Effective potential and quadratic divergences

M. B. Einhorn
Institute for Theoretical Physics, University of California, Santa Barbara, California 93106-4030

D. R. T. Jones*
Institute for Theoretical Physics, University of California, Santa Barbara, California 93106-4030

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We use the effective potential to give a simple derivation of Veltman's formula for the quadratic divergence in the Higgs self-energy. We also comment on the effect of going beyond the one-loop approximation.

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There has been some interest recently [1–8] in the nature of the quadratic divergences present in renormalizable field theories, both in general and in the standard model (SM). The quadratic divergences in the SM in the Higgs self-energy are indicative of the fact that the "natural" order of magnitude of the Higgs-boson mass is at least $\sim M$, where M represents the scale of new physics beyond the standard model [9]. Much of the interest in supersymmetric theories (from a phenomenological point of view) derives from the fact that [excluding U(1) D terms] they are quite generally free of quadratic divergences [10]. They thus admit the possibility of "naturally" light scalar particles.

Quadratic divergences in the SM context were first studied by Veltman [9] in the context of dimensional regularization. He showed that, as long as regularization by dimensional reduction [10] (DRED) is employed rather than conventional dimensional regularization (DREG), the requirement of the absence of quadratic divergences at one loop in the standard model can be expressed by the formula

$$3m_H^2+6m_W^2+3m_Z^2-12m_t^2=0 \ . \tag{1}$$

Here m_H, m_W, m_Z, and m_t are the masses of the Higgs boson, W boson, Z boson, and top quark, respectively. For simplicity, we have dropped contributions from the lighter quarks and leptons. The generalization of this relation to an arbitrary renormalizable gauge theory may be found in Ref. [1]. In the same reference, it was noted that imposing both Eq. (1) itself and that it be renormalization group (RG) invariant leads to two constraints which cannot be simultaneously satisfied for any m_t and m_H, while in Ref. [4] it was shown that, if strong-interaction contributions to the RG evolution are ignored, then the predictions $m_t\approx 115$ GeV and $m_H\approx 180$ GeV are obtained. We will comment later on the effect of higher orders on these predictions.

Although originally derived in the context of DRED, Eq. (1) is reproduced by any straightforward regularization method that does not involve continuation in dimension, for example, nonlocal regularization [4] or point splitting [8]. We shall see why below, when we provide a particularly simple derivation of Eq. (1). With DREG, on the other hand, a different expression is obtained [9]. This arises as follows. The coefficients of the m_W^2 term and the m_Z^2 term in Eq. (1) are in fact $2(d'-1)$ and $(d'-1)$, respectively, where $d'=g_{\mu\nu}g^{\mu\nu}$ and $g_{\mu\nu}$ is the metric tensor. With DRED, $d'=4$ because the continuation to d dimensions is done by compactification, while, in DREG, $d'=d$ since the whole Lagrangian is continued to d dimensions. Then the fact that with either DRED or DREG quadratic divergences are manifested as poles at $d=4-2/L$ (where L is the number of loops) leads with DRED to Eq. (1), but with DREG to

$$3m_H^2+2m_W^2+m_Z^2-12m_t^2=0 \ . \tag{2}$$

DRED is a scheme in which the number of spin degrees of freedom is not changed, there being three physical states associated with a massive vector and four associated with a massive Dirac fermion. DREG implicitly varies the number of vector-boson degrees of freedom by altering the polarization sum. This is unphysical, as is further illustrated by the fact that it changes in each loop order. As our discussion below will show, any cutoff scheme that retains the correct number of spin degrees of freedom for each particle, as any truly physical cutoff would do, will lead to Eq. (1) rather than Eq. (2).

We turn now to the derivation of Eq. (1) promised above. Consider the formula for the one-loop correction to the effective potential in an arbitrary gauge theory:

$$V_1(\phi)=(64\pi^2)^{-1}\int d^4k \ \mathrm{STr}\{\ln[k^2+M^2(\phi)]\} \ , \tag{3}$$

where

$$\mathrm{STr}\equiv \sum_{\text{scalars}} +3\sum_{\text{vectors}} -2\sum_{\text{fermions}} \ . \tag{4}$$

The coefficient for fermions reflects use of a Weyl basis. Appropriate counterterms must be added to render $V_1(\phi)$ finite.

We are using the Landau gauge; $V_1(\phi)$ is in general gauge dependent, but it is easy to verify that the quadratically divergent part of it is gauge invariant in R_ξ gauges, at least at one loop [11]. Writing

*Present address: DAMTP, University of Liverpool, P.O. Box 147, Liverpool L69 3BX United Kingdom.

$$\ln[k^2+M^2]=\ln k^2+\frac{M^2}{k^2}+\cdots \ , \quad (5)$$

it is clear that requiring the absence of quadratic divergences amounts to imposing the relation[1]

$$\mathrm{STr}M^2=0 \ . \quad (6)$$

The regulator ambiguities referred to above have been finessed by postponing the evaluation of the Feynman integrals; evidently, if a cutoff is used, then the choice we have made is a common cutoff for all loops. Now it was shown by Ferrara, Giradello, and Palumbo [12] that Eq. (6) is automatically satisfied in supersymmetric theories. What about nonsupersymmetric theories? Naive application of Eqs. (4) and (6) to the SM leads to the relation

$$m_H^2+6m_W^2+3m_Z^2-12m_t^2=0 \ , \quad (7)$$

which differs from Eq. (1). This apparent paradox was noted but left unresolved in Ref. [8], and so we felt it worthwhile to provide an explanation and incidentally a derivation of Eq. (1) from Eq. (6).

