Isgur-Wise symmetry in two dimensions

Matthias Burkardt and Eric S. Swanson
Center for Theoretical Physics, Laboratory for Nuclear Science, and Department of Physics, Massachusetts Institute of Technology, Cambridge, Massachusetts 02139
(Received 11 May 1992)

The approach to the heavy-quark Isgur-Wise limit is examined in (1+1)-dimensional QCD where exact calculations are possible. To facilitate the discussion, an explicit scaling equation for the mesonic Bethe-Salpeter amplitude is derived. We find that the leading finite heavy-quark mass corrections to the decay constant and the elastic form factor are comparable in strength. Nonrelativistic estimates of both of the quantities work reasonably well and generalize to 3+1 dimensions. We find that the corrections to the (3+1)-dimensional form factor at large $v \cdot v'$ are comparable to those estimated for the decay constant by lattice gauge theory and QCD sum rules. Furthermore, the structure function $f_-(v \cdot v')$ is found to be quite large at $v \cdot v'=1$ and we argue that this will remain true in 3+1 dimensions. This implies that one must exercise caution when attempting to extract Cabibbo-Kobayashi-Maskawa matrix elements from semileptonic decays. Finally, it is shown that the Isgur-Wise symmetry never breaks down in two-dimensional QCD for heavy-quark to heavy-quark transitions.

PACS number(s): 11.30.Hv, 12.38.Lg, 13.20.−v, 14.40.Jz

I. INTRODUCTION

The analysis of hadronic form factors simplifies considerably when one of the relevant quarks is heavy ($m_Q \gg \Lambda_{\rm QCD}$) because the spin and flavor degrees of freedom of the heavy quark decouple from the system. This notion has been formalized as an approximate "SU($2N_h$)⊗ Lorentz" symmetry of QCD by Isgur and Wise [1]. (N_h is the number of heavy-quark flavors.) In particular, Isgur and Wise have used this symmetry to relate heavy-quark transition matrix elements. For example, in the limit $m_{Q_i}, m_{Q_j} \to \infty$,

$$\langle P_j(v')|\bar{Q}_j \gamma_\mu Q_i|P_i(v)\rangle = \sqrt{\mu_i \mu_j} C_{ij} \xi_R(v \cdot v')(v_\mu + v'_\mu) + O\left[\frac{1}{m_{Q_i}}\right] \tag{1.1}$$

and

$$\langle P_j^*(v',\epsilon)|\bar{Q}_j \gamma_\mu Q_i|P_i(v)\rangle = \sqrt{\mu_i \mu_j^*} C_{ij} \xi_R(v \cdot v') i\epsilon_{\mu\nu\alpha\beta}\epsilon^{*\nu} v^\alpha v'^\beta + O\left[\frac{1}{m_{Q_i}}\right] . \tag{1.2}$$

In these expressions, $P_i(v)[P_i^*(v,\epsilon)]$ represents a pseudoscalar [vector] meson of mass m_i [m_i^*] and velocity v_μ (and polarization ϵ_μ) containing a heavy quark of mass m_{Q_i}. The states are normalized as

$$\langle P_j(v')|P_i(v)\rangle = 2E_i \delta_{ij} \delta^3(\bar{p} - \bar{p}') .$$

The C_{ij} are renormalization-group factors which map the effective heavy-quark theory at the scale μ_i to the scale μ_j. The explicit form of C_{ij} does not concern us here. The final factor ξ_R is the universal renormalized Isgur-Wise function (once again the specifics of the renormalization do not concern us). The Isgur-Wise function contains all of the physics carried by the light degrees of freedom in the decay and may not be calculated in perturbation theory. However, it is known that the Isgur-Wise function is normalized at zero recoil because the vector current is conserved: $\xi_R(v \cdot v'=1)=1$.

The relationships between the heavy-quark transition matrix elements and the normalization condition greatly facilitate the extraction of Cabibbo-Kobayashi-Maskawa (CKM) matrix elements from measurements of the semileptonic decays of heavy mesons. Thus it is important to assess the size of the $1/m_{Q_i}$ corrections to the Isgur-Wise limit to determine the region of applicability of the heavy-quark methodology. At present, the only feasible ways to test the symmetry limit are with QCD sum-rule techniques [2] or numerically on the lattice [3]. In this paper we adopt the more modest program of examining the heavy-quark limit of two-dimensional SU(N_c) gauge theory in the large-N_c limit (the 't Hooft model). The goal is to examine the approach to the heavy-quark limit in a theory which may be solved exactly. The hope is that this will serve to illustrate the nature of the heavy-quark symmetry in 3+1 dimensions.

The 't Hooft model and its heavy-quark and nonrelativistic limits are described in Sec. II. We discuss the Isgur-Wise limit and its corrections in Sec. III. We also speculate on generalizations to 3+1 dimensions and discuss the possible breakdown of Isgur-Wise symmetry in 1+1 dimensions. Conclusions are presented in Sec. IV.

II. ISGUR-WISE SYMMETRY IN THE 't HOOFT MODEL

A. 't Hooft model

Two-dimensional Yang-Mills theory was introduced by 't Hooft in 1974 [4] as a relativistic field theory which exhibits confinement. The theory simplifies considerably in light-cone coordinates $v_\pm = (1/\sqrt{2})(v_0 \pm v_1)$, where v_μ is any vector $(x_\mu, \gamma_\mu, A_\mu, \ldots)$. The light-cone gauge $A_- = A^+ = 0$ is particularly useful because ghosts decouple and there are no gluon self-interactions in this gauge. Furthermore, the gluons may be eliminated from the theory, leaving in their stead a linearly rising "Coulomb" potential [5]:

$$\mathcal{L} = \sum_f \bar{\psi}_f \left[i\partial\!\!\!/ - e\frac{\lambda^a}{2} A_a^- \gamma^+ - m_f \right] \psi_f + \frac{e^2}{4} \int dy^- V^+(x^+, x^-) |x^- - y^-| V^+(x^+, y^-) . \tag{2.1}$$

Because the theory is two dimensional, rotations are impossible and hence there is no notion of spin. Furthermore, the γ matrices are 2×2 and hence only four independent components may be formed out of fermionic bilinears:

$$S = \bar{\psi}\psi , \tag{2.2a}$$

$$P = \bar{\psi}\gamma_5\psi , \tag{2.2b}$$

$$V_\mu = \bar{\psi}\gamma_\mu\psi , \tag{2.2c}$$

$$A_\mu = \bar{\psi}\gamma_\mu\gamma_5\psi = -\epsilon_{\mu\nu}V^\nu , \tag{2.2d}$$

$$T_{\mu\nu} = \bar{\psi}\sigma_{\mu\nu}\psi = -i\epsilon_{\mu\nu}P . \tag{2.2e}$$

