λ and σ and the couplings $\gamma_A^{(1)}$ and $\gamma_A^{(2)}$ may be treated as more or less arbitrary. The frequencies Ω_A satisfy

$$\Omega_A^2(\eta)=\omega_A^2-(1-6\xi)\frac{a''}{a}\ , \tag{29}$$

where a prime denotes a conformal time derivative $\partial/\partial\eta$.

Suppose now that, in terms of cosmic time t, the scale factor a evidences a simple power-law time dependence $a=t^p$, with $p\geq\frac{1}{2}$. It then follows that

$$a\propto(\eta-\eta_0)^{p/(1-p)}\ , \tag{30}$$

where η_0 denotes an integration constant, so that

$$\Omega_A^2(\eta)=\omega_A^2-\frac{\nu^2}{(\eta-\eta_0)^2}\ , \tag{31}$$

where

$$\nu^2=(1-6\xi)\frac{p(2p-1)}{(1-p)^2} \tag{32}$$

is intrinsically positive when $p>\frac{1}{2}$ and $1-6\xi\geq0$. The quantities ω_A^2 denote eigenvalues of the spatial Laplacian.

To proceed further, one needs to determine the mode functions Ξ_A for the time-dependent frequencies. These can clearly be evaluated in terms of Bessel functions. However, in so doing there are at least two possible ways in which to proceed, namely, considering (1) complex modes involving Hänkel functions or (2) real modes involving ordinary Bessel and Neumann functions. This paper will adopt the (less conventional) second choice, since it provides a more direct connection with earlier work on Langevin equations: in the absence of a time-dependent expansion, the mode functions reduce to sines and cosines. Straightforward algebra reveals that the equation

$$\Xi_A''+\left[\omega_A^2-\frac{\nu^2}{(\eta-\eta_0)^2}\right]\Xi_A=0 \tag{33}$$

is solved by a general

$$\Xi_A=[\omega_A(\eta-\eta_0)]^{1/2}Z_\mu(\omega_A(\eta-\eta_0))\ , \tag{34}$$

where Z_μ denotes an arbitrary solution to Bessel's equation of order

$$\mu^2=(1-6\xi)\frac{p(2p-1)}{(1-p)^2}+\frac{1}{4}=\nu^2+\frac{1}{4}\ . \tag{35}$$

Note that, for the special cases $\xi=\frac{1}{6}$ (conformal coupling) and/or $p=\frac{1}{2}$ (a universe dominated by conformal electromagnetic radiation), $\mu=\frac{1}{2}$ and the solutions Ξ_A reduce to sines and cosines. For these particular values, the bath Hamiltonian δH_B is time-independent in the conformal frame. This implies that the physical frequencies are simply redshifted uniformly as the Universe expands.

Presuming that the initial value problem for the Langevin equation is specified at time $\eta=0$, the appropriate solutions will be C_A and S_A, combinations of Bessel and Neumann functions J_ν and N_ν, modulated by factors $\omega_A(\eta-\eta_0)$, satisfying $C_A(0)=dS_A(0)/d\eta=1$ and $S_A(0)=dC_A(0)/d\eta=0$. If $\mu\neq\frac{1}{2}$, the functions C_A and S_A are, for long wavelengths with $|\omega_A(\eta-\eta_0)|\ll\mu$, very different from sines and cosines. Indeed, they are not even oscillatory. However, to the extent that such longer wavelengths do not couple significantly to the system, one can approximate the Ξ_A's by the forms appropriate when $|\omega_A(\eta-\eta_0)|\gg\mu$. In this case, the mode functions reduce to

$$\Xi_A(\eta)\approx\left[\frac{2}{\pi}\right]^{1/2}\sin\left[\omega_A(\eta-\eta_0)-\frac{\pi}{2}\left[\mu+\frac{1}{2}\right]\right], \tag{36}$$

$$\Xi_A(\eta)\approx\left[\frac{2}{\pi}\right]^{1/2}\cos\left[\omega_A(\eta-\eta_0)-\frac{\pi}{2}\left[\mu+\frac{1}{2}\right]\right], \tag{37}$$

i.e., ordinary sines and cosines, modulated by phase shifts which can of course be absorbed in the normalizations.

