Nonperturbative solution of two-body Dirac equations for quantum electrodynamics and related field theories

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In quark-model calculations of the meson spectrum, fully covariant two-body Dirac equations dictated by Dirac's relativistic constraint mechanics gave a good fit to the entire meson mass spectrum (excluding flavor mixing) with constituent world scalar and vector potentials depending on just one or two parameters. In this paper, we investigate the properties of these equations that made them work so well by solving them numerically for quantum electrodynamics (QED) and related field theories. The constraint formalism generates a relativistic quantum mechanics defined by two coupled Dirac equations on a 16-component wave function which contain Lorentz-covariant constituent potentials that are initially undetermined. An exact Pauli reduction leads to a second-order relativistic Schrödinger-like equation for a reduced eight-component wave function determined by an effective interaction—the quasipotential. We first determine perturbatively to lowest order the relativistic quasipotential for the Schrödinger-like equation by comparing that form with one derived from the Bethe-Salpeter equation. Insertion of this perturbative information into the minimal interaction structures of the two-body Dirac equations then completely determines their interaction structures. Then we give a procedure for constructing the full 16-component solution to our coupled first-order Dirac equations from a solution of the second-order equation for the reduced wave function. Next, we show that a perturbative treatment of these equations yields the standard spectral results for QED and related interactions. The relativistic potentials in our exact Schrödinger-like equations incorporate detailed minimal interaction and dynamical recoil effects characteristic of field theory yet, unlike the approximate Fermi-Breit forms, do not lead to singular wave functions for any angular momentum states. Hence, we are able to solve them numerically and compare the resultant nonperturbative energy eigenvalues to their perturbative counterparts and hence to standard field-theoretic results. We find that nonperturbative solution of our equation produces energy levels that agree with the perturbative spectrum through order α^4. Surprisingly, this agreement depends crucially on inclusion of coupling between upper-upper and lower-lower components of our 16-component Dirac wave functions and on the short-distance behavior of the relativistic quasipotential in the associated Schrödinger-like equation. To examine speculations that the effective potentials (including the angular momentum barrier) for some states in the e^+e^- system may become attractive for small separations, we study whether our equations predict pure QED resonances in the e^+e^- system which might correspond to the anomalous positron peaks in the yield of e^+e^- pairs seen in heavy-ion collisions. For the $^3P_0$ state we find that, even though the quasipotential becomes attractive at separations near 10 fm and overwhelms the centrifugal barrier, the attraction is not strong enough to hold a resonance. This result contradicts recent predictions of such states by other authors based on numerical solutions of three-dimensional truncations of the Bethe-Salpeter equation for which the QED bound-state wave equation has been treated successfully only by perturbation theory.

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I. INTRODUCTION

Recent quark-model calculations of the meson spectrum [1-2] using fully covariant two-body Dirac equations [3–5] derived by Crater and Van Alstine from Dirac's relativistic constraint dynamics [6–10] gave a good description of the light-quark as well as the heavy-quark meson masses resulting from world scalar and vector potentials. Although static potentials that have a close connection with quantum chromodynamics (QCD) such as the Adler-Piran potential [11] or the cruder Richardson potential [12] were responsible for the quality of the fit to the heavy mesons, the good quality of the simultaneous fit to the lighter mesons (with the same one

or two potential parameters used for the entire spectrum) was due to exact two-body relativistic kinematics combined with the minimal interaction structure of these equations for vector and scalar potentials. In particular the structure of the vector potentials in these equations was originally abstracted from the classical electrodynamics of Wheeler and Feynman by two of us [13]. We shall show in this paper that this structure may be obtained from QED, by first deriving the Todorov quasipotential equation [13,14] from the Bethe-Salpeter equation and then comparing it to the two-body Dirac equations. One may formulate these quantum-mechanical equations for semiphenomenological meson studies with interactions taken from QCD or for electrodynamic bound-state calculations with interactions dictated by QED. Since the Abelian vector structure [15] of electrodynamics carries over to the short-distance structure of QCD, in order that the equations be appropriate for QCD bound-state calculations, they must give correct answers to the appropriate order in the fine-structure constant α when applied to QED bound states. In previous work, Crater and Van Alstine have been able to solve analytically the full 16-component coupled Dirac equations for the electrodynamic case to obtain a family of exact solutions for parapositronium [16], with energy spectrum in agreement with standard approaches to QED through order α^4. Does the agreement with QED extend to unequal masses and to all angular momentum states? If so, this agreement would constitute the first successful test of the strong potential structure of two-body relativistic wave equations for QED for states of arbitrary angular momentum. It is imperative that such a test be done in order to discover whether a nonperturbative treatment of these or any other candidate equations faithfully represents the field-theoretic dynamics obtained rigorously from perturbation theory as in QED or semiphenomenologically from QCD. In order to carry out this check, we first treat our "minimal interaction constraint equations" perturbatively for the electromagnetic interaction. We show that they yield the correct two-body spectrum through order α^4 when one treats as perturbations to the static Coulomb potential the various corrections of order $1/c^2$ generated by the spin structure of the Dirac equations alone. Unlike the equations produced by other approaches, the Schrödinger-like form of our two-body Dirac equations possesses local spin-dependent and Darwin terms that are quantum-mechanically well defined. Since our equations are devoid of highly singular effective potential terms that appear in most three-dimensional truncations of the Bethe-Salpeter equation and in the Breit equation [17], we can go on to solve our equations nonperturbatively.

We shall demonstrate in this paper that a numerical solution of the two-body Dirac equations of constraint dynamics yields energies for the $n=1,2,3$ levels of fermion-antifermion systems in QED that agree through order α^4 with those produced by a perturbative treatment of these equations and with those produced by standard perturbative approaches to QED. Furthermore, as a check on the scalar and timelike vector interactions for our equations, we shall demonstrate nonperturbatively that our equations yield no hyperfine splitting for those interactions, in agreement with a perturbative treatment. In each case we shall treat the general unequal mass system, including only the potentials that arise from the single exchange diagram and ignoring the contribution of the virtual annihilation diagram to the equal-mass case.

Crater and Van Alstine originally abstracted the electrodynamic vector interaction in these equations from classical electrodynamics in order to describe the semiphenomenological short-range interactions of QCD [1,2]. Because numerical solution of our equations reproduces the standard perturbative bound-state spectrum of QED, we have a set of two-body relativistic wave equations for electrodynamics whose nonperturbative predictions for other phenomena in QED ought to be taken as seriously if not more seriously than those of other field theoretic equations that have not been similarly checked. Just such a situation presents itself in the interpretation of recent results in heavy-ion physics. Wong and Becker [18] have speculated that the unexplained anomalous peaks in the yield of e^+e^- pairs in heavy-ion collisions [19–24] might result from purely QED resonances [25,26] in the e^+e^- system produced by strong potential electrodynamic structures in the appropriate two-body wave equation. If there is such a resonance, first one must study it using a wave equation [27], and second this wave equation must be treated nonperturbatively and covariantly, not by perturbative, semirelativistic means. In this paper we investigate what the two-body Dirac equations have to say about such states. The fact that in our equations (in Schrödinger-like form) each term of the quasipotential is quantum-mechanically well defined all the way into the origin is critical to our investigation. We make a numerical search for resonances in the $^3P_0$ continuum states of positronium. We show that numerical calculations of the phase shift for energies of $1.4-1.8$ MeV agree with perturbatively computed phase shifts. Thus, we find theoretical evidence that no such resonances exist in our electrodynamic constraint equations. We find that even though the local QED quasipotential for the $^3P_0$ state becomes attractive at small distances and overwhelms the centrifugal barriers as in the model of Wong and Becker [18], the QED quasipotential is not deep or wide enough to hold a resonance. This result directly contradicts the results obtained by Vary and Spence [28] from standard nonlocal truncations of the Bethe-Salpeter equation.

As we shall see in this paper, when we solve the two-body Dirac equations numerically, we find that relativistic potential structures that do not contribute in the usual perturbation theory play a significant role. What is the origin of these structures in two-body Dirac equations? The basic relativistic interaction in our equations is determined by the Bethe-Salpeter equation via the Feynman scattering amplitudes of the relevant quantum field theory. The resulting two-body Dirac equations then assume different forms depending in part on the Lorentz character of the chosen field-theoretic interaction and in part on the spin structure dictated by the mathematical compatibility of the two coupled wave equations. Together these nonperturbative requirements completely

specify spin dependence. Our equations inherit the basic potential structure of the single-particle Dirac equation corrected by recoil terms dictated by compatibility (a relativistic version of Newton's third law [5]). The requirement of compatibility also automatically controls the relative time by forcing its elimination from the invariant potential in the center-of-momentum (c.m.) frame.

Later in this paper, we shall recast the two-body Dirac equations for electrodynamics into the Schrödinger-like form:

$$[p^2+\Phi_w(\mathcal{A})-(\epsilon_w^2-m_w^2)]\psi=0 \tag{1.1}$$

in which $\Phi_w(\mathcal{A})$ is a (16×16)-component c.m.-energy-(w-) dependent, relativistic quasipotential matrix, dependent on an invariant function $\mathcal{A}$ derived from field theory at the lowest order. Those terms in Eq. (1.1) in Φ_w beyond the collective minimal (Todorov) form [29] $2\epsilon_w\mathcal{A}-\mathcal{A}^2$ (see Sec. II for a definition of m_w and ϵ_w) we will call "strong potential" terms. The role played by these terms can be fully investigated only by nonperturbative means (for example, through numerical solution of the resultant eigenvalue equation). In past work [1,2] on two-body Dirac equations, we had tacitly assumed (along with authors of all other treatments of the Bethe-Salpeter equation of which we are aware) that a full nonperturbative, numerical treatment of the equations would yield standard spectral results since the "weak-potential" form of the equations (including the usual $1/r^3$ and δ-function potentials) reduced to a form [14,30] known to generate the standard spectral results. But, in view of the failure of another two-body equation, the Breit equation, to generate its own perturbative results when some of the Breit terms are included nonperturbatively [31,32], can we trust this assumption? If it were not true for a particular equation when applied to the vector interaction of perturbative QED, how could we trust results produced by that particular equation in a purely nonperturbative application (dominated by a related vector interaction) such as to the quark-antiquark bound states of QCD. Any candidate two-body wave equation, applied to QCD with such an interaction, must reproduce, if applied to QED, the perturbative QED spectrum when that equation is treated nonperturbatively regardless of the agreement of its semiphenomenological spectrum with the meson spectrum.

The ordinary one-body Dirac equation with external Coulomb potential certainly yields agreement between nonperturbative solution and perturbative evaluation. In that case, the exact solution produces a spectrum that agrees through order α^4 with that given by perturbative treatment of the Darwin and spin-orbit terms obtained from the usual Pauli reduction of the Dirac equation. As two of us found in a previous paper [16], the two-body Dirac equations of constraint dynamics for the e^+e^- system in the ${}^1J_J$ states also possess a family of exact solutions with total c.m. energy w given by a Sommerfeld formula

$$w=m\left[2+2\left[1+\frac{\alpha^2}{[n+\sqrt{(l+\frac{1}{2})^2-\alpha^2}-l-\frac{1}{2}]^2}\right]^{1/2}\right]^{1/2}$$
$$=2m-\frac{m\alpha^2}{4n^2}-\frac{m\alpha^4}{2n^3(2l+1)}+\frac{11}{64}\frac{m\alpha^4}{n^4}+O(\alpha^6)\ . \tag{1.2}$$

These energies are in agreement through order α^4 with those of the perturbative solution of the same equation and also with those of standard approaches to QED. As we shall see the two-body Dirac equations of QED extend this agreement to the $n=1,2,3$ levels for all allowable j and unequal masses. This agreement has not been demonstrated for the traditional three-dimensional truncations of the Bethe-Salpeter equation [17]. Such truncations do yield the correct QED spectrum for fermion-antifermion systems through order α^4 (from single-photon exchange) when treated perturbatively. In all of these traditional treatments, one starts with a bound-state Coulomb wave function (whether nonrelativistic or relativistic) and uses first-order perturbation theory to compute Breit corrections corresponding to Darwin, spin-orbit, spin-spin, and tensor interactions. However, these three-dimensional truncations have not been solved analytically or numerically for QED [33] with enough accuracy to demonstrate agreement with a perturbative treatment of these equations through order α^4.

Our paper is organized as follows. In Sec. II we review the constraint formalism for the two-body Dirac equations containing mutual world scalar and vector potentials. We suggest that the reader who is already familiar with constraint dynamics and wishes to skim through the detailed presentations of our new results read Secs. VI and VII first and then return to the earlier sections of the paper for details. In Sec. III we begin our presentation of new results by showing how we obtain the relativistic interactions of our equations from the appropriate perturbative quantum field theory in concert with the minimal interaction structures of the two-body Dirac equations in both their constituent Dirac and collective Schrödinger forms. This procedure determines the quasipotential Φ_w of Eq. (1.1).

In Sec. IV, from the coupled Dirac equations, we derive an eight-component Schrödinger-like form of the equations, which we later solve numerically. In the process we show how to use the solutions of the Schrödinger-like equations to construct the full 16-component solutions of the two original Dirac equations. In Sec. V we give a perturbative treatment of the weak potential form of these equations for later comparison with the nonperturbative solution. In Sec. VI we arrive at the first nonperturbative numerical result of this paper. There, we examine the eigenvalues obtained from numerical solution of the Schrödinger-like forms derived in Sec. IV and compare these with the corresponding perturbative results of Sec. V. In each case, we find that the nonperturbative bound-state spectrum produced by solution of the fully coupled system of equations yields the perturbative results within an error of order α^6. We find that the coupling between upper-upper and lower-lower parts

of the 16-component wave functions in our equations is crucial to this agreement. This dependence is unexpected since that coupling does not contribute through order α^4 in the perturbative evaluation of these equations. Moreover, we find that the parts of the quasipotential essential for agreement with the perturbative results become significant only at separations on the order of a few fermis. Thus, insofar as the order-α^4 spectral results are concerned, these two-body Dirac equations give correct results when used well below the Compton wavelength. This agreement allows us to test with confidence the hypothesis of possible e^+e^- resonances in the $^3P_0$ state. In Sec. VII we use a further decoupling of the equations, derived in Appendix D of Ref. [46], to compute phase shifts using both perturbative and nonperturbative treatments. We find no evidence for a pure QED resonance in the e^+e^- system, in direct contradiction to the results of Spence and Vary [28]. Finally in Sec. VIII we compare various properties of our two-body Dirac approach with those of other relativistic two-body wave equations.

II. REVIEW OF TWO-BODY DIRAC EQUATIONS FOR TWO SPIN-$\frac{1}{2}$ PARTICLES FOR WORLD SCALAR AND FOUR-VECTOR INTERACTIONS

A. "External potential" or "Minimal interaction" forms of the two-body Dirac equations

We begin by examining explicit covariant forms of the two-body Dirac equations [3–5] that two of us have developed for use in semiphenomenological meson-spectroscopy calculations [1,2] and for investigations of the electromagnetic positronium system [16]. For two relativistic spin-$\frac{1}{2}$ particles interacting through scalar and vector potentials, the two compatible 16-component (or 4×4 matrix) Dirac equations [1–5] of constraint dynamics are

$$\mathcal{S}_1\psi\equiv\gamma_{51}[\gamma_1\cdot(p_1-A_1)+m_1+S_1]\psi=0\ , \tag{2.1a}$$

$$\mathcal{S}_2\psi\equiv\gamma_{52}[\gamma_2\cdot(p_2-A_2)+m_2+S_2]\psi=0\ . \tag{2.1b}$$

The subscript $i=1,2$ stands for the ith particle so that m_1 and m_2 are the masses of the interacting fermions. In Eqs. (2.1) the potentials A_i^μ and S_i introduce the interactions that the ith particle experiences due to the presence of the other particle. (Thus we will refer to these forms of the two-body Dirac equations either as the "external potential forms" or the "minimal interaction forms.") In meson calculations motivated by QCD the Lorentz-invariant scalar potentials S_i are semiphenomenological while the vector potentials A_i^μ are composed of two independent covariant parts: one semiphenomenological (long range and confining) like the scalar interactions, and the other (short range) closely tied to perturbative quantum field theory. The first part contains only long range timelike pieces (parallel to the total four-momentum of the two particles), while the second is electromagneticlike (short range), containing field theoretically specified portions of timelike and spacelike pieces (transverse to the total four-momentum of the two particles). The specific forms of the covariant spin-dependent terms in the interactions are consequences of the necessary compatibility of the two Dirac equations

$$[\mathcal{S}_1,\mathcal{S}_2]\psi=0\ . \tag{2.2}$$

In detail [2,5,16] the vector potentials A_i^μ are given in terms of three invariant functions G, E_1, and E_2 by

$$A_1^\mu=\left[(\epsilon_1-E_1)-i\frac{G}{2}\gamma_2\cdot\left(\frac{\partial E_1}{E_2}+\partial\ln G\right)\gamma_2\cdot\hat{P}\right]\hat{P}^\mu+(1-G)p^\mu-\frac{i}{2}\partial G\cdot\gamma_2\gamma_2^\mu\ , \tag{2.3a}$$

$$A_2^\mu=\left[(\epsilon_2-E_2)+i\frac{G}{2}\gamma_1\cdot\left(\frac{\partial E_2}{E_1}+\partial\ln G\right)\gamma_1\cdot\hat{P}\right]\hat{P}^\mu-(1-G)p^\mu+\frac{i}{2}\partial G\cdot\gamma_1\gamma_1^\mu\ , \tag{2.3b}$$

while the scalar potentials S_i are functions of G and two additional invariant functions M_1 and M_2:

$$S_1=M_1-m_1-\frac{i}{2}G\gamma_2\cdot\frac{\partial M_1}{M_2} \tag{2.4a}$$

$$S_2=M_2-m_2+\frac{i}{2}G\gamma_1\cdot\frac{\partial M_2}{M_1}\ . \tag{2.4b}$$

In the case of lowest-order QED, $S_i=0$, and the spacelike and timelike vectors are not independent but combine into the electromagneticlike four-vectors

$$A_1^\mu=\left[\epsilon_1-\frac{G(\epsilon_1-\epsilon_2)}{2}+\frac{\epsilon_1+\epsilon_2}{2G}\right]\hat{P}^\mu+(1-G)p^\mu-\frac{i}{2}\partial G\cdot\gamma_2\gamma_2^\mu\ , \tag{2.5a}$$

$$A_2^\mu=\left[\epsilon_2-\frac{G(\epsilon_2-\epsilon_1)}{2}+\frac{\epsilon_1+\epsilon_2}{2G}\right]\hat{P}^\mu-(1-G)p^\mu+\frac{i}{2}\partial G\cdot\gamma_1\gamma_1^\mu\ . \tag{2.5b}$$

In Eqs. (2.3) and (2.5) the variable

$$P=p_1+p_2 \tag{2.6}$$

is the total four-momentum. In our metric $-P^2=w^2$ is the c.m. energy squared so that $\hat{P}^2=-1$ where $\hat{P}\equiv P/w$. The variables ϵ_i are the conserved c.m. energies of the constituent particles given by

$$\epsilon_1=(w^2+m_1^2-m_2^2)/2w \ ,$$
$$\epsilon_2=(w^2+m_2^2-m_1^2)/2w \tag{2.7}$$

so that $\epsilon_1+\epsilon_2=w$. In terms of these energies the usual relative momentum defined by $p_1=\epsilon_1\hat{P}+p$, $p_2=\epsilon_2\hat{P}-p$ becomes

$$p=(\epsilon_2 p_1-\epsilon_1 p_2)/w \ . \tag{2.8}$$

In order that Eqs. (2.1a) and (2.1b) be compatible [i.e., satisfy (2.2)] it is necessary that the invariant functions E_1, E_2, G, M_1, and M_2 depend on the relative separation $x=x_1-x_2$ only through the spacelike coordinate four-vector [7–9]

$$x_\perp^\mu=x^\mu+\hat{P}^\mu(\hat{P}\cdot x) \tag{2.9}$$

which is perpendicular to the total four-momentum P. In general E_1, E_2, G, M_1, and M_2 may depend on

$$x_\perp^2\equiv r^2, \quad l^2=l_\mu l^\mu, \quad \text{and} \quad p_\perp^2 \tag{2.10}$$

where $l_\mu=\epsilon_{\mu\nu\kappa\lambda}\hat{P}^\nu x_\perp^\kappa p_\perp^\lambda$. Note that the invariant r is the interparticle separation in the c.m. system. In this paper we shall assume that the invariant functions depend only on r.

In general E_1, E_2, and G are related to each other [13,5] and for QCD applications are functions of only two invariant functions $\mathcal{V}(r)$ and $\mathcal{A}(r)$, whose forms we take to be

$$E_1^2(\mathcal{A},\mathcal{V})=G^2[(\epsilon_1-\mathcal{A})^2-2\epsilon_w\mathcal{V}+\mathcal{V}^2] \ , \tag{2.11a}$$
$$E_2^2(\mathcal{A},\mathcal{V})=G^2[(\epsilon_2-\mathcal{A})^2-2\epsilon_w\mathcal{V}+\mathcal{V}^2] \ , \tag{2.11b}$$

and

$$G^2=\frac{1}{1-2\mathcal{A}/w} \ . \tag{2.11c}$$

From the expressions (2.3) and (2.5) of the vector potentials we see that the invariant function $\mathcal{A}(r)$ is responsible for the covariant electromagneticlike parts of A_i^μ while $\mathcal{V}$ is responsible for the additional independent covariant timelike parts of A_i^μ. Even though the dependences of E_1, E_2, and G on $\mathcal{A}$ and $\mathcal{V}$ are not unique, they are constrained by the requirement that they yield the correct nonrelativistic and semirelativistic limits. Demanding that the Schrödinger form of the two-body Dirac equations incorporate the collective minimal (Todorov) interaction structures [29] of Eq. (1.1), we find the simple forms given in Eqs. (2.11) satisfy these requirements. (The details of this argument are given in Refs. [5,13,34].) In general M_1 and M_2 are related to each other [4,5] and for QCD applications are functions of the two invariant functions $\mathcal{A}(r)$ and $S(r)$:

$$M_1^2(\mathcal{A},S)=m_1^2+G^2(2m_wS+S^2) \ , \tag{2.12a}$$
$$M_2^2(\mathcal{A},S)=m_2^2+G^2(2m_wS+S^2) \ . \tag{2.12b}$$

The invariant function $S(r)$ is primarily responsible for the scalar potentials since $S_i=0$ if $S(r)=0$ while $\mathcal{A}(r)$ contributes to the S_i [if $S(r)\neq 0$] as well as to the vector potentials A_i^μ. Demanding that the Schrödinger form of the two-body Dirac equations incorporate the collective minimal (Todorov) interaction structures, we find that the simple forms given in Eqs. (2.12) give the correct nonrelativistic and semirelativistic limits. (The details of this argument are likewise given in Refs. [4,5,34]). Thus the five invariant functions M_1, M_2, E_1, E_2, and G are constrained to depend on three independent invariant functions S, $\mathcal{A}$, and $\mathcal{V}$. (In QED applications, $\mathcal{V}=0$ and in lowest order $S=0$.) The kinematical variables

$$m_w=m_1m_2/w \ , \tag{2.13a}$$
$$\epsilon_w=(w^2-m_1^2-m_2^2)/2w \tag{2.13b}$$

are the relativistic reduced mass and energy of a fictitious particle of relative motion. The corresponding value of the on-mass-shell relative momentum squared then takes the form

$$b^2(w)\equiv[w^4-2w^2(m_1^2+m_2^2)+(m_1^2-m_2^2)^2]/4w^2$$
$$=\epsilon_w^2-m_w^2 \ . \tag{2.14}$$

For the electromagneticlike vector interactions the minimal interaction form of the two-body Dirac equations (2.1) is a consequence of gauge invariance. In any one-body wave equation, gauge invariance exhibits itself in two related ways. For the system of particle and field, (Abelian) gauge invariance manifests itself as invariance under change of the vector field by the addition of the gradient of an arbitrary scalar combined with local phase variation of the wave function. This is achieved through the derivative structure of the field equations in concert with the minimal coupling of the potential to particle. However, once one eliminates the vector potentials in terms of source motions in a fixed gauge, the minimal structures persist as dynamical structures of the resulting particle equations. For example, in the case of a single charged particle interacting with an infinitely heavy massive charge, the resulting Klein-Gordon or Dirac equation, with Coulomb potential obtained from the full particle-plus-field problem by elimination of the field potential in a fixed gauge, contains the dynamical potential as a minimal subtraction from the energy and retains the phase change with a compensating addition of the gradient of a scalar to the vector potential minimally subtracted from the momentum. Thus, "gauge invariance" of the resulting particle equations is a dynamical symmetry inherited from the original system of particle and field through the elimination of the vector field in a fixed gauge.

Similarly, the two-body Dirac equations (2.1) contain vector potentials (one for each particle) obtained from quantum field theory from the Bethe-Salpeter equation in the Feynman gauge [35] (see Sec. III below) or from clas-

sical field theory in the Lorentz gauge. Thus, the two-body Dirac equations are two-body counterparts of the one-body particle equations with eliminated field and should possess an analogous inherited dynamical "gauge invariance" if they retain any invariance at all. In fact, we find that since our equations are two simultaneous wave equations on one wave function with two (albeit related) four-potentials, Eqs. (2.1) turn out to be invariant under any gauge transformation of the form $A_i^\mu \to A_i^\mu + \partial_i^\mu \chi(x_\perp)$ with χ the phase change of the single wave function. The origin of the two dynamical potentials A_1^μ and A_2^μ as solutions for vector fields in the Lorentz gauge shows up as the property

$$\frac{\partial}{\partial x^\mu}(A_1^\mu + A_2^\mu) = 0 , \tag{2.15}$$

which is a consequence of the fact that $A_1^\mu + A_2^\mu \propto \hat{P}^\mu f$ where $f = f(x_\perp)$. This property of the potentials is forced upon us by the compatibility (2.2) of the constraints.