' t Hooft has shown that only planar graphs contribute to matrix elements in the large-N_c limit. This, combined with the absence of gluon self-interactions, means that the ladder approximation becomes exact so that the Bethe-Salpeter amplitude for a meson satisfies the simple equation (the "'t Hooft equation")

$$\mu_n^2 \phi_n(x) = \left[\frac{m_R^2}{x} + \frac{M_R^2}{1-x} \right] \phi_n(x) - \frac{g^2}{\pi} ⨍_0^1 \frac{\phi_n(y)}{(x-y)^2} dy , \tag{2.3}$$

with the boundary condition

$$\phi_n(0) = \phi_n(1) = 0 . \tag{2.4}$$

The momenta of the meson (mass μ_n), the constituent antiquark (mass m_R), and the constituent quark (mass M_R) are taken to be p_μ, q_μ, and $p_\mu - q_\mu$, respectively. The momentum fraction carried by the antiquark is then $x \equiv q_-/p_-$ (and $1-x$ for the quark). The coupling is defined in terms of the large-N_c limit:

$$g^2 = \lim_{N_c \to \infty} e^2 \frac{N_c^2 - 1}{N_c} . \tag{2.5}$$

There are no sea quarks present in the large-N_c limit. Fortunately, this is not a problem since the amplitude for the production of a virtual $Q\bar{Q}$ pair scales as $1/m_Q^2$. Thus sea quarks do not affect the Isgur-Wise limit or the $1/M_Q$ corrections to it. The "renormalized" quark mass is defined as

$$m_R^2 = m^2 - \frac{g^2}{\pi} , \tag{2.6}$$

where m is the parameter appearing in the Lagrangian and the last term represents the exact quark self-energy. The slash notation in the integral signifies that the infrared divergences in the integrand have been regulated with the principal-value prescription [6]

$$\frac{1}{p^2} = \lim_{\epsilon \to 0} \frac{1}{2} \left[\frac{1}{(p+i\epsilon)^2} + \frac{1}{(p-i\epsilon)^2} \right] . \tag{2.7}$$

Since the model is two dimensional, it is super-renormalizable and the coupling carries units of energy. We have fixed the scale by solving 't Hooft's equation (2.3) and equating the slope of μ_n^2 with respect to n to the leading Regge trajectory value of 0.18 GeV^2. The result is $g = 608$ MeV. Henceforth, we set $g^2/\pi = 1$, so that the bare quark masses are roughly $m_u = 0.02$, $m_d = 0.03$, $m_s = 0.5$, $m_c = 4.3$, and $m_b = 14.5$.

Callan, Coote, and Gross [7] have derived the independent current matrix elements as follows:[1]

$$\langle 0 | V_- | n(p) \rangle = p_- \left[\frac{N_c}{\pi} \right]^{1/2} \int_0^1 \phi_n(x) dx , \tag{2.8}$$

$$\langle 0 | V_+ | n(p) \rangle = -p_+ \frac{mM}{\mu_n^2} \left[\frac{N_c}{\pi} \right]^{1/2} \int_0^1 \frac{\phi_n(x)}{x(1-x)} dx , \tag{2.9}$$

$$\langle 0 | S | n(p) \rangle = \frac{1}{2} \left[\frac{N_c}{\pi} \right]^{1/2} \int_0^1 \left[\frac{m}{x} - \frac{M}{1-x} \right] \phi_n(x) dx , \tag{2.10}$$

$$\langle 0 | P | n(p) \rangle = \frac{1}{2} \left[\frac{N_c}{\pi} \right]^{1/2} \int_0^1 \left[\frac{m}{x} + \frac{M}{1-x} \right] \phi_n(x) dx . \tag{2.11}$$

Vector-current decay constants are defined by

$$\langle 0 | V_\mu | n(p) \rangle = \frac{f_n}{\sqrt{2\mu_n}} \begin{cases} p_\mu, & \text{even-parity meson ,} \\ \epsilon_{\mu\nu} p^\nu, & \text{odd-parity meson .} \end{cases} \tag{2.12}$$

Meson form factors are defined in terms of the matrix elements given below:

$$\langle m(v') | V_\mu | n(v) \rangle = \sqrt{\mu_n \mu_m} [f_+^{nm}(v \cdot v')(v_\mu + v'_\mu) + f_-^{nm}(v \cdot v')(v_\mu - v'_\mu)] , \tag{2.13}$$

---

[1]We have restored factors of $\frac{1}{2}$ in the matrix elements of the scalar and pseudoscalar currents which were missed in Ref. [7]. The factor $\sqrt{N_c/\pi}$ will be suppressed in the following discussion.

$$\langle m(v')|S|n(v)\rangle=\sqrt{\mu_n\mu_m}f_S^{nm}(v\cdot v') , \quad (2.14)$$

$$\langle m(v')|P|n(v)\rangle=\sqrt{\mu_n\mu_m}f_P^{nm}(v\cdot v')\epsilon_{\mu\nu}v^{\mu}v'^{\nu} . \quad (2.15)$$

Each current is understood to annihilate a heavy quark of mass M and create one of mass M', and we have chosen to label the mesons in terms of four-velocities for later convenience. These definitions hold for like parity mesons; similar ones hold for opposite-parity mesons. The explicit expressions for these matrix elements are rather long and hence are given in the Appendix.

B. Heavy-quark limit

One may apply the effective-field theory trace formalism of Falk *et al.* [8] to obtain the Isgur-Wise limits of the meson form factors:

$$f_+^{nm}(v\cdot v')\to\xi^{nm}(v\cdot v') , \quad (2.16)$$

$$f_-^{nm}(v\cdot v')\to 0 , \quad (2.17)$$

$$f_S^{nm}(v\cdot v')\to\xi^{nm}(v\cdot v')(1+v\cdot v') , \quad (2.18)$$

$$f_P^{nm}(v\cdot v')\to\xi^{nm}(v\cdot v') . \quad (2.19)$$

Once again, the theory is superrenormalizable and so there are no renormalization factors in these relationships.

The expressions for the decay constant and form factors must be evaluated by numerically solving 't Hooft's equation and then performing the appropriate integrations. To do this we employ the method outlined in Ref. [9]. Unfortunately, the numerics become increasingly difficult for large M; thus, it is useful to derive an equation which describes the heavy-quark limit directly. To do this we set $\mu_n^2=M_R^2+2M_R\delta\mu_n+O(M_R^0)$, expand the right-hand side of 't Hooft's equation (2.3) to order M_R, and equate to get

$$2\delta\mu_n\chi_n(z)=\left[\frac{m^2}{z}+z\right]\chi_n(z)+\frac{g^2}{\pi}\int_0^{\infty}\frac{\chi_n(z)-\chi_n(y)}{(z-y)^2}dy , \quad (2.20)$$

where the scaling wave function is defined by

$$\chi_n(z)=\lim_{M_R\to\infty}\frac{1}{\sqrt{M_R}}\phi_n\left[\frac{z}{M_R}\right] . \quad (2.21)$$