It is clear that, in this limit, one recovers a relatively simple nonlocal Langevin equation, for which the only explicit time dependences are in the potential U and the couplings between the system and the environment. It thus follows that, in this approximation, a fluctuation-dissipation theorem holds, so that one would anticipate an evolution towards some steady-state solution at late times. The fluctuation-dissipation theorem is of course exact when $\xi=\frac{1}{6}$ and/or $p=\frac{1}{2}$.

It should be stressed that the condition $|\omega_A(\eta-\eta_0)|\gg\mu$ has a very simple physical interpretation. Reexpressed in terms of the physical cosmic time t, this condition becomes $(\omega_A/a)t\gg\mu|1-p|$, this corresponding, for μ and p of order unity, to the demand that the physical period of the oscillation be short compared with the time scale t_H associated with the expansion of the Universe. In other words, the wavelength must be short compared with the horizon length.

In general, however, a fluctuation-dissipation theorem does not hold. Recall that the memory kernel $K(t,s)$ satisfies

$$K(t,s)=\sum_A\Omega_A^2(t)A_A(s,t)W_A(s,t), \tag{38}$$

in terms of the Wronksian of Eq. (12). Alternatively, a thermal average of the noise autocorrelation, taken at the initial time $\eta=0$, satisfies

$$(k_BT)^{-1}\langle F_s(t)F_s(s)\rangle=\sum_A\Omega_A^2(t)A_A(s,t)Q_A(s,t), \tag{39}$$

where

$$Q_A(s,t)=\Omega_A^2(s)\left[\frac{C_A(t)C_A(s)}{\Omega_A^2(0)}+S_A(t)S_A(s)\right]. \tag{40}$$

When the frequencies Ω_A are all time independent ($\xi=\frac{1}{6}$ or $p=\frac{1}{2}$), $W_A(s,t)$ and $Q_A(s,t)$ are in fact equal so that one recovers the fluctuation-dissipation theorem (25). In general, however, this is clearly not so; and the typical size of the fractional deviation between these two quantities provides a concrete measure of the degree to which the fluctuation-dissipation theorem fails. Suppose that $\omega_{\rm cr}$ denotes a typical frequency associated with the coupling of the system and the environment. It then follows straightforwardly that the fractional amplitude of the deviation is of order

$$\frac{1}{\omega_{\rm cr}(\eta-\eta_0)}\sim\frac{a}{\omega_{\rm cr}t}\ . \tag{41}$$

The fractional deviation from a fluctuation-dissipation theorem scales as the ratio of a characteristic oscillator period $\sim(\omega_{\rm cr}/a)^{-1}$ to the expansion time scale t_H.

It remains to consider the circumstances under which a Markov approximation can be justified. Note first of all that, to the extent that the bath oscillators are interpreted as representing the modes of some free field, one would anticipate a spectral distribution $\propto\omega_A^2$, so that, given some cutoff $\omega_{\rm max}$, one can pass to a continuum limit

$$\sum_A\rightarrow\int d\omega\,\omega^2\ . \tag{42}$$

Consistent with the equation for δH_I, now separate out the explicit Ω dependence in the system-bath couplings and write $\gamma_A^{(n)}(\eta)=c_A^{(n)}(\eta)/\Omega_A^2(\eta)$.

As a particularly simple first approximation, suppose that $c_A^{(n)}$ is essentially independent of frequency. And, moreover, assume that the longest wavelength modes are not very important in the coupling, so that one can neglect the frequency shift associated with the expansion of the Universe and set $\Omega_A\approx\omega_A$. Given these approximations, one has, e.g., that

$$K(\eta,s)\approx\int d\omega\sum_{m,n}c^{(n)}(s)c^{(m)}(\eta)x^{n-1}(s)x^{m-1}(\eta)\times\cos\omega(\eta-s)\ . \tag{43}$$

Suppose, however, that one can also neglect the time dependence of the c's, a reasonable assumption, e.g., *if* the c's change only on an expansion time scale t_H. One can then effect the $d\omega$ integration explicitly to obtain (for large $\omega_{\rm max}$) a delta function $\delta_D(\eta-s)$. It follows that the memory of $K(t,s)$ is indeed very short, so that, presuming that the time scale t_S associated with U is not too short, one can approximate

$$K(\eta,s)\approx\left[\sum_n c^{(n)}(t)x^{n-1}(\eta)\right]^2\delta_D(\eta-s)\equiv2\mathcal{H}(x,\eta)\delta_D(\eta-s)\ . \tag{44}$$

An analogous expression holds for the other memory kernel $M(\eta,s)$.