B. Hyperbolic forms of the two-body Dirac equations

The expansions (2.11) and (2.12) for the five invariant functions in terms of the three invariants $\mathcal{A}(x_\perp)$, $\mathcal{V}(x_\perp)$, $S(x_\perp)$ are important for semiphenomenological and other applications that emphasize the relationship of the interactions in our equations to external potentials of the two associated one-body problems. However, for applications in which the identification of these five invariants in terms of either a perturbative or semiphenomenological field theoretic scattering amplitude is desirable, two of us have found a hyperbolic representation [36] of these five invariants in terms of three other invariants, L, J, and $\mathcal{G}$. This representation is

$$M_1 = m_1 \cosh L + m_2 \sinh L , \tag{2.16a}$$

$$M_2 = m_2 \cosh L + m_1 \sinh L , \tag{2.16b}$$

$$E_1 = \epsilon_1 \cosh J + \epsilon_2 \sinh J , \tag{2.16c}$$

$$E_2 = \epsilon_2 \cosh J + \epsilon_1 \sinh J , \tag{2.16d}$$

$$G = e^{\mathcal{G}} . \tag{2.16e}$$

$L(x_\perp)$, $J(x_\perp)$, and $\mathcal{G}(x_\perp)$ generate scalar, timelike vector and spacelike vector interactions respectively. As shown in the next section, this representation puts the two-body equations in a form whose interactions are simply related to the Bethe-Salpeter equation via the Feynman scattering amplitude. If we use (2.16) and the "theta" matrices

$$\theta_i^\mu \equiv i\sqrt{\tfrac{1}{2}}\gamma_{5i}\gamma_i^\mu , \quad \mu = 0,1,2,3, \ i = 1,2 \tag{2.17a}$$

$$\theta_{5i} \equiv i\sqrt{\tfrac{1}{2}}\gamma_{5i} \tag{2.17b}$$

we can rewrite (2.1) as

$$\mathcal{S}_1\psi = [G\theta_1\cdot p + E_1\theta_1\cdot\hat{P} + M_1\theta_{51} + iG(\theta_2\cdot\partial\mathcal{G}\theta_{1\perp}\cdot\theta_{2\perp} + \theta_2\cdot\partial J\theta_1\cdot\hat{P}\theta_2\cdot\hat{P} - \theta_2\cdot\partial L\theta_{51}\theta_{52})]\psi = 0 , \tag{2.18a}$$

$$\mathcal{S}_2\psi = [-G\theta_2\cdot p + E_2\theta_2\cdot\hat{P} + M_2\theta_{52} - iG(\theta_1\cdot\partial\mathcal{G}\theta_{1\perp}\cdot\theta_{2\perp} + \theta_1\cdot\partial J\theta_1\cdot\hat{P}\theta_2\hat{P} - \theta_1\cdot\partial L\theta_{51}\theta_{52})]\psi = 0 . \tag{2.18b}$$

Simplification of these equations results if we introduce the following invariant matrix functions $\Delta_k(x_\perp)$, with $k = L, J, \mathcal{G}$. For scalar interactions

$$\Delta_L = -\frac{\mathcal{O}_1 L(x_\perp)}{2} = -\frac{I_1 I_2 L(x_\perp)}{2}\mathcal{O}_1 , \tag{2.19}$$

where I_1 and I_2 are the identity operators and $\mathcal{O}_1 = 2\theta_{51}\theta_{52}$. For timelike vector interactions

$$\Delta_J = \frac{\mathcal{O}_2 J(x_\perp)}{2} = \frac{\gamma_1\cdot\hat{P}\gamma_2\cdot\hat{P}J(x_\perp)}{2}\mathcal{O}_1 , \tag{2.20}$$

where $\mathcal{O}_2 = 2\theta_1\cdot\hat{P}\theta_2\cdot\hat{P}$, and for spacelike vector interactions

$$\Delta_{\mathcal{G}} = \frac{\mathcal{O}_3\mathcal{G}(x_\perp)}{2} = \frac{\gamma_{1\perp}\cdot\gamma_{2\perp}\mathcal{G}(x_\perp)}{2}\mathcal{O}_1 , \tag{2.21}$$

where $\mathcal{O}_3 = 2\theta_{1\perp}\cdot\theta_{2\perp}$. For convenience we define

$$\Delta_T = \Delta_J + \Delta_L + \Delta_{\mathcal{G}} = \tfrac{1}{2}[\mathcal{O}_2 J(x_\perp) - \mathcal{O}_1 L(x_\perp) + \mathcal{O}_3\mathcal{G}(x_\perp)] . \tag{2.22}$$

In terms of these matrix functions, the compatible two-body Dirac equations become

$$\mathcal{S}_1\psi = [e^{\mathcal{G}}\theta_1\cdot p + \cosh(2\Delta_J)\epsilon_1\theta_1\cdot\hat{P} + \sinh(2\Delta_J)\epsilon_2\theta_2\cdot\hat{P} + m_1\cosh(2\Delta_L)\theta_{51} + m_2\sinh(2\Delta_L)\theta_{52} + e^{\mathcal{G}}i\theta_2\cdot\partial\Delta_T]\psi = 0 \tag{2.23a}$$

$$\mathcal{S}_2\psi = [-e^{\mathcal{G}}\theta_2\cdot p + \cosh(2\Delta_J)\epsilon_2\theta_2\cdot\hat{P} + \sinh(2\Delta_J)\epsilon_1\theta_1\cdot\hat{P} + m_2\cosh(2\Delta_L)\theta_{52} + m_1\sinh(2\Delta_L)\theta_{51} - e^{\mathcal{G}}i\theta_1\cdot\partial\Delta_T]\psi = 0 . \tag{2.23b}$$

Remarkably, the linear combinations

$$\mathbf{S}_1\psi=[\cosh(\Delta)\mathcal{S}_1-\sinh(\Delta)\mathcal{S}_2]\psi=0\ , \tag{2.24a}$$

$$\mathbf{S}_2\psi=[\cosh(\Delta)\mathcal{S}_2-\sinh(\Delta)\mathcal{S}_1]\psi=0 \tag{2.24b}$$

of the constraint equations given in (2.23) have very simple forms. Since $\mathcal{O}_1^2=\mathcal{O}_2^2=\frac{1}{4}(\mathcal{O}_1\mathcal{O}_2-\mathcal{O}_3)^2=1$ we are able to use various hyperbolic identities to simplify (2.24). In particular, by bringing the matrices on the left of each $\mathcal{S}_i$ to the right we find that [36]

$$\mathbf{S}_1\psi=[\mathcal{S}_{10}\cosh(\Delta)+\mathcal{S}_{20}\sinh(\Delta)]\psi=0\ , \tag{2.25a}$$

$$\mathbf{S}_2\psi=[\mathcal{S}_{20}\cosh(\Delta)+\mathcal{S}_{10}\sinh(\Delta)]\psi=0\ . \tag{2.25b}$$

(One can even start from free Dirac equations in the form of Eqs. (2.25) with constant Δ and introduce interactions by "gauging," i.e., letting Δ become point dependent [36].) In Eqs. (2.25) $\mathcal{S}_{10}$ and $\mathcal{S}_{20}$ are the free Dirac operators [37]

$$\mathcal{S}_{10}=\theta_1\cdot p_1+m_1\theta_{51}=\theta_1\cdot p+\epsilon_1\theta_1\cdot\hat{P}+m_1\theta_{51}\ , \tag{2.26a}$$

$$\mathcal{S}_{20}=\theta_2\cdot p_2+m_2\theta_{52}=-\theta_2\cdot p+\epsilon_2\theta_2\cdot\hat{P}+m_2\theta_{52}\ . \tag{2.26b}$$

In (2.1) [or (2.18)] the relativistic potentials are two-body analogs of, and in the limit $m_1\to\infty$ (or $m_2\to\infty$) go over to, the ordinary external potentials of the one-body Dirac equation. The Lorentz character of these interactions is apparent from the "external potential" or minimal interaction form of the equations. On the other hand, the hyperbolic forms (2.25) display the Lorentz character of the interaction through the γ matrix structures of the scalar Δ. These matrix structures of its Lorentz-invariant terms are dictated either by the perturbative agreement of the hyperbolic interactions with the corresponding interactions of the Bethe-Salpeter equation or by phenomenological considerations. Equations (2.25) are closely related to another form of the two-body Dirac equations introduced by Sazdjian [38]. In the notation used here his equations are

$$(\mathcal{S}_{10}+\mathcal{S}_{20}\Delta)\psi=0\ , \tag{2.27a}$$

$$(\mathcal{S}_{20}+\mathcal{S}_{10}\Delta)\psi=0\ . \tag{2.27b}$$

The Sazdjian equations are equivalent to ours in the weak-potential limit [39].

We use the forms (2.25) to relate the matrix potentials Δ to a given field theoretic or semiphenomenological 16×16 matrix Feynman amplitude. For example, a matrix amplitude proportional to $\gamma_1^\mu\gamma_{2\mu}$ corresponding to an electromagneticlike interaction would according to (2.20) and (2.21) dictate that $J=-\mathcal{G}$ (see Sec. III below). Matrix amplitudes proportional to either I_1I_2 or $\gamma_1\cdot\hat{P}\gamma_2\cdot\hat{P}$ would correspond to semiphenomenological scalar or timelike vector interactions. The hyperbolic forms (2.25) of the two-body Dirac equations lead to a particularly simple version [36] for the norm of the 16-component Dirac spinor. On the other hand the minimal interaction or "external potential" forms (2.1) [or (2.18)] of the two-body Dirac equations are simpler to reduce to the Schrödinger-like forms most useful for numerical calculations of bound and scattering states.

III. FIELD THEORETIC IDENTIFICATION OF THE QUASIPOTENTIAL

In the quark-model calculations for meson spectroscopy described in Refs. [1,2] the identifications of the invariant forms $\mathcal{V}$, $\mathcal{S}$, and $\mathcal{A}$ or L, J, and $\mathcal{G}$ were taken from static potentials obtained from an educated guess, (Richardson's potential [12]) or from an effective nonlinear classical field theory based on mean field approximations to QCD (the Adler-Piran potential [11]). In contrast, for QED we obtain the invariant form of the quasipotential Φ_w directly from field theory. In this section we show how the invariant function $\mathcal{A}$ contained within Φ_w is obtained from lowest-order QED. Before doing this for the Dirac equations we review the constraint equations for spinless bosons to guide our effort.

In recent work [2] two of us used Sazdjian's "quantum mechanical transform" [40] of the Bethe-Salpeter wave function to derive the "quasipotential equation" of Todorov [14] from a field theory for spinless particles. The Todorov quasipotential equation is an inhomogeneous integral equation which relates the quasipotential Φ_w appearing in a Schrödinger-like, three-dimensional equation

$$[p_\perp^2+\Phi_w(x_\perp,p_\perp)]\psi_w(x_\perp)=b^2(w)\psi_w(x_\perp) \tag{3.1}$$

to certain matrix elements of the off-mass-shell, field theoretic, relativistic scattering amplitude. It is closely connected to the present work through Eq. (3.1) which it shares with constraint dynamics.

A. The quasipotential equation for spinless particles

The two, coupled, Klein-Gordon equations of constraint dynamics [7–10,5,13] can be written as

$$\mathcal{H}_i\psi_w\equiv(p_i^2+m_i^2+\Phi_i)\psi_w=0,\quad i=1,2 \tag{3.2}$$

with

$$P^2\psi_w=-w^2\psi_w\ . \tag{3.3}$$

The compatibility requirement $[\mathcal{H}_1,\mathcal{H}_2]\psi_w=0$ implies [7–10,4,5,13] that if $\Phi_1=\Phi_2\equiv\Phi_w$ then

$$\begin{aligned}\tfrac{1}{2}(\mathcal{H}_1-\mathcal{H}_2)\psi_w&=\tfrac{1}{2}[(p_1^2-p_2^2)-(m_2^2-m_1^2)]\psi_w\\&\equiv P\cdot p\,\psi_w=0\ .\end{aligned} \tag{3.4}$$

Thus, even though ψ_w is off mass shell it does satisfy $(p_1^2+m_1^2)\psi_w=(p_2^2+m_2^2)\psi_w$ [8,13]. The right-hand side of (3.4) implies that in the c.m. system the wave function is independent of the relative time $[x_\perp=(0,\mathbf{r})]$. A second independent combination of the constraints $\mathcal{H}_1$ and $\mathcal{H}_2$,

$$\begin{aligned}&w^{-1}(\epsilon_2\mathcal{H}_1+\epsilon_1\mathcal{H}_2)\psi_w\\&\quad=[p_\perp^2+\Phi_w(x_\perp,p_\perp)-b^2(w)]\psi_w(x_\perp)=0\ ,\end{aligned} \tag{3.5}$$

determines this off-shell behavior through the quasipotential Φ_w. For scattering states in an arbitrary Lorentz frame

$$\psi^{(+)}_{P,q_\perp}(X,x)=\frac{e^{iP\cdot X}}{(2\pi)^4}\psi^{(+)}_{w,q_\perp}(x_\perp) \tag{3.6}$$

with

$$\psi^{(+)}_{w,q_\perp}(x_\perp)=\psi^{(0)}_{w,q_\perp}(x_\perp)-[2w(p_\perp^2-b^2(w)-i\epsilon)]^{-1}\times V_w(x_\perp,p_\perp)\psi^{(+)}_{w,q_\perp}(x_\perp)\ , \tag{3.7}$$

$w \geq m_1+m_2$

where

$$\psi^{(0)}_{w,q_\perp}(x_\perp)\equiv(2\pi)^{-3}e^{iq_\perp\cdot x_\perp} \tag{3.8}$$

with $q_\perp^2=b^2(w)$ and $V_w\equiv 2w\Phi_w$. The corresponding momentum space wave function is given by the four-dimensional Fourier transform

$$\int d^4x\,\psi^{(+)}_{w,p_\perp}(x_\perp)e^{-ik\cdot x}=2\pi\delta(\hat{P}\cdot k)\tilde{\psi}^{(+)}_{w,q_\perp}(k_\perp)\ , \tag{3.9}$$

where $\tilde{\psi}^{(+)}_{w,q_\perp}(k_\perp)$ is itself given in terms of the position space wave function by the covariant three-dimensional transform

$$\tilde{\psi}^{(+)}_{w,p_\perp}(k_\perp)=\int d^4x\ \delta(\hat{P}\cdot x)\psi^{(+)}_{w,q_\perp}(x_\perp)e^{-ik\cdot x} \tag{3.10}$$

whose inverse transform is

$$\psi^{(+)}_{w,q_\perp}(x_\perp)=\int\frac{d^4k}{(2\pi)^3}\delta(\hat{P}\cdot k)\tilde{\psi}^{(+)}_{w,p_\perp}(k_\perp)e^{ik\cdot x}\ . \tag{3.11}$$

Equation (3.7) then yields

$$\tilde{\psi}^{(+)}_{w,q_\perp}(k_\perp)=\delta^3(k_\perp-q_\perp)-\{2w[k_\perp^2-b^2(w)-i\epsilon]\}^{-1}\int\frac{d^3k'_\perp}{(2\pi)^3}\tilde{\mathbf{V}}_w(k_\perp,k'_\perp)\tilde{\psi}^{(+)}_{w,q_\perp}(k'_\perp) \tag{3.12}$$

in which

$$\delta^3(p_\perp-q_\perp)\equiv\int d(\hat{P}\cdot p)\delta^4(p-q) \tag{3.13a}$$

and

$$\int d^3k'_\perp\equiv\int d^4k'\delta(\hat{P}\cdot k') \tag{3.13b}$$

with $\tilde{\mathbf{V}}$ related to the Fourier transform of $V_w(x_\perp,p_\perp)$ by

$$\tilde{\mathbf{V}}_w(k_\perp,k'_\perp)=\tilde{V}_w(k_\perp-k'_\perp,k'_\perp)\ . \tag{3.14}$$

Note that the momentum space constraint wave function is not $\psi^{(+)}_{w,q_\perp}(k_\perp)$ but rather $\delta(\hat{P}\cdot k)\tilde{\psi}^{(+)}_{w,q_\perp}(k_\perp)$.

If we define the scattering amplitude $\mathbf{T}_w(p_\perp,q_\perp)$ in the usual way

$$\mathbf{T}_w(p_\perp,q_\perp)\equiv-\int d^3p'_\perp\tilde{\mathbf{V}}_w(p_\perp,p'_\perp)\tilde{\psi}^{(+)}_{w,q_\perp}(p'_\perp)\ , \tag{3.15}$$

multiply Eq. (3.12) by $-\tilde{\mathbf{V}}_w$, and integrate we are led to a Lippmann-Schwinger equation for this amplitude in terms of the quasipotential $\tilde{\mathbf{V}}_w$:

$$\mathbf{T}_w(p_\perp,q_\perp)+\tilde{\mathbf{V}}_w(p_\perp,q_\perp)+\int\frac{d^3p'_\perp}{(2\pi)^3}\frac{\tilde{\mathbf{V}}_w(p_\perp,p'_\perp)}{2w(p'^2_\perp-b^2(w)-i\epsilon)}\mathbf{T}_w(p'_\perp,q_\perp)=0\ . \tag{3.16}$$

Symbolically this equation is of the form

$$\mathbf{T}_w+\mathbf{V}_w+\mathbf{V}_wG_\perp\mathbf{T}_w=0 \tag{3.16'}$$

in which $G_\perp$ stands for

$$G_w^{(+)}(p_\perp)=\frac{1}{(2\pi)^3 2w(p_\perp^2-b^2-i\epsilon)}\ . \tag{3.17}$$

The scattering amplitude then automatically satisfies the elastic two-body unitarity condition [14]

$$\mathbf{T}_w(p_\perp,q_\perp)-\mathbf{T}_w^*(q_\perp,p_\perp)=\frac{\pi i}{w}\int\frac{d^3k_\perp}{(2\pi)^3}\mathbf{T}_w^*(k_\perp,p_\perp)\delta(k_\perp^2-b^2)\mathbf{T}_w(k_\perp,q_\perp)\ . \tag{3.18}$$

The Lippmann-Schwinger equation (3.16) gives the relativistic quantum mechanical scattering amplitude $\mathbf{T}_w$ in terms of a prescribed quasipotential $\tilde{\mathbf{V}}_w$. Equation (3.16) is Todorov's inhomogeneous quasipotential equation. However, that equation is usually solved for $\tilde{\mathbf{V}}_w$ in terms of a $\mathbf{T}_w$ which is identified with the field theoretic scattering amplitude $T_w(p_\perp,q_\perp)$. In Appendix A we present an explicit momentum space derivation of the Todorov inhomogeneous quasipotential from the inhomogeneous Bethe-Salpeter equation

$$T_w(p;q)=K_w(p;q)-\frac{i}{(2\pi)^4}\int d^4k\,K_w(p;k)G_1^{(+)}(\epsilon_1\hat{P}+k)G_2^{(+)}(\epsilon_2\hat{P}-k)T_w(k;q) \tag{3.19}$$

[relating the Bethe-Salpeter kernel $K_w(p_\perp,q_\perp)$ that plays the role of the potential in the homogeneous Bethe Salpeter equation to $T_w(p_\perp,q_\perp)$]. Like the formal operator derivation given earlier [2] by two of us, it uses Sazdjian's quantum mechanical transform of the Bethe-Salpeter wave function. However, the new derivation shows the connection with earlier three dimensional approaches [41,42] and emphasizes the role of elastic two-body unitarity.

To summarize, the two constraint equations (3.4) and (3.5) play two different roles. Equation (3.4) forces the relative energy (in the c.m. system) to vanish while the Schrödinger-like equation (3.5) describes the effect of the dynamics and puts the system on a collective mass shell (of total energy w in the c.m.). Other than the requirement that the constraint potential Φ_w depend on x only through $x_\perp$, the constraint equations give no further restriction on the dynamical content of the constraint potential (for spinless particles). When constraint dynamics is being used in conjunction with quantum field theory, the potential Φ_w can be determined from an appropriate quantum field theory by way of (3.16) [Eq. (A19) in Appendix A or in terms of Eqs. (A17) and (A18)]. When the field theoretic starting point is the Bethe-Salpeter equation, the connection must be made through an object, the Sazdjian projection, in which the relative time (about which nothing is said in the Bethe-Salpeter equation) is eliminated as in (3.4) or (A12). Thus, one starts from the Bethe-Salpeter equation (3.19) and ends with the constraint equation (3.1) with the constraint potential Φ_w determined from (3.16), or (A18) and (A19).

Transformations from the two-time four-dimensional Bethe-Salpeter equation to one-time three-dimensional quasipotential equations have a long history dating back to early work of Logunov and Tavkhelidze [41] and Blankenbecler and Sugar [42]. In subsequent papers Yaes [17] and Gross [17] pointed out that there are in fact an infinite number of such three-dimensional reductions of the Bethe-Salpeter equation. The equations presented here and that of Sazdjian are particular cases, motivated by constraint dynamics, that lead to simple Schrödinger-like wave equations.

B. The quasipotential equation for two spin-$\frac{1}{2}$ particles

1. Constraint dynamics

When spin is included we describe the quantum system in terms of two compatible Dirac equations (2.25). At this stage we are only interested in first-order field-theoretic amplitudes. For these, our equations are approximately the weak potential forms

$$\mathbf{S}_1\psi=(\mathcal{S}_{10}+\mathcal{S}_{20}\Delta^{(1)})\psi=0 , \tag{3.20a}$$

$$\mathbf{S}_2\psi=(\mathcal{S}_{20}+\mathcal{S}_{10}\Delta^{(1)})\psi=0 , \tag{3.20b}$$

which are Sazdjian's forms of the two-body Dirac equations [38].

Now we obtain from these two equations forms analogous to those we used in the spinless case. First, because $[\mathcal{S}_{10},\mathcal{S}_{20}]=0$, we find that

$$(\mathcal{S}_{10}\mathbf{S}_1-\mathcal{S}_{20}\mathbf{S}_2)\psi=-P\cdot p\psi=0 . \tag{3.21}$$

Now since $P\cdot p\psi=0$, we have

$$-2\mathcal{S}_{10}^2\psi=-2\mathcal{S}_{20}^2\psi=(p_\perp^2-b^2)\psi . \tag{3.22}$$

As a result, we find that

$$\begin{aligned}-2\mathcal{S}_{10}\mathbf{S}_1\psi&=-2\mathcal{S}_{20}\mathbf{S}_2\psi\\&=(p_\perp^2-b^2+\Phi_w^{(1)})\psi=0 ,\end{aligned} \tag{3.23}$$

where

$$\Phi_w^{(1)}=-2\mathcal{S}_{10}\mathcal{S}_{20}\Delta^{(1)} , \tag{3.24}$$

which relates Sazdjian's $\Delta^{(1)}$ to the quasipotential of our relativistic Schrödinger equation.

2. Field theory

In order to determine $\Phi_w^{(1)}$ and from it the corresponding $\Delta^{(1)}$ from field theory, we consider the inhomogeneous Bethe-Salpeter equation for two spin-$\frac{1}{2}$ particles:

$$\begin{aligned}T&=K+K[(\gamma_1p_1+m_1)(\gamma_2p_2+m_2)]^{-1}T\\&=K+K\mathcal{S}_{10}^{-1}\mathcal{S}_{20}^{-1}\theta_{51}\theta_{52}T .\end{aligned} \tag{3.25}$$

We remove the γ matrices from the denominator. We let

$$\mathcal{T}=4\mathcal{S}_{10}\mathcal{S}_{20}\theta_{51}\theta_{52}T , \tag{3.26}$$

and

$$\mathcal{K}=4\mathcal{S}_{10}\mathcal{S}_{20}\theta_{51}\theta_{52}K . \tag{3.27}$$

Thus we have

$$\begin{aligned}\mathcal{T}&=\mathcal{K}+\mathcal{K}\tfrac{1}{4}\mathcal{S}_{10}^{-2}\mathcal{S}_{20}^{-2}4\mathcal{S}_{10}\mathcal{S}_{20}\theta_{51}\theta_{52}T\\&=\mathcal{K}+\mathcal{K}G_1^+G_2^+\mathcal{T} ,\end{aligned} \tag{3.28}$$

where $G_i^+\equiv(p_i^2+m_i^2-i\epsilon)^{-1}$ is the Feynman propagator for the spinless case. Because of the similarity between the spinless equations (3.1) and (3.19) and the second-order form of the spin-$\frac{1}{2}$ equations (3.23) and (3.28), we can use the derivation of the Todorov inhomogeneous quasipotential equation for the spinless case given in Appendix A to prove that the (lowest-order) Sazdjian transform of the Bethe-Salpeter wave function for the spin-$\frac{1}{2}$, spin-$\frac{1}{2}$ case is

$$(p_\perp^2+\Phi_w^{(1)})\Psi=b^2(w)\Psi , \tag{3.29}$$

but with an $x_\perp$ and spin-dependent quasipotential

$$\Phi_w^{(1)}=-\frac{\mathcal{T}^{(1)}}{2w} . \tag{3.30}$$

Comparison of this with (3.24) identifies $\Delta^{(1)}$ as

$$\Delta^{(1)}=\frac{\theta_{51}\theta_{52}T^{(1)}}{w} . \tag{3.31}$$

We have shown how, in lowest order, the four-dimensional Bethe-Salpeter equation can be transformed into the three-dimensional Eq. (3.29). This equation is identical to the one obtained from the weak-potential

constraint equations (3.20). However, if we regard the field-theoretic connection not as a rigid one to the weak-potential Sazdjian form (3.20) but instead to the strong-potential constraint form (2.25) [related to the "external potential" or "minimal interaction" constraint form by way of (2.24)], those equations [Eqs. (2.25)] clothe the (perturbative) field-theoretic interactions in their own peculiar quantum-mechanical structures. These "strong-potential" structures appear in the dynamics of our wave equations (2.25) through two-body potentials that treat each particle as though it were minimally coupled to an external potential (or potentials) generated by the other particle and in the hyperbolic structure of our equation through the occurrence of simple forms for the corresponding quantum-mechanical norm [36] of the wave function. These strong-potential structures (2.25) induce two different sorts of terms beyond those that appear in Sazdjian's (3.20). First, the nonlinear Δ terms in $\cosh\Delta$ and $\sinh\Delta$ produce additional spin dependences. Second, the quantity Δ, through its dependence on the invariant potentials L, J, and $\mathcal{G}$, differs from $\Delta^{(1)}$ calculated in first-order perturbation theory using (3.31). The invariants that appear in $\Delta^{(1)}$ are perturbative approximations of those that appear in Δ. When one attempts to extrapolate the perturbative invariants above the order of approximation justified through comparison with the perturbative Bethe-Salpeter equation, those extrapolations are merely provisional—subject to change when higher-order field-theoretic corrections in Δ are included [see (A23) in Appendix A]. However, nonperturbative principles like gauge invariance and our related demands of both constituent and collective forms of minimal interaction will constrain the forms of Δ that can appear in (2.25). We shall show this for QED below.

