Large-order perturbation theory for the electromagnetic current-current correlation function

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The constraints imposed by asymptotic freedom and analyticity on the large-order behavior of perturbation theory for the electromagnetic current-current correlation function are examined. By suitably applying the renormalization group, the coefficients of the asymptotic expansion in the deep Euclidean region may be expressed explicitly in terms of the perturbative coefficients of the Minkowski space discontinuity (the R ratio in e^+e^- scattering). This relation yields a "generic" prediction for the large-order behavior of the Euclidean perturbation series and suggests the presence of nonperturbative $1/q^2$ correction in the Euclidian correlation function. No such "generic" prediction can be made for the physically measurable R ratio. A novel functional method is developed to obtain these results.

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I. INTRODUCTION

The renormalization group relates changes in the value of the renormalization point to equivalent changes in the values of renormalized couplings. For any dimensionless physical quantity depending on a single momentum variable, $\mathcal{F}(q)$, the renormalization group may be used to reexpress a perturbative expansion with a fixed coupling $g^2(\mu^2)$ and momentum-dependent coefficients,

$$\mathcal{F}(q) \sim \sum_n c_n(q^2/\mu^2)\, g(\mu^2)^{2n}\,, \tag{1.1}$$

as an asymptotic expansion[1] in powers of a running coupling $g^2(q^2)$ with fixed (purely numerical) coefficients,

$$\mathcal{F}(q) \sim \sum_n c_n(1)\, g(q^2)^{2n}\,. \tag{1.2}$$

In this form, one is using the renormalization group to relate changes in the magnitude of q^2 to changes in the value of the running coupling.

The renormalization group can also be used to express a change in the phase of q^2 as an equivalent change in the complex value of the running coupling. Consequently, when combined with analyticity, the renormalization group may be used to relate a Euclidean space perturbation series [in powers of $g^2(q^2)$, with q^2 real and positive] to the corresponding Minkowski space perturbation series [in powers of $g^2(-q^2)$].

In this paper, using only the renormalization group and analyticity, we examine the precise relation between the Euclidean space asymptotic expansion of the electromagnetic current-current correlation function (the "dispersive part") and the corresponding asymptotic behavior of its Minkowski space discontinuity, which is the R ratio in e^+e^- annihilation (the "absorptive part"). We find that the exact relation between these expansions takes an extremely simple form when expressed in terms of a natural generalization of the Borel transform of the perturbative series. The implications of this relation on the possible large-order behavior of perturbation theory are discussed. In particular, we find that the perturbative coefficients of the Euclidean correlation function and the Minkowski discontinuity need not show the same behavior at large orders. The renormalization group and analyticity constraints alone are *not* sufficient to uniquely determine the large-order behavior of perturbation theory, in contradiction to a recent claim by West [1]. Moreover, as discussed in detail in Appendix A, the asymptotic forms presented by West are inconsistent with the renormalization group and do not obey the exact relation between the dispersive and absorptive parts which we derive.

Our general renormalization group relation implies that the Borel transform of the Euclidean perturbation series will exhibit an infinite set of regularly spaced singularities. These include the infrared and ultraviolet "renormalon" singularities found from the examination of individual Feynman diagrams [2–5]. However, for QCD the renormalization group analysis also suggests the presence of one further singularity which does not correspond to an expected renormalon singularity. As we shall discuss, this extra singularity may be interpreted as indicating the presence of nonperturbative $1/q^2$ corrections in the Euclidean correlation function. Such corrections will have significant implications (which we do not explore) for phenomenological applications of the operator-product expansion such as QCD sum rules and heavy-quark expansions. This new Borel transform singularity must be present *unless* the perturbative coefficients of the R ratio conspire to produce an exact cancella-

[1] We use the symbol $\sim$ to denote an asymptotic series in the precise sense employed by mathematicians: $f(z) \sim \sum_n f_n z^n$ if, for any N, $f(z) - \sum_{n=0}^{N} f_n z^n = O(z^{N+1})$ as $z \to 0$.

tion in a (modified) Borel transform of the absorptive series. Whether or not this cancellation might occur is not known; however, as we discuss in Appendix E, no hint of this cancellation is seen using the known terms in the expansion of the R ratio.