The key is in the realization that M^2 is a function of ϕ. In a supersymmetric theory, Eq. (6) is true *for all* ϕ and thus represents simultaneous satisfaction of three sets of constraints, corresponding to terms in ϕ^2, ϕ, and ϕ^0, respectively. The ϕ^0 constraint reflects the cancellation of the zero-point energies of the fields (when quantized with respect to the state $\phi=0$) [13].

Turning to the SM, we take the tree-level potential to be

$$V_0=-\frac{1}{2}m^2\phi^2+\frac{\lambda}{24}\phi^4 \quad (8)$$

(we have exploited gauge invariance to replace the scalar doublet by a real scalar ϕ).

Then we have

$$\mathrm{STr}M^2(\phi)=H+3G+6W+3Z-12T \ , \quad (9)$$

where $H=-m^2+\frac{1}{2}\lambda\phi^2$, $G=-m^2+\frac{1}{6}\lambda\phi^2$, $W=\frac{1}{4}g^2\phi^2$, $Z=\frac{1}{4}(g^2+g'^2)\phi^2$, and $T=\frac{1}{2}h^2\phi^2, h$ being the top-quark Yukawa coupling. When $\phi=\langle\phi\rangle$, H, W, Z, and T become m_H^2, m_W^2, m_Z^2, and m_t^2, respectively, and $G=0$. Note that Eq. (9) does not involve a term linear in ϕ, because the SM does not admit a cubic scalar invariant.

Now it is easy to see that we cannot impose Eq. (6) for general ϕ, since the m^2 terms in Eq. (9) do not cancel. This is not, however, a divergence in the two-point function and is irrelevant to naturalness considerations. It is the quadratic divergences in the Higgs-boson mass that concern us; evidently, requiring them to be absent amounts to requiring

$$\frac{1}{2}\frac{\partial^2}{\partial\phi^2}\mathrm{STr}M^2=0 \ , \quad (10)$$

which amounts, of course, to cancellation of the term quadratic in ϕ. It is important to realize that this is a relation among the dimensionless couplings of the theory. It does not depend, for example, on the sign of m^2. If, however, we multiply Eq. (10) by $\langle\phi\rangle^2$ and use the relation

$$\lambda\langle\phi\rangle^2=3m_H^2 \ , \quad (11)$$

then it is very easy to see that Eq. (10) leads to Eq. (1). This derivation of Veltman's formula is elegant and also useful in showing that the result is not dependent on the use of DRED.

Finally, we should explain the origin of Eq. (7). If we take Eq. (9) and evaluate at $\phi=\langle\phi\rangle$, where $\langle\phi\rangle$ is given by Eq. (11), then we indeed obtain

$$\mathrm{STr}M^2(\langle\phi\rangle)=m_H^2+6m_W^2+3m_Z^2-12m_t^2 \ . \quad (12)$$

So requiring Eq. (7) amounts to requiring that the *value* of the effective potential at the minimum be free from quadratic divergences. As we have emphasized above, the *value* of the effective potential (even at the extremum) is not relevant to the issue of whether the Higgs boson can be naturally light. To reiterate, imposing Eq. (1) renders the radiative corrections to the Higgs-boson mass free of quadratic divergences (at one loop). Imposing Eq. (7) would remove the quadratic divergence in the vacuum energy, which, however is *quartically* divergent in the SM, because its particle *content* differs from that of a supersymmetric theory (see Ref. [13]).

Finally, we comment briefly on the issue of higher-order corrections. If one requires the absence of quadratic divergences order by order, it is clear that an infinite number of constraints must be satisfied, which is seemingly improbable in the absence of an identifiable symmetry. But what if one admits the possibility of cancellation between different orders of perturbation theory; might one then have that Eq. (1) merely suffered radiative corrections of generic order g^2m^2? In the context of DRED (or DREG), this appears difficult to implement since, for example, at two loops one finds poles at both $d=2$ and 3, the former (and, it was conjectured, the latter [1–4]) being related to the one-loop $d=2$ pole via a generalized RG. Let us see, however, what happens in some kind of cutoff scheme. (The following argument is heuristic inasmuch as issues of scheme and gauge dependence are ignored, but we believe the essential points made would survive a more careful treatment.) Consider a theory with many couplings λ_i but a single mass parameter m^2. Then the bare parameter m_B^2 is given by

$$m_B^2=m^2+\Lambda^2\sum_{n=0}^{\infty}f_n(\lambda_i)\ln^n\Lambda/\mu+\cdots \ , \quad (13)$$

where μ is the RG scale and we have kept only quadratically divergent terms. From the fact that $\mu\partial m^2/\partial\mu$ is finite, it is easy to deduce that

$$(n+1)f_{n+1}=\mu\frac{\partial}{\partial\mu}f_n=\beta_i\frac{\partial}{\partial\lambda_i}f_n \ . \quad (14)$$

The absence of quadratic divergences amounts to requiring that $f_n=0$ for *all* n; but note that, of course, the

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[1]The fact that in dimensional regularization $\int d^dk/k^2=0$ is due to a cancellation between ultraviolet and infrared divergences specific to the method; the ir divergences must be separately regulated if dimensional regularization is employed.

vanishing of f_n does not of itself imply the vanishing of f_{n+1}. Thus an infinite number of constraints are indeed required to ensure the absence of quadratic divergences to all orders (contrary to a recent claim) [14]. The predictions of m_t and m_H mentioned earlier (see also Ref. [14]) were obtained by solving the equations $f_0=f_1=0$ at leading order for the SM. It is straightforward to evaluate f_2 to leading order using Eq. (14) and to show that it is nonzero when the one-loop β functions are substituted. Thus the scepticism evinced in Ref. [4] with respect to the predictions would seem to be justified, even if one were prepared to swallow the opportunistic suppression of the strong-interaction terms.

It remains conceivable, of course, that the *new physics* whereby the SM resolves its naturalness problem is such that Eq. (1) remains intact.

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