C. The case of quantum electrodynamics

For the electromagnetic interaction, the T matrix in momentum space in the Feynman gauge is

$$T_w^{(1)}(p,q)=\frac{e_1e_2}{(p-q)^2-i\epsilon}\gamma_1^\mu\gamma_{2\mu} \tag{3.32}$$

where p and q are relative momenta. The simple form of the resulting constraint equations eliminates the practical necessity in other approaches (for example the formalism of Caswell and Lepage [35]) of working in the Coulomb gauge. Because of the constraint (3.21), the coordinate space form of (3.29) becomes

$$\left[p_\perp^2-b^2-2\mathcal{S}_{10}\mathcal{S}_{20}\theta_{51}\theta_{52}\frac{\gamma_1^\mu\gamma_{2\mu}e_1e_2}{4\pi w|x_\perp|}\right]\Psi=0\ . \tag{3.33}$$

Direct comparison of this form of the Bethe-Salpeter equation with the Schrödinger-like constraint form given in (3.23) and (3.24) yields

$$\Delta^{(1)}=\theta_{51}\theta_{52}\gamma_1^\mu\gamma_{2\mu}\frac{e_1e_2}{4\pi w|x_\perp|}=\frac{e_1e_2}{4\pi w|x_\perp|}\theta_1\cdot\theta_2\ . \tag{3.34}$$

Comparison of (3.34) with the definition of Δ_T in (2.22) shows (1) $L^{(1)}(x_\perp)=0$ as expected, since a vector field theory cannot generate scalar potentials in lowest order; (2) $J^{(1)}=-\mathcal{G}^{(1)}$, which just tells us how the spacelike and timelike vector portions are related for electromagnetic interactions; and finally (3)

$$\mathcal{G}^{(1)}(x_\perp)=\frac{e_1e_2}{4\pi w|x_\perp|}\ . \tag{3.35}$$

Equations (2.11c) and (2.16e) of Sec. II imply that $\mathcal{A}^{(1)}=w\mathcal{G}^{(1)}$ so that

$$\mathcal{A}^{(1)}(x_\perp)=\frac{e_1e_2}{4\pi|x_\perp|}\ . \tag{3.36}$$

The nonperturbative extension of $\mathcal{G}$, or equivalently of the invariant function

$$\mathcal{A}=\frac{w}{2}[1-\exp(-2\mathcal{G})] \tag{3.37}$$

[see (2.11c) and (2.16e)] is not determined by this comparison at orders beyond Eq. (3.35) or (3.36). However, $\mathcal{G}$ is restricted through gauge invariance as realized through the introduction of interactions through minimal substitution as done in the two-body Dirac equations (2.1) of Sec. II. When restricted to electromagneticlike interactions ($\mathcal{V}=S=0$), these equations become the electromagnetic two-body Dirac equations

$$\mathcal{S}_1\psi=(\pi_1\cdot\theta_1+m_1\theta_{51})\psi=0\ , \tag{3.38a}$$

$$\mathcal{S}_2\psi=(\pi_2\cdot\theta_2+m_2\theta_{52})\psi=0\ . \tag{3.38b}$$

previously derived by two of us [5,16] and solved analytically for the equal-mass singlet case—positronium. In these equations the constituent vector potentials appear through the minimal substitutions

$$p_1^\mu\rightarrow\pi_1^\mu\equiv p_1^\mu-A_1^\mu=G[p^\mu+(\epsilon_1-\mathcal{A})\hat{P}^\mu+i\theta_2\cdot\partial\mathcal{G}\theta_2^\mu]\ , \tag{3.39a}$$

$$p_1^\mu\rightarrow\pi_1^\mu\equiv p_2^\mu-A_2^\mu=G[-p^\mu+(\epsilon_2-\mathcal{A})\hat{P}^\mu-i\theta_1\cdot\partial\mathcal{G}\theta_1^\mu] \tag{3.39b}$$

[see Eqs. (2.3) and (2.11)] so that the squared forms of the constraints take the simple one-body electromagneticlike forms

$$\mathcal{H}_1\psi\equiv-\tfrac{1}{2}[\mathcal{S}_1,\mathcal{S}_1]_+\psi=(\pi_1^2-\tfrac{1}{2}\sigma_1^{\mu\nu}F_{1\mu\nu}+m_1^2)\psi=0\ , \tag{3.40a}$$

$$\mathcal{H}_2\psi\equiv-\tfrac{1}{2}[\mathcal{S}_2,\mathcal{S}_2]_+\psi=(\pi_2^2-\tfrac{1}{2}\sigma_2^{\mu\nu}F_{2\mu\nu}+m_2^2)\psi=0\ , \tag{3.40b}$$

where $F_{a\mu\nu}\equiv(1/i)[\pi_{\mu a},\pi_{\nu a}]$. The difference of these two equations is $(\mathcal{H}_1-\mathcal{H}_2)\psi=2P\cdot p\psi=0$, just as in the spinless case. If we identify

$$\mathcal{U}^\mu\equiv\frac{1}{i}\partial^\mu\mathcal{G}\ , \tag{3.41}$$

and write out only the spinless part of π_i, we find (see de-

tails in Sec. IV) that the weighted sum

$$\mathcal{H}\psi=\left[\frac{\epsilon_2}{w}\mathcal{H}_1+\frac{\epsilon_1}{w}\mathcal{H}_2\right]\psi=0$$

yields

$$[-(\epsilon_w-\mathcal{A})^2+(p-\mathcal{U})^2+m_w^2+\text{spin-dependent}+\text{Darwin terms}]\psi=0\ . \quad (3.42)$$

The spin-independent terms at the first part of the equation display Todorov's interpretation of system potentials as minimal extensions of the four-momentum of relative motion $(\epsilon_w,\mathbf{p})$. In fact, if we define [43]

$$\mathcal{P}^\mu=\epsilon_w\hat{P}+p^\mu,\quad A^\mu=\mathcal{A}\hat{P}^\mu+\mathcal{U}^\mu\ , \quad (3.43)$$

then the first part can be completely written in the collective minimal (Todorov) interaction form $(\mathcal{P}^\mu-A^\mu)^2$ for the effective particle of relative motion.

What are the additional restrictions on $\mathcal{G}$ or $\mathcal{A}$ that arise from this collective minimal interaction form? The first restriction follows from the fact that the portion $2\epsilon_w\mathcal{A}-\mathcal{A}^2$ of the quasipotential must be quantum mechanically well defined for a Schrödinger-like equation. This restricts $\mathcal{A}$ so that $-\mathcal{A}^2$ must not be singular $(<-1/4r^2)$ as $r\to 0$. For example, the simple choice $\mathcal{G}=\mathcal{G}^{(1)}=e_1e_2/wr$ [corresponding to $\mathcal{A}=(w/2)(1-\exp(-2e_1e_2/wr)$] would produce a $-\mathcal{A}^2$ term that grows exponentially as $r\to 0$ for $e_1e_2<0$ yielding an unacceptable singular behavior in the effective Schrödinger equation. The second restriction is that

$$2\epsilon_w\mathcal{A}-\mathcal{A}^2=2\epsilon_w\mathcal{A}^{(1)}-(\mathcal{A}^{(1)})^2+O((\mathcal{A}^{(1)})^3)\ . \quad (3.44)$$

must be satisfied when $\mathcal{A}$ is expanded in powers of $\mathcal{A}^{(1)}$. Classically this restriction implies that when one carries out an expansion through order $1/c^2$ by solving the minimal Todorov equation $p^2-(\epsilon_w-\mathcal{A}^{(1)})^2+m_w^2=0$ for w, then one obtains an expansion that includes not only the standard relativistic corrections to the non-relativistic kinetic energy but also relativistic corrections [13,34] to the nonrelativistic potential that are canonically equivalent to the Darwin interaction. Thus this collective minimal interaction structure incorporates in a covariant way the Darwin interaction corrections to the nonrelativistic potential $-\alpha/r$ without a complicated momentum dependence. These simple structures occur in Todorov's closely related quasipotential equation [14] and in quantum constraint dynamics for spinless particles under vector interactions [13]. Using a scale transformation developed by Schwinger [44], two of us have shown that [5,13] this collective minimal structure yields an $O(1/c^2)$ expansion that is canonically equivalent to the standard $O(1/c^2)$ momentum-dependent Darwin interaction. In fact, the Todorov equation can be derived from the Darwin interaction [76].

One solution to these two perturbative conditions on $\mathcal{A}$ is the naive identification

$$\mathcal{A}=\mathcal{A}^{(1)}\ . \quad (3.45)$$

This particular $\mathcal{A}$ provides a considerable simplification of the calculation of the semirelativistic (order-α^4) corrections to the QED bound-state spectrum [13,14] for spinless particles over the standard Breit related approaches. For this $\mathcal{A}$, Todorov *et al.* [30] have also shown how the $-\mathcal{A}^2$ terms correspond to higher-order ladder, cross-ladder, and iterated exchange contributions to Φ_w (corresponding to segments of $\mathcal{G}^{(2)}$ in the notation of this paper). These constituent and collective minimal interaction requirements are nonperturbative ones beyond the strictly perturbative field-theoretic restriction of (3.35) on $\mathcal{G}$. With the naive choice Eq. (3.45), the nonperturbative extension of $\mathcal{G}$ implied by Eq. (3.37) is

$$\mathcal{G}=-\frac{1}{2}\ln\left[1-\frac{2e_2e_2}{wr}\right]\ . \quad (3.46)$$

[Note again that the simple choice $\mathcal{G}=\mathcal{G}^{(1)}$ gives the correct lowest-order $\mathcal{G}$ but would lead through Eq. (3.37) to an $\mathcal{A}$ that would produce a singular quasipotential.]

In summary we have found three restrictions that $\mathcal{A}$ must satisfy. First, it must generate the correct lowest-order interaction $\mathcal{G}^{(1)}$. Second, it must generate the minimal Todorov form, Eq. (3.44), in order to give the correct $O(1/c^2)$ dynamics. Third, it must satisfy the nonperturbative restriction that $\mathcal{A}^2$ must not be too singular at $r=0$. As we anticipate (see Sec. V), the above restrictions will guarantee that these nonperturbative potential forms will yield the correct results if the equations are treated perturbatively. A more crucial test of these nonperturbative or strong potential structures will be to determine if a nonperturbative (numerical) treatment of these equations will yield the correct spectrum to the appropriate order (see Sec. VI).

D. Phenomenological scalar and vector interactions

To carry out semiphenomenological applications of the constraint equations such as to the quark models of mesons, one does not perturbatively determine Δ from field theory as we did in (3.32) and (3.34). Two of us in Ref. [2] divided up the nonrelativistic static quark potential $U(r)$ in terms of the three invariants $\mathcal{A}(r)$, $\mathcal{V}(r)$, and $S(r)$ of Eqs. (2.11) and (2.12) chosen so that

$$\mathcal{A}(r)+\tfrac{1}{2}[\mathcal{V}(r)+S(r)]=U(r)\ . \quad (3.47)$$

This division of $U(r)$ was arbitrary, guided primarily by phenomenological considerations. However, this choice has consequences far beyond its nonrelativistic roots since the matrix structure to which it is attached dictates different relativistic Darwin and spin-dependent corrections depending on the corresponding matrix Lorentz-invariant structures that appear in Δ. In Ref. [2] we used

$$\Delta_T=\Delta_J+\Delta_L+\Delta_{\mathcal{G}}=\tfrac{1}{2}[\mathcal{O}_2J(x_\perp)-\mathcal{O}_1L(x_\perp)+\mathcal{O}_3\mathcal{G}(x_\perp)]\ . \quad (2.22')$$

in which, from (2.16),

$$L=L(\mathcal{A},S)=\ln\frac{(M_1+M_2)}{m_1+m_2}\ , \quad (3.48a)$$

$$J=J(\mathcal{A},\mathcal{V})=\ln\frac{(E_1+E_2)}{\epsilon_1+\epsilon_2}\ , \tag{3.48b}$$

$$\mathcal{G}=\mathcal{G}(\mathcal{A})=\ln(G)\ . \tag{3.48c}$$

[We remind the reader that the "minimal interaction" form (2.18) (depending on $\mathcal{A}$, $\mathcal{V}$, and S) of the two-body Dirac equations is equivalent to the hyperbolic form (2.25) (depending on L, J, and $\mathcal{G}$), related by (2.24).] Note that for models with $\mathcal{V}=0$, our vector interaction is that used for the Abelian interactions of lowest-order QED. In the quark-model applications, we used directly the "external potential" form (2.18) with $\mathcal{A}$, $\mathcal{V}$, and S identified as in Eq. (3.47). In contrast, for QED the form of $\mathcal{A}$ was dictated by the match between the Δ of the weak potential form, Eq. (3.20), on the one hand and the field-theoretically derived quasipotential equation on the other.

IV. REDUCTION OF TWO-BODY DIRAC EQUATIONS TO SECOND-ORDER RELATIVISTIC SCHRÖDINGER-LIKE EQUATIONS

We wish to determine the total energy eigenvalues for the stationary states of two interacting spin-$\frac{1}{2}$ particles using the "external potential" or "minimal interaction" form of the two coupled Dirac equations (2.18). For this purpose, we have at our disposal the analogs of all of the decoupling procedures and simplifications resulting from special choices of Dirac matrix representations that one uses to solve the one-body Dirac equation. For example, two of us [16] used the fact that $\sigma_{\mu\nu}$ is diagonal in the chiral representation to obtain exact solutions for bound electromagnetic equal-mass singlet states. In meson work [2], two of us decomposed the second-order equations corresponding to Eqs. (2.18) (with gamma matrices in the Dirac representation) into four decoupled four-component second-order equations. Subsequently, Sazdjian pointed out [45] to us that our reduction in the Dirac representation could only be carried out for singlet and $j=l$ triplet states, not for $j=l\pm1$ triplet states (we review this development in Appendix D of Ref. [46]). Since the validity of that reduction turned out to be state dependent, we replace it in this paper by a generally valid reduction of the coupled 16-component first-order equations (2.18) to two decoupled eight-component second-order Schrödinger-like equations.

Since we shall work in a general frame, for convenience we define the covariant versions of the standard Dirac $\boldsymbol{\alpha}$ and β and $\boldsymbol{\Sigma}$ matrices for the two particles:

$$\beta_i\equiv-\gamma_i\cdot\hat{P}=2\theta_{5i}\theta_i\cdot\hat{P}\ , \tag{4.1}$$

$$\alpha_i^\mu\equiv\beta_i\gamma_{i\perp}^\mu=2\theta_{i\perp}^\mu\theta_i\cdot\hat{P}\ , \tag{4.2}$$

and

$$\Sigma_i^\mu\equiv2\sqrt{2}i\theta_{5i}\theta_i\cdot\hat{P}\theta_{\perp i}^\mu \tag{4.3}$$

in which the subscript i is the particle label. In the c.m. system, α and Σ have no time components. In a general frame their components parallel to the total four-momentum P are zero. Using these, we obtain

$$\theta_{\perp i}^\mu=-i\sqrt{\tfrac{1}{2}}\beta_i\Sigma_i^\mu \tag{4.4}$$

$$\theta_i\cdot\hat{P}=i\sqrt{\tfrac{1}{2}}\beta_i\gamma_{5i} \tag{4.5}$$

$$\theta_{5i}=i\sqrt{\tfrac{1}{2}}\gamma_{5i}\ . \tag{4.6}$$

The two-body Dirac equations (2.18) then take the form

$$\mathcal{S}_1\psi=\left[-G\beta_1\boldsymbol{\Sigma}_1\cdot\mathcal{P}_2+E_1\beta_1\gamma_{51}+M_1\gamma_{51}+G\frac{i}{2}\boldsymbol{\Sigma}_2\cdot\partial(J\beta_1-L\beta_2)\gamma_{51}\gamma_{52}\right]\psi=0\ , \tag{4.7}$$

$$\mathcal{S}_2\psi=\left[G\beta_2\boldsymbol{\Sigma}_2\cdot\mathcal{P}_1+E_2\beta_2\gamma_{52}+M_2\gamma_{52}-G\frac{i}{2}\boldsymbol{\Sigma}_1\cdot\partial(J\beta_2-L\beta_1)\gamma_{51}\gamma_{52}\right]\psi=0 \tag{4.8}$$

in which

$$\mathcal{P}_i\equiv p-\frac{i}{2}\boldsymbol{\Sigma}_i\cdot\partial\ln G\boldsymbol{\Sigma}_i\ . \tag{4.9}$$

Thus, $\mathcal{S}_1^2\psi=0$ becomes

$$\left\{(G\boldsymbol{\Sigma}_1\cdot\mathcal{P}_2)^2-[G\beta_1\boldsymbol{\Sigma}_1\cdot\mathcal{P}_2,E_1\beta_1+M_1]\gamma_{51}-E_1^2+M_1^2+G^2\frac{1}{4}[\partial(J-L)]^2\right.$$
$$\left.-\left[G\beta_1\boldsymbol{\Sigma}_1\cdot\mathcal{P}_2,G\frac{i}{2}\boldsymbol{\Sigma}_2\cdot\partial(J\beta_1-L\beta_2)\right]\gamma_{51}\gamma_{52}-iG(E_1\partial J+\beta_1\beta_2M_1\partial L)\cdot\boldsymbol{\Sigma}_2\gamma_{52}\right\}\psi=0\ . \tag{4.10}$$

Now we substitute the Dirac equations (4.7) and (4.8) in the forms

$$\gamma_{51}\psi=\frac{1}{E_1\beta_1+M_1}G\left[\beta_1\boldsymbol{\Sigma}_1\cdot\mathcal{P}_2-\frac{i}{2}\partial(J\beta_1-L\beta_2)\cdot\boldsymbol{\Sigma}_2\gamma_{51}\gamma_{52}\right]\psi\ , \tag{4.11a}$$

$$\gamma_{52}\psi=-\frac{1}{E_2\beta_2+M_2}G\left[\beta_2\Sigma_2\cdot\mathcal{P}_1-\frac{i}{2}\partial(J\beta_2-L\beta_1)\cdot\Sigma_1\gamma_{51}\gamma_{52}\right]\psi\ , \tag{4.11b}$$

into the respective singly odd parts of (4.10) (those parts that contain only one γ_{5i} factor) and evaluate the commutator appearing in one of the singly odd terms. This yields a single second-order Schrödinger-like equation [4,5]

$$\left\{(G\Sigma_1\cdot\mathcal{P}_2)^2-E_1^2+M_1^2+G^2\frac{1}{4}[\partial(J-L)]^2+iG\Sigma_1\cdot\partial\ln(E_1\beta_1+M_1)(G\Sigma_1\cdot\mathcal{P}_2)+iG\Sigma_2\cdot\partial\ln(E_2\beta_2+M_2)(G\Sigma_2\cdot\mathcal{P}_1)\right.$$
$$+\left[-\left[G\Sigma_1\cdot\mathcal{P}_2,G\frac{i}{2}\Sigma_2\cdot\partial(J-L\beta_1\beta_2)\right]+\frac{1}{2}G^2\Sigma_1\cdot\partial\ln(E_1\beta_1+M_1)\partial(J-L\beta_1\beta_2)\cdot\Sigma_2\right.$$
$$\left.\left.+\frac{1}{2}G^2\Sigma_2\cdot\partial\ln(E_2\beta_2+M_2)\partial(J-L\beta_1\beta_2)\cdot\Sigma_1\right]\gamma_{51}\gamma_{52}\right\}\psi=0\ . \tag{4.12}$$

The 16-component Dirac spinor in (4.12) we write as

$$\psi=\psi_1\begin{bmatrix}1\\0\end{bmatrix}_1\begin{bmatrix}1\\0\end{bmatrix}_2+\psi_2\begin{bmatrix}0\\1\end{bmatrix}_1\begin{bmatrix}1\\0\end{bmatrix}_2+\psi_3\begin{bmatrix}1\\0\end{bmatrix}_1\begin{bmatrix}0\\1\end{bmatrix}_2+\psi_4\begin{bmatrix}0\\1\end{bmatrix}_1\begin{bmatrix}0\\1\end{bmatrix}_2$$
$$\equiv\sum_{\kappa\lambda}\psi_{\kappa\lambda}\eta_\kappa^{(1)}\eta_\lambda^{(2)} \tag{4.13a}$$

or simply

$$\psi=\begin{bmatrix}\psi_1\\\psi_2\\\psi_3\\\psi_4\end{bmatrix}\ , \tag{4.13b}$$

where ψ_i are four-component spinors (see Ref. [4] or Appendix B in Ref. [46] for the convention we use in defining the 16×16 gamma matrices for the product space). Note that since there are no terms in Eq. (4.12) that contain an odd number of γ_5 matrices, the upper-upper components couple only to the lower-lower components. Equation (4.12) also couples the lower-upper to the upper-lower components but we will not need the resulting equations since Eqs. (4.11) determine these components in terms of the upper-upper and lower-lower components.

Even though in principle the squaring procedure used to construct these Schrödinger-like equations could introduce spurious solutions, it turns out that the equations we obtain by this procedure are identical to those Schrödinger-like equations obtained by simply manipulating the Dirac equations without squaring them [47]. In Refs. [4,46] we first review the analogous property for the one-body Dirac equation and then perform similar manipulations (in Appendix B of Ref. [46]) on the two-body Dirac equations. There we obtain (without squaring) the upper-upper component of (4.12) (the other components could be obtained by an analogous procedure), thus showing the equivalence of the two approaches (not demonstrated in Refs. [4,5]). Comparison of the two procedures leads to no further constraints beyond the original two-body Dirac equations. One could construct the full 16-component solutions of Eqs. (4.7) and (4.8) by solving the upper-upper and lower-lower portions of Eqs. (4.12) for ψ_1 and ψ_4 and then using Eqs. (4.11) to obtain ψ_2 and ψ_3. Having constructed the full solutions to the coupled 16-component Dirac equations we could in principle use them in conjunction with the inner product [36] derived from these Dirac equations for the 16-component wave functions and apply them to the computation of decay and other current matrix elements.

In Appendix B of Ref. [46] we also perform simplifying Pauli matrix algebra on the coupled upper-upper and lower-lower components of Eq. (4.12). We find that the upper-upper component of (4.12) becomes ($\hat{r}\equiv\chi_1/r$)

$$\left\{p^2+2m_wS+S^2+2\epsilon_w\mathcal{A}-\mathcal{A}^2+2\epsilon_w\mathcal{V}-\mathcal{V}^2+\frac{1}{4}(\partial J-\partial L)^2+i\ln'(\chi_1\chi_2)\hat{r}\cdot p-\frac{1}{2}\partial^2\mathcal{G}+\frac{3}{4}(\mathcal{G}')^2+\frac{1}{2}\ln'(\chi_1\chi_2)\mathcal{G}'\right.$$
$$-\frac{\ln'\chi_1}{r}L\cdot\sigma_1-\frac{\ln'\chi_2}{r}L\cdot\sigma_2+\left[\frac{1}{3}\partial^2\mathcal{G}-\frac{1}{2}(\mathcal{G}')^2-\frac{1}{3}\ln'(\chi_1\chi_2)\mathcal{G}'\right]\sigma_1\cdot\sigma_2+\left[-\frac{1}{6}\left[\mathcal{G}''-\frac{\mathcal{G}'}{r}\right]+\frac{1}{6}\ln'(\chi_1\chi_2)\mathcal{G}'\right]S_T\Big\}\psi_1$$
$$+\left\{\left[+\frac{1}{6}\ln'(\chi_1\chi_2)(J-L)'+\frac{1}{2}\mathcal{G}'(J-L)'-\frac{1}{6}\partial^2(J-L)\right]\sigma_1\cdot\sigma_2\right.$$
$$\left.+\left[+\frac{1}{6}\ln'(\chi_1\chi_2)(J-L)'-\frac{1}{6}\left[(J-L)''-\frac{(J-L)'}{r}\right]\right]S_T\right\}\psi_4=b^2(w)\psi_1 \tag{4.14a}$$

in which $\chi_i=(E_i+M_i)/G$. This couples to the lower-lower component of (4.12) which becomes

$$\left\{p^2+2m_w S+S^2+2\epsilon_w\mathcal{A}-\mathcal{A}^2+2\epsilon_w\mathcal{V}-\mathcal{V}^2+\frac{1}{4}(\partial J-\partial L)^2+i\ln'(\bar{\chi}_1\bar{\chi}_2)\hat{r}\cdot p-\frac{1}{2}\partial^2\mathcal{G}+\frac{3}{4}(\mathcal{G}')^2+\frac{1}{2}\ln'(\bar{\chi}_1\bar{\chi}_2)\mathcal{G}'\right.$$
$$-\frac{\ln'\bar{\chi}_1}{r}L\cdot\sigma_1-\frac{\ln'\bar{\chi}_2}{r}L\cdot\sigma_2+\left[\frac{1}{3}\partial^2\mathcal{G}-\frac{1}{2}(\mathcal{G}')^2-\frac{1}{3}\ln'(\bar{\chi}_1\bar{\chi}_2)\mathcal{G}'\right]\sigma_1\cdot\sigma_2+\left[-\frac{1}{6}\left[\mathcal{G}''-\frac{\mathcal{G}'}{r}\right]+\frac{1}{6}\ln'(\bar{\chi}_1\bar{\chi}_2)\mathcal{G}'\right]S_T\Big\}\psi_4$$
$$+\left\{\left[+\frac{1}{6}\ln'(\bar{\chi}_1\bar{\chi}_2)(J-L)'+\frac{1}{2}\mathcal{G}'(J-L)'-\frac{1}{6}\partial^2(J-L)\right]\sigma_1\cdot\sigma_2\right.$$
$$\left.+\left[+\frac{1}{6}\ln'(\bar{\chi}_1\bar{\chi}_2)(J-L)'-\frac{1}{6}\left[(J-L)''-\frac{(J-L)'}{r}\right]\right]S_T\right\}\psi_1=b^2(w)\psi_4 \tag{4.14b}$$

in which $\bar{\chi}_i=(E_i-M_i)/G$. Equations (4.14) are the two coupled eight-component Schrödinger-like equations that we shall use in Secs. V and VI for our bound-state calculations. The same two equations would have been obtained if we had started with $\mathcal{S}_2^2$ instead of $\mathcal{S}_1^2$ since their difference is $\sim\hat{P}\cdot p\psi=0$. Although not given here, the corresponding equations that couple the upper-lower and lower-upper components can be combined with Eqs. (4.14) into the general form

$$(p^2+\Phi_w)\psi=(p^2+\Phi_{\rm SI}+\Phi_D+\Phi_{\rm SO}+\Phi_{\rm SS}+\Phi_T+\Phi_{\rm DO}\gamma_{51}\gamma_{52})\psi=b^2(w)\psi \tag{4.15}$$

in which ψ is the full 16-component wave function. The forms of the parts of the quasipotential are given as in Eq. (4.14a) but with $\chi_i=E_i+\beta_i M_i$. $\Phi_{\rm SI}$ is the spin-independent minimal (Todorov) portion of the quasipotential, Φ_D denotes the Darwin terms [the last five terms of the first lines of Eqs. (4.14a) and (4.14b)], $\Phi_{\rm SO}$ the spin-orbit, $\Phi_{\rm SS}$ the spin-spin, and Φ_T the tensor portions of the quasipotential. $\Phi_{\rm DO}$ is the doubly odd part which couples the upper-upper and lower-lower, or the upper-lower and low-upper portions of the wave function [48]. As discussed in Appendix D of Ref. [46] a further decoupling of these equations can occur for special angular momentum states.