If the c_A's depend strongly on frequency and/or the spectral distribution is considerably different, the analysis becomes more complicated. It is, however, evident that, whenever the couplings are such that the lowest frequency modes are not very important, so that $t_C \ll t_R$, and t_H, and the distribution of modes is approximately ohmic, a Markov approximation should in fact be justified, leading to a viscosity $\propto \mathcal{H}v$.

Under these circumstances, the nonlocal Langevin equation can be well approximated by a local equation of the form

$$v' = -\frac{\partial U}{\partial x} - \mathcal{M}(\eta)x(\eta) - \mathcal{H}(\eta)v(\eta) + F_s(\eta) , \tag{45}$$

where, for an initial thermal ensemble,

$$\langle F_s(\eta)F_s(s)\rangle = 2k_B T\mathcal{H}(x,\eta)\delta_D(\eta - s) . \tag{46}$$

This local Langevin equation leads immediately to a Fokker-Planck equation of the form

$$\frac{\partial f}{\partial \eta} + \frac{\partial}{\partial x}(vf) + \frac{\partial}{\partial v}\left[\left[-\frac{\partial U}{\partial x} - \mathcal{M}x - \mathcal{H}v\right]f\right] - k_B T\mathcal{H}\frac{\partial^2 f}{\partial v^2} = 0 . \tag{47}$$

Translated back into the physical frame, the Langevin equation (45) becomes

$$\dot{V} = -\frac{1}{a^4}\frac{\partial U}{\partial X} - \mathbf{M}(t)X - \left[3\frac{\dot{a}}{a} + \mathbf{K}(t)\right]V + \mathbf{F}_s(t) , \tag{48}$$

where the overdot denotes differentiation with respect to cosmic time, and

$$X = x/a , \tag{49}$$

$$V = \dot{X} = v/a^2 - (\dot{a}/a^2)x , \tag{50}$$

$$\mathbf{K}(t) = \mathcal{H}/a , \tag{51}$$

$$\mathbf{M}(t) = \frac{\mathcal{M}}{a^2} + \mathbf{K}(t)\left[\frac{\dot{a}}{a}\right] + \left[\frac{\dot{a}}{a}\right]^2 + \frac{\ddot{a}}{a} , \tag{52}$$

$$\mathbf{F}_s(t) = F_s/a^3 . \tag{53}$$

The noise autocorrelator is now

$$\langle \mathbf{F}_s(t)\mathbf{F}_s(t')\rangle = 2\frac{k_B T}{a^4}\mathbf{K}(t)\delta_D(t - t') . \tag{54}$$

Note that, in the physical frame, there are two sources of damping, namely the viscosity $\propto \mathbf{K}V$ and the cosmological frame dragging $\propto H \equiv \dot{a}/a$. The corresponding Fokker-Planck equation is now

$$\frac{\partial f_p}{\partial t} + \frac{\partial}{\partial X}(Vf_p) + \frac{\partial}{\partial V}\left\{\left[-\frac{1}{a^4}\frac{\partial U}{\partial X} - \mathbf{M}(t)X - \left[3\frac{\dot{a}}{a} + \mathbf{K}(t)\right]V\right]f_p\right\} - \frac{k_B T}{a^4}\mathbf{K}(t)\frac{\partial^2 f_p}{\partial V^2} = 0 . \tag{55}$$

The form of this equation is consistent with the Fokker-Planck equation analyzed by Brandenberger and co-workers [1]. Given, however, that the approach followed here is entirely different from that of Ref. [1] a more detailed comparison is appropriate: The present formalism is intended primarily to describe the behavior of a homogeneous degree of freedom (dilaton mode). However, for the special case of a free field with no mode couplings, the Fokker-Planck equation (55) holds equally well for *any* single field mode coupled to a heat bath, which is precisely the model problem considered by Brandenberger and co-workers. Comparing Eq. (55) with Eq. (20) of Ref. [1], one finds that, even though the latter equation was arrived upon in a rather different way, the diffusion terms are consistent in that they scale the same way with a, and that in both cases the "Hubble damping" term is present. However, there *is* one important difference. While the present model incorporates back-reaction effects due to the heat bath, namely, the normal viscosity $\mathbf{K}$ and the potential renormalization $\mathbf{M}$, these physical effects are not taken into account in Ref. [1]. A quantitative assessment of the importance of these effects will be presented elsewhere.