Note that the scaling equation is just as complicated as the full 't Hooft equation. The end-point behavior of the scaling wave function is given by

$$\chi_n(z)\to c_n z^{\beta} \text{ for } z\to 0 \quad (2.22)$$

and

$$\chi_n(z)\to d_n z^{-3} \text{ for } z\to\infty . \quad (2.23)$$

The power in Eq. (2.22) is defined by the solution to

$$m_R^2+g^2\beta\cot\beta\pi=0 , \quad (2.24)$$

for $\beta\in(0,1)$. To illustrate the approach of the ground-state wave function to the scaling wave function, we have plotted

$$\tilde{\phi}_0^M(x)\equiv\frac{1}{\sqrt{M}}\phi_0\left[\frac{1}{M}\frac{x}{1-x}\right] , \quad (2.25)$$

for various M in Fig. 1. The scaling functions are related by

$$\chi_n(z)=\lim_{M\to\infty}\tilde{\phi}_n^M\left[\frac{z}{1+z}\right] . \quad (2.26)$$

The Isgur-Wise limit of the heavy-quark decay constant is given in terms of the scaling wave function by

$$f_n^{\infty}\equiv\lim_{M_R\to\infty}f_n=\sqrt{2}\int_0^{\infty}\chi_n(z)dz . \quad (2.27)$$

Note that the scaling equation (2.20) and the boundary condition (2.23) imply that

$$f_n^{\infty}=\sqrt{2}\frac{\pi}{g^2}d_n . \quad (2.28)$$

The Isgur-Wise function is given by

$$\xi^{nm}(v\cdot v')=\frac{2\omega}{1+\omega}\int_0^{\infty}\chi_n(\omega z)\chi_m(z)dz , \quad (2.29)$$

where

$$\omega_{\pm}=(v\cdot v')\pm[(v\cdot v')^2-1]^{1/2} . \quad (2.30)$$

Note that either root may be used in Eq. (2.29) (see the discussion in Sec. III C). At zero recoil, $v\cdot v'=1$, so that $\omega=1$ and hence $\xi^{nm}(1)=\delta^{nm}$, as expected.

It is also instructive to consider the nonrelativistic limit of the 't Hooft model. The 't Hooft equation becomes

$$-\frac{1}{2\mu}\frac{d^2}{dy^2}\phi_n^{\rm NR}(y)+\frac{g^2}{2}|y|\phi_n^{\rm NR}(y)=E_n\phi_n^{\rm NR}(y) , \quad (2.31)$$

where $\mu=m_RM_R/(m_R+M_R)$, $\phi_n^{\rm NR}(y)$ is the nonrelativistic wave function in configuration space, and E_n is the binding energy of the meson. The solutions to Eq. (2.31) are Airy functions

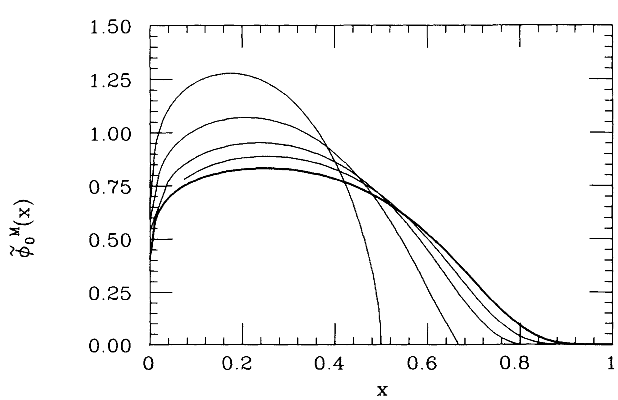

FIG. 1. Approach of the wave function to the scaling limit. From top to bottom, the curves represent wave functions with $M=1$, 2, 4, 8, and ∞, respectively. The light-quark mass has been fixed at 0.1.

$$\phi_n^{\rm NR}(y)=\begin{cases} N_n \mathrm{Ai}[(\mu g^2)^{1/3}|y|-\lambda_n] , & \phi_n^{\rm NR} \text{ even} , \\ \epsilon(y) N_n' \mathrm{Ai}[(\mu g^2)^{1/3}|y|-\rho_n], & \phi_n^{\rm NR} \text{ odd} , \end{cases} \tag{2.32}$$

where

$$\epsilon(y)=\begin{cases} +1, & y>0 \\ -1, & y<0 , \end{cases} \tag{2.33}$$

N_n and N_n' are renormalization constants, λ_n satisfies $\mathrm{Ai}'(-\lambda_n)=0$, and ρ_n satisfies $\mathrm{Ai}(-\rho_n)=0$ ($\lambda_1=1.018\,79$; $\rho_1=2.338\,20$). The eigenenergies are given by

$$E_n=\left[\frac{g^4}{8\mu}\right]^{1/3}\begin{cases} \lambda_n, & \phi_n^{\rm NR} \text{ even} , \\ \rho_n, & \phi_n^{\rm NR} \text{ odd} . \end{cases} \tag{2.34}$$

III. APPROACH TO THE SCALING LIMIT

A. Decay constant

We now use the exact expressions for the current matrix elements of Sec. II to examine the approach to the Isgur-Wise scaling limit. Consider first the large-M behavior of the decay constant defined by

$$f_n(m;M)=f_n^\infty\left[1+\frac{c_n(m)}{M}+\frac{d_n(m)}{M^2}+\cdots\right] . \tag{3.1}$$

The approach to the scaling limit is displayed in Fig. 2. Figure 3 shows f_0^∞ and the nonrelativistic approximation to this as a function of light-quark mass. The nonrelativistic approximation is very accurate for $m \gtrsim 300$ MeV, with roughly 100% error at $m=m_u$. Note that f_0^∞ is practically constant for $m \lesssim 400$ MeV; this is a reflection of flavor symmetry for quark species with $m \ll g$.

Values for $c_n(m)$ were calculated by extrapolating the finite differences $2M/f_n^\infty[f_n(M)-f_n(2M)]$ to infinite heavy-quark mass. Typical values of M used were $M=16$, 32, 64, and 128. The results were verified by means of a perturbative expansion of Eq. (2.3) in $1/M$. Statistical and systematic errors are roughly at the 1% level.

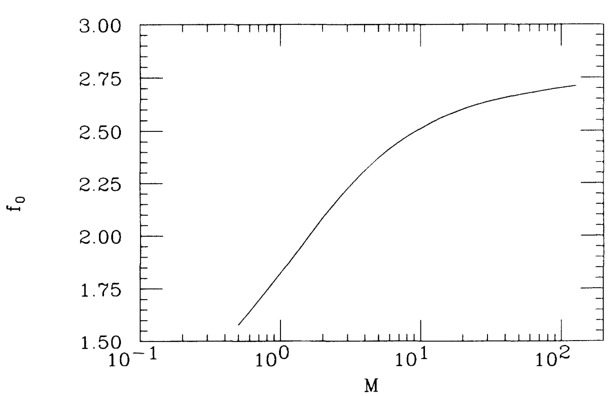

FIG. 2. Scaling limit of the decay constant. The light-quark mass has been fixed at 0.1.