V. PERTURBATIVE TREATMENT OF THE TWO-BODY DIRAC EQUATIONS

Here we examine the bound-state energies produced by our constraint equations (4.14) in the weak-potential limit in order to obtain the energy spectra of our relativistic two-body system analytically through order α^4 by means of perturbation theory. For the case of the purely electromagnetic interaction ($\mathcal{A}=-\alpha/r$, $\mathcal{V}=S=0$), we compare the spectra with those obtained in perturbative QED from the Fermi-Breit reduction of the Bethe-Salpeter equation [31]. Since scalar and purely timelike interactions were also important in the meson spectroscopy work of Refs. [1,2], we will also compute the energy spectra analytically by using perturbation theory for the scalar and timelike four-vector interactions ($S=-\alpha/r$, $\mathcal{V}=\mathcal{A}=0$ for the scalar and $\mathcal{V}=-\alpha/r$, $S=\mathcal{A}=0$ for the timelike four-vector interaction) through order α^4. In Sec. VI we compare the perturbative eigenvalues with those obtained from a nonperturbative numerical solution of the unapproximated constraint equations (4.14). The comparison with the standard spectral results from the Bethe-Salpeter equation for the case of the purely electromagneticlike interactions will provide a critical test at both the perturbative and nonperturbative levels of the capability of the two-body Dirac equations of constraint dynamics to generate accurate spin-dependent as well as spin-independent relativistic recoil corrections for QED, from a static input potential $\mathcal{A}(r)$. The comparisons between the perturbative and nonperturbative solutions of the two-body Dirac equations for the scalar and timelike vector interactions are also important since they will reveal whether or not the unapproximated equations yield the perturbative relativistic (primarily short-distance) corrections of lowest order for those interactions. For example, in the case of a purely scalar interaction, there should be no hyperfine splitting through order α^4. A glance at Eqs. (4.14) shows that these equations contain terms (e.g., $\partial^2 L\sigma_1\cdot\sigma_2$) that could contribute to such a splitting. If a numerical treatment of the unapproximated equations with these interactions showed a larger hyperfine splitting than that predicted from a perturbative treatment of the equations, then these equations would not be trustworthy relativistic equations for meson spectroscopy calculations in which more general effective scalar and timelike four-vector interactions are used.

A. General equations

In order to perturbatively evaluate the energy eigenvalues we first obtain the weak-potential form that we will use from Eq. (4.14a) for the upper-upper component ψ_1 of the wave function, coupled to ψ_4. In the weak-potential limit we ignore the coupling to the lower-lower wave function ψ_4. Furthermore in the weak potential limit, in terms of the total mass $M=m_1+m_2$ we make the replacements [see Eqs. (2.11), (2.12), and (3.48)]

$$\mathcal{G}\approx\frac{\mathcal{A}}{M}\ , \tag{5.1}$$

$$L\approx\frac{S}{M}\ , \tag{5.2}$$

$$J \approx -\frac{\mathcal{A}+\mathcal{V}}{M} , \tag{5.3}$$

$$\chi_1 \approx 2m_1\left[1-\frac{\mathcal{A}}{M}\right]+m_2\frac{S-\mathcal{A}-\mathcal{V}}{M} , \tag{5.4}$$

and

$$\chi_2 \approx 2m_2\left[1-\frac{\mathcal{A}}{M}\right]+m_1\frac{S-\mathcal{A}-\mathcal{V}}{M} . \tag{5.5}$$

In this approximation Eq. (4.14a) simplifies to

$$\left\{p^2+2m_w S+S^2+2\epsilon_w\mathcal{A}-\mathcal{A}^2+2\epsilon_w\mathcal{V}-\mathcal{V}^2-\frac{1}{2}\partial_\perp^2\frac{\mathcal{A}}{M}+i\frac{\mathcal{A}'(-1-M/2\mu)+(S'-\mathcal{V}')(-1+M/2\mu)}{M}\hat{r}\cdot p\right.$$
$$+\left[\frac{\mathcal{A}'}{M}-\frac{1}{2}\frac{m_2}{m_1}\frac{(S-\mathcal{A}-\mathcal{V})'}{M}\right]\frac{L\cdot\sigma_1}{r}+\left[\frac{\mathcal{A}'}{M}-\frac{1}{2}\frac{m_1}{m_2}\frac{(S-\mathcal{A}-\mathcal{V})'}{M}\right]\frac{L\cdot\sigma_2}{r}$$
$$\left.+\frac{1}{3}\frac{\partial_\perp^2\mathcal{A}}{M}\sigma_1\cdot\sigma_2+\left[-\frac{1}{6}\frac{(\mathcal{A}''-\mathcal{A}'/r)}{M}\right]S_T\right\}\psi_1=b^2(w)\psi_1 \tag{5.6}$$

in which $\mu=m_1m_2/M$.

We now specialize to three cases.

(1) $\mathcal{V}=S=0$ leads to the weak-potential form of the equation containing only an electromagnetic interaction generated by the invariant $\mathcal{A}$:

$$\left\{p^2+2\epsilon_w\mathcal{A}-\mathcal{A}^2-\frac{1}{2}\partial_\perp^2\frac{\mathcal{A}}{M}+i\frac{\mathcal{A}'(-1-M/2\mu)}{M}\hat{r}\cdot p+\frac{\mathcal{A}'}{M}\left[1+\frac{1}{2}\frac{m_2}{m_1}\right]\frac{L\cdot\sigma_1}{r}\right.$$
$$\left.+\frac{\mathcal{A}'}{M}\left[1+\frac{1}{2}\frac{m_1}{m_2}\right]\frac{L\cdot\sigma_2}{r}+\frac{1}{3}\frac{\partial_\perp^2\mathcal{A}}{M}\sigma_1\cdot\sigma_2-\frac{1}{6}\frac{\mathcal{A}''-\mathcal{A}'/r}{M}S_T\right\}\psi_1=b^2(w)\psi_1 . \tag{5.7}$$

For lowest-order electrodynamics we found in Sec. III that the quasipotential reduction of QED led to

$$\Delta^{(1)}=\theta_{51}\theta_{52}\gamma_1^\mu\gamma_{2\mu}\frac{e_1e_2}{4\pi w|x_\perp|}=\frac{e_1e_2}{4\pi w|x_\perp|}\theta_1\cdot\theta_2 . \tag{5.8}$$

This corresponds to $\mathcal{G}^{(1)}=-J^{(1)}=-\alpha/wr$, and $L^{(1)}=0$ where $\alpha=-e_1e_2/4\pi$. To the order we are considering, we can set $w=M$ so that (3.48c) implies $\mathcal{A}=-\alpha/r$. In terms of the dimensionless Coulomb variable $\mathbf{x}=\epsilon_w\alpha\mathbf{r}$, Eq. (5.7) becomes [49] (in c.m.)

$$\left\{\left[-\frac{1}{x}\frac{d^2}{dx^2}x+\frac{\mathbf{L}^2}{x^2}-\frac{2}{x}\right]+\alpha^2\left[-\frac{1}{x^2}-\left[\frac{\mu}{M}+\frac{1}{2}\right]\frac{1}{x^2}\frac{d}{dx}+\pi\delta^3(\mathbf{x})\frac{\mu}{M}\left[-2+\frac{4}{3}\sigma_1\cdot\sigma_2\right]\right.\right.$$
$$\left.\left.+\frac{1}{x^3}\mathbf{L}\cdot(\sigma_1+\sigma_2)\left[\frac{\mu}{2M}+\frac{1}{4}\right]+\frac{1}{x^3}\mathbf{L}\cdot(\sigma_1-\sigma_2)\frac{m_2-m_1}{4M}+\frac{\mu}{2Mx^3}S_T\right]\right\}\psi_1=-\lambda^2\psi_1 , \tag{5.9}$$

in which $\lambda^2=-b^2(w)/(\epsilon_w\alpha)^2$.

(2) $\mathcal{A}=\mathcal{V}=0$ leads to the weak-potential form of the equation, containing only a scalar interaction generated by the invariant S:

$$\left\{p^2+2m_wS+S^2+\frac{S'(-1+M/2\mu)}{M}\hat{r}\cdot p-\frac{1}{2}\frac{m_2}{m_1}\frac{S'}{M}\frac{L\cdot\sigma_1}{r}-\frac{1}{2}\frac{m_1}{m_2}\frac{S'}{M}\frac{L\cdot\sigma_2}{r}\right\}\psi_1=b^2(w)\psi_1 . \tag{5.10}$$

For our model scalar interaction, the analog for scalar field theory of the quasipotential reduction of Sec. III leads to

$$\Delta^{(1)}=\theta_{51}\theta_{52}\frac{g_1g_2}{4\pi w|x_\perp|} . \tag{5.11}$$

This corresponds to $\mathcal{G}^{(1)}=0$, and $L^{(1)}=-\alpha/wr$, where $\alpha=g_1g_2/4\pi$. To the order we are considering we can set $w=M$, so that Eq. (3.48a) implies $S=-\alpha/r$. In terms of the dimensionless variable $\mathbf{x}=m_w\alpha\mathbf{r}$, Eq. (5.10) becomes (in c.m.)

$$\left\{\left[-\frac{1}{x}\frac{d^2}{dx^2}x+\frac{\mathbf{L}^2}{x^2}-\frac{2}{x}\right]+\alpha^2\left[\frac{1}{x^2}-\left[\frac{\mu}{M}-\frac{1}{2}\right]\frac{1}{x^2}\frac{d}{dx}-\frac{1}{4x^3}\mathbf{L}\cdot(\boldsymbol{\sigma}_1+\boldsymbol{\sigma}_2)-\frac{1}{x^3}\mathbf{L}\cdot(\boldsymbol{\sigma}_1-\boldsymbol{\sigma}_2)\frac{m_2-m_1}{4M}\right]\right\}\psi_1=-\lambda^2\psi_1\ , \tag{5.12}$$

in which $\lambda_2=-b^2(w)/(m_w\alpha)^2$.

(3) $\mathcal{A}=S=0$ leads to the weak-potential form of the equation containing only timelike vector interactions generated by the invariant $\mathcal{V}$:

$$\left\{p^2+2\epsilon_w\mathcal{V}-\mathcal{V}^2-i\frac{\mathcal{V}'(-1+M/2\mu)}{r}\hat{r}\cdot p+\frac{1}{2}\frac{m_2}{m_1}\frac{\mathcal{V}'}{M}\frac{L\cdot\sigma_1}{r}+\frac{1}{2}\frac{m_1}{m_2}\frac{\mathcal{V}'}{M}\frac{L\cdot\sigma_2}{r}\right\}\psi_1=b^2(w)\psi_1\ . \tag{5.13}$$

If we had separated the timelike and spacelike parts of the electromagnetic interaction Eq. (5.8) and retained only the time-like part we would have obtained

$$\Delta^{(1)}=\theta_1\cdot\hat{P}\theta_2\cdot\hat{P}\frac{e_1e_2}{4\pi w|x_\perp|}\ . \tag{5.14}$$

This choice for our model timelike vector interaction corresponds to $\mathcal{G}^{(1)}=L^{(1)}=0$ and $J^{(1)}=-\alpha/wr$, where $\alpha=-e_1e_2/4\pi$. Again we can set $w=M$ so that Eq. (3.48b) implies $\mathcal{V}=-\alpha/r$. In terms of the dimensionless Coulomb variable $\mathbf{x}=\epsilon_w\alpha\mathbf{r}$, Eq. (5.13) becomes (in c.m.)

$$\left\{\left[-\frac{1}{x}\frac{d^2}{dx^2}x+\frac{\mathbf{L}^2}{x^2}-\frac{2}{x}\right]+\alpha^2\left[-\frac{1}{x^2}+\left[\frac{\mu}{M}-\frac{1}{2}\right]\frac{1}{x^2}\frac{d}{dx}+\frac{1}{4x^3}\mathbf{L}\cdot(\boldsymbol{\sigma}_1+\boldsymbol{\sigma}_2)+\frac{1}{x^3}\mathbf{L}\cdot(\boldsymbol{\sigma}_1-\boldsymbol{\sigma}_2)\frac{m_2-m_1}{4M}\right]\right\}\psi_1=-\lambda^2\psi_1\ , \tag{5.15}$$

in which $\lambda^2=-b^2(w)/(\epsilon_w\alpha)^2$.

In all three cases we have, to the lowest order, $-\lambda^2=-1/n^2$ with the unperturbed wave function given by

$$\psi_{nlsjm}=R_{nl}\mathcal{Y}_{lsjm}\ , \tag{5.16}$$

in which

$$R_{nl}=\frac{u_{nl}}{x}=\left[\left[\frac{2}{n}\right]^3\frac{(n-l-1)!}{2n(n+l)!}\right]^{1/2}\times e^{-x/n}\left[\frac{2x}{n}\right]^l L_{n-l-1}^{2l+1}\left[\frac{2x}{n}\right] \tag{5.17}$$

and $\mathcal{Y}_{lsjm}$ is the total angular momentum eigenfunction. The quantum numbers *jlsm* given here refer to those of the upper-upper component of the wave function. In general, j, m, and parity are the only good quantum numbers for the wave function as a whole. (For the equal mass case, charge parity is also a valid quantum number.) The α^2 terms will be treated as first-order perturbations; they eliminate some of the $2n^2$-fold degeneracy of the unperturbed state. For the $n=1$ states, the twofold degeneracy between the $^1S_0$ and $^3S_1$ states is removed only by the electromagnetic interaction since the scalar and timelike vector interactions do not have any spin-spin terms to this order. We note also that the tensor term does not produce any first-order shift between singlet and triplet ground states, since $l=0$.

For the $n=2$ level, every α^2 term contributes to the removal of the degeneracy between the $l=0$ and $l=1$ states ($^1S_0$, $^3S_1$, $^1P_1$, $^3P_0$, $^3P_1$, $^3P_2$). Again, only the spin-spin term in the electromagnetic interaction can remove the spin degeneracy in the $l=0$ states to this order. For the $l=1$ states, the spin-orbit interaction $[\mathbf{L}\cdot(\boldsymbol{\sigma}_1+\boldsymbol{\sigma}_2)]$ and also the tensor term produce a fourfold splitting. Furthermore, for unequal masses the spin degeneracy is removed between the $^1P_1$ and $^3P_1$ states by a diagonalization of the spin-orbit difference term $[\mathbf{L}\cdot(\boldsymbol{\sigma}_1-\boldsymbol{\sigma}_2)]$. This spin-mixing term is crucial in merging the four P states into two P states in the limit that one of the particles becomes very heavy (see spectral results below). The tensor term does not mix spin, but mixes the $l=0$ and the $l=2$ states. However, as in the case of the ground state, this mixing will not produce any first-order perturbative shifts in the spectrum, since the lowest $l=2$ state has $n=3$ and is not degenerate with the $n=2$ (or 1), $l=0$ state. The same comment applies to the $l=1$ and $l=3$ mixing.

For the levels with $n\geq 3$ one might expect that the tensor force would provide an additional first-order splitting beyond that appearing for the $n=2$ level. However, there is no such additional splitting from the l mixing since the radial matrix element $\langle nl|1/x^3|nl'\rangle$ vanishes for $|l-l'|=2$ (see Appendix C of Ref. [46]).

B. General spectra

The results of the perturbative calculations are summarized below. For details see Appendix C of Ref. [46].

(1) For the electromagnetic interactions, the binding energy through order α^4 is

$$\epsilon_B \equiv w-M=-\frac{\mu\alpha^2}{2n^2}+\frac{\mu\alpha^4}{2n^3}\eta+\frac{\mu\alpha^4}{8n^4}\left[3-\frac{\mu}{M}\right], \tag{5.18}$$

in which

$$\eta=-\frac{2}{2l+1}+\delta_{l0}+\frac{[j(j+1)-l(l+1)-2](\mu/M+\frac{1}{2})+\kappa_{lj}\mu/M}{(2l+1)l(l+1)}(1-\delta_{l0})(1-\delta_{s0})+\frac{8\mu}{3M}\delta_{l0}[s(s+1)-\tfrac{3}{2}], \tag{5.19}$$

with

$$\kappa_{lj}\equiv-\frac{2l}{2l+3}\delta_{j\,l+1}+2\delta_{jl}-\frac{2(l+1)}{(2l-1)}\delta_{j\,l-1}, \tag{5.20}$$

for all states except $j=l\geq 1$. For the $j=l\geq 1$ states, the spin mixing term produces the split spectrum

$$\epsilon_B=w-M=-\frac{\mu\alpha^2}{2n^2}+\frac{\mu\alpha^4}{2n^3}\eta_{\pm}+\frac{\mu\alpha^4}{8n^4}\left[3-\frac{\mu}{M}\right],\quad j=l\geq 1, \tag{5.21}$$

in which

$$\eta_{\pm}=a+c\pm\sqrt{(a-c)^2+4b^2}, \tag{5.22}$$

with

$$a=-\frac{2}{2l+1}, \tag{5.23}$$

$$b=\frac{1}{(2l+1)\sqrt{l(l+1)}}\frac{m_2-m_1}{M}, \tag{5.24}$$

and

$$c=-\frac{2}{2l+1}-\frac{1}{(2l+1)l(l+1)}. \tag{5.25}$$

These are the standard results [50] of Ref. [31].

(2) For the scalar interaction, the binding energy through order α^4 is

$$\epsilon_B\equiv w-M=-\frac{\mu\alpha^2}{2n^2}+\frac{\mu\alpha^4}{2n^3}\eta+\frac{\mu\alpha^4}{8n^4}\left[-1-\frac{\mu}{M}\right], \tag{5.26}$$

in which

$$\eta=\frac{2}{2l+1}-\delta_{l0}\left[1-\frac{2\mu}{M}\right]+\frac{[j(j+1)-l(l+1)-2](\mu/M-\frac{1}{2})}{(2l+1)l(l+1)}\times(1-\delta_{l0})(1-\delta_{s0}), \tag{5.27}$$

for all states except the $j=l\geq 1$. For the $j=l\geq 1$ states, the spin mixing term produces the split spectrum

$$\epsilon_B=w-M=-\frac{\mu\alpha^2}{2n^2}+\frac{\mu\alpha^4}{2n^3}\eta_{\pm}+\frac{\mu\alpha^4}{8n^4}\left[-1-\frac{\mu}{M}\right],\quad j=l\geq 1, \tag{5.28}$$

in which

$$\eta_{\pm}=a+c\pm\sqrt{(a-c)^2+4b^2}, \tag{5.29}$$

$$a=\frac{2}{2l+1}, \tag{5.30}$$

$$b=-\frac{1}{(2l+1)\sqrt{l(l+1)}}\frac{m_2-m_1}{M}, \tag{5.31}$$

and

$$c=\frac{2}{2l+1}+\frac{1}{(2l+1)l(l+1)}\left[1-\frac{2\mu}{M}\right]. \tag{5.32}$$

This is the spectrum that would come from the Breit equation for spinless photon exchange [51].

(3) For the timelike vector interaction, the binding energy through order α^4 is

$$\epsilon_B\equiv w-M=-\frac{\mu\alpha^2}{2n^2}+\frac{\mu\alpha^4}{2n^3}\eta+\frac{\mu\alpha^4}{8n^4}\left[3-\frac{\mu}{M}\right], \tag{5.33}$$

in which

$$\eta=-\frac{2}{2l+1}+\delta_{l0}\left[1-\frac{2\mu}{M}\right]+\frac{[-l(l+1)-2](\mu/M-\frac{1}{2})}{(2l+1)l(l+1)}\times(1-\delta_{l0})(1-\delta_{s0}), \tag{5.34}$$

for all states except the $j=l\geq 1$. For the $j=l\geq 1$ states, the spin mixing term produces the split spectrum

$$\epsilon_B = w - M = -\frac{\mu\alpha^2}{2n^2} + \frac{\mu\alpha^4}{2n^3}\eta_\pm + \frac{\mu\alpha^4}{8n^4}\left[3 - \frac{\mu}{M}\right], \quad j = l \geq 1 , \tag{5.35}$$

in which

$$\eta_\pm = a + c \pm \sqrt{(a-c)^2 + 4b^2} , \tag{5.36}$$

$$a = -\frac{2}{2l+1} , \tag{5.37}$$

$$b = \frac{1}{(2l+1)\sqrt{l(l+1)}} \frac{m_2 - m_1}{M} , \tag{5.38}$$

and

$$c = -\frac{2}{2l+1} - \frac{1}{(2l+1)l(l+1)}\left[1 - \frac{2\mu}{M}\right] . \tag{5.39}$$

These spectral results agree with those of QED to lowest order if the effects of the transverse photon are omitted.

C. One-body limit

In the limit $m_2 \to \infty$, all these results should reduce to the corresponding one-body Dirac spectra. In that limit, the electromagnetic and timelike vector results reduce to the common form (where $m \equiv m_1$)

$$\epsilon_B = -\frac{m\alpha^2}{2n^2} + \frac{m\alpha^4}{2n^3}\eta + \frac{3m\alpha^4}{8n^4} , \tag{5.40}$$

in which

$$\eta = -\frac{2}{2l+1} + \delta_{l0} + \frac{j(j+1) - l(l+1) - 2}{2(2l+1)l(l+1)}(1 - \delta_{l0}) , \tag{5.41}$$

for $j = 0$ and $j = l \pm 1$ states. The binding energies of the $j = l \geq 1$ states, split by the spin mixing term, reduce as $m_2 \to \infty$ to

$$\epsilon_B = -\frac{m\alpha^2}{2n^2} + \frac{m\alpha^4}{2n^3}\eta_\pm + \frac{3m\alpha^4}{8n^4} \quad \text{for } j = l \geq 1 , \tag{5.42}$$

in which

$$\eta_\pm = a + c \pm \sqrt{(a-c)^2 + 4b^2} , \tag{5.43}$$

$$a = -\frac{2}{2l+1} , \tag{5.44}$$

$$b = \frac{1}{(2l+1)\sqrt{l(l+1)}} , \tag{5.45}$$

and

$$c = -\frac{2}{2l+1} - \frac{1}{(2l+1)l(l+1)} . \tag{5.46}$$

For the singlet and triplet $l = 0$ states, the binding energies go into a single expression given by

$$\epsilon_B = -\frac{m\alpha^2}{2n^2} - \frac{m\alpha^4}{n^3}\left[\frac{1}{2} - \frac{3}{8n}\right] \quad \text{for } l = 0 . \tag{5.47}$$

The binding energies of the $l = j \pm 1 \geq 1$ states become

$$\epsilon_B = -\frac{m\alpha^2}{2n^2} - \frac{m\alpha^4}{n^3}\left[\frac{1}{2l} - \frac{3}{8n}\right] \quad \text{for } l = j+1 , \tag{5.48}$$

and

$$\epsilon_B = -\frac{m\alpha^2}{2n^2} - \frac{m\alpha^4}{n^3}\left[\frac{1}{2l+2} - \frac{3}{8n}\right] \quad \text{for } l = j-1 , \tag{5.49}$$

respectively, and coincide with the spin-mixed $l = j \geq 1$ energies.

The exact one-body Dirac spectrum for hydrogen with an infinitely heavy pointlike proton, when expanded out through terms of order α^4, is

$$\epsilon_B = -\frac{m\alpha^2}{2n^2} - \frac{m\alpha^4}{n^3}\left[\frac{1}{2|k|} - \frac{3}{8n}\right] , \tag{5.50}$$

in which $k = l+1$ for $j = l + \frac{1}{2}$ and $k = -l$ for $j = l - \frac{1}{2}$, with l the angular momentum of the large-component wave function. In this case Eqs. (5.40)–(5.49), which are the two-body constraint results in the limit $m_2 \to \infty$, produce exactly this one-body result.

For scalar interactions in the static limit, our perturbative treatment gives

$$\epsilon_B = -\frac{m\alpha^2}{2n^2} + \frac{m\alpha^4}{2n^3}\eta - \frac{m\alpha^4}{8n^4} , \tag{5.51}$$

in which

$$\eta = \frac{2}{2l+1} - \delta_{l0} - \frac{j(j+1) - l(l+1) - 2}{2(2l+1)l(l+1)}(1 - \delta_{l0}) , \tag{5.52}$$

for $j = 0$ and $j = l \pm 1$ states. The binding energies of the $j = l \geq 1$ states, which are split by the spin mixing term, are

$$\epsilon_B = -\frac{m\alpha^2}{2n^2} + \frac{m\alpha^4}{2n^3}\eta_\pm - \frac{m\alpha^4}{8n^4}, \quad \text{for } j = l \geq 1 , \tag{5.53}$$

in which

$$\eta_\pm = a + c \pm \sqrt{(a-c)^2 + 4b^2} , \tag{5.54}$$

$$a = \frac{2}{2l+1} , \tag{5.55}$$

$$b = -\frac{1}{(2l+1)\sqrt{l(l+1)}} , \tag{5.56}$$

and

$$c = \frac{2}{2l+1} + \frac{1}{(2l+1)l(l+1)} . \tag{5.57}$$

In this case, the binding energies for the singlet and triplet $l = 0$ states become

$$\epsilon_B = -\frac{m\alpha^2}{2n^2} - \frac{m\alpha^4}{n^3}\left[-\frac{1}{2} + \frac{1}{8n}\right], \quad \text{for } l = 0 , \tag{5.58}$$

while the binding energies for the $l = j \pm 1 \geq 1$ states become

$$\epsilon_B=-\frac{m\alpha^2}{2n^2}-\frac{m\alpha^4}{n^3}\left[-\frac{1}{2l}+\frac{1}{8n}\right], \quad \text{for } l=j+1 , \tag{5.59}$$

$$\epsilon_B=-\frac{m\alpha^2}{2n^2}-\frac{m\alpha^4}{n^3}\left[-\frac{1}{2l+2}+\frac{1}{8n}\right], \quad \text{for } l=j-1\geq 1 . \tag{5.60}$$

These expressions coincide with those for the spin-mixed $l=j\geq 1$ states. The exact one-body Dirac spectrum for "hydrogen" with an infinitely heavy point "proton" bound by only a scalar interaction, when expanded out through terms of order α^4, is

$$\epsilon_B=-\frac{m\alpha^2}{2n^2}-\frac{m\alpha^4}{n^3}\left[-\frac{1}{2|k|}+\frac{1}{8n}\right], \tag{5.61}$$

in which $k=l+1$ for $j=l+\frac{1}{2}$ and $k=-l$ for $j=l-\frac{1}{2}$. In the case of scalar interaction, in the limit $m_2\rightarrow\infty$, the two-body constraint results (5.52)–(5.60) produce exactly this one-body result.

D. Summary

In this section we have used a perturbative treatment of the Schrödinger form of the two-body Dirac equations to derive the energy spectra through order α^4 for relativistic two-body fermion-antifermion bound states in which the two masses are not necessarily the same, for electromagnetic, scalar, or timelike interactions arising from the corresponding Born amplitudes. Historically, the electromagnetic result was first derived with the use of the Breit equation. We have seen in this section that our perturbative treatment of the weak potential form of our Eq. (4.14a) yields results for QED at this order that agree with the standard results given in Ref. [31]. In the next section we show that a nonperturbative treatment of the general (unapproximated) equations leads to the same perturbative spectral results within an error on the order of $\mu 10^{-4}\alpha^4$, $\mu 10^{-5}\alpha^4$ or, roughly, of order $\mu\alpha^6$. Thus, the results of Sec. V in combination with those of VI will show the agreement of our nonperturbative treatment of Eq. (4.14) with the standard field theoretic results through this order.