In an earlier paper[2] [7], the special case of a theory with a one-term β function was examined. In this paper, we show how to extend the analysis of [7] to the case of a general β function, and we examine the applications of these results at greater length. A detailed summary and discussion of our results appears in the next section. This starts with a review of the perturbative expansion of the electromagnetic current-current correlation function, a brief summary of the one-term β function results in [7], and a discussion of the choice of renormalization scheme which will be most convenient for the general case. We introduce a "modified" Borel transform and summarize its properties and its connection with the ordinary Borel transform. Our results on the relation between Euclidean and Minkowski space asymptotic behavior are then presented, followed by a discussion of the implications of this relation on the possible large-order behavior of the perturbative series. The connection between our work and previous renormalon results is described, followed by a review of the relation between Borel transform singularities and the operator-product expansion. Finally, we discuss the significance of our "extra" renormalon singularity and its implications concerning nonperturbative $1/q^2$ corrections in the Euclidean correlation function.

The actual derivation of our results follows this lengthy summary. By using a functional approach, we find a very simple form for the exact solution to the renormalization group equations relating the perturbative coefficients. This (slightly abstract) approach exploits familiar quantum-mechanical techniques. In this setting, the modified Borel transformation emerges naturally as the representation of the series in a convenient overcomplete basis, just as the usual Borel transform corresponds to the standard coherent state representation of the series.

Appendix B examines the effect of a coupling redefinition on a Borel transform. This appendix shows that the position and nature of the leading singularities of a Borel transform are not changed by any "reasonable" coupling redefinition. Some details concerning the modified Borel transforms are relegated to Appendix C. Alternative methods (more traditional but less convenient) for deriving the results presented in the text are sketched in Appendix D. The calculations of the perturbative coefficients of the current-correlation function are scattered throughout the literature. These results are collected and summarized in our notation in Appendix E to enable any comparison with the asymptotic results of this paper that the reader may wish to make.

[2]Note added. After this paper was submitted for publication, we became aware of related work previously presented in the lectures of Bjorken [6] at the 1989 Cargèse school. Bjorken investigated the relationship between the dispersive and absorptive parts of the current-correlation function using a one-term β function and found the basic result of paper [7] described in Sec. II A below, including the simple Borel transform relation (2.13).

II. SUMMARY OF RESULTS AND DISCUSSION

The electromagnetic current-current correlation function

$$K^{\mu\nu}(q) \equiv i \int (d^4x)\, e^{-iqx} \langle 0|\mathcal{T}(j^\mu(x) j^\nu(0))|0\rangle = (g^{\mu\nu}q^2 - q^\mu q^\nu)\, K(-q^2) \tag{2.1}$$

involves a single scalar function $K(t)$ which is analytic in the entire $t = -q^2$ plane save for a cut along the positive real axis. The discontinuity across this cut is related to the high-energy limit of the total e^+e^- hadronic cross section if one neglects the Z^0 exchange contribution. In terms of the R ratio, defined as the ratio of the total cross section for $e^+e^- \to$ hadrons to that for $e^+e^- \to$ muon pairs, we have

$$R(s) = 12\pi \operatorname{Im} K(s + i0^+) \,. \tag{2.2}$$

In an asymptotically free theory such as QCD, renormalized perturbation theory, plus the renormalization group, may be used to compute the asymptotic behavior of $K(t)$ as $|t| \to \infty$. The discontinuity $\operatorname{Im} K(s + i0^+)$ has an asymptotic expansion[3] in powers of the running coupling $g^2(s)$,

$$\operatorname{Im} K(s + i0^+) \sim \sum_{n=0}^{\infty} a_n\, g^{2n}(s) \,, \tag{2.3}$$

while the asymptotic behavior along the negative real t-axis (corresponding to the Euclidean space correlation function) is given by

$$K(t) \sim \kappa(\mu^2) + \tilde{c}_{-1}\, g^2(-t)^{-1} + \tilde{c}_0 \ln g^2(-t) + \sum_{n=1}^{\infty} c_n\, g^{2n}(-t) \,. \tag{2.4}$$