Having formulated the Fokker-Planck equation, it is worth recalling once again the critical assumptions that went into its derivation. (1) The decay time t_C must be short compared with the time scales t_R and t_S on which the system changes in response either to its potential U or in response to the environment *and* with the expansion time t_H. (2) For $\omega(\eta - \eta_0) \ll \mu$, the modes are nonoscillatory. One must also assume that the coupling of the system with these infrared modes is negligible, so that there is no significant long time tail to $K(\eta,s)$ to prevent the existence of a Markov limit. (3) The spectral distribution must be approximately "ohmic," with $g(\omega) \propto \omega^2$, so that the nonlocal contribution involving $K(\eta,s)v(s)$ gives rise to an ordinary viscosity. Fortunately, this is precisely what one expects of a heat bath of thermal photons.

V. AN APPROXIMATE SOLUTION

Given the assumed form (26) for the Hamiltonian (with the system-bath coupling assumed to be weakly time dependent), the coefficients $\mathcal{H}$ and $\mathcal{M}$ may be written as

$$\mathcal{H} = \lambda_0 + 2\lambda_1 x + \lambda_2 x^2 \tag{56}$$

and

$$\mathcal{M} = \mu_0 + 2\mu_1 x + \mu_2 x^2 \tag{57}$$

and the noise correlator

$$\langle F_s(\eta)F_s(s)\rangle = 2k_B T\mathcal{H}\delta_D(\eta-s)$$
$$= 2k_B T(\lambda_0+2\lambda_1 x+\lambda_2 x^2)\delta_D(\eta-s)\ , \quad (58)$$

where the λ_n's and μ_n's are independent of x ad v. Note that λ_0 and λ_2 are necessarily positive definite, but that λ_1 and the μ_n's are of indeterminate sign. $\mathcal{H}$ gives rise to an x-dependent viscosity, whereas $\mathcal{M}$ induces a further renormalization of the potential U in terms of new coefficients Λ, Θ, and Σ:

$$\mathcal{U}(x,\eta) = \tfrac{1}{4}\lambda(\eta)x^4+\tfrac{1}{3}\theta(\eta)x^3+\tfrac{1}{2}\sigma(\eta)x^2+\tfrac{1}{2}\mu_0 x^2$$
$$+\tfrac{2}{3}\mu_1 x^3+\tfrac{1}{4}\mu_2 x^4$$
$$\equiv \tfrac{1}{4}\Lambda(\eta)x^4+\tfrac{1}{3}\Theta(\eta)x^3+\tfrac{1}{2}\Sigma(\eta)x^2\ . \quad (59)$$

By inserting Eqs. (58) into (47), one obtains an explicit Fokker-Planck equation, which one may hope to solve. Unfortunately, however, in general such a multivariable Fokker-Planck equation cannot be solved exactly except via numerical techniques. Nevertheless, one *can* at least obtain an approximate solution for the expectation value $\langle E\rangle$ of the system energy E using the so-called "energy envelope" technique introduced by Stratonovich [15] and further developed by Lindenberg and Seshadri [16].

The idea underlying this approximation is in fact straightforward: In many cases of physical interest, such as that considered here, one can visualize the system as exhibiting (nonlinear) oscillations on the time scale $t_S \sim \omega_0^{-1}$ associated with the potential $\mathcal{U}$, oscillations which are eventually altered by the coupling with the environment on the damping time scale t_R. To the extent that the damping time t_R and the expansion time t_H are both long compared with t_S, one can then assume that the system energy E is nearly conserved on time scales $\sim t_S$, and varies only on a time scale much longer than the time associated with variations in x. It is thus natural to transform from x and v to new variables x and E, to treat E as an adiabatic invariant, and to implement an "orbit average" of the transformed E-x Fokker-Planck equation to extract an equation involving only E and η.