VI. NONPERTURBATIVE NUMERICAL SOLUTION OF THE TWO-BODY DIRAC EQUATIONS

We obtain the radial forms of the coupled constraint equations (4.14) needed for our numerical solution for the general fermion-antifermion system by using standard matrix elements of spin-dependent operators (see Appendix D of Ref. [46]). We take the general wave function to be of the form

$$\psi_{ijm}=\sum_{l,s} c_{ils}R_{ilsj}\mathcal{Y}_{lsjm}, \quad i=1,2,3,4 \tag{6.1}$$

in which $R_{ilsj}=u_{ilsj}/r$ is the associated radial wave function and $\mathcal{Y}_{lsjm}$ is the total angular momentum eigenfunction. The resultant Schrödinger-like equation (4.14a) for the singlet states ($j=l$, $s=0$) u_{1j0j} which couples this upper-upper component to u_{1j1j} and u_{4j0j} is given by

$$\left\{-\frac{d^2}{dr^2}+\frac{j(j+1)}{r^2}+2m_w S+S^2+2\epsilon_w\mathcal{A}-\mathcal{A}^2+2\epsilon_w\mathcal{V}-\mathcal{V}^2\right.$$
$$\left.+\ln'(\chi_1\chi_2)\frac{d}{dr}-\frac{3}{2}\partial^2\mathcal{G}+\frac{9}{4}(\mathcal{G}')^2+\frac{3}{2}\ln'(\chi_1\chi_2)\mathcal{G}'+\frac{1}{4}(J'-L')^2-\frac{\ln'(\chi_1\chi_2)}{r}\right\}u_{1j0j}$$
$$-\frac{\ln'(\chi_1/\chi_2)}{r}\sqrt{j(j+1)}u_{1j1j}+\left[-\frac{1}{2}\ln'(\chi_1\chi_2)(J-L)'-\frac{3}{2}\mathcal{G}'(J-L)'+\frac{1}{2}\partial^2(J-L)\right]u_{4j0j}=b^2(w)u_{1j0j} . \tag{6.2a}$$

The corresponding equation (4.14b) for the lower-lower component u_{4j0j} which couples it to u_{4j1j} and u_{1j0j} takes the form

$$\left\{-\frac{d^2}{dr^2}+\frac{j(j+1)}{r^2}+2m_w S+S^2+2\epsilon_w\mathcal{A}-\mathcal{A}^2+2\epsilon_w\mathcal{V}-\mathcal{V}^2\right.$$
$$\left.+\ln'(\bar{\chi}_1\bar{\chi}_2)\frac{d}{dr}-\frac{3}{2}\partial^2\mathcal{G}+\frac{9}{4}(\mathcal{G}')^2+\frac{3}{2}\ln'(\bar{\chi}_1\bar{\chi}_2)\mathcal{G}'+\frac{1}{4}(J'-L')^2-\frac{\ln'(\bar{\chi}_1\bar{\chi}_2)}{r}\right\}u_{4j0j}$$
$$-\frac{\ln'(\bar{\chi}_1/\bar{\chi}_2)}{r}\sqrt{j(j+1)}u_{4j1j}+\left[-\frac{1}{2}\ln'(\bar{\chi}_1\bar{\chi}_2)(J-L)'-\frac{3}{2}\mathcal{G}'(J-L)'+\frac{1}{2}\partial^2(J-L)\right]u_{1j0j}=b^2(w)u_{4j0j} . \tag{6.2b}$$

For $l=0$ states or equal mass systems, these equations decouple from those for u_{1j1j} and u_{4j1j} since $\sqrt{j(j+1)}\ln(\chi_1/\chi_2)=\sqrt{j(j+1)}\ln(\bar{\chi}_1/\bar{\chi}_2)=0$. However, for the general unequal-mass case, these equations are coupled to those for the $j=l$, $s=1$ components u_{1j1j} and u_{4j1j}. For those triplet states the coupled Schrödinger-like equations are

$$\left\{-\frac{d^2}{dr^2}+\frac{j(j+1)}{r^2}+2m_w S+S^2+2\epsilon_w\mathcal{A}-\mathcal{A}^2+2\epsilon_w\mathcal{V}-\mathcal{V}^2+\ln'\chi_1\chi_2\frac{d}{dr}-\frac{1}{2}\partial^2\mathcal{G}+\frac{1}{4}(\mathcal{G}')^2\right.$$
$$\left.+\frac{1}{2}\ln'(\chi_1\chi_2)\mathcal{G}'+\frac{1}{4}(J'-L')^2+\frac{\mathcal{G}'}{r}\right\}u_{1j1j}-\frac{\ln'(\chi_1/\chi_2)}{r}\sqrt{j(j+1)}u_{1j0j}$$
$$+\left\{+\frac{1}{2}\ln'(\chi_1\chi_2)(J-L)'+\frac{1}{2}\mathcal{G}'(J-L)'-\frac{1}{2}\partial^2(J-L)+\frac{(J-L)'}{r}\right\}u_{4j1j}=b^2(w)u_{1j1j} \quad (6.2c)$$

and

$$\left\{-\frac{d^2}{dr^2}+\frac{j(j+1)}{r^2}+2m_w S+S^2+2\epsilon_w\mathcal{A}-\mathcal{A}^2+2\epsilon_w\mathcal{V}-\mathcal{V}^2+\ln'\bar{\chi}_1\bar{\chi}_2\frac{d}{dr}-\frac{1}{2}\partial^2\mathcal{G}+\frac{1}{4}(\mathcal{G}')^2\right.$$
$$\left.+\frac{1}{2}\ln'(\bar{\chi}_1\bar{\chi}_2)\mathcal{G}'+\frac{1}{4}(J'-L)^2+\frac{\mathcal{G}'}{r}\right\}u_{4j1j}-\frac{\ln'(\bar{\chi}_1/\bar{\chi}_2)}{r}\sqrt{j(j+1)}u_{4j0j}$$
$$+\left\{+\frac{1}{2}\ln'(\bar{\chi}_1\bar{\chi}_2)(J-L)'+\frac{1}{2}\mathcal{G}'(J-L)'-\frac{1}{2}\partial^2(J-L)+\frac{(J-L)'}{r}\right\}u_{1j1j}=b^2(w)u_{4j1j}\ . \quad (6.2d)$$

Next we write out the four coupled equations for the two triplet states $j=l\pm1$. Equation (4.14a) for the triplet states ($s=1$, $l=j-1$) u_{1j-11j}, which couples this upper-upper component wave function to u_{1j+11j}, u_{4j+11j}, and u_{4j-11j}, becomes (note that the subscripts on u are in the order i, l, s, j)

$$\left\{-\frac{d^2}{dr^2}+\frac{j(j-1)}{r^2}+2m_w S+S^2+2\epsilon_w\mathcal{A}-\mathcal{A}^2+2\epsilon_w\mathcal{V}-\mathcal{V}^2+\ln'(\chi_1\chi_2)\frac{d}{dr}-\frac{1}{2(2j+1)}\partial^2\mathcal{G}+\frac{1}{4}(\mathcal{G}')^2\right.$$
$$\left.+\frac{1}{2(2j+1)}\ln'(\chi_1\chi_2)\mathcal{G}'+\frac{1}{4}(J'-L')^2-j\frac{\ln'(\chi_1\chi_2)}{r}-\frac{(j-1)\mathcal{G}'}{(2j+1)r}\right\}u_{1j-11j}$$
$$+\frac{\sqrt{j(j+1)}}{2j+1}\left[-\left[\mathcal{G}''-\frac{\mathcal{G}'}{r}\right]+\ln'(\chi_1\chi_2)\mathcal{G}'\right]u_{1j+11j}$$
$$+\left\{\left[\frac{1}{2j+1}\ln'(\chi_1\chi_2)+\mathcal{G}'\right]\frac{(J-L)'}{2}-\frac{1}{2(2j+1)}\partial^2(J-L)-\frac{j-1}{2j+1}\frac{(J-L)'}{r}\right\}u_{4j-11j}$$
$$+\frac{\sqrt{j(j+1)}}{2j+1}\left[\ln'(\chi_1\chi_2)(J-L)'-\left[(J-L)''-\frac{(J-L)'}{r}\right]\right]u_{4j+11j}=b^2(w)u_{1j-11j}\ . \quad (6.3a)$$

The corresponding equation (4.14b) for the lower-lower component becomes

$$\left\{-\frac{d^2}{dr^2}+\frac{j(j-1)}{r^2}+2m_w S+S^2+2\epsilon_w\mathcal{A}-\mathcal{A}^2+2\epsilon_w\mathcal{V}-\mathcal{V}^2+\ln'(\bar{\chi}_1\bar{\chi}_2)\frac{d}{dr}-\frac{1}{2(2j+1)}\partial^2\mathcal{G}+\frac{1}{4}(\mathcal{G}')^2\right.$$
$$\left.+\frac{1}{2(2j+1)}\ln'(\bar{\chi}_1\bar{\chi}_2)\mathcal{G}'+\frac{1}{4}(J'-L')^2-j\frac{\ln'(\bar{\chi}_1\bar{\chi}_2)}{r}-\frac{(j-1)\mathcal{G}'}{(2j+1)r}\right\}u_{4j-11j}$$
$$+\frac{\sqrt{j(j+1)}}{2j+1}\left[-\left[\mathcal{G}''-\frac{\mathcal{G}'}{r}\right]+\ln'(\bar{\chi}_1\bar{\chi}_2)\mathcal{G}'\right]u_{4j+11j}$$
$$+\left\{\left[\frac{1}{2j+1}\ln'(\bar{\chi}_1\bar{\chi}_2)+\mathcal{G}'\right]\frac{(J-L)'}{2}-\frac{1}{2(2j+1)}\partial^2(J-L)-\frac{j-1}{2j+1}\frac{(J-L)'}{r}\right\}u_{1j-11j}$$
$$+\frac{\sqrt{j(j+1)}}{2j+1}\left[\ln'(\bar{\chi}_1\bar{\chi}_2)(J-L)'-\left[(J-L)''-\frac{(J-L)'}{r}\right]\right]u_{1j+11j}=b^2(w)u_{4j-11j}\ . \quad (6.3b)$$

Equations (6.3) are coupled to the corresponding two equations for the triplet $s=1$, $l=j+1$ states given by

$$\left\{-\frac{d^2}{dr^2}+\frac{(j+1)(j+2)}{r^2}+2m_w S+S^2+2\epsilon_w\mathcal{A}-\mathcal{A}^2+2\epsilon_w\mathcal{V}-\mathcal{V}^2+\ln'(\chi_1\chi_2)\frac{d}{dr}+\frac{1}{2(2j+1)}\partial^2\mathcal{G}+\frac{1}{4}(\mathcal{G}')^2\right.$$
$$\left.-\frac{1}{2(2j+1)}\ln'(\chi_1\chi_2)\mathcal{G}'+\frac{1}{4}(J'-L')^2+(j+1)\frac{\ln'(\chi_1\chi_2)}{r}-\frac{(j+2)\mathcal{G}'}{(2j+1)r}\right\}u_{1j+11j}$$
$$+\frac{\sqrt{j(j+1)}}{2j+1}\left[-\left[\mathcal{G}''-\frac{\mathcal{G}'}{r}\right]+\ln'(\chi_1\chi_2)\mathcal{G}'\right]u_{1j-11j}$$
$$+\left\{\left[-\frac{1}{2j+1}\ln'(\chi_1\chi_2)+\mathcal{G}'\right]\frac{(J-L)'}{2}+\frac{1}{2(2j+1)}\partial^2(J-L)-\frac{j+2}{2j+1}\frac{(J-L)'}{r}\right\}u_{4j+11j}$$
$$+\frac{\sqrt{j(j+1)}}{2j+1}\left[\ln'(\chi_1\chi_2)(J-L)'-\left[(J-L)''-\frac{(J-L)'}{r}\right]\right]u_{4j-11j}=b^2(w)u_{1j+11j} \tag{6.3c}$$

and

$$\left\{-\frac{d^2}{dr^2}+\frac{(j+1)(j+2)}{r^2}+2m_w S+S^2+2\epsilon_w\mathcal{A}-\mathcal{A}^2+2\epsilon_w\mathcal{V}-\mathcal{V}^2+\ln'(\bar\chi_1\bar\chi_2)\frac{d}{dr}+\frac{1}{2(2j+1)}\partial^2\mathcal{G}+\frac{1}{4}(\mathcal{G}')^2\right.$$
$$\left.-\frac{1}{2(2j+1)}\ln'(\bar\chi_1\bar\chi_2)\mathcal{G}'+\frac{1}{4}(J'-L')^2+(j+1)\frac{\ln'(\bar\chi_1\bar\chi_2)}{r}-\frac{(j+2)\mathcal{G}'}{(2j+1)r}\right\}u_{4j+11j}$$
$$+\frac{\sqrt{j(j+1)}}{2j+1}\left[-\left[\mathcal{G}''-\frac{\mathcal{G}'}{r}\right]+\ln'(\bar\chi_1\bar\chi_2)\mathcal{G}'\right]u_{4j-11j}$$
$$+\left\{\left[-\frac{1}{(2j+1)}\ln'(\bar\chi_1\bar\chi_2)+\mathcal{G}'\right]\frac{(J-L)'}{2}+\frac{1}{2(2j+1)}\partial^2(J-L)-\frac{j+2}{2j+1}\frac{(J-L)'}{r}\right\}u_{1j+11j}$$
$$+\frac{\sqrt{j(j+1)}}{2j+1}\left[\ln'(\bar\chi_1\bar\chi_2)(J-L)'-\left[(J-L)''-\frac{(J-L)'}{r}\right]\right]u_{1j-11j}=b^2(w)u_{4j+11j}\ . \tag{6.3d}$$

For the $^3P_0$ states there are only two coupled equations:

$$\left\{-\frac{d^2}{dr^2}+\frac{2}{r^2}+2m_w S+S^2+2\epsilon_w\mathcal{A}-\mathcal{A}^2+2\epsilon_w\mathcal{V}-\mathcal{V}^2+\ln'(\chi_1\chi_2)\frac{d}{dr}+\frac{1}{2}\partial^2\mathcal{G}\right.$$
$$\left.+\frac{1}{4}(\mathcal{G}')^2-\frac{1}{2}\ln'(\chi_1\chi_2)\mathcal{G}'+\frac{1}{4}(J'-L')^2+\frac{\ln'(\chi_1\chi_2)}{r}-\frac{2\mathcal{G}'}{r}\right\}u_{1110}$$
$$-\left\{\frac{1}{2}\ln'(\chi_1\chi_2)(J-L)'+\frac{1}{2}\mathcal{G}'(J-L)'+\frac{1}{2}\partial^2(J-L)-2\frac{(J-L)'}{r}\right\}u_{4110}=b^2(w)u_{1110} \tag{6.4a}$$

and

$$\left\{-\frac{d^2}{dr^2}+\frac{2}{r^2}+2m_w S+S^2+2\epsilon_w\mathcal{A}-\mathcal{A}^2+2\epsilon_w\mathcal{V}-\mathcal{V}^2+\ln'(\bar\chi_1\bar\chi_2)\frac{d}{dr}+\frac{1}{2}\partial^2\mathcal{G}\right.$$
$$\left.+\frac{1}{4}(\mathcal{G}')^2-\frac{1}{2}\ln'(\bar\chi_1\bar\chi_2)\mathcal{G}'+\frac{1}{4}(J'-L')^2+\frac{\ln'(\bar\chi_1\bar\chi_2)}{r}-\frac{2\mathcal{G}'}{r}\right\}u_{4110}$$
$$-\left\{\frac{1}{2}\ln'(\bar\chi_1\bar\chi_2)(J-L)'+\frac{1}{2}\mathcal{G}'(J-L)'+\frac{1}{2}\partial^2(J-L)-2\frac{(J-L)'}{r}\right\}u_{1110}=b^2(w)u_{4110}\ . \tag{6.4b}$$

Note that in each of the equations (6.2)–(6.4), the quasipotential couples the upper-upper component to the lower-lower component. In Ref. [46], we bring these coupled equations to a compact form suitable for numerical solution, emphasizing the importance of scale transformations in the dependent and independent variables.

In the tables below we give the numerical results for electromagnetic interactions (QED) obtained from (6.2)–(6.4) with $\mathcal{A}=-\alpha/r$, and $S=\mathcal{V}=0$ so that $\mathcal{G}=-J=-\frac{1}{2}\ln[1+2\alpha/(wr)]$, $L=0$ where α is

1/137.035 989 5. We present results for numerical calculations for muonium ($e^-\mu^+$) and positronium (e^-e^+). We do not, however, include the effects of the annihilation diagram for the e^-e^+ system in the tests presented below. A Fierz transformation of the annihilation diagram to quasipotential form would include pseudoscalar and pseudovector couplings in addition to scalar and vector couplings which would require a generalization of the interactions contained in the two-body Dirac equations considered in this paper [52].

In Table I we present the binding energies in electron volts for the $n=1,2,3$ levels for the e^+e^- system [53]. The quantum numbers are those of the upper-upper component of the system wave function ψ_{ijm}. In the first four columns are the values of the quantum numbers l, s, j, n. In the fifth column is the number of coupled equations N_c that are included in the numerical test. When $N_c=1$, we use just one equation, the one for the upper-upper component, with the couplings to the lower-lower component and l mixing neglected. The case $N_c=2$ corresponds to the fully coupled system (upper-upper and lower-lower) of equations for the singlet states, the $^3P_0$ states, and the triplet $j=l$ states. For the other triplet states in the $N_c=2$ case we neglect either the coupling due to angular momentum mixing or that due to the coupling between ψ_1 and ψ_4. To distinguish between these $N_c=2$ cases for the nonperturbative (numerical) test, we let M stand for the neglect of the $l=j+1$, $j-1$ coupling while C stands for the neglect of the coupling between the upper-upper and lower-lower components. The $N_c=4$ case corresponds to the fully coupled triplet states for $l=j+1$, $j-1$, $j\neq 0$ in which couplings between the upper-upper and lower-lower component as well as the l mixing are not neglected. In the next column are the energy levels in units of eV obtained from the perturbative expansions given in Sec. V by Eqs. (5.18)–(5.25) [which involve only a single, uncoupled equation (5.9)]. In the next-to-last column are the nonperturbative numerical results from Eqs. (6.2)–(6.4), the most important results of the paper. The last column gives the differences between the perturbative and numerical calculations divided by $\mu\alpha^4/n^3$. Since we are not including radiative corrections, these differences should be on the order of $\mu\alpha^6$ (as opposed to radiative corrections on the order of $\mu\alpha^5$ or $\mu\alpha^5\ln\alpha$) when all of the couplings in Eqs. (6.2)–(6.4) are included. Thus the entries in this difference column for the full coupled equations (two or four depending on the quantum numbers) should be on the order of α^2 or 10^{-4}, 10^{-5}.

Table II gives the binding energies in electron volts for $n=1,2,3$ levels for muonium ($e^-\mu^+$). The columns are labeled as before except that M stands for the neglect of l mixing (for $l=j+1$, $j-1$) or s mixing ($l=j\geq 1$) in the triplet $N_c=2$ cases. Note that for muonium the $N_c=2$ case corresponds to the fully coupled system only for the $j=0$ states (singlet or triplet). For all other states the fully coupled constraint equations correspond to $N_c=4$ with the combined coupling for upper-upper and lower-lower components, and for l mixing (for $l=j+1$, $j-1$) or s mixing ($l=j$).

For the cases of scalar and purely time-like interactions, we present just the ground-state results for equal masses. In Table III we give the numerical results for scalar interactions obtained from (6.2)–(6.4) with $S=-\alpha/r$, and $\mathcal{V}=\mathcal{A}=0$ so that $\mathcal{G}=-J=0$ with L given by (3.48a). The results in the perturbative column are the energy levels obtained from the perturbative expansions given in Sec. V by Eqs. (5.26)–(5.32) [which employ only a single, uncoupled equation (5.12)].

In Table IV we give the numerical results for timelike interactions obtained from (6.2)–(6.4) with $\mathcal{V}=-\alpha/r$, and $S=\mathcal{A}=0$ so that $\mathcal{G}=L=0$ with J given by (3.48b). The results in the perturbative column are the energy levels obtained from the perturbative expansions given in Sec. V by Eqs. (5.33)–(5.39) [which employ only a single, uncoupled equation (5.15)].

Let us examine these results more closely. Note that in all cases for given values of l, s, j, and n, the agreement between the nonperturbative numerical result for the fully coupled system of equations (highest value of N_c) and the perturbative result is excellent. Typically, the differences are on the order of $\mu\alpha^6$. As we neglect couplings in the strong potential form of Eqs. (6.2)–(6.4), this agreement is spoiled to one degree or another, except in cases when the agreement between the single uncoupled equation and the perturbative result is already excellent. Thus, when effects due to these couplings are significant, they conspire in a complex fashion to produce agreement with the perturbative result. In the case of the electromagnetic interaction for the $^1S_0$ positronium states, the coupling between the upper-upper and lower-lower wave functions is crucial in order to obtain agreement through order α^4 with the perturbatively computed spectral results (or for that matter with the exact solution obtained by other methods [16]). Without them, the error is on the order of 5% of $\mu\alpha^4$, much larger than the order $\mu\alpha^6$ error expected. The same phenomenon occurs for the $^3S_1$ states in positronium except that in this case the (off-diagonal) tensor coupling is needed in conjunction with the coupling between the upper-upper and lower-lower components. (Note that in the perturbative treatment of the equation, the latter coupling to the lower-lower component played no role whatsoever.) Neither coupling by itself is sufficient to produce a result accurate enough so that the errors are on the order of α^6. For the equal mass $l>0$ states however, it is not necessary to include the effects of either the (off diagonal) tensor coupling or that between the upper-upper and lower-lower components to obtain good agreement. The uncoupled upper-upper equation by itself is sufficient.

For muonium, the coupling between the upper-upper and lower-lower components is crucial for the $l=0$ states just as it was in the equal-mass case. Again the agreement improves by two or three orders of magnitude when the coupling to the lower-lower component is included. This improvement might at first not seem significant since the relative error starts at an already respectable $10^{-5}\sim\alpha^2$ so that further improvement to $\sim 10^{-7}$ may appear meaningless. However, a glance at the perturbative spectrum reveals that the smallest corrections are the

TABLE I. Comparison between perturbatively and numerically calculated energy levels for positronium. The first four columns list the quantum numbers l, s, j, n. In the fifth column is the number of coupled equations N_c that were used to perform the numerical test. In the next column are the energy levels (in units of eV) obtained from the perturbative expansions given in Sec. V. The last column gives the difference between the perturbative and numerical calculations divided by $\mu\alpha^4/n^3$. We use the symbol M to indicate that we are neglecting the $l=j+1$, $j-1$ coupling while we use C to indicate we are neglecting the coupling between the upper-upper and lower-lower components for the triplet equations. These symbols have the same meaning in Tables II–IV.

l	s	j	n	N_c		Perturbative	Numerical	Diff $/\frac{\mu\alpha^4}{n^3}$
0	0	0	1	1		−6.803 325 627 9	−6.803 286 157 9	5.45×10^{-2}
0	0	0	1	2		−6.803 325 627 9	−6.803 325 671 9	-6.08×10^{-5}
0	1	1	1	1		−6.802 842 613 2	−6.802 807 499 0	4.84×10^{-2}
0	1	1	1	2	*M*	−6.802 842 613 2	−8.802 808 219 5	4.75×10^{-2}
0	1	1	1	2	*C*	−6.802 842 613 2	−6.802 823 949 9	2.58×10^{-2}
0	1	1	1	4		−6.802 842 613 2	−6.802 842 663 6	-6.97×10^{-5}
0	0	0	2	1		−1.700 787 539 4	−1.700 782 606 8	5.45×10^{-2}
0	0	0	2	2		−1.700 787 539 4	−1.700 787 546 7	-8.04×10^{-5}
0	1	1	2	1		−1.700 727 162 6	−1.700 722 774 1	4.85×10^{-2}
0	1	1	2	2	*M*	−1.700 727 162 6	−1.700 722 864 2	4.75×10^{-2}
0	1	1	2	2	*C*	−1.700 727 162 6	−1.700 724 830 6	2.57×10^{-2}
0	1	1	2	4		−1.700 727 162 6	−1.700 727 170 0	-8.09×10^{-5}
1	0	1	2	1		−1.700 727 162 6	−1.700 727 163 0	-5.20×10^{-6}
1	0	1	2	2		−1.700 727 162 6	−1.700 727 163 0	-5.02×10^{-6}
1	1	0	2	1		−1.700 757 351 0	−1.700 757 353 8	-3.08×10^{-5}
1	1	0	2	2		−1.700 757 351 0	−1.700 757 356 3	-5.82×10^{-5}
1	1	1	2	1		−1.700 734 709 7	−1.700 734 710 2	-5.26×10^{-6}
1	1	1	2	2		−1.700 734 709 7	−1.700 734 710 3	-6.83×10^{-6}
1	1	2	2	1		−1.700 716 596 6	−1.700 716 596 9	-3.00×10^{-6}
1	1	2	2	2	*M*	−1.700 716 596 6	−1.700 716 596 9	-3.01×10^{-6}
1	1	2	2	2	*C*	−1.700 716 596 6	−1.700 716 596 9	-3.11×10^{-6}
1	1	2	2	4		−1.700 716 596 6	−1.700 716 596 9	-3.27×10^{-6}
0	0	0	3	1		−0.755 895 999 4	−0.755 894 539 7	5.44×10^{-2}
0	0	0	3	2		−0.755 895 999 4	−0.755 896 003 4	-1.47×10^{-4}
0	1	1	3	1		−0.755 878 110 0	−0.755 876 811 6	4.84×10^{-2}
0	1	1	3	2	*M*	−0.755 878 110 0	−0.755 876 838 3	4.74×10^{-2}
0	1	1	3	2	*C*	−0.755 878 110 0	−0.755 877 420 9	2.57×10^{-2}
0	1	1	3	4		−0.755 878 110 0	−0.755 878 114 0	-1.51×10^{-4}
1	0	1	3	1		−0.755 878 110 0	−0.755 878 111 5	-5.71×10^{-5}
1	0	1	3	2		−0.755 878 110 0	−0.755 878 111 5	-5.71×10^{-5}
1	1	0	3	1		−0.755 887 054 7	−0.755 887 056 9	-8.26×10^{-5}
1	1	0	3	2		−0.755 887 054 7	−0.755 887 057 7	-1.12×10^{-4}
1	1	1	3	1		−0.755 880 346 2	−0.755 880 347 7	-5.71×10^{-5}
1	1	1	3	2		−0.755 880 346 2	−0.755 880 347 8	-6.02×10^{-5}
1	1	2	3	1		−0.755 874 979 3	−0.755 874 980 8	-5.47×10^{-5}
1	1	2	3	2	*M*	−0.755 874 979 3	−0.755 874 980 8	-5.47×10^{-5}
1	1	2	3	2	*C*	−0.755 874 979 3	−0.755 874 980 8	-5.48×10^{-5}
1	1	2	3	4		−0.755 874 979 3	−0.755 874 980 8	-5.50×10^{-5}
2	0	2	3	1		−0.755 874 532 1	−0.755 874 532 4	-1.16×10^{-5}
2	0	2	3	2		−0.755 874 532 1	−0.755 874 532 4	-1.16×10^{-5}
2	1	1	3	1		−0.755 876 768 3	−0.755 876 768 6	-1.29×10^{-5}
2	1	1	3	2	*M*	−0.755 876 768 3	−0.755 876 768 6	-1.29×10^{-5}
2	1	1	3	2	*C*	−0.755 876 768 3	−0.755 876 768 6	-1.29×10^{-5}
2	1	1	3	4		−0.755 876 768 3	−0.755 876 768 6	-1.29×10^{-5}
2	1	2	3	1		−0.755 874 979 3	−0.755 874 979 7	-1.17×10^{-5}
2	1	2	3	2		−0.755 874 979 3	−0.755 874 979 7	-1.17×10^{-5}
2	1	3	3	1		−0.755 873 254 3	−0.755 873 254 6	-1.13×10^{-5}
2	1	3	3	2	*M*	−0.755 873 254 3	−0.755 873 254 6	-1.13×10^{-5}
2	1	3	3	2	*C*	−0.755 873 254 3	−0.755 873 254 6	-1.13×10^{-5}
2	1	3	3	4		−0.755 873 254 3	−0.755 873 254 6	-1.14×10^{-5}