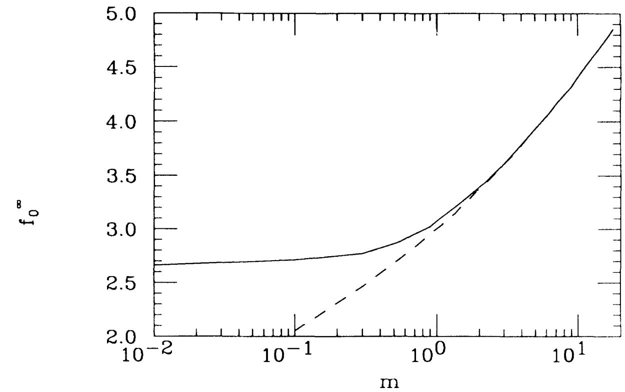

FIG. 3. Scaling decay constant vs light-quark mass. The solid line is the exact result, while the dashed line is the nonrelativistic expression $f_0^\infty=\sqrt{2\pi/\lambda_1}(mg^2)^{1/6}$.

We plot c_0 versus the light-quark mass in Fig. 4. Once again, $c_0(m)$ is roughly independent of the light-quark mass for $m \lesssim 300$ MeV. The sign and magnitude of the $1/M$ correction may be understood in the nonrelativistic limit of the 't Hooft model ($g \ll m,M$). In this regime the $1/M$ correction reflects the dependence of the wave function at the origin on the reduced mass $mM/(m+M)$. This yields $c_{\rm NR}=-m/6$, which is shown as a dashed line in Fig. 4. The convergence of c to $c_{\rm NR}$ is rather slow. Perhaps this is not surprising since an annihilation process is intrinsically relativistic.[2] In 3+1 dimensions a similar argument yields $\sqrt{m+M}f \sim [mM/(m+M)]^{3/2}$, which gives $c_{\rm NR}^{3+1}=9c_{\rm NR}^{1+1}$. (Neubert [10] has argued that this reduced mass effect does indeed comprise most of the $1/M$ coefficient in 3+1 dimensions.)

The subleading correction to f_0^∞ is also constant for $m \lesssim 300$ MeV and grows rapidly in m for $m \gtrsim 400$ MeV. Numerically d_0 is roughly $(0.5)^2$ for $m<1$ and $d_0 \sim (0.3m)^2$ for large m. Thus c_0 and d_0 are comparable for all light-quark masses. The decay constant and its corrections have also been calculated by Grinstein and Mende [11]. Their result for $f_0^\infty(m=0.55)$ agrees with ours; however, their value for c_0 disagrees with ours by a factor of 2.

Lattice gauge theory [3] and QCD sum-rule calculations [2] in four dimensions yield $c_0^{3+1} \simeq -800$ MeV, roughly 4 times larger than the value in two dimensions.

B. Elastic form factor

We now turn our attention to the elastic form factor. The expressions for the form factors given in the Appendix are written in terms of the light-cone momentum fraction

$$x=\frac{q_-}{p_-} . \tag{3.2}$$

[2]Note that introducing an effective light-quark mass such that $m_{\rm eff}>m$ would improve the correspondence between c and $c_{\rm NR}$.

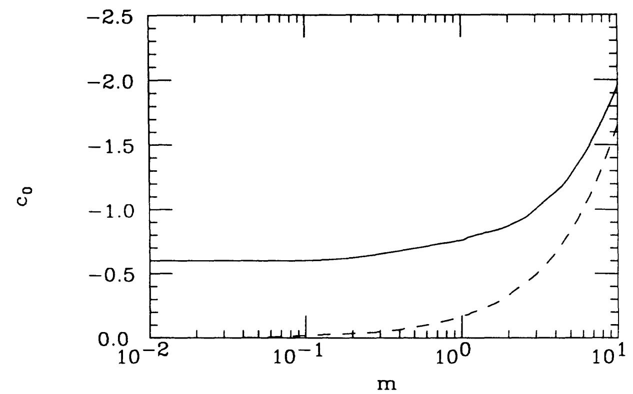

FIG. 4. $1/M$ correction to the decay constant vs light-quark mass. The solid line is the exact result, while the dashed line is the nonrelativistic expression.

When q^2 is spacelike, there exists an $x\in[0,1]$ such that conservation of momentum may be expressed in the covariant form

$$\mu_n^2=\frac{q^2}{x}+\frac{\mu_m^2}{1-x} . \tag{3.3}$$

The reader is referred to Fig. 5 for the definitions of the various momenta. In this figure the heavy (light) lines signify heavy- (light-) quark propagators, while mesons are represented by double lines. In the case of the elastic form factor, x may be expressed in terms for $v\cdot v'$ as

$$x=1-v\cdot v'\pm[(v\cdot v')^2-1]^{1/2} . \tag{3.4}$$

The ground-state Isgur-Wise function is shown in Fig. 6 for several light-quark masses. Note that the Isgur-Wise functions have a power-law dependence on $v\cdot v'$ for $v\cdot v\gg 1$. This may be understood as follows. The limit $v\cdot v'\to\infty$ corresponds to $q^2\to-\infty$, which implies that $x\to 1$, and hence the momentum fraction of the initial heavy quark is also driven to unity $[y\equiv x+(1-x)z$ (see Fig. 5)]. Now, as $y\to 1$, $\phi_n(y)\to c(1-y)^{\beta_m}$, where β_m is defined in terms of the *light*-quark mass via the equation

$$m^2-\frac{g^2}{\pi}+g^2\beta_m\cot\beta_m\pi=0 , \tag{3.5}$$

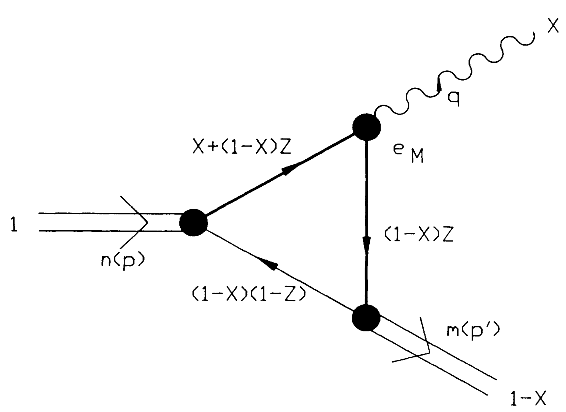

FIG. 5. Meson form-factor kinematics.

in analogy with Eq. (2.24). This yields the large $v\cdot v'$ behavior of the form factor [6]

$$\langle n(p')|V_-|n(p)\rangle\to 2e_M c[2(v\cdot v'-1)]^{-1-\beta_m} \times\left\{\int_0^1\phi_n(z)(1-z)^{\beta_m}dz+O(M^{-2})\right\}. \tag{3.6}$$

Thus we expect

$$\xi^{nm}(m;v\cdot v')\to G(m)(v\cdot v')^{-1-\beta_m} , \tag{3.7}$$

where G is a function of the light-quark mass only. This is indeed the behavior seen in Fig. 6.