Implement, therefore, a change of variables from (x,v) to (x,E), where, explicitly. $E=\frac{1}{2}v^2+\mathcal{U}(x,\eta)$, to obtain a new Fokker-Planck equation for $W(x,E,t)$ satisfying

$$f(x,v,\eta)dx\,dv = W(x,E,\eta)dx\,dE\ . \quad (60)$$

To the extent that the energy is approximately conserved during a single oscillation of the system, one can then assume that

$$W(x,E,\eta) = \{2\Phi'(E)[E-\mathcal{U}(x,\eta)]^{1/2}\}^{-1}W_1(E,\eta)\ , \quad (61)$$

where

$$\Phi(E) = \int dx\,[E-\mathcal{U}(x,\eta)]^{1/2} \quad (62)$$

and a prime now denotes a $\partial/\partial E$ derivative. Here the integration extends over the values of x along the unperturbed orbit associated with E. Note that the prefactor of W_1 is simply the relative amount of time that, for fixed E, the system spends at each point x.

By integrating the Fokker-Planck equation for $W(x,E,\eta)$ over x, one obtains the desired equation for $W_1(E,\eta)$, which takes the form

$$\frac{\partial}{\partial\eta}W_1(E,\eta) = -\left[\frac{\partial}{\partial E}\left[\frac{1}{\chi'(E)}\{\lambda_0[\Phi(E)-k_BT\Phi'(E)]+2\lambda_1[\chi(E)-k_BT\chi'(E)]+\lambda_2[\Psi(E)-k_BT\Psi'(E)]\}\right]\right.$$
$$\left.+k_BT\frac{\partial^2}{\partial E^2}\left[\frac{1}{\chi'(E)}[\lambda_0\Phi(E)+2\lambda_1\chi(E)+\lambda_2\Psi(E)]\right]\right]W_1(E,\eta)\ , \quad (63)$$

where

$$\chi(E) = \int dx\,x\,[E-\mathcal{U}(x,\eta)]^{1/2} \quad (64)$$

and

$$\Psi(E) = \int dx\,x^2[E-\mathcal{U}(x,\eta)]^{1/2}\ . \quad (65)$$

Note that this differs from Eq. (3.8) in [16], which assumes implicitly that the system's unperturbed orbit is symmetric about $x=0$, so that Eq. (64) vanishes identically.

Unfortunately, for a generic potential $\mathcal{U}$ the functions $\Phi(E)$, $\chi(E)$, and $\Psi(E)$ cannot be evaluated analytically, so that one cannot realize the right-hand side of Eq. (63) explicitly in terms of simple functions of E. Thus, e.g., for the quartic potential (60), these functions can only be expressed as elliptic integrals, which must be evaluated numerically. There *is*, however, one limit in which one can proceed analytically, namely when the energy E is small and the system is oscillating about a local minimum of $\mathcal{U}$. The obvious point is that, in this limit, one can evaluate the orbit integrals, assuming that the system is effectively evolving in a simple harmonic oscillator potential.

Suppose for simplicity that $\Theta=0$, so that the dressed potential $\mathcal{U}$ is itself of the Landau-Ginzburg form. When Σ is positive, one can then approximate the system as oscillating with squared frequency $\Omega_0^2=\Sigma$ about the origin. And similarly, when Σ is negative, and $\mathcal{U}$ is a "Mexican hat" potential, the system can be assumed to oscillate with squared frequency $\Omega_0^2=2|\Sigma|$ about one of the two minima at $x_0=\pm(\Lambda/|\Sigma|)^{-1/2}$.

In either case, one can evaluate Φ, χ, and Ψ, to realize the right-hand side of Eq. (63) in terms of polynomials at

most quadratic in E. And, given such an explicit representation, it is straightforward to derive from the Fokker-Planck equation a transport equation involving the time derivative of the first energy moment:

$$\langle E(\eta)\rangle \equiv \int dE\, E W_1(E,\eta) \ . \tag{66}$$

Suppose in the first instance that Σ is positive, and that the system is executing small oscillations about the ground state $x=0$ with $\omega_0^2=\Sigma$. In this approximation, one verifies that, to the extent that the time dependence of E, ω_0, and the λ_n's may be ignored, the moment equation takes the form

$$\frac{\partial}{\partial\eta}\langle E(\eta)\rangle = k_B T\lambda_0 - \left[\lambda_0 - \frac{k_B T\lambda_2}{\omega_0^2}\right]\langle E(\eta)\rangle - \frac{\lambda_2}{\omega_0^2}\langle E^2(\eta)\rangle \ . \tag{67}$$

Note that, because of the reflection symmetry $x\rightarrow -x$ for the potential $\mathcal{U}$, the functions $\chi=\chi'=0$, so that the contributions involving λ_1 vanish identically.