TABLE II. Comparison between perturbatively and numerically calculated energy levels for muonium. The first four columns list the quantum numbers l, s, j, n. In the fifth column is the number of coupled equations N_c that were used to perform the numerical test. In the next column are the energy levels (in units of eV) obtained from the perturbative expansions given in Sec. V. The last column gives the difference between the perturbative and numerical calculations divided by $\mu\alpha^4/n^3$.

l	s	j	n	N_c		Perturbative	Numerical	Diff $/\frac{\mu\alpha^4}{n^3}$
0	0	0	1	1		−13.540 410 157 8	−13.540 410 298	8.87×10^{-5}
0	0	0	1	2		−13.540 410 157 8	−13.540 410 158 1	-2.44×10^{-7}
0	1	1	1	1		−13.540 391 738 1	−13.540 391 556 8	1.26×10^{-4}
0	1	1	1	2	*M*	−13.540 391 738 1	−13.540 391 564 6	1.20×10^{-4}
0	1	1	1	2	*C*	−13.540 391 738 1	−13.540 391 641 7	6.69×10^{-5}
0	1	1	1	4		−13.540 391 738 1	−13.540 391 737 3	5.52×10^{-7}
0	0	0	2	1		−3.385 111 916 9	−3.385 111 903 7	7.32×10^{-5}
0	0	0	2	2		−3.385 111 916 9	−3.385 111 919 7	-1.59×10^{-5}
0	1	1	2	1		−3.385 109 614 4	−3.385 109 594 5	1.11×10^{-4}
0	1	1	2	2	*M*	−3.385 109 614 4	−3.385 109 595 4	1.05×10^{-4}
0	1	1	2	2	*C*	−3.385 109 614 4	−3.385 109 605 1	5.18×10^{-5}
0	1	1	2	4		−3.385 109 614 4	−3.385 109 617 0	-1.45×10^{-5}
1	0	1	2	1		−3.385 065 318 1	−3.385 080 147 5	-8.23×10^{-2}
1	0	1	2	2	*M*	−3.385 065 318 1	−3.385 080 147 5	-8.23×10^{-2}
1	0	1	2	2	*C*	−3.385 065 318 1	−3.385 065 318 7	-3.35×10^{-6}
1	0	1	2	4		−3.385 065 318 1	−3.385 065 318 7	-3.35×10^{-6}
1	1	0	2	1		−3.385 110 765 7	−3.385 110 767 8	-1.20×10^{-5}
1	1	0	2	2		−3.385 110 765 7	−3.385 110 767 8	-1.22×10^{-5}
1	1	1	2	1		−3.385 109 997 3	−3.385 095 169 4	8.23×10^{-2}
1	1	1	2	2	*M*	−3.385 109 997 3	−3.385 095 169 4	8.23×10^{-2}
1	1	1	2	2	*C*	−3.385 109 997 3	−3.385 109 999 4	-1.15×10^{-5}
1	1	1	2	4		−3.385 109 997 3	−3.385 109 999 4	-1.16×10^{-5}
1	1	2	2	1		−3.385 065 010 2	−3.385 065 010 8	-3.32×10^{-6}
1	1	2	2	2	*M*	−3.385 065 010 2	−3.385 065 010 8	-3.32×10^{-6}
1	1	2	2	2	*C*	−3.385 065 010 2	−3.385 065 010 8	-3.32×10^{-6}
1	1	2	2	4		−3.385 065 010 2	−3.385 065 010 8	-3.32×10^{-6}
0	0	0	3	1		−1.504 488 908 9	−1.504 488 908 8	3.43×10^{-6}
0	0	0	3	2		−1.504 488 908 9	−1.504 488 913 5	-8.57×10^{-5}
0	1	1	3	1		−1.504 488 226 7	−1.504 488 224 6	4.08×10^{-5}
0	1	1	3	2	*M*	−1.504 488 226 7	−1.504 488 224 8	3.54×10^{-5}
0	1	1	3	2	*C*	−1.504 488 226 7	−1.504 488 227 7	-1.80×10^{-5}
0	1	1	3	4		−1.504 488 226 7	−1.504 488 231 2	-8.43×10^{-5}
1	0	1	3	1		−1.504 475 101 9	−1.504 479 498 6	-8.23×10^{-2}
1	0	1	3	2	*M*	−1.504 475 101 9	−1.504 479 498 6	-8.23×10^{-2}
1	0	1	3	2	*C*	−1.504 475 101 9	−1.504 475 104 8	-5.51×10^{-5}
1	0	1	3	4		−1.504 475 101 9	−1.504 475 104 8	-5.47×10^{-5}
1	1	0	3	1		−1.504 488 567 8	−1.504 488 571 3	-6.42×10^{-5}
1	1	0	3	2		−1.504 488 567 8	−1.504 488 571 3	-6.44×10^{-5}
1	1	1	3	1		−1.504 488 340 2	−1.504 483 949 5	8.22×10^{-2}
1	1	1	3	2	*M*	−1.504 488 340 2	−1.504 483 949 5	8.22×10^{-2}
1	1	1	3	2	*C*	−1.504 488 340 2	−1.504 488 343 6	-6.37×10^{-5}
1	1	1	3	4		−1.504 488 340 2	−1.504 488 343 6	-6.37×10^{-5}
1	1	2	3	1		−1.504 475 010 7	−1.504 475 013 6	-5.52×10^{-5}
1	1	2	3	2	*M*	−1.504 475 010 7	−1.504 475 013 6	-5.52×10^{-5}
1	1	2	3	2	*C*	−1.504 475 010 7	−1.504 475 013 6	-5.52×10^{-5}
1	1	2	3	4		−1.504 475 010 7	−1.504 475 013 6	-5.52×10^{-5}
2	0	2	3	1		−1.504 470 614 5	−1.504 472 374 9	-3.30×10^{-2}
2	0	2	3	2	*M*	−1.504 470 614 5	−1.504 472 374 9	-3.30×10^{-2}
2	0	2	3	2	*C*	−1.504 470 614 5	−1.504 470 615 1	-1.13×10^{-5}
2	0	2	3	4		−1.504 470 614 5	−1.504 470 615 1	-1.13×10^{-5}
2	1	1	3	1		−1.504 475 078 9	−1.504 475 079 5	-1.22×10^{-5}
2	1	1	3	2	*M*	−1.504 475 078 9	−1.504 475 079 5	-1.22×10^{-5}
2	1	1	3	2	*C*	−1.504 475 078 9	−1.504 475 079 5	-1.22×10^{-5}
2	1	1	3	4		−1.504 475 078 9	−1.504 475 079 5	-1.22×10^{-5}

TABLE II. (*Continued*).

l	s	j	n	N_c		Perturbative	Numerical	Diff $/\frac{\mu\alpha^4}{n^3}$
2	1	2	3	1		−1.504 475 024 2	−1.504 473 265 1	3.29×10^{-2}
2	1	2	3	2	M	−1.504 475 024 2	−1.504 473 265 1	3.29×10^{-2}
2	1	2	3	2	C	−1.504 475 024 2	−1.504 475 024 9	-1.21×10^{-5}
2	1	2	3	4		−1.504 475 024 2	−1.504 475 024 9	-1.21×10^{-5}
2	1	3	3	1		−1.504 470 579 3	−1.504 470 579 9	$-1..14\times10^{-5}$
2	1	3	3	2	M	−1.504 470 579 3	−1.504 470 579 9	-1.14×10^{-5}
2	1	3	3	2	C	−1.504 470 579 3	−1.504 470 579 9	-1.14×10^{-5}
2	1	3	3	4		−1.504 470 579 3	−1.504 470 579 9	-1.14×10^{-5}

recoil corrections $\sim\mu^2\alpha^4/M$. Relative to these the correction due to the coupling to the lower-lower component becomes as significant as in the equal-mass case. Note also that the spin-mixing coupling between the ^1L_l and ^3L_l states is crucial to obtain agreement between the nonperturbative and perturbative treatments of our equations. This coupling was also important to obtain agreement between our perturbative results and standard treatments. Further, in both the equal- and unequal-mass cases, we have ignored the coupling to the lower-lower component in the perturbative calculations. Its nonperturbative importance for S states thus comes as a surprise. We find that couplings to different components of the wave equation that are numerically important for the nonperturbative calculation are not important for the perturbative calculation [54].

The perturbative treatment of our equations for the scalar and purely timelike interactions generates no hyperfine splitting through order α^4. The nonperturbative numerical results are consistent with this but again only if *all* couplings are included. (The lack of hyperfine splitting holds as well for the nonperturbative treatment of the unequal-mass cases.)

Besides the coupling, what other relativistic strong-potential structures are crucial for the excellent agreement we have obtained? We have referred earlier to strong potential terms as those in Eqs. (4.14) beyond the collective minimal (Todorov) part (e.g., $+2\epsilon_w\mathcal{A}-\mathcal{A}^2$ for electromagnetic interaction). "Relativistic strong-potential structures" refers collectively to these terms, in particular to the potential energy dependences in denominators appearing in those terms of the form E_i+M_i, G^2. In the weak-potential limit [Eqs. (5.1)–(5.6)], for $\mathcal{A}=-\alpha/r$, these terms become singular potentials (ones more attractive than $-1/4r^2$ near the origin) but are themselves nonsingular in the constraint equation Eq. (4.14). Singular potentials appear in our formalism only as a result of perturbative approximation, when the strong-potential terms such as $-\ln'\chi_1/r$, $\ln'\chi_1 d/dr$, $-\partial^2\mathcal{G}$ that appear in (4.14) are treated as weak. In such a perturbative approximation, attractive potential energy terms with a radial dependence of the form $1/r^3$, $(1/r^2)d/dr$, and $\delta(\mathbf{r})$ arise as typical relativistic weak-potential limits of the relativistic strong-potential terms. Those limits can only be treated perturbatively, using well-behaved unperturbed wave functions. Otherwise a nonperturbative treatment of these singular potentials would lead to nonnormalizable singular wave functions. However, the unapproximated strong-potential terms $-\ln'(\chi_1)/r$, $\ln'(\chi_1)d/dr$, $-\partial^2\mathcal{G}$, etc. in our equation, from which the singular potentials originate, are well behaved for small r, since the logarithmic derivatives generate denominators that moderate the small-distance behavior. Thus the unapproximated terms do not lead to singular wave functions. For example, in the case of QED, consider terms with a radial dependence of the form

$$-\frac{\ln'\chi_1}{r}=\frac{\mathcal{A}'}{r}\frac{1+m_1/(wG^3)}{\epsilon_1-\mathcal{A}+m_1/G}\ . \tag{6.5}$$

For $\mathcal{A}=-\alpha/r$ and G^2 defined in (2.11c), this becomes $1/r^2$ for small r instead of $\alpha/(2m_1r^3)$ as it does in the weak-potential approximation in which the potential

TABLE III. Comparison between perturbatively and numerically calculated energy levels for scalar interactions. The first four columns list the quantum numbers l, s, j, n. In the fifth column is the number of coupled equations N_c that were used to perform the numerical test. In the next column are the energy levels (in units of eV) obtained from the perturbative expansions given in Sec. V. The last column gives the difference between the perturbative and numerical calculations divided by $\mu\alpha^4/n^3$.

l	s	j	n	N_c		Perturbative	Numerical	Diff $/\frac{\mu\alpha^4}{n^3}$
0	0	0	1	1		−6.802 419 975 3	−6.802 346 112 4	0.10
0	0	0	1	2		−6.802 419 975 3	−6.802 419 788 6	2.58×10^{-4}
0	1	1	1	1		−6.802 419 975 3	−6.802 346 112 4	0.11
0	1	1	1	2	M	−6.802 419 975 3	−6.802 353 026 4	9.24×10^{-2}
0	1	1	1	2	C	−6.802 419 975 3	−6.802 346 112 4	0.11
0	1	1	1	4		−6.802 419 975 3	−6.802 419 744 6	3.18×10^{-4}

TABLE IV. Comparison between perturbatively and numerically calculated energy levels for time-like interactions. The first four columns list the quantum numbers l, s, j, n. In the fifth column is the number of coupled equations N_c that were used to perform the numerical test. In the next column are the energy levels (in units of eV) obtained from the perturbative expansions given in section V. The last column gives the difference between the perturbative and numerical calculations divided by $\mu\alpha^4/n^3$.

l	s	j	n	N_c		Perturbative	Numerical	Diff $/\frac{\mu\alpha^4}{n^3}$
0	0	0	1	1		−6.803 144 497 3	−6.803 084 988 4	8.21×10^{-2}
0	0	0	1	2		−6.803 144 497 3	−6.803 144 305 9	2.64×10^{-4}
0	1	1	1	1		−6.803 144 497 3	−6.803 084 988 4	8.21×10^{-2}
0	1	1	1	2	M	−6.803 144 497 3	−6.803 090 700 8	7.43×10^{-2}
0	1	1	1	2	C	−6.803 144 497 3	−6.803 084 988 4	8.24×10^{-2}
0	1	1	1	4		−6.803 114 497 3	−6.803 144 288 0	2.89×10^{-4}

dependence in the denominator is left out. The $1/r^2$ behavior gives acceptable nonperturbative numerical solutions when combined with the centrifugal barrier term, whether the sign of this term is positive or negative. On the other hand the $1/r^3$ dependence would not give any convergent nonperturbative numerical solution when the coefficient is negative, as can happen for QED interactions in the $^3P_0$ case. So the nonsingular short-distance behavior [55] is crucial for every term in the quasipotential that appears in Eqs. (4.1) since using the weak-potential approximation in any one of the terms could render the equation as a whole ill defined quantum mechanically. Those terms, which include Darwin and spin-dependent and relativistic recoil terms (ones that vanish when one of the masses $\to\infty$), yield important contributions to the calculated spectra.

Just what are the distance scales at which the potentials in the various denominator terms become important? For the equal-mass case the invariant $\mathcal{A}$ becomes comparable to the electron mass and the energy terms in the denominator at distance scales of the order of the classical electron radius, well inside the Compton wavelength. Such short-distance behavior of the quasipotential is important since without this radial dependence in the denominator the correct spectral results could not be reproduced in a nonperturbative treatment. For example, if one artificially replaces $\mathcal{A}$ by a constant in the non-Coulombic part of the quasipotential at a distance r less than a Compton wavelength, then the S-state nonperturbative spectral results will no longer agree with their perturbative counterparts to the required accuracy, even though the P-state results will. Thus the minimal interaction constraint equations provide a natural cutoff mechanism that is essential for a nonperturbative treatment of the equation. An analogous effect occurs in the one-body Dirac equation [54].

VII. THEORETICAL PREDICTIONS FOR THE $^3P_0$ SCATTERING STATES

A few years ago, several groups [19–24] observed anomalous positron peaks at positron kinetic energies of 250–400 keV in heavy-ion collisions with a united-atom charge $Z > 163$ for collision energies near the Coulomb barrier. The energies of the peaks seem to be nearly independent of the projectile and target combinations. Electrons were found in coincidence with the positrons with about the same energy as the positrons at the anomalous peak [19]. These observations were interpreted as resulting from the formation of a neutral particle or composite with subsequent decay into a positron and an electron [21,56–58]. Such an intermediate state could either be the product of new (nonelectromagnetic) forces or hidden features of old (electromagnetic) forces. The authors of Refs. [56–58] attributed the anomaly to the nonelectromagnetic production and decay of a pseudoscalar axion. Other authors [59,60] proposed the participation of new phases of QED with a larger coupling constant. Not wishing to invoke new forces, Wong and Becker [18] speculated that short-distance, strong-potential, relativistic effects in QED might generate a resonant composite state of the e^+e^- system and investigated the possible origin of such resonances using an assumed electromagnetic mechanism.

Recently, measurements of Bhabha scattering [26] have failed to show the presence of such resonances with lifetimes in the range from 10^{-13}–-10^{-9} sec. That is, the Bhabha scattering results so far are consistent with the results of perturbative QED. To reconcile the electromagnetic part of these results theoretically what is needed, then, is a relativistic calculation of the phase shift at c.m. energies in the neighborhood of peaks seen in the heavy-ion collisions. Recently, Spence and Vary [28] have carried out just such a calculation using several truncated versions of the Bethe-Salpeter equation. They find a family of resonances "of zero width" in the region between 1.4 and 2.2 MeV. On the other hand we shall show that our relativistic two-body Dirac equations, despite all of their short-distance, strong-potential, relativistic structures, predict no resonances in the $^3P_0$ state in the relevant energy range—no deviation from ordinary Bhabha scattering.

We must now decide on the form of the equations that will best display the origins of the physics in the relativistic quasipotential. In principle we could use Eqs. (6.4) (as they stand) for the $^3P_0$ state in order to compute the phase shift. However, further manipulation of these equations using the first-order form of the two-body Dirac equations (see Appendix D of Ref. [46]) leads to an

equation for the $j=l$ states as well as the $^3P_0$ states in which the upper-upper components are completely decoupled from the lower-lower components. This allows us to see graphically whether the effective potential develops a pocket or other structure that could produce a resonance. The equation for the $^3P_0$ upper-upper component wave function which we derive in Appendix D of Ref. [46] is

$$\left\{-\frac{d^2}{dr^2}+\frac{2}{r^2}-\frac{2\epsilon_w\alpha}{r}-\left[\frac{\alpha}{r}\right]^2-\frac{\alpha}{r^3 w}\frac{8}{1+2\alpha/(rw)}+\left[\frac{\alpha}{r^2 w}\right]^2\frac{8}{(1+2\alpha/rw)^2}+16\pi\frac{\delta^3(\mathbf{r})}{w}\frac{1}{1+2\alpha/(rw)}\right\}u_{110}=b^2(w)u_{110}\ . \quad (7.1)$$

(Note that the δ-function term will not contribute because its coefficient vanishes at $r=0$.) This is the equation from which we will obtain our numerical results. We remind the reader that this equation is dictated by the combination of two-body relativistic quantum mechanics of the constraint formalism with a field theoretic Δ of the Bethe-Salpeter equation. This equation has the following special features. First, this covariant Schrödinger-like form displays exact relativistic kinematics. Second, the local potential structure of Eq. (7.1) generated by $\mathcal{A}=-\alpha/r$ is determined by perturbative QED in concert with the minimal interaction form that follows from quantum-mechanical gauge invariance. Third, we have shown that the short-distance strong-potential structure of these equations (albeit in an equivalent form) was crucial for the accurate numerical determination of the bound-state spectrum demonstrating the validity of the equations down to distances of the order of α/m. Fourth, because the kinetic and Darwin terms in Eq. (7.1) are local [unlike the three-dimensional Salpeter equation or its $O(1/c^2)$ Fermi-Breit reduction], our approach provides a graphical as well as covariant way of examining the short distance behavior directly. Fifth, the effective potential (including the centrifugal potential barrier) in Eq. (7.1) is *attractive* and nonsingular near the origin, having the limiting behavior $\sim-\alpha/r^2$ as $r\to 0$, in contrast with the more singular terms appearing in the standard $O(1/c^2)$ Fermi-Breit reduction of the Salpeter equation.

The corresponding decoupled equation (see Ref. [16] and Appendix D of Ref. [46]) for the upper-upper component for the ^1J_j states of the e^+e^- system is

$$\left\{-\frac{d^2}{dr^2}+\frac{j(j+1)}{r^2}-\frac{2\epsilon_w\alpha}{r}-\left[\frac{\alpha}{r}\right]^2\right\}u_{j0j}=b^2(w)u_{j0j}\ . \quad (7.2)$$

In both Eqs. (7.1) and (7.2), the effective potential is nonsingular near the origin.

In order to determine whether there are any purely electromagnetic resonances or other nonperturbative effects in the $^3P_0$ states as described by our equation (7.1) for the e^+e^- system, we need to compute the phase shifts as a function of energy and compare them with the perturbatively computed phase shifts. In Secs. V and VI we performed a successful test of our formalism which found agreement between the $^3P_0$ bound-state spectral results computed perturbatively and numerically. Do we obtain agreement here between the two types of computations of the phase shifts? The general form of our equation (7.1) is

$$\left[-\frac{d^2}{dr^2}+\frac{l(l+1)}{r^2}-\frac{2\epsilon_w\alpha}{r}+\Delta\Phi\right]u(r)=b^2u(r) \quad (7.3)$$

where $\Delta\Phi$ consists of the short-range parts of the effective potential. Because of the long-range nature of the effective potential in Eq. (7.3), the asymptotic form of its wave function is

$$u(r\to\infty)\to\text{const}\times\sin(br-\eta\ln 2br+\Delta) \quad (7.4)$$

in which

$$\Delta=\delta_l+\sigma_l-l\pi/2\ , \quad (7.5)$$

while $\sigma_l=\arg\Gamma(l+1+i\eta)$ is the Coulomb phase shift (with $\eta=-\alpha\epsilon_w/b$). For the $^3P_0$ state, the phase shift δ_l is due to $\Delta\Phi$ for the $^3P_0$ state:

$$\Delta\Phi=-\frac{\alpha^2}{r^2}-\frac{8\alpha}{wr^3(1+2\alpha/wr)}+8\left[\frac{\alpha}{r^2 w(1+2\alpha/wr)}\right]^2. \quad (7.6)$$

Before computing this nonperturbatively, we evaluate δ_l in perturbation theory for a few representative values of the c.m. energy w. In analogy to the perturbative expression for the phase shift for short range ($\alpha=0$) potentials,

$$\delta_l=-b\int_0^\infty j_l^2(br)\Delta\Phi(r)r^2dr\ , \quad (7.7a)$$

when we treat $\Delta\Phi$ as a perturbation we find

$$\delta_l=-b\int_0^\infty\frac{F_l^2(\eta,br)}{b^2r^2}\Delta\Phi(r)dr \quad (7.7b)$$

when $\alpha\neq 0$. In this distorted Born approximation, $F_l(\eta,\rho)$ is the radial Coulomb wave function. For perturbative purposes we use the unperturbed solution which can be expanded as

$$F_l(\eta,\rho)=(2l+1)!!\rho C_l(\eta)\sum_{k=l}^{\infty} b_k j_k(\rho) \tag{7.8}$$

where

$$b_l=1,\quad b_{l+1}=\frac{2l+3}{l+1}\eta\ , \tag{7.9}$$

$$b_k=\frac{2k+1}{k(k+1)-l(l+1)}\times\left\{2\eta b_{k-1}-\frac{(k-1)(k-2)-l(l+1)}{2k-3}b_{k-2}\right\}\ , \tag{7.10}$$

$j_k(\rho)$ is a spherical Bessel function, and

$$C_l(\eta)=\frac{2^l\sqrt{\Gamma(l+1+\eta)e^{-\pi\eta}\Gamma(l+1+i\eta)}}{(2l+1)!}\ . \tag{7.11}$$

In our case, we approximate $\Delta\Phi$ as the weak potential form $\Delta\Phi_0$ given by

$$\Delta\Phi_0\cong-\frac{\alpha^2}{r^2}-\frac{8\alpha}{wr^3}\ . \tag{7.12}$$

Substitution into Eq. (7.7b) followed by the indicated integration yields

$$\delta_l=\alpha^2[(2l+1)!!]^2[C_l(\eta)]^2\sum_{n=2l}^{\infty}\sum_{k=l}^{n-l}\frac{b_k b_{n-k}}{n+1}\frac{\sin[(2k-n+\epsilon)\pi/2]}{2k-n+\epsilon}$$
$$+\frac{8\alpha b}{w}[(2l+1)!!]^2[C_l(\eta)]^2\sum_{n=2l}^{\infty}\sum_{k=l}^{n-l}\frac{b_k b_{n-k}}{n(n+2)}\frac{\sin[(2k-n+1+\epsilon)\pi/2]}{2k-n+1+\epsilon}\frac{2}{n-2k+1+\epsilon}\ , \tag{7.13}$$

in which ϵ is a positive infinitesimal.