Note that the form factor gets harder for decreasing light mass. For small $v\cdot v'$ the reason is purely kinematical; indeed,

$$\xi^{nm}(v\cdot v')\sim 1+\tfrac{1}{2}\langle r_H^2\rangle_n q^2 , \tag{3.8}$$

where r_h is the heavy-quark radius. Nonrelativistically, $r_H=[m/(m+M)]r$, where r is the relative coordinate and

$$\langle r^2\rangle_n=\kappa_n\left[g^2\frac{mM}{m+M}\right]^{-2/3} , \tag{3.9}$$

with

$$\kappa_n=\left[\frac{8}{15}\lambda_n^2+\frac{1}{5\lambda_n}\right] , \tag{3.10}$$

for even wave functions [12]. Thus

$$\xi^{nm}(v\cdot v')\sim 1-\kappa_n\left[\frac{m}{g}\right]^{4/3}(v\cdot v'-1) \tag{3.11}$$

and the Isgur-Wise function does get larger for smaller light-quark mass. Equation (3.11) also agrees very well with the numerical results of Fig. 6. The situation is different in four dimensions:[3] $\langle r^2\rangle\sim m^{-2}$ and $\xi\sim 1-k(v\cdot v'-1)$, where k is a dimensionless constant. Thus the Isgur-Wise function is insensitive to the light-quark mass when it is heavy. However, as the light-quark mass decreases, the QCD scale will make its presence felt and this argument no longer applies.

Figure 7 shows the $1/M$ correction to the ground-state Isgur-Wise function, which is defined by

$$f^{nn'}(m,M;v\cdot v')\sim\xi^{nn'}(m;v\cdot v') \times\left[1+\frac{\delta\xi^{nn'}}{M}(m;v\cdot v')+\cdots\right] . \tag{3.12}$$

For $v\cdot v'\gg 1$, $\delta\xi^{00}$ is independent of $v\cdot v'$. This follows simply from Eq. (3.7)—for large $v\cdot v'$, the velocity and quark mass dependence factor in the current matrix element and hence $\delta\xi^{nn'}$ is independent of $v\cdot v'$ for

[3]This is because the kinetic energy scales as $m^{-1/3}$ in 1+1 dimensions and m in 3+1.

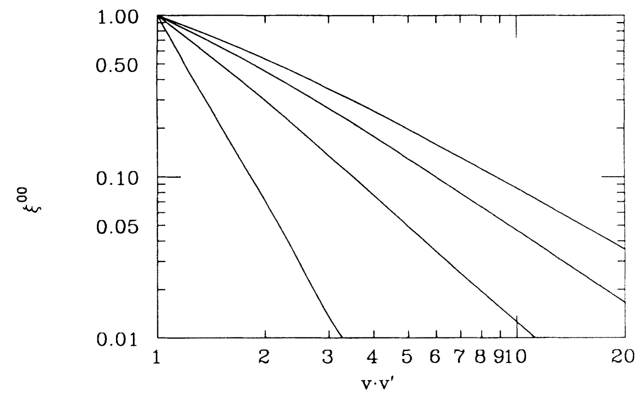

FIG. 6. ξ^{00} vs $v\cdot v'$. From top to bottom, the curves correspond to $m=\frac{1}{2}$, 1, 2, and 5, respectively.

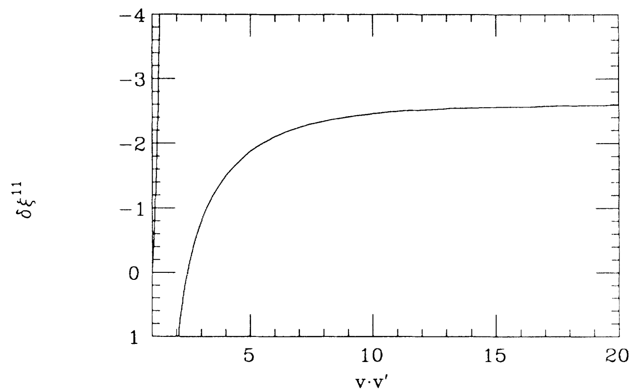

FIG. 8. $\delta\xi^{11}$ vs $v\cdot v'$ for $m=1$.

sufficiently large $v\cdot v'$.

The $1/M$ correction for the "p-wave" mesons is displayed in Fig. 8. Note that $\delta\xi^{11}$ diverges near zero recoil. This is due to the zero in ξ^{11}, which must appear. Thus one should not expect Isgur-Wise symmetry to give accurate relative predictions near zero recoil for excited mesons. Even though ξ^{11} is negative for large $v\cdot v'$, it is approached from below (in M) for $v\cdot v'\lesssim 2.4$ and from above for $v\cdot v'\gtrsim 2.4$, and hence $\delta\xi^{11}$ becomes a negative constant for sufficiently large $v\cdot v'$.

The general behavior of $\delta\xi^{nn}$ may be elucidated by considering the nonrelativistic limit of the heavy-quark form factor. In this limit the heavy-quark form factor is simply the Fourier transform of the heavy-quark density. This may be related to the quark-antiquark density through the relation $r_H=[m/(m+M)]r$:

$$f_{\rm NR}^{nn}(Q^2)=\int dr_H e^{iQr_H}\rho_n^H(r_H)$$
$$=\int dr\, e^{i[Qm/(m+M)]r}\rho_n(r) \qquad (3.13)$$

[we have absorbed a factor of $m/(m+M)$ in the definition of $\rho_n(r)$]. Now $\rho_n(r)=|\phi_n^{\rm NR}(r)|^2$, where $\phi_n^{\rm NR}$ are the wave functions given in Eq. (2.32). Thus, in the nonrelativistic limit, the only heavy-quark mass dependence in f^{nn} arises from the reduced mass in the Schrödinger equation (2.31). Dimensional analysis shows that

$$\phi_n^{\rm NR}(r)=(\mu g)^{1/6}\hat{\phi}_n[(\mu g^2)^{1/3}r] ,$$

where the function $\hat{\phi}_n$ has no implicit scale dependence (μ is the reduced mass here). Thus we have the relation

$$f_{\rm NR}^{nn}(v\cdot v')=\hat{f}_n[m^2(\mu g^2)^{-2/3}(v\cdot v'-1)] , \qquad (3.14)$$

where $\hat{f}_n$ does not depend on m, μ, or g^2. Expanding this in powers of $1/M$ yields the result

$$\delta\xi_{\rm NR}^{nn}=\frac{2}{3}m(v\cdot v'-1)\frac{d}{d(v\cdot v')}\ln\xi^{nn}(v\cdot v') . \qquad (3.15)$$

Note that Luke's theorem [13] (there are no $1/M$ corrections at zero recoil to order α_s/M) is incorporated in this result since $\delta\xi$ vanishes for $v\cdot v'=1$.[4]

We may use Eqs. (3.10), (3.11), and (3.15) to estimate the slope of $\delta\xi^{nn}$ at zero recoil:

$$\lim_{v\cdot v'\to 1}\frac{d\delta\xi^{nn}}{d(v\cdot v')}\simeq-\kappa_n\left[\frac{m}{g}\right]^{4/3}\frac{2}{3}m . \qquad (3.16)$$

Comparison with Fig. 7 shows that this expression works very well for $m=5$, but becomes much too small for $m<1$.