Unfortunately, this equation still cannot be solved exactly for $\langle E(\eta)\rangle$, as it involves the unknown function $\langle E^2(\eta)\rangle$. To obtain a formula for $\langle E^2(\eta)\rangle$, one must consider the second moment equation, which in turn relates $d\langle E^2(\eta)\rangle/d\eta$ to the third moment $\langle E^3(\eta)\rangle$. In the spirit of, e.g., the Bogoliubov-Born-Green-Kirkwood-Yvon (BBGKY) hierarchy, one requires a truncation approximation.

As in Refs. [15] and [16], suppose therefore that

$$\langle E^2(t)\rangle \approx \kappa\langle E(t)\rangle^2 \ , \tag{68}$$

with $\kappa=2$. One knows that, when the system is at equilibrium, with energy $E=k_BT$, this equation is satisfied identically for $\kappa=2$, and one might expect on physical grounds that, before the system is "at equilibrium," the energy distribution will be narrower and $\kappa<2$. As emphasized by Lindenberg and Seshadri [16], this truncation approximation thus leads to an upper limit on the time scale on which the system "equilibrates" with the bath. Given this truncation, one can immediately write down the solution [16]

$$\langle E(\eta)\rangle = \frac{k_BT(E_0+Ak_BT)-AkT(k_BT-E_0)\exp\{-[(A+1)/A]\lambda_0\eta\}}{(E_0+Ak_BT)-(k_BT-E_0)\exp\{-[(A+1)/A]\lambda_0\eta\}} \ , \tag{69}$$

where

$$A \equiv \frac{\lambda_0\omega_0^2}{\lambda_2 k_B T} \ . \tag{70}$$

In the limit that $\lambda_2\rightarrow 0$, the multiplicative noise "turns off" and the system approaches an "equilibrium" with $\langle E\rangle = k_BT$ on a time scale $t_R\sim\lambda_0^{-1}$. If $\lambda_2\neq 0$, the system still evolves towards an equilibrium with $\langle E\rangle = k_BT$, but the time scale t_R can be altered significantly. Indeed, in the limit that $\lambda_0\rightarrow 0$, the additive noise "turns off" and $t_R\sim\omega_0^2/(\lambda_2 k_BT)$. It thus follows that, when the nonlinear coupling is sufficiently strong, the system may be driven towards equilibrium, *not* by the ordinary additive noise associated with the linear coupling, but primarily by the multiplicative noise associated with the nonlinear coupling.

The only point that remains to be checked is that one is still assuming, as is implicit in this "envelope" approximation, that the time scale ω_0^{-1} is much shorter than the damping time. This, however, is clearly the case when λ_0 and λ_2 are not too large. Indeed, one verifies that (a) the weak damping approximation *is* legitimate but (b) multiplicative noise dominates the evolution towards an equilibrium whenever [16]

$$\frac{\lambda_0}{\omega_0} \ll \frac{\lambda_0\omega_0^2}{\lambda_2 k_B T} \ll 1 \ . \tag{71}$$

Turn now to the case when $\Sigma<0$ and the system is oscillating about one of the two minima of the potential $x_0\neq 0$. This is the case relevant to first-order phase transitions. Here the terms involving χ and its energy derivative do not vanish, and the formula for Φ and Ψ acquire additional terms involving the location x_0 of the new minimum. However, one still recovers a relatively simple exact equation for $\partial\langle E\rangle/\partial\eta$. Specifically, one finds that, in this case, Eq. (67) is replaced by

$$\frac{\partial}{\partial\eta}\langle E(\eta)\rangle = k_BTL - \left[L - \frac{k_BT\lambda_2}{\omega_0^2}\right]\langle \mathcal{E}(\eta)\rangle - \frac{\lambda_2}{2\omega_0^2}\langle \mathcal{E}^2(\eta)\rangle \ , \tag{72}$$

where now

$$\mathcal{E} = E - \mathcal{U}(x_0) \tag{73}$$

denotes the system energy defined relative to the minimum of the potential, and

$$L = \lambda_0 + \lambda_1 x_0 + \lambda_2 x_0^2 \tag{74}$$

plays the role of a "dressed" λ_0. The obvious point here is that, since one is effectively expanding in a Taylor series around the point $x_0=\pm(\Lambda/|\Sigma|)^{-1/2}$, the coupling terms $\propto\lambda_1$ and λ_2 will induce $(x-x_0)$-independent effects.