For our nonperturbative calculations (with unapproximated $\Delta\Phi$), we use the variable phase method [61] generalized here to include long-range interactions. Consider the two differential equations

$$u''+(b^2-W-\overline{W})u=0 \tag{7.14}$$

and

$$\bar{u}_i''+(b^2-\overline{W})\bar{u}_i=0,\quad i=1,2 \tag{7.15}$$

in which $u(0)=\bar{u}_1(0)=0$. Choose

$$\overline{W}(r)=-2\epsilon_w/r\ , \tag{7.16}$$

$$W(r)=l(l+1)/r^2+\Delta\Phi \tag{7.17}$$

so that

$$\bar{u}_1(r\to\infty)\to\text{const}\times\sin(br-\eta\ln 2br+\bar{\Delta})\ , \tag{7.18}$$

$$\bar{u}_2(r\to\infty)\to\text{const}\times\cos(br-\eta\ln 2br+\bar{\Delta}) \tag{7.19}$$

in which

$$\bar{\Delta}=\sigma_0 \tag{7.20}$$

and

$$u(r\to\infty)\to\text{const}\times\sin(br-\eta\ln 2br+\Delta)\ , \tag{7.21}$$

$$\Delta=\delta_l+\sigma_l-l\pi/2\ . \tag{7.22}$$

In the variable-phase method, one obtains a nonlinear first-order differential equation for the phase-shift function $\delta_l(r)$ such that

$$\delta_l(\infty)=\delta_l \tag{7.23}$$

and $\delta_l(0)=0$. This is done by rewriting $u(r)$ as

$$u(r)=\alpha(r)[\cos\gamma(r)\bar{u}_1(r)+\sin\gamma(r)\bar{u}_2(r)] \tag{7.24}$$

so that

$$\Delta=\bar{\Delta}+\gamma(\infty)\ . \tag{7.25}$$

Since we have written $u(r)$ in terms of two arbitrary functions we are free to impose a condition relating them:

$$u'(r)=\alpha'(r)[\cos\gamma(r)\bar{u}_1'(r)+\sin\gamma(r)\bar{u}_2'(r)]\ . \tag{7.26}$$

Combination of these two equations leads to

$$\gamma(r)=-\arctan\left[\frac{u(r)\bar{u}_1'(r)-u'(r)\bar{u}_1(r)}{u(r)\bar{u}_2'(r)-u'(r)\bar{u}_2(r)}\right] \tag{7.27}$$

in which $\gamma(0)=0$. In our case

$$u_1(r)=F_0(\eta,br)\ , \tag{7.28a}$$

$$u_2(r)=G_0(\eta,br)\ . \tag{7.28b}$$

From Eqs. (7.27)–(7.28), and the Wronskian $F_0G_0'-F_0'G_0=b$ we obtain by simple differentiation the differential equation

$$\gamma'(r)=-W(r)[\cos\gamma(r)F_0(\eta,br)+\sin\gamma(r)G_0(\eta,br)]^2/b\ . \tag{7.29}$$

Note that for $W(r\to 0)\to\lambda(\lambda+1)/r^2$ [since $F_0(\eta,br\to 0)\to C_0 br$ and $G_0(\eta,br\to 0)\to 1/C_0$], we obtain the relation

$$\gamma'(0)=-C_0^2 b\lambda/(\lambda+1)\ . \tag{7.30}$$

Note also that if we were to exclude the angular momentum barrier term $l(l+1)/r^2$ from $W(r)$ and include it in $\overline{W}(r)$ instead, λ would become complex in the case of Eqs. (7.3) and (7.6). With our division however, we must

integrate the wave function to very large distances because of the long tail of the barrier term. However, we can overcome this difficulty by letting

$$\gamma(r)=\beta(r)+\epsilon(r) \tag{7.31}$$

with $\beta(r)$ defined so that

$$\beta'(r)=-\frac{l(l+1)}{r^2}[\cos\beta(r)F_0(r)+\sin\beta(r)G_0(r)]^2/b\ . \tag{7.32}$$

This equation has the exact solution

$$\beta(r)=-\arctan\left[\frac{F_l(r)F_0'(r)-F_l'(r)F_0(r)}{F_l(r)G_0'(r)-F_l'(r)G_0(r)}\right] \tag{7.33}$$

with $\beta(0)=0$ and $\beta'(0)=-C_0^2bl/(l+1)$ and

$$\beta(\infty)=\sigma_l-l\pi/2-\sigma_0\ . \tag{7.34}$$

Combining (7.20), (7.22), (7.25), (7.31), (7.34) then leads to

$$\delta_l=\epsilon(\infty)\ . \tag{7.35}$$

Thus, if we solve

$$\epsilon'(r)=-[l(l+1)/r^2+\Delta\Phi]\{\cos[\beta(r)+\epsilon(r)]F_0(\eta,br)+\sin[\beta(r)+\epsilon(r)]G_0(\eta,br)\}^2/b$$
$$+[l(l+1)/r^2][\cos\beta(r)F_0(r)+\sin\beta(r)G_0(r)]^2/b \tag{7.36}$$

subject to the boundary condition $\epsilon(0)=0$ and the condition Eq. (7.30), transcribed to the form

$$\epsilon'(0)=-C_0^2b\lambda/(\lambda+1)+C_0^2bl/(l+1)\ ,$$

we obtain the additional phase shift (above the Coulomb phase shift) by integration to $\epsilon(\infty)$.

As a first application of Eq. (7.36) we compute δ_l for the spin singlet equation in which

$$\Delta\Phi=-\frac{\alpha^2}{r^2}\ . \tag{7.37}$$

This provides a particularly strong test of our procedure since δ can be computed analytically by incorporating the term $-\alpha^2/r^2$ with $l(l+1)/r^2$ into $\lambda(\lambda+1)/r^2$ and using

$$\sigma_l-l\pi/2=\arg\Gamma(l+1+i\eta)-l\pi/2\to\sigma_\lambda-\lambda\pi/2$$
$$=\arg\Gamma(\lambda+1+i\eta)-\lambda\pi/2 \tag{7.38}$$

where $\lambda(\lambda+1)=l(l+1)-\alpha^2$. We are interested in the phase shift δ_l produced by $\Delta\Phi$ beyond the Coulomb phase shift σ_l which in this case is given by

$$\delta_l=\sigma_\lambda-\sigma_l\ . \tag{7.39}$$

Hence, with $\eta=-\alpha\epsilon_w/b$ we find

$$\delta_l=\arg\frac{\Gamma(\lambda+1+i\eta)}{\Gamma(l+1+i\eta)}-(\lambda-l)\pi/2\ . \tag{7.40}$$

For $l=0,1$ and $w=1.6$ MeV we obtain $\delta_l=8.391\times10^{-5}$, 2.794×10^{-5}. Using the first term in the perturbative expression (7.13) for comparison we obtain $\delta_l=8.431\times10^{-5}$, 2.797×10^{-5}. The corresponding numerical results are $\delta_l=8.396\times10^{-5}$, 2.770×10^{-5}. Thus we find agreement between the perturbative and numerically calculated values of the phase shifts for the singlet states.

We are now ready to perform the same calculations and comparisons for the $^3P_0$ states in order to test our Eq. (7.1) and to see whether it predicts any resonant states. We compute the results for the $^3P_0$ states at several energies. At

$$w=1.4,\ 1.6,\ 1.8,\ 2.0\ \text{MeV} \tag{7.41}$$

we find that the nonperturbatively computed numerical values are

$$\delta_l=2.529\times10^{-3},\ 2.847\times10^{-3},\ 3.038\times10^{-3},\ 3.175\times10^{-3}\ . \tag{7.42}$$

These agree well with the perturbatively computed values using the full Eq. (7.13):

$$\delta_l=2.556\times10^{-3},\ 2.876\times10^{-3},\ 3.075\times10^{-3},\ 3.210\times10^{-3}\ . \tag{7.43}$$

Thus our two-body Dirac equations predict no resonances in the $^3P_0$ states in the above energy range, and no significant deviation from ordinary Bhabba scattering [62].

Let us examine now how the various parts of the quasipotential in Eq. (7.1) conspire to produce the turnover of the total effective potential (including the angular momentum barrier) for the $^3P_0$ state while keeping the potential narrow and shallow enough to forbid a resonance (see Fig. 1). At very long distances the Coulomb term dominates. As the interparticle separation goes to zero, the angular momentum barrier $l(l+1)/r^2$ $(=2/r^2)$

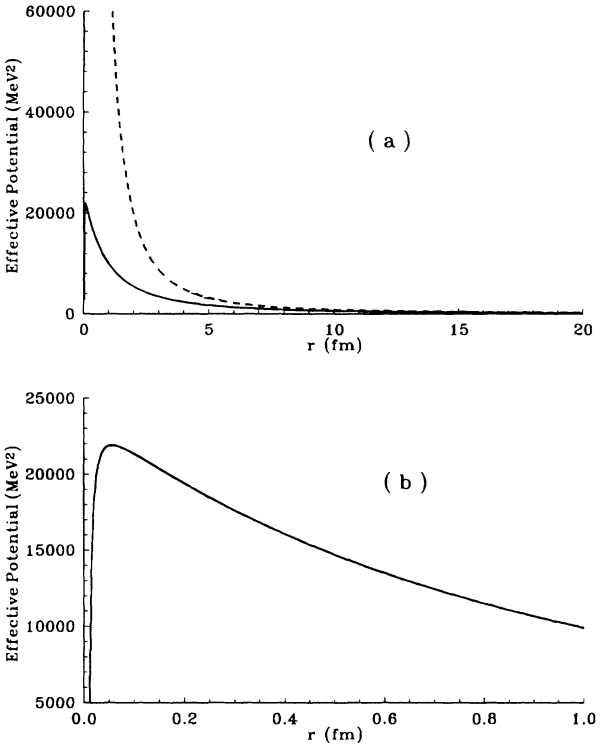

FIG. 1. (a) The effective potentials, including the angular momentum barrier $(2/r^2)$, for the $^1P_1$ state (dashed curve) and the $^3P_0$ state (solid curve) in units of MeV^2 as functions of r in fermis. (b) The effective potential for the $^3P_0$ state in units of MeV^2 versus r in fermis in an expanded radial scale.

becomes dominant at about an Angstrom. At this distance the spin-orbit and tensor terms (combining to give the last term in the first line) have an attractive $1/r^3$ behavior, whereas the spin-spin and Darwin terms (combining to give the first term in the second line) yield a repulsive $1/r^4$ behavior. The attractive $1/r^3$ terms counteract the angular momentum barrier reducing that barrier by about a factor of $\frac{1}{2}$ for $r \sim 0.5$ fm, eventually causing the potential to turn over at around 0.06 fm. But by this distance, the G^2 factor $1/(1+2\alpha/wr)$ approaches $wr/2\alpha$ so that it moderates the attractive $1/r^3$ spin-orbit part, leading to $-4/r^2$. At about the same distance, the spin-spin and Darwin terms of the potential have their repulsive $1/r^4$ behavior modified by the factor G^4 $(\rightarrow w^2r^2/4\alpha^2)$ to the form $2/r^2$. The net result is the $-\alpha^2/r^2$ behavior given by (7.1) for the $^3P_0$ state. This behavior and the attendant phase shift are a direct consequence of the matrix Δ we obtained from QED and the minimal interaction structure for incorporating QED into our two-body Dirac equations [63].

VIII. CONCLUSIONS AND RELATIONSHIPS TO OTHER APPROACHES

In this paper, we have solved a system of coupled Dirac equations previously formulated by two of us for electrodynamic and related two-body systems [5,16]. These equations, which are spin-dependent strong-potential versions of an equation originally developed for QED by Todorov [14], contain local but nonsingular potentials and so may be solved nonperturbatively for bonafide relativistic wave functions. Yet, they contain effects in their wave functions that are traditionally obtained from perturbation theory. Two of us had previously found 16-component exact analytic solutions for singlet states of positronium with energies agreeing with the field theoretic spectrum through order α^4. In this paper, we have shown for a representative set of radial, orbital, and spin states that nonperturbative numerical solution for the wave function yields the correct field-theoretic spectrum through order α^4. As far as we know, this sort of spectral agreement has never been obtained before from numerical solution of a relativistic wave equation. Even though we had originally applied these equations (with appropriate potentials $\mathcal{A}$, $\mathcal{V}$, and S) to calculations of the meson spectrum, their suitability for electrodynamics is not a total surprise since two of us had originally abstracted the form of the vector interactions appearing in them from (the field-theoretic) Wheeler-Feynman electrodynamics [13].

Comparison of the structures of our equations with those of selected traditional approaches to QED and with those of recent alternatives and applications will help to clear up the origins and possible physical significance of our results. All of the equations that we will consider share the property that when treated perturbatively they reproduce the correct QED spectral results through order α^4 that arise from the field-theoretic Born diagram alone.

Relativistic wave equations have been used in electrodynamics primarily in three ways. First, such equations have been solved both numerically and analytically as wave equations when the absence of singularities and nonlocal terms permitted, leading to "nonperturbative" spectra (Balmer formulas). (However, such solutions are not guaranteed to agree with the results of quantum field theory—witness the erroneous results for parapositronium produced by nonperturbative treatment of the local Breit equation [31,32].) Second, such equations, as they stand, have been used as perturbative forms that are divided into a nonrelativistic wave equation with well behaved solutions and a singular remainder to be used only in low-order perturbation theory. Third, such equations have been used purely as springboards for field-theoretic perturbative treatments. Typically, one selects a relativistic wave equation with simple wave functions that generate the correct lowest-order (α^2 and parts of the order α^4 and higher order) spectrum directly through the wave function and then systematically treats the remaining order α^4 and higher effects dictated by the Bethe-Salpeter equation as field-theoretic perturbations built around the analytic solutions of the wave equation.

A. "Nonperturbative" features of wave equations and solution where possible

Properly our numerical wave functions and spectra ought to be compared directly with their counterparts from numerical solution of the Bethe-Salpeter equation.

However, as far as we know there have been no numerical tests (nonperturbative solutions) of any of the traditional three-dimensional rearrangements of the four-dimensional Bethe-Salpeter equation of QED. (This situation has occurred because perturbative treatment of the weak potential forms of those equations (see subsection B below) are sufficient for QED and because treatments of nonlocal bound-state equations has been technically difficult.) To see why this is so consider the most widely used rearrangement: the Salpeter equation [17]. That equation for single-photon exchange in the instantaneous approximation is

$$[E-H_1-H_2]\phi_{12}(\mathbf{p})=\Lambda(\mathbf{p})\frac{e^2}{(2\pi)^3}\int d^3k\frac{\gamma_{01}\gamma_{\mu 1}\gamma_{02}\gamma_2^{\mu}}{|\mathbf{k}|^2}\phi_{12}(\mathbf{p}+\mathbf{k}) \tag{8.1}$$

where

$$H_1(\mathbf{p})=m_1\beta_1+\mathbf{p}\cdot\boldsymbol{\alpha}_1 ,$$

$$H_2(\mathbf{p})=m_2\beta_2-\mathbf{p}\cdot\boldsymbol{\alpha}_2 ,$$

$$\Lambda=[\Lambda_+^1(\mathbf{p})\Lambda_+^2(\mathbf{p})-\Lambda_-^1(\mathbf{p})\Lambda_-^2(\mathbf{p})] ,$$

$$\Lambda_{\pm}^i(\mathbf{p})=[E_i(\mathbf{p})\pm H_i(\mathbf{p})]/2E_i(\mathbf{p}) ,$$

and

$$E_i(\mathbf{p})=\sqrt{m_i^2+\mathbf{p}^2}$$

in which the three-dimensional Salpeter wave function ϕ is given in terms of the four-dimensional Bethe-Salpeter wave function χ by

$$\phi_{12}(\mathbf{p})=\int dp^0\chi_{12}(p^0,\mathbf{p})$$

and

$$\phi_{12}^{+-}(\mathbf{p})=\phi_{12}^{-+}(\mathbf{p})=0$$

where $\phi_{12}^{\kappa\lambda}=\Lambda_{\kappa}^1\Lambda_{\lambda}^2\phi_{12}$, for $\kappa,\lambda=\pm$. The particular (but *ad hoc*) elimination of the relative time and the relative energy in the derivation of Eq. (8.1) forces on the user nonlocal (in coordinate space) free-particle energies $E_i(\mathbf{p})$. In contrast, the corresponding role is played in our equations by the local but c.m. energy-dependent ϵ_i of Todorov. Furthermore, the compatibility of our two 16-component Dirac equations automatically restricts their dependence on the relative time in such a way as to permit an exact reduction (with no truncations) to two coupled (and in some angular momentum cases, one), Schrödinger-like equations with a total c.m. energy-dependent (but not necessarily) momentum-dependent effective potential, each involving two four-component wave functions [see Eqs. (4.14)]. Not only are our local minimal-interaction constraint equations much easier to handle numerically, but also (for momentum-independent interactions), they permit a direct covariant examination of the short-distance behavior (see Fig. 1). Such an examination cannot be performed as directly on the momentum space form or on the necessarily nonlocal coordinate space form of the three-dimensional Salpeter equation without an $O(1/c^2)$ expansion (which we shall examine below).

Recently, Mandelzweig and Wallace [64] presented a new covariant approach to the two-body problem. Instead of the two two-body Dirac equations of (2.1) or (3.38) they employ a single 16-component "sum" Dirac equation of the form

$$[(\gamma_1\cdot p_1+m_1)\lambda_1+(\gamma_2\cdot p_2+m_2)\lambda_2-V]\psi=0 \tag{8.2}$$

(in our metric) in which

$$\lambda_i=\frac{\gamma_i\cdot p_{i\perp}+m_i}{\sqrt{m_i^2+p_{i\perp}^2}} , \tag{8.3}$$

accompanied by a spin-independent constraint

$$(p_1^2+m_1^2)\psi=(p_2^2+m_2^2)\psi , \tag{8.4}$$

on the relative energy. [In our approach Eq. (8.4) is a consequence of our two Dirac equations (see (3.21) and also Refs. [4,5]).]

Like our equations, the Mandelzweig-Wallace equations yield the correct single-particle Dirac equation with an external potential when one particle becomes infinitely heavy. However, each set of equations achieves this result in a different way. Mandelzweig and Wallace noted that in the Bethe-Salpeter formalism both the single-particle Dirac limit and the high-energy eikonal limit depend on cancellations between crossed and uncrossed Feynman graphs when the kernel in the Bethe-Salpeter equation is truncated. Consequently, in deriving their equation, they included crossed graphs using a form of the eikonal approximation in such a way that the high-energy limit and the heavy-particle limits are preserved despite truncation of the kernel. In contrast our spin-dependent equations and the closely related spinless Todorov quasipotential equation [in the form (3.1)] achieve the heavy-particle limits automatically through their classical relativistic kinetic and potential structures without further manipulation of the potential. (In fact starting from only the Born term, the Todorov equation sums up all cross ladder and ladder diagrams in the limit of small exchanged mass and momentum transfer [14,65,66].)

The two-body Dirac equations Eqs. (2.1) and the Mandelzweig-Wallace equation differ substantially in their spin-dependent structures through the full 16×16 matrix potentials. Just like the free-particle sub-energies $E_i(\mathbf{p})$ of the Salpeter equation, the Mandelzweig-Wallace

equation contains the free-particle Dirac projectors λ_i [which contain the $E_i(\mathbf{p})$] as coefficients in the "sum" form Eq. (8.2). These render the Mandelzweig-Wallace equation nonlocal in the Born approximation, for which the coupled Dirac equations [Eqs. (2.1) or (4.14)] are local. Nonetheless Wallace and Thayyullathil [33] have been able to solve the Mandelzweig-Wallace equations numerically for the ground-state hyperfine splittings in QED. Thus, some results given by both sets of equations are available for comparison. These results show that one consequence of the difference in spin-dependent structures in the two approaches is their different dependences on and sensitivities to the four four-component pieces of the 16-component wave function. For example, for the hyperfine splitting of muonium, the Mandelzweig-Wallace results with the lower-lower parts of the wave function excluded are comparable to our results through order α^4 when our equations are fully coupled. However, inclusion of the lower-lower portion of the wave function in the Mandelzweig-Wallace equation produces large deviations from the field-theoretic values through order α^4. This contrasts sharply with our results of Sec. VI for which inclusion of the coupling to the lower-lower portion of the wave function was crucial for agreement with the field theoretic values through order α^4. A complete comparison of the two approaches awaits calculations in the Mandelzweig-Wallace approach of the counterparts to the fine-structure splitting and radial excitations given by our equations in Sec. VI.

B. Weak-potential perturbative form

Traditionally relativistic wave equations which cannot be or have not been solved numerically or analytically have been rearranged as corrections to the nonrelativistic Schrödinger equation with Coulomb potential. For example the $O(1/c^2)$ Fermi-Breit expansion [67] of the Salpeter equation, yields

$$H\psi = w\psi \tag{8.5a}$$

in which w is the total c.m. energy and

$$\begin{aligned} H = {} & \left[m_1 c^2 + \frac{\mathbf{p}^2}{2m_1} - \frac{(\mathbf{p}^2)^2}{8m_1^3 c^2} \right] + \left[m_2 c^2 + \frac{\mathbf{p}^2}{2m_2} - \frac{(\mathbf{p}^2)^2}{8m_2^3 c^2} \right] \\ & + e_1 e_2 \left\{ \left[\left[1 - \frac{p^2}{m_1 m_2 c^2} \right] \frac{1}{r} - \frac{1}{2m_2 m_2 c^2 r} \mathbf{p}\cdot(1+\hat{r}\hat{r})\cdot\mathbf{p} \right]_{\text{ordered}} - \frac{\hbar^2}{2c^2} \left[\frac{1}{m_1^2} + \frac{1}{m_2^2} \right] \delta(\mathbf{r}) \right. \\ & - \frac{\hbar}{4c^2} \frac{\mathbf{L}}{r^3} \cdot \left[\left[\frac{1}{m_1^2} + \frac{2}{m_1 m_2} \right] \boldsymbol{\sigma}_1 + \left[\frac{1}{m_2^2} + \frac{2}{m_1 m_2} \right] \boldsymbol{\sigma}_2 \right] \\ & \left. + \frac{\hbar^2}{4m_1 m_2 c^2} \left[-\frac{8\pi}{3} \boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2 \delta(\mathbf{r}) + \frac{\boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2}{r^3} - \frac{3\boldsymbol{\sigma}_1\cdot\mathbf{r}\boldsymbol{\sigma}_2\cdot\mathbf{r}}{r^5} \right] \right\} . \end{aligned} \tag{8.5b}$$

This equation (to which our equation is canonically equivalent in order of $O(1/c^2)$ for weak potentials [68]) contains terms after the Coulomb term that are too singular at the origin to be treated nonperturbatively. On the other hand, the unapproximated counterparts of these terms in the covariant, Schrödinger-like form, Eqs. (4.14), of the two-body Dirac equations are quantum mechanically well defined for all tested angular momentum states. This means that in contrast with the Fermi-Breit equation [and Eq. (5.9)], the wave functions for all angular momentum states are affected by all terms. Despite this fact, our solution of the unapproximated covariant minimal interaction constraint form, (4.14), reproduces the correct perturbative spectral results for fine and hyperfine splittings. Because we are able to solve our unapproximated equation numerically, we are able to carry out a double cross-check of its nonperturbative spectral results with its own perturbative spectral results [from Eq. (5.9)] and with the corresponding results of perturbative quantum field theory [from Eq. (8.5)]. The fact that they all agree shows that (i) the weak potential form Eq. (5.9) yields an accurate perturbative evaluation of the exact equation (4.14) (mathematical property), (ii) the unusual (though local) short-distance structure of the exact equation (4.14) (and hence of the wave function) does not disrupt the perturbative spectrum (mathematical property), and (iii) the unapproximated coupled two-body Dirac equations (2.1) and their Schrödinger-like rearrangement Eqs. (4.14) yield a spectrum from single-photon exchange in agreement with that of perturbative QED for all angular momentum states through order α^4 (physical property). Although this agreement exists in the one-body Dirac equation, as far as we know because of difficulty of nonperturbative solution, such agreement has never been demonstrated for any other two-body equation with spin [69,70].

C. Relativistic wave equations as anchors for field-theoretic perturbation theory

In order to carry out any perturbative solution of the Bethe-Salpeter equation of QED, one must first specify a

lowest-order equation. As noted by Barbieri and Remiddi [71], any such equation must contain a kinetic term and a Coulomb-like interaction term, and must be able to produce bound states and contain the largest part of the full Bethe-Salpeter kernel. It must reproduce the correct nonrelativistic dynamics with corrections of second order in momenta so that the usual Balmer formula appears in lowest order (α^2) with no corrections of order α^3. In addition, it must yield the relativistic propagator of two free fermions when the interaction vanishes (at high momentum). Finally, for purposes of perturbative calculation, its wave functions must be known analytically or numerically. By carrying out the numerical solution of the two-body Dirac equations (2.1) in this paper, we have completed the demonstration that they possess all these properties.

In the work of Barbieri and Remiddi [71] and in the work of Caswell and Lepage [35] (for fermions of comparable mass), the fact that "no equation for two fermions is known that can be solved exactly and which gives the correct α^4 structure" forced those authors to confine the dynamical contributions of their lowest-order equations to the relativistic Coulomb potential alone. This restriction does not destroy the perturbative procedure, however, as long as "the correct α^4 terms, as well as all higher-order corrections to the energy levels, are obtained in the systematic perturbative expansion to be built starting from the lowest-order equation in question."

Why then would anyone propose to replace the basic relativistic wave equation including Coulomb potential with an equation with additional dynamical structures? The advantages are twofold. First, in most mathematical structures, increase in accuracy of the unperturbed piece pays dividends in the form of increased rate of convergence and the ability to dispense with the treatment of terms whose only function is to build up some basic structure of the unperturbed term. If one includes a basic nonperturbative structure, one gets all of the higher-order perturbative terms corresponding to it as a bonus. For example, Barbieri and Remiddi, and Caswell and Lepage pass from the nonrelativistic Schrödinger equation with nonrelativistic Coulomb potential to a relativistic wave equation with relativistic potential to reap the benefit of inclusion of some of the α^4 terms in their lowest-order equation and to ensure correct relativistic kinematical contributions in higher order. In addition, Caswell and Lepage, in their first work on systems with one heavy particle and one light particle took as their lowest-order equation the one-body Dirac equation with the second particle on the mass shell in order to include fine structure (i.e., Dirac spin structure) in their unperturbed solutions. Second, the lowest-order equation of perturbative QED with its particular structures is abstracted by many authors for use elsewhere in QED, nuclear, and particle physics as a wave equation for the bound-state calculation. In those applications structures of little consequence in perturbative QED (e.g., short-distance or strong-potential behaviors) may play a significant role. Thus, accurate knowledge of QED structures of this sort serves as a check on the uses and abuses of such equations.

As we have mentioned, as first shown by Yaes and Gross [17], there are an infinite set of "equivalent" three-dimensional reductions of the Bethe-Salpeter equation, differing in form and therefore in ease of application and interpretation. The electromagnetic constraint equation employed in this paper, with its characteristic energy dependences and potential structures, permits nonperturbative solution to higher-order in α than has yet been possible for others in the set. To illustrate this point, we consider in some detail the work of Caswell and Lepage [35]. Caswell and Lepage reformulated the Bethe-Salpeter equation in two different ways: the first in terms of a one-body Dirac equation with the second particle on the mass shell, the second in terms of an effective Schrödinger equation (in the c.m. frame). In the first approach Caswell and Lepage were able to incorporate "fine structure of levels with differing angular momenta" in the unperturbed QED solutions. However, the unsymmetric nature of this solution restricted its application to cases in which the binding was weak or the mass ratio large. To remedy this defect, Caswell and Lepage developed an effective Schrödinger equation to treat the case of comparable masses. The price they had to pay was loss of unperturbed solutions containing fine structure of levels with differing angular momenta. They were able only to retain their version of a relativistic Coulomb potential in their unperturbed Schrödinger equation. They attributed the different structures of their treatments to the physical fact that "the fine structure of atoms with constituents of equal mass is quite different in character from that of atoms with a large mass ratio." They observed that "it is difficult to create a formalism which naturally accommodates both cases and still admits analytic solutions comparable in simplicity to those presented" (in their second paper).