For large $v\cdot v'$, $\delta\xi^{00}$ may be approximated with the aid of Eq. (3.7) with the result

$$\delta\xi^{00}-\tfrac{2}{3}m(1+\beta_m) . \qquad (3.17)$$

To be consistent with the nonrelativistic limit, we should use $\beta_m\simeq 1$ for $m\gg g$ and hence $\delta\xi^{00}\simeq-\frac{4}{3}m$. Comparison with Fig. 7 shows that this result works quite well over the range of light-quark masses which we tested. (It is somewhat peculiar that $\delta\xi=-\frac{3}{2}m$ works extremely well.)

The same sort of analysis may be carried out for $\delta\xi^{11}$. In this case, $\xi^{11}\sim\kappa(v\cdot v'-1)^{-3}$ and hence

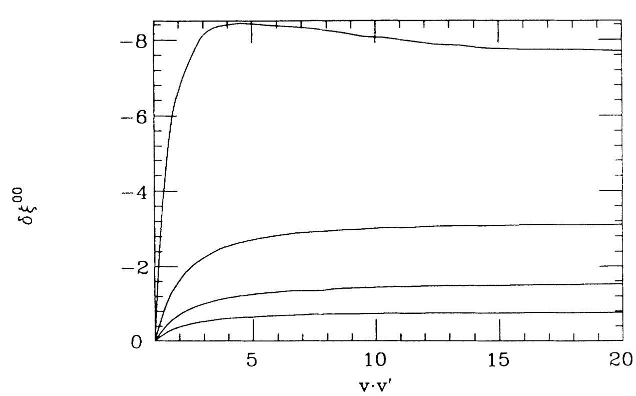

FIG. 7. $\delta\xi^{00}$ vs $v\cdot v'$. From top to bottom, the curves correspond to $m=5$, 2, 1, and $\frac{1}{2}$, respectively.

[4]This is true unless ξ' diverges at $v\cdot v'=1$. However, the condition $\xi(1)=1$ then implies that ξ' is discontinuous—a physically unreasonable situation.

$$\delta\xi^{11} \simeq -2m \ . \tag{3.18}$$

The actual value for $m=1$ is roughly -2.6 (see Fig. 8) so that the nonrelativistic approximation seems to serve as a reasonable estimate of the $1/M$ corrections to the Isgur-Wise limit even though $m=1$ is still far from the nonrelativistic regime.

It is interesting to carry out the same analysis in $3+1$ dimensions. For $m, M \gg \Lambda_{\rm QCD}$, the only scale is the reduced mass so that $\phi_{3+1}^{\rm NR} \sim \mu^{3/2}\phi(\mu r)$ and

$$\delta\xi_{3+1}^{nn} \sim 2m(v\cdot v'-1)\frac{d}{d(v\cdot v')}\ln\xi_{3+1}^{nn}(v\cdot v') \ . \tag{3.19}$$

Once again, this expression incorporates Luke's theorem. For large $v\cdot v'$ power counting [14] implies that $\xi_{3+1}^{00} \sim \kappa(v\cdot v'-1)^{-2}$ and hence

$$\delta\xi_{3+1}^{00} \sim -4m \ . \tag{3.20}$$

In this case one should interpret m as being the mass of the light constituents of the meson. If the $1+1$ calculations serve as a guide to the importance of relativistic effects on these predictions, we may reasonably say that the $1/M$ corrections in $3+1$ dimensions roughly follow the prediction of Eq. (3.20). Thus we estimate the corrections to the decay constant to be

$$\frac{\delta\xi_{3+1}}{M} \sim \begin{cases} -30\% & \text{for } B \text{ mesons ,} \\ -80\% & \text{for } D \text{ mesons,} \\ -130\% & \text{for } D_s \text{ mesons ,} \\ -200\% & \text{for } K \text{ mesons .} \end{cases} \tag{3.21}$$

Of course, smaller corrections are expected near zero recoil.

We have already seen that the $1/M$ corrections to the decay constant and form factor are comparable in $1+1$ dimensions. Thus it is interesting to compare the predictions of Eq. (3.21) to nonperturbative estimates of the correction to the decay constant in $3+1$ dimensions. Results from lattice gauge theory [3] indicate corrections of 15–30 % for f_B and 40–100 % for f_D. The agreement is satisfactory. This suggests that Eq. (3.19) may serve as a useful guide in $3+1$ dimensions and that the rough equality of the decay-constant and form-factor corrections seen in $1+1$ dimensions may also hold in $3+1$ dimensions. (The reader may recall that the nonrelativistic estimate of the $1/M$ correction to the decay constant in $3+1$ dimensions is 9 times larger than in $1+1$ dimensions. However, $\delta\xi_{3+1}$ is only 3 times larger than $\delta\xi_{1+1}$. This seems to contradict the conclusion which has just been reached. The resolution is that we have been discussing the case of small light-quark mass where the nonrelativistic estimates of the correction to the decay constant are not accurate.)

C. Inelastic form factor

The structure function $f_-(v\cdot v')$ is not constrained to be zero for inelastic transitions. This represents a slight technical difficulty as two independent quantities are required to evaluate $f_+(v\cdot v')$ and $f_-(v\cdot v')$. Of course, one could use the expression for F_+ given in the Appendix; however, this is the "bad" component of F_μ and hence is numerically difficult to evaluate [15]. Instead, we make use of the fact that f_+ and f_- are functions of $v\cdot v'$ only. Now Eq. (2.30) implies that $2v\cdot v' = \omega + 1/\omega$. [Equation (2.30) is for elastic scattering; however, it still holds for the inelastic case if we define $\omega = (M/M')(1-x)$.] This, along with the relationship $\omega_+\omega_- = 1$, implies that f_+ and f_- are independent of the root chosen for the variable ω. However, $F_- = F_-(x_\pm; M, M')$, where $x_\pm$ are the roots of Eq. (3.3). This allows one to extract f_+ and f_- solely from F_- in the timelike region. [Physically, the amplitudes $F_-(x_\pm; M, M')$ represent left- or right-moving final meson states.] Thus one has

$$f_+^{nm}(v\cdot v') = \frac{\omega_+ F_-^{nm}(x_-; M, M') + F_-^{nm}(x_+; M, M')}{2(1+\omega_+)p_-\sqrt{M'/M}} \tag{3.22}$$

and

$$f_-^{nm}(v\cdot v') = \frac{F_-^{nm}(x_+; M, M') - \omega_+ F_-^{nm}(x_-; M, M')}{2(1-\omega_+)p_-\sqrt{M'/M}} \ . \tag{3.23}$$

In the infinite heavy-quark mass limit, f_-^{nm} is zero. Furthermore,

$$F_-^{nm}(x; M, M') \to 2\left[\frac{M'}{M}\right]^{1/2} p_-\omega\int_0^\infty \chi_n(\omega z)\chi_m(z)dz \tag{3.24}$$

(see the Appendix). Thus one has the relationship

$$\int_0^\infty \chi_n(\omega z)\chi_m(z)dz = (-)^{m+n}\int_0^\infty \chi_n(z)\chi_m(\omega z)dz \ , \tag{3.25}$$

which is valid for all $\omega > 0$. This is one of the remarkable identities of $1+1$ QCD which must hold in order that physical observables be invariant under discrete Lorentz symmetries.