Note in particular that, if $|x_0|$ is large, as will be the case when $\Sigma\ll\Lambda$, the dressed L can be much larger than λ_0. This implies that, in this case, the nonlinear cou-

plings can reduce the overall equilibration time *both* through the introduction of a new term $\propto\lambda_2\langle E^2\rangle$ *and* through an increase in the effective linear coupling.

In any event, to the extent that $\mathcal{U}(x_0,\eta)$ is only slowly varying in time, the derivative $\partial\langle E(\eta)\rangle/\partial\eta$ can be replaced by $\partial\langle\mathcal{E}(\eta)\rangle/\partial n$. And, to the extent that the coefficients L and λ_2 may be approximated as time independent, Eq. (72) can be solved analytically. The result is an expression identical to Eq. (65), except that E is replaced by the shifted $\mathcal{E}=E-\mathcal{U}_0$ and

$$A=\frac{L\omega_0^2}{\lambda_2 k_B T} . \tag{75}$$

If the energy E and the couplings λ_m cannot be treated as independent of time, the analysis becomes more complicated, but, at least when E is small, one can again formulate an analogue of Eq. (67). If E depends explicitly on η, the moment equation will of course acquire an additional term $\langle\partial E/\partial\eta\rangle$. Suppose that the system may be approximated as simply oscillating with squared frequency $\Sigma(\eta)$ about $x_0=0$. One then concludes that

$$\frac{\partial E}{\partial\eta}=\frac{1}{2}\langle\Sigma x^2\rangle\frac{d}{d\eta}\ln\Sigma . \tag{76}$$

Consistent, however, with the ansatz (61), $\Sigma x^2/2$, can be replaced by its "orbit averaged" value $E/2$ [this is the exception value associated with the W of Eq. (61)], so that the moment equation may be written in the form

$$\frac{\partial}{\partial\eta}\langle E(\eta)\rangle=\frac{1}{2}\langle E(\eta)\rangle\frac{d}{d\eta}\ln\Sigma+k_B T\lambda_0 - \left[\lambda_0-\frac{k_B T\lambda_2}{\Sigma}\right]\langle E(\eta)\rangle-\frac{\lambda_2}{\Sigma}\langle E^2(\eta)\rangle . \tag{77}$$

Suppose now that the time dependence of Σ is relatively unimportant, i.e., that the time scale on which Σ changes is long compared with the time scale on which the environment effects the system. In this case, it makes sense to speak of (at least) an approximate approach towards equilibrium on a time scale set by the time-dependent couplings. If the coupling between the system and the environment is dominated by λ_0, one thus infers a decay of initial conditions and an approach towards an equilibrium, driven by the additive noise, proceeding as $\exp[-\int d\eta/\lambda_0(\eta)]$. And similarly, if the coupling is dominated by λ_2, one has an approach towards equilibrium, driven by the multiplicative noise, proceeding as

$$\exp\{-\int d\eta[\Sigma/\lambda_2(k_B T\eta)]\} .$$

Suppose, however, that the time dependence of Σ *is* important, and that Σ changes appreciably on time scales $\ll t_R$. In this case, one can no longer speak of a simple approach towards equilibrium, since the form of the system Hamiltonian is actually changing on a time scale $\ll t_R$. One now concludes that, in a first approximation

$$E(\eta)/E(0)=[\Sigma(\eta)/\Sigma(0)]^{1/2} ,$$

and that the coupling with the environment is only a perturbation on this simple power-law evolution.

VI. CONCLUSION

The principal thrusts of this paper have been (a) the motivation of a relatively general phenomenological Hamiltonian, in terms of which to characterize the evolution of a "system" degree of freedom with its surrounding environment, (b) the rigorous derivation of Langevin and Fokker-Planck equations for a system described by this model Hamiltonian, and (c) a qualitative analysis of the new effects incorporated in these equations which are absent from other, more heuristic, descriptions. The model considered in this paper has several potential applications, the most obvious being to the study of cosmological phase transitions. Unfortunately, however, the Langevin and Fokker-Planck equations derived here are, except in a few simple cases, very difficult to solve analytically. For this reason, a numerical study applying the model to several cosmological problems of interest, such as the onset of new inflation and first order cosmological phase transitions, is currently under way.

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