However, the electromagnetic two-body Dirac equations (4.14) are a solution to the problem posed by Caswell and Lepage. Since these describe the system symmetrically through two coupled Dirac equations whose potential structures do not change discontinuously from high mass ratio to comparable masses, they can be solved nonperturbatively in both regimes using the same numerical techniques. Furthermore, for comparable masses, they can be solved numerically without truncation to the simple Coulomb potential of Caswell and Lepage. Thus, their solutions contain the fine structure lost by Caswell and Lepage.

We may see this explicitly by making use of the fact that in one case, the equal-mass singlet, our equations possess an exact analytic solution (see Ref. [16] and Appendix D in Ref. [46]). In fact, in that case, the second-order form of our equations reduces to the minimal Todorov equation [72] on a singlet wave function:

$$[\mathbf{p}^2+2\epsilon_w\mathcal{A}-\mathcal{A}^2-\epsilon_w^2+m_w^2]\psi=0 \tag{8.6}$$

in which ϵ_w and m_w are Todorov's reduced energy and mass of a relativistic particle of relative motion introduced so that the second-order two-body equation takes the mass-shell form. On the other hand, Caswell and Lepage write their second-order equation in the form of

an effective nonrelativistic Schrödinger equation:

$$\left[\frac{\mathbf{p}^2}{2\tilde{m}}+\mathcal{A}\right]\psi=\tilde{\epsilon}\psi \tag{8.7}$$

in which $\tilde{\epsilon}$ and $\tilde{m}$ are an effective binding energy and mass given by

$$\tilde{\epsilon}=\frac{w^2-(m_1+m_2)^2}{2w},$$
$$\tilde{m}=\frac{w^2-(m_1-m_2)^2}{2w}. \tag{8.8}$$

In fact, we see that $\tilde{\epsilon}$ is simply the difference

$$\tilde{\epsilon}=\epsilon_w-m_w, \tag{8.9}$$

while $\tilde{m}$ is the average

$$\tilde{m}=\tfrac{1}{2}(\epsilon_w+m_w). \tag{8.10}$$

Using these facts, we may rewrite the Caswell-Lepage equation in the Todorov form:

$$[\mathbf{p}^2+(\epsilon_w+m_w)\mathcal{A}-\epsilon_w^2+m_w^2]\psi=0. \tag{8.11}$$

We see that, for the singlet state, the unperturbed equation that we solve shares its relativistic kinetic structure with that of Caswell and Lepage. On the other hand, the two equations differ in their dependence on the relativistic Coulomb potential and on the energy-dependent ϵ_w and m_w. Despite its more elaborate structure, the minimal Todorov equation still permits exact solution [16]. In each case, the unperturbed eigenvalue may be found by mapping the relativistic equation to the nonrelativistic Schrödinger equation with Coulomb potential ($\mathcal{A}=-\alpha/r$). Rewritten in the Todorov variables, the Caswell-Lepage "relativistic Balmer formula" takes the form

$$\epsilon_w^2-m_w^2=-\frac{\alpha^2}{n^2}\tilde{m}^2 \tag{8.12}$$

whereas the minimal Todorov equation's "relativistic Balmer formula" is

$$\epsilon_w^2-m_w^2=-\frac{\alpha^2}{(n-\delta_l)^2}\epsilon_w^2 \tag{8.13}$$

in which δ_l is the relativistic shift of the angular momentum given by

$$\delta_l=l+1/2-[(l+1/2)^2-\alpha^2]^{1/2}. \tag{8.14}$$

Solution of each of these for the total energy w followed by expansion in α leads to

$$w\sim 2m-\frac{\alpha^2 m}{4n^2}+\frac{3}{64}\frac{\alpha^4 m}{n^4} \tag{8.15}$$

for the Caswell-Lepage equation and

$$w\sim 2m-\frac{\alpha^2 m}{4n^2}-\frac{\alpha^4 m}{2n^3(2l+1)}+\frac{11}{64}\frac{\alpha^4 m}{n^4} \tag{8.16}$$

for the minimal Todorov equation, respectively. Note that the singlet eigenvalue for our equation (the minimal Todorov equation) already contains the correct angular-momentum-dependent fine structure as well as the correct angular-momentum-independent fine-structure correct through order α^4. On the other hand, Caswell and Lepage's unperturbed fine structure must be perturbatively corrected by "relativistic corrections to single Coulomb exchange" and "single transverse photon exchange" in the Coulomb gauge to yield the singlet spectrum correct to order α^4.

Now that we have a new lowest-order equation, how are we to go on to higher order perturbation theory? One could use that equation for a new approach to perturbative QED calculations for bound states. We have shown in Ref. [2], Appendix A, and in Sec. III how one may use the projection of Sazdjian to obtain Todorov's inhomogeneous quasipotential equation from the Bethe-Salpeter equation. One could use that equation to correct the interactions that appear in the two-body Dirac equations perturbatively, and then one could solve the resulting corrected wave equation nonperturbatively just as we solved the lowest-order equation in this paper. (This would avoid the necessity of using higher-order quantum mechanical perturbation theory.)

D. Nonperturbative application of relativistic wave equations to e^+e^- and $q\bar{q}$ composites

In the past, many authors have transported the relativistic wave equations and relativistic correction structures of perturbative electrodynamics far from their origins in perturbation theory. In the process, strong potential structures of these equations which were of no consequence (to a given order) in perturbation theory may come to play an important role. This has several consequences. First, equations whose agreement with quantum field theory for bound states has been checked perturbatively but not when solved as wave equations may be used in the (sometimes mistaken) belief that solution works. The danger of this is illustrated by the local Breit equation whose nonperturbative treatment produces erroneous results for parapositronium and which, as has been pointed out by Childers, leads to singular potentials for other e^+e^- states [73]. The agreement must be checked, as we have done for two-body Dirac equations (4.14) in Sec. VI. Second, different wave equations that gave equivalent results to a given order (either when treated perturbatively or nonperturbatively) for bound or scattering states in the semirelativistic region may yield inequivalent results when solved nonperturbatively for bound or scattering states in a highly relativistic region.

As we have shown, the two-body Dirac equations (2.1) and (4.14) provide an alternative treatment of the two-fermion bound-state problem. These equations use their own characteristic local potential structures to produce the same spectra for perturbative QED that are produced by more complicated momentum structures in standard equations. Thus, when such equations are extended to other problems, we may find disagreements of the first or second types. For their part, the two-body Dirac equations yield straightforward numerical solutions for (eventual) comparison with the other methods.

Recently, motivated by the work [18] of two of us, Spence and Vary [28] have nonperturbatively solved three different three-dimensional truncations of the Bethe-Salpeter equation with single-photon exchange and in each instance obtained "zero-width e^+e^- resonances" (continuum bound states) at 1.351, 1.498, 1.659, 1.830, 2.009, and 2.195 MeV in direct contradiction to our results of Sec. VII. The reader should note that Spence and Vary restrict themselves to the same field-theoretic dynamics (single-photon exchange) that we do. They use the Tamm-Dancoff equation, the no-pair form of the Breit equation (the Salpeter equation), and the Blankenbecler-Sugar equation along with a nonperturbative treatment of the corresponding Lippmann-Schwinger equations. All of the equations that they employ are nonlocal; they claim that this feature is crucial for generating their continuum bound-state solutions. They point out that one of these equations—the no pair form of the Breit equation—is known to produce a good description of the ordinary bound states of positronium. However, the bound-state calculation that they refer to is actually a perturbative calculation and thus lends no support to their nonperturbative solution of that equation. Thus, if one were to trust their results, one would first have to rule out a disagreement of the first type by carrying out a nonperturbative treatment of each of these truncations to obtain the standard QED energy levels through order α^4 (just as we have done for Eqs. (4.14) in Sec. VI [74]). Interestingly, we were originally motivated to study the nonperturbative treatment of the QED bound-state spectra in our equation by the possibility that the potential structures responsible for disagreements of the second type could lead to highly relativistic resonant e^+e^- states.

Thus far there is no direct evidence for e^+e^- resonances in Bhabha scattering experiments. Recent searches for both short-lived and long-lived low-mass couplings in the e^+e^- system [26] have found no evidence for deviations from the nonresonant Bhabha scattering background within statistical uncertainties of $0.2\%(\sigma)$ in the invariant energy range from 1500 to 1850 keV. Of course if the authors of Ref. [28] calculate corrections to their zero-width predictions, they may find lifetimes outside the range looked for in the experiments. On the other hand, if our treatment of this problem resembles the full Bethe-Salpeter solution, such states do not exist, so that one must look beyond the two-particle sector of pure QED to explain the relativistic resonances seen in heavy-ion collisions.

In yet another area of relativistic two-body physics, various authors have borrowed (sometimes innocently and sometimes with additional cutoffs [75]) interactions from nonrelativistic and relativistic electrodynamics for use in models of quark-antiquark bound systems. In previous papers [1,2], two of us have applied two-body Dirac equations with potential structures motivated by QCD to calculate the mass spectra of mesons composed of light quarks along with those composed of heavy quarks. The goodness of the resulting fit to the full meson spectrum was due in no small part (especially for the "hyperfine" and "fine-structure" splittings), to the peculiar short-distance vector interaction structure of our equations inherited from both the constituent and collective (Todorov) minimal interaction structures contained within them. As we have shown, this structure, when applied to electrodynamics itself, reproduces the two-body spectrum of QED. These results, taken together, argue that any competing approach to QCD which solves wave equations in which short distance dynamics is dominated by effective Abelian replacements for the Coulomb potential inserted into elaborations of the Darwin interaction (or equivalently the Breit interaction) or which are based on truncations of the Bethe-Salpeter equation of QED should be judged on their ability to reproduce the spectra of QED when treated numerically or analytically before being applied to QCD. Measures to avoid singularities in the interactions borrowed from certain approaches to QED, such as the use of cutoffs, may invalidate the equations for QED applications, introducing spurious dynamics.

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APPENDIX A: DERIVATION OF TODOROV'S INHOMOGENEOUS QUASIPOTENTIAL EQUATION FOR SPINLESS PARTICLES FROM THE BETHE SALPETER EQUATION

Written in terms of the constituent c.m. energies ϵ_1,ϵ_2, the inhomogeneous Bethe-Salpeter equation in an arbitrary Lorentz frame takes the form

$$T_w(p;q)=K_w(p;q)-\frac{i}{(2\pi)^4}\int d^4k K_w(p;k)G_1^{(+)}(\epsilon_1\hat{P}+k)G_2^{(+)}(\epsilon_2\hat{P}-k)T_w(k;q) \tag{A1}$$

or symbolically,

$$T_w=K_w+K_wG_{12,w}^{(+)}T_w,\quad G_{12,w}^{(+)}=-i(2\pi)^{-4}G_1^{(+)}G_2^{(+)}\ . \tag{A2}$$

Equation (A1) relates the off mass-shell scattering amplitude $T_w(p;q)$ to the two-particle-irreducible kernel $K_w(p;q)$. ($G_i^{(+)}$ is the Feynman propagator.) For an incident-free (on-mass-shell) plane wave, given in relative momentum space by

$$\tilde{\chi}^{(0)}_{w,q\perp}(p)=\delta(\hat{P}\cdot p)\delta^3(p_\perp-q_\perp)\ , \tag{A3}$$

we construct the Bethe-Salpeter "wave function" $\tilde{\chi}^{(+)}_{w,q\perp}$,

$$T_w(p;q_\perp)=\int d^4p' T_w(p;p')\tilde{\chi}^{(0)}_{w,q_\perp}(p')\equiv\int d^4p' K_w(p;p')\tilde{\chi}^{(+)}_{w,q_\perp}(p')\ , \tag{A4}$$

in which $q=-\hat{P}(\hat{P}\cdot q)+q_\perp$. Then (A1) is reproduced for $\hat{P}\cdot q=0$, if

$$\begin{aligned}\tilde{\chi}^{(+)}_{w,q_\perp}(p')&=\delta(\hat{P}\cdot p')\delta^3(p'_\perp-q_\perp)+G^{(+)}_{12,w}(p')T_w(p';q_\perp)\\&=\tilde{\chi}^{(0)}_{w,q_\perp}(p')+G^{(+)}_{12,w}(p')\int d^4k K_w(p';k)\tilde{\chi}^{(+)}_{w,q_\perp}(k)\end{aligned} \tag{A5}$$

In this equation K_w plays the role of the potential [the role played by $\tilde{V}_w$ in (3.12)]. We now write the two-particle, off-mass shell, Feynman propagator as a sum;

$$G^{(+)}_{12,w}(p)=\mathcal{G}^{(+)}_{w,f}(p)+R^{(+)}_{w,f}(p)\ , \tag{A6}$$

of a "minimally off the mass shell" Green's function:

$$\mathcal{G}^{(+)}_{w,f}(p)=\frac{1}{(2\pi)^4}\frac{\pi}{w}\frac{f(p_\perp^2,w)\delta(\hat{P}\cdot p)}{p_\perp^2-b^2(w)-i\epsilon}\equiv\delta(\hat{P}\cdot p)G^{(+)}_{w,f}(p_\perp) \tag{A7}$$

and a residual, R. Like the Todorov Green's function, $G^{(+)}_{w,f}$ satisfies elastic unitarity provided that $f(p_\perp^2,w)=1$ on mass shell. $G^{(+)}_{w,f}$ reduces to the Todorov Green function when $f=1$. For equal masses we obtain the Logunov-Tavkhelidze Green function when $f=w(2\sqrt{p_\perp^2+m^2}+w)/[8(p_\perp^2+m^2)]$ and Blankenbecler and Sugar Green function when $f=w/2\sqrt{p_\perp^2+m^2}$. Following the work of Blankenbecler and Sugar [42] we write T in terms of the Green function $\mathcal{G}^{(+)}_{w,f}$ and an effective interaction W defined by

$$W_w=K_w+K_wR_wW_w=K_w+W_wR_wK_w\ , \tag{A8a}$$

$$K=(1-K_wR_w)W_w=W_w(1-R_wK_w)\ . \tag{A8b}$$

Then (A2), (A6), and (A7) imply that

$$(1-KR)T=K+K\mathcal{G}^{(+)}_{w,f}T\ , \tag{A9}$$

which is satisfied, according to (A8b), when

$$T=W+W\mathcal{G}^{(+)}_{w,f}T\ . \tag{A10}$$

Equation (A8a) does not restrict elements of W to the mass shell. We wish to determine the quasipotential $V(=2w\Phi)$ in terms of W. We begin by performing a general Sazdjian projection [40,2] of the Bethe-Salpeter wave function $\tilde{\chi}^{(+)}$:

$$\phi^{(+)}_{w,f,q_\perp}(p)=\tilde{\chi}^{(0)}_{w,q_\perp}(p)+\delta(\hat{P}\cdot p)G^{(+)}_{w,f}(p_\perp)\int d^4k K_w(p_\perp;k)\tilde{\chi}^{(+)}_{w,q_\perp}(k)\ . \tag{A11}$$

This removes the c.m. relative energy dependence of $\tilde{\chi}^{(+)}_{w,q_\perp}(p)$, so that like the constraint wave function ϕ satisfies $\hat{P}\cdot p\phi^{(+)}_{w,f,q_\perp}(p)=0$. Then, we define the wave function $\tilde{\Psi}$ such that

$$\phi^{(+)}_{w,f,q_\perp}(p)=\delta(\hat{P}\cdot p)\tilde{\Psi}^{(+)}_{w,f,q_\perp}(p_\perp)\ . \tag{A12}$$

We can eliminate the general factor f by multiplying Eq. (A11) by f^{-1} and using the fact that $f^{-1}\tilde{\chi}^{(0)}_{w,q_\perp}(p)=\tilde{\chi}^{(0)}_{w,p_\perp}(p)$. This produces the Todorov wave function

$$\phi^{(+)}_{w,q_\perp}(p)\equiv f(p_\perp^2,w)^{-1}\phi^{(+)}_{w,f,q_\perp}(p)\equiv\delta(\hat{P}\cdot p)\tilde{\Psi}^{(+)}_{w,q_\perp}(p_\perp)\ . \tag{A13}$$

Hence we see that all the Sazdjian wave functions $\phi^{(+)}_{w,f,q_\perp}(p)$ (and associated Green functions) are related to the Todorov choice $\phi^{(+)}_{w,q_\perp}(p)$ by a scale transformation. Todorov's choice, the simplest one ($f=1$), yields the Schrödinger-like equation of constraint dynamics; hence we use it in this paper. We use (A5) and (A6) to rewrite the transform as

$$\phi^{(+)}_{w,q_\perp}(p)=\int d^4k[\delta(p-k)-R^{(+)}_w(p)K_w(p;k)]\tilde{\chi}^{(+)}_{w,q_\perp}(k)\ . \tag{A14}$$

From (A8b) and (A14) we obtain, without employing formal inverses,

$$\int d^4k K_w(p;k)\tilde{\chi}^{(+)}_{w,q_\perp}(k)=\int d^4k' W_w(p;k')\phi^{(+)}_{w,q_\perp}(k')=\int d^3k'_\perp W_w(p;k'_\perp)\tilde{\Psi}^{(+)}_{w,q_\perp}(k'_\perp) \tag{A15}$$

so that (A11) can be written as

$$\tilde{\Psi}^{(+)}_{w,q_\perp}(p)=\delta^3(p_\perp-q_\perp)+G^{(+)}_w(p_\perp)\int d^3k_\perp W_w(p_\perp;k_\perp)\tilde{\Psi}_{w,q_\perp}(k)\ . \tag{A16}$$

Equation (A16) has the same form as the quantum mechanical momentum space integral equation (3.12) for the constraint wave function. If we rewrite Eq. (A8a) in terms of

$$\tilde{V}_w(p_\perp,k_\perp)=-W_w(p_\perp,k_\perp)\ , \tag{A17}$$

then

$$\tilde{V}_w(p_\perp,k_\perp)+K_w(p_\perp;k_\perp)+\int d^4k' K_w(p_\perp;k')R_w(k')W_w(k';k_\perp)=0 \tag{A18}$$

so that (A10) implies the Todorov quasipotential equation

$$T_w(p_\perp,k_\perp)+\tilde{V}_w(p_\perp,k_\perp)+\int\frac{d^3k'_\perp}{(2\pi)^3}\tilde{V}_w(p_\perp,k'_\perp)\frac{1}{2w(k'^2_\perp-b^2(w)-i\epsilon)}T_w(k'_\perp,k_\perp)=0\ . \tag{A19}$$

The difference between this field-theoretic equation and the formally equivalent quantum mechanical Lippmann-Schwinger equation (3.16) is that Eq. (A19) gives $\tilde{V}_w$ in terms of T_w whereas (3.16) gives $\mathbf{T}_w$ in terms of $\tilde{\mathbf{V}}_w$. Since the homogeneous form of Eq. (A19) would be identical in form to the constraint equation Eq. (3.1) (with $\Phi=V/2w$) we use the field-theoretic $\tilde{V}_w$ as the $\tilde{\mathbf{V}}_w$ in our quantum constraint equation.

As a simple example, consider a scalar Yukawa field theory with a momentum space Born amplitude

$$T^{(1)}_P(p',q')=\delta(p'_1+p'_2-q'_1-q'_2)T^{(1)}_w(p',q')\ . \tag{A20}$$

Hence, the corresponding momentum form of our constraint potential is

$$\tilde{\mathbf{V}}^{(1)}_w(p'_\perp,q'_\perp)=-T^{(1)}_w(p'_\perp,p'_\perp)=-\frac{g_1g_2}{(p'_\perp-q'_\perp)^2+\mu^2-i0}\ , \tag{A21}$$

so that the coordinate space form of Φ_w, the quasipotential to be used in our constraint equation (3.1), would be

$$\Phi^{(1)}_w(x_\perp)=\frac{\mathbf{V}_w(x_\perp)}{2w}=-\frac{g_1g_2}{8\pi w}\frac{e^{-\mu|x_\perp|}}{|x_\perp|}=-2m_w\alpha\frac{e^{-\mu|x_\perp|}}{|x_\perp|}\ , \tag{A22}$$

in which $g_1g_2=16\pi m_1m_2\alpha$ and $m_w=m_1m_2/w$. In order to determine the constraint potential to a higher order (say $\mathbf{V}^{(2)}$) we would first have to evaluate the corresponding single-loop diagrams $T^{(2)}_w$ (appropriately renormalized). In that case (A19) leads to

$$\tilde{V}^{(2)}_w(p_\perp,k_\perp)=-T^{(2)}w(p_\perp,k_\perp)+\int\frac{d^3k'_\perp}{(2\pi)^3}\tilde{\mathbf{V}}^{(1)}_w(p_\perp,k'_\perp)\frac{1}{2w[k'^2_\perp-b^2(w)-i\epsilon]}\tilde{\mathbf{V}}^{(1)}_w(k_\perp,k_\perp)\ . \tag{A23}$$

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$$[m/(E+2m-V)][(r_e/r^3)(1+\mathbf{s}\cdot\mathbf{l}) -(r_e^2/6r^4)(S_{12}-2\sigma_1\cdot\sigma_2)],$$

the term $\mathbf{s}\cdot\mathbf{l}$ was inadvertently programmed as $r_e\mathbf{s}\cdot\mathbf{l}$ where r_e is in fermis. The result of this reference was obtained by using this potential. When the erroneous factor is removed, although the potential shape is similar, there are no resonance states.

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[48] The doubly odd part of the quasipotential is distinctly different from the quasipotential in Refs. [4,5] (valid only for $j=l$) which involved no coupling. Equation (4.15) gives one possible correction to Eq. (172) in Ref. [5]. Other possibilities are presented in Appendix D of Ref. [46].

[49] This scaling (just like the ones below for scalar and timelike vector interactions) is a relativistic version of that occurring for the nonrelativistic Schrödinger equation which allows the decoupling of effects of scale m from those at the scale αm.

[50] For positronium our results agree with the standard results of Ref. [31] without the annihilation diagram. Reference [31] also gives the spectrum for the general hydrogenic (unequal mass) case but in the *j-j* coupling scheme, rather than the *LS* coupling scheme that we present here. This scheme is given in Ref. [30] but with a spectrum that

does not have the correct $j=l$ spin mixing fine structure even though the weak-potential bound-state equations given there are correct. The corrected spectrum appears in J. Connell, Phys. Rev. D **43**, 1393 (1991). Note that the equations Connell uses are similar to the Sazdjian equations in the Coulomb gauge. Our equation [Eq. (5.7)] uses the Feynman gauge.

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[52] Reference [30] presents a perturbative treatment of this based on the Todorov equation. The constraint equation for this will be derived in a future publication which will include a nonperturbative as well as a perturbative treatment.

[53] Note that this table and the one for muonium include many more states than are actually observed. The reason that we include so many states is that we wish to test the validity of the strong potential form of the equations for the rich variety of quantum numbers observed for quarkonia. The actual experimental spectra for positronium and muonium are, unfortunately, not as rich as those presented above. Of course for the $^3S_1$ positronium states the experimental spectrum would include the effect of the annihilation diagram (which would avoid the artificial degeneracy in the table between the $^3S_1$ and $^1P_1$ states with the same n).

[54] Alternatively, the fact that perturbative results do not require any coupling to the lower-lower components may seem surprising. Note that even for the ordinary one-body Dirac equation one can find an example for which terms that do not contribute to the spectrum in a perturbative treatment are crucial for producing the same results when the equation is treated nonperturbatively (see Sec. VI of Ref. [46]).

[55] In this paper we do not address the question of the short-distance behavior of the invariant $\mathcal{A}(r)$ which to lowest order is simply $-\alpha/r$. The additional corrections are dictated by higher-order QED. Given $\mathcal{A}$, the way in which that $\mathcal{A}$ affects the strong-potential structures of the quasipotential in the wave equations (4.14) is fixed by the original form (2.1) of the wave equation through its dependence on $G(\mathcal{A})$.

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[62] We only test the $^3P_0$ state for possible resonances since this is the only angular momentum state in which the spin-orbit term (behaving at long distance like $[j(j+1)-l(l+1)-2]/wr^3$ and at very short distance like $[j(j+1)-l(l+1)-2]/r^2$) can overwhelm the angular momentum barrier term $[l(l+1)/r^2]$.

[63] Note that this short-distance strong-potential behavior is distinctly different from the short-distance behavior produced by the effective potential used by two of us in Ref. [18]. Our effective potential (including the angular momentum barrier) in the Schrödinger-like form Eq. (7.1) of the two-body Dirac equations dictated by QED, does have an attractive $1/r^3$ part and a repulsive $1/r^4$ part but at distances intermediate between a Fermi and an Angstrom, not at still shorter distances (as appears in the effective potential hypothesized in Ref. [18]).

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[65] Note that the Todorov equation does not exactly sum up all ladder and cross ladder diagrams. It sums those contributions that survive in the eikonal limit. Consequently it does not include any radiative parts of the diagrams which would give rise to α^5 and $\alpha^5 \ln\alpha$ corrections to the spectrum. That this is so can be seen by rewriting our bound-state equations in terms of Coulomb variables ($\mathbf{x}=\epsilon_w \alpha \mathbf{r}$). α then appears quadratically guaranteeing that only even orders of α appear in the spectrum.

[66] Moreover, E. D. Cooper, and B. K. Jennings [Nucl. Phys. **A483**, 601 (1988)] have recently rederived the two-body constraint equation (2.1) using an eikonal argument.

[67] By Fermi-Breit expansion we mean, for example, that obtained by applying the Chraplyvy transformation (a two-body Foldy-Wouthuysen transformation) to the three-dimensional Bethe-Salpeter equation. See Stroscio in Ref. [17].

[68] The Schrödinger-like form of the covariant constraint equations (4.14) for the fermion-antifermion system has a weak-potential approximation given in Eq. (5.9). A canonical transformation of the $O(1/c^2)$ expansion of Eq. (5.9) (see Refs. [13] and [34]) then reproduces the Fermi-Breit equation (8.5).

[69] This has been done for two spinless particles in Ref. [13].

[70] In Ref. [16], two of us found that in the case of singlet positronium solution of our full coupled sixteen component two-body Dirac equations [Eqs. (2.1)] reduces to the solution of the minimal Todorov equation Eq. (7.2). Independently, Bohnert *et al.* [see G. Bohnert, R. Decker, A. Hornberg, H. Pilkuhn, and H. G. Schlaile, Z. Phys. D **2**, 23 (1986)] found that their method leads to the minimal Todorov equation as a second-order wave equation for singlet positronium.

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[76] H. Crater and P. Van Alstine, Phys. Rev. D **46**, 766 (1992).