Note that Eq. (3.24) implies that ξ^{nm} is a function of ω only [see Eq. (2.29)] and hence does not depend on the heavy-quark masses M or M' or their ratio. This has the interesting consequence that the Isgur-Wise symmetry never breaks down for heavy-quark to heavy-quark transitions regardless of the value of $v\cdot v'$. One may not have expected this since the heavy quark may experience large recoil in decays with $M/M' \gg 1$, thereby ruining the static color source picture of Isgur-Wise symmetry. (In other works, ξ^{nm} could depend parametrically on M/M'.) In $3+1$ dimensions, the mass- and spin-dependent hyperfine structure of one gluon exchange would also be probed, once again running the flavor and spin decoupling one expects for heavy quarks. In our case the effect of the heavy-quark recoil is included in the mesonic wave function and this does *not* cause a breakdown of the Isgur-Wise symmetry in $1+1$ dimensions. In a similar vein, Isgur [16] has argued that parametric

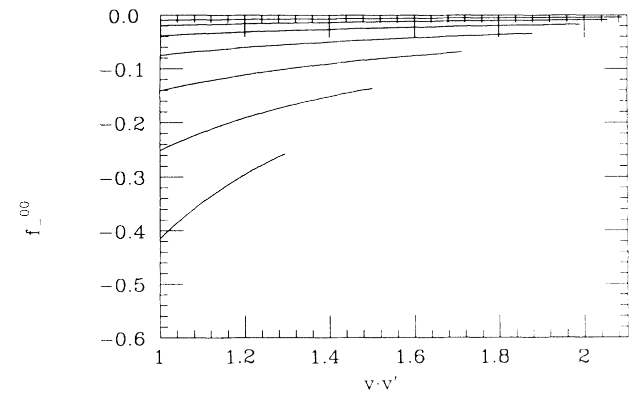

FIG. 9. f_-^{00} vs $v \cdot v'$ for $M/M'=4$ and $m=1$. The curves correspond to $M'=1$, 2, 4, 8, 16, 32, and 64 from bottom to top.

dependence of the transverse momentum distribution on the heavy-quark mass will not upset the heavy-quark symmetry in $3+1$ dimensions. Of course, our results do not have anything to say about the damaging effects of the hyperfine interaction in $3+1$ dimensions.

Figure 9 is a plot of f_-^{00} for $M/M'=4$ (roughly appropriate for $\bar{B}\to Dl\bar{\nu}$) over the kinematic range of $v\cdot v'$ appropriate to decays (timelike q^2).[5] In general, f_-^{00} becomes larger as the ratio M/M' becomes larger or as m becomes larger. For $m \lesssim 300$ MeV, f_-^{00} loses its dependence on the light-quark mass just as we have seen for the decay constant and as is demanded by flavor symmetry for light quarks.

One sees that f_-^{00} is quite large at zero recoil. This may be attributed to the proximity of the pole in the form factor at $v\cdot v'=0$ which exists when $M/M'\gg 1$. This pole does not affect f_+ because it is protected by the normalization condition and Luke's theorem [13].

The preceding discussion applies equally well in $3+1$ dimensions so that one expects f_- to be quite large near zero recoil. This means that one must be careful in using the normalization condition $\xi(1)=1$ to extract CKM matrix elements from semileptonic decays. In particular, a decay such as $\bar{B}\to Dl\bar{\nu}_l$ will have a large contribution from f_- at zero recoil. This is *not* true, however, for a decay like $\bar{B}\to D^*l\bar{\nu}_l$, which kinematically suppresses the contribution from f_- at zero recoil [17].

IV. CONCLUSIONS

We have calculated decay constants and elastic and inelastic form factors in the 't Hooft model (which is a fully relativistic field theory in the large-N_c limit). The calculation and elucidation of the results were facilitated by the introduction of the explicit scaling limit of the 't Hooft model introduced in Sec. II. It was found that the $1/M$ corrections to the heavy-quark limit of the ground-state decay constant and elastic form factor are comparable in size, obey flavor symmetry, and are reasonably well approximated by simple nonrelativistic scaling relations. Furthermore, the explicit form of the Isgur-Wise functions shows that Isgur-Wise symmetry never breaks down in $1+1$ dimensions for heavy-quark to heavy-quark transitions. Finally, it was demonstrated that $f_-(v\cdot v'=1)$ can be quite large—with immediate consequences for attempts at extracting V_{cb} from $\bar{B}\to D$ decays in $3+1$ dimensions.

We have argued that it is possible to abstract several of the features found in $1+1$ QCD to $3+1$ dimensions. Thus, for example, we expect that the corrections to the decay constant and Isgur-Wise function will be of comparable strength and that these may be roughly estimated with a nonrelativistic scaling argument. Thus we estimate -30% corrections for B mesons, -80% for D mesons, and -200% corrections for K mesons. These numbers are in accord with lattice calculations. Finally, it was argued that parametric dependence of the mesonic wave functions on the large quark mass will not upset Isgur-Wise symmetry in $3+1$ dimensions for large $v\cdot v'$ (however, we have nothing to say about hyperfine effects).

Note added in proof. In the "note added" of Ref. [11] it is stated that the contribution of f_- to $\bar{B}\to De\bar{\nu}_e$ is suppressed by m_e^2/m_B^2. As we emphasize in Sec. III C and as is made clear by Eq. (7) of Ref. [17] this is not true. The point is that f_- *does* contribute to $\bar{B}\to De\bar{\nu}_e$ while it does not contribute to $\bar{B}\to D^*e\bar{\nu}_e$ at zero recoil. Furthermore, the authors of Ref. [11] draw conclusions based on the quantity $f\sqrt{M}$ rather than $f\sqrt{\mu}$. The use of the former expression to extract $1/M$ corrections is misleading because it is not experimentally accessible and because spurious $1/M$ terms are generated by the factor $\sqrt{M/\mu}$.

ACKNOWLEDGMENTS

We thank N. Isgur, S. Sharpe, P. Mende, and B. Grinstein for discussions on this work. The financial support of the Alexander von Humboldt-Stiftung and the Natural Sciences and Engineering Research Council of Canada is gratefully acknowledged. This work was supported in part by funds provided by the U.S. Department of Energy (DOE) under Contract No. DE-AC02-76ER03069.

APPENDIX

The expressions for meson transition matrix elements have been derived in Refs. [6] and [11] and are reproduced below for convenience. The relevant kinematic variables are defined in Fig. 5, from which we have $p^2=\mu_n^2$, $p'^2=\mu_m^2$, $q=p-p'$, and $x=q_-/p_-$. Momentum conservation implies that $\mu_n^2=q^2/x+\mu_m^2/(1-x)$. Setting $F_\mu^{nm}\equiv\langle n(p')|V_\mu|m(p)\rangle$ and $F_S^{nm}\equiv\langle n(p')|S|m(p)\rangle$ gives

$$F_-^{nm}=2p_-(1-x)(f_1^{nm}+f_2^{nm}+f_3^{nm})\ , \qquad (A1)$$

with

$$f_1^{nm}=\int_0^1\phi_n(x+(1-x)z)\phi_m(z)dz\ , \qquad (A2)$$

[5]We have not shown f_+^{00} since its behavior is similar to that of the elastic form factor.

$$f_2^{nm} = -x^2 \frac{g^2}{\pi} \int_0^1 \int_0^1 \frac{\phi_n(x+(1-x)z)\phi_m(z) G(u;q^2)}{[x(1-u)+(1-x)z]^2} du\, dz \ , \tag{A3}$$

$$f_3^{nm} = x^2 \frac{g^2}{\pi} \int_0^1 \int_0^1 \frac{\phi_n(xu)\phi_m(z) G(u;q^2)}{[x(1-u)+(1-x)z]^2} du\, dz \ , \tag{A4}$$

and

$$G(u;q^2) = \int_0^1 \sum_{n=0}^{\infty} \frac{\phi_n(u)\phi_n(v)}{q^2-\mu_n^2+i\epsilon} dv \ . \tag{A5}$$

The term f_1 is the valence quark approximation to F_- where the current couples directly to the heavy quark; f_2 and f_3 represent the cases where the current couples to an intermediate meson, which then couples to the heavy quark. These last two terms vanish as $1/M^2$ in the heavy-quark limit, which therefore yields the simple expression for the Isgur-Wise function given in Eq. (2.29). The "bad" component of F_μ^{nm} is given by

$$F_+^{nm} = \frac{1}{p_-}(1-x)(g_1^{nm}+g_2^{nm}+g_3^{nm}) \ , \tag{A6}$$

with

$$g_1^{nm} = \frac{M'M}{(1-x)} \int_0^1 \frac{\phi_n(x+(1-x)z)\phi_m(z)}{[x+(1-x)z]z} dz \ , \tag{A7}$$

$$g_2^{nm} = -q^2 \int_0^1 \int_0^1 \frac{\phi_n(xu)-\phi_n(x+(1-x)z)}{[x(1-u)+(1-x)z]^2} G(u;q^2)\phi_m(z) du\, dz \ , \tag{A8}$$

$$g_3^{nm} = \int_0^1 \int_0^1 \frac{\phi_n(xu)-\phi_n(x+(1-x)z)}{[x(1-u)+(1-x)z]^2} \phi_m(z) du\, dz \ . \tag{A9}$$

Finally,

$$F_S^{nm} = (1-x)(h_1^{nm}+h_2^{nm}) \ , \tag{A10}$$

with

$$h_1^{nm} = \int_0^1 \phi_n(x+(1-x)z)\phi_m(z) \left[\frac{M}{x+(1-x)z} + \frac{M'}{(1-x)z} \right] dz \ , \tag{A11}$$

$$h_2^{nm} = -2x \frac{g^2}{\pi} \int_0^1 \int_0^1 \frac{\phi_n(xu)-\phi_n(x+(1-x)z)}{[x(1-u)+(1-x)z]^2} \phi_m(z) G_S(u;q^2) du\, dz \ , \tag{A12}$$

and

$$G_S(u;q^2) = \frac{1}{2} \int_0^1 \left[\frac{M}{v} - \frac{M'}{1-v} \right] \sum_{n=0}^{\infty} \frac{\phi_n(u)\phi_n(v)}{q^2-\mu_n^2+i\epsilon} dv \ . \tag{A13}$$

[1] N. Isgur and M. B. Wise, Phys. Lett. B **232**, 113 (1989); **237**, 527 (1990); H. Georgi, *ibid.* **240**, 447 (1990); E. Eichten and B. Hill, *ibid.* **234**, 511 (1990).

[2] M. Neubert, Phys. Rev. D **45**, 2451 (1992); A. V. Radyushkin, Phys. Lett. B **271**, 218 (1991); see also E. V. Shuryak, Nucl. Phys. **B198**, 83 (1982).

[3] P. Boucaud *et al.*, Phys. Lett. B **220**, 219 (1989); C. R. Allton *et al.*, Nucl. Phys. **B249**, 598 (1991); Stephen Sharpe (private communication).

[4] G. 't Hooft, Nucl. Phys. **B72**, 461 (1974); **B75**, 461 (1974). See F. Lenz, M. Thies, S. Levit, and K. Yazaki, Ann. Phys. (N.Y.) **208**, 1 (1991), for a discussion of the regularization used by 't Hooft.

[5] S. Coleman, in *Aspects of Symmetry* (Cambridge University Press, New York, 1985).

[6] M. B. Einhorn, Phys. Rev. D **14**, 3451 (1976).

[7] C. G. Callan, Jr., N. Coote, and D. J. Gross, Phys. Rev. D **13**, 1649 (1976).

[8] A. Falk, H. Georgi, B. Grinstein, and M. B. Wise, Nucl. Phys. **B343**, 1 (1990).

[9] A. J. Hanson, R. D. Peccei, and M. K. Prasad, Nucl. Phys. **B121**, 447 (1977); S. Huang, J. W. Negele, and J. Polonyi, *ibid.* **B307**, 669 (1986); R. L. Jaffe and P. F. Mende, *ibid.* **B369**, 189 (1992).

[10] M. Neubert, presented at the Topical Conference on Heavy Quark Symmetry, Santa Barabara, California, 1992 (unpublished).

[11] B. Grinstein and P. F. Mende, Phys. Rev. Lett. **69**, 1018 (1992).

[12] M. Burkardt, Ph.D. thesis, Univ. Erlangen-Nürnberg, 1989.

[13] M. E. Luke, Phys. Lett. B **252**, 447 (1990).

[14] J. G. Körner and P. Kroll, DESY Report No. DESY-91-131, 1991 (unpublished).

[15] S. D. Drell, D. J. Levy, and T.-M. Yan, Phys. Rev. **187**, 2159 (1969).

[16] N. Isgur, Phys. Rev. D **43**, 810 (1991).

[17] M. Neubert, Phys. Lett. B **264**, 455 (1991).