The origin of the nonanalytic $1/g^2$ and $\ln g^2$ terms is reviewed in Sec. III. Only the constant term $\kappa(\mu^2)$ depends on the renormalization point μ; the perturbative coefficients $\{\tilde{c}_{-1}, \tilde{c}_0, c_n\}$ and $\{a_n\}$ are pure numbers, independent of μ^2 and all mass parameters. All mass-dependent terms, such as m^2/t, vanish faster as $|t| \to \infty$ than any power of the running coupling, and hence they may be neglected. We will refer to the coefficients $\{a_n\}$ in the expansion of the discontinuity as the "absorptive" coefficients, and the coefficients $\{\tilde{c}_{-1}, \tilde{c}_0, c_n\}$ in the Euclidean expansion as the "dispersive" coefficients.

[3]We assume that $K(t)$ does not contain unexpected, pathological terms such as $\exp\{i\sqrt{t/\Lambda^2}\}$ which, away from the positive real axis, make no contribution to the asymptotic expansion (2.4).

The absorptive and dispersive coefficients are not independent; one may express the absorptive coefficients in terms of dispersive coefficients, or vice versa. The precise relation between the coefficients depends on the behavior of the running coupling as determined by the β function

$$\beta(g^2(\mu^2)) \equiv \mu^2 \frac{dg^2(\mu^2)}{d\mu^2}, \tag{2.5}$$

with the explicit form governed by the coefficients of its perturbative expansion,

$$\beta(g^2) \sim -b_0\, g^4 - b_1\, g^6 - b_2\, g^8 - \cdots . \tag{2.6}$$

A. One-term β function results

In the earlier paper [7], the special case of a one-term β function,

$$\beta(g^2) = -b_0\, g^4 , \tag{2.7}$$

was studied. The results are remarkably simple when expressed in terms of the Borel transforms of the perturbative coefficients, defined as

$$A(z) \equiv \sum_{n=1}^{\infty} \frac{n\, a_n}{\Gamma(n+1)}\, z^n , \tag{2.8}$$

and

$$C(z) \equiv \tilde{c}_0 + \sum_{n=1}^{\infty} \frac{n\, c_n}{\Gamma(n+1)}\, z^n . \tag{2.9}$$

As shown in [7], these satisfy

$$A(z) = \sin(\pi b_0 z)\, C(z) . \tag{2.10}$$

Expanding both sides of this result in powers of z generates explicit expressions for the absorptive coefficients as linear combinations of the dispersive coefficients,

$$a_{2n} = \sum_{k=0}^{n-1} \frac{(-)^k\, (2n-1)!}{(2k+1)!\, (2n-2k-2)!}\, (\pi b_0)^{2k+1}\, c_{2(n-k)-1} , \tag{2.11}$$

and

$$a_{2n+1} = \sum_{k=0}^{n-1} \frac{(-)^k\, (2n)!}{(2k+1)!\, (2n-2k-1)!}\, (\pi b_0)^{2k+1}\, c_{2(n-k)} + \frac{(-)^n}{2n+1}\, (\pi b_0)^{2n+1}\, \tilde{c}_0 . \tag{2.12}$$

For $n = 0$, Eq. (2.11) is replaced by $a_0 = -\pi b_0\, \tilde{c}_{-1}$. Note that the absorptive coefficient at any given order is determined by the dispersive coefficients at lower orders.

Conversely, expanding

$$C(z) = A(z)/\sin(\pi b_0 z) \tag{2.13}$$

yields the inverse relations

$$n\, c_n = \sum_{k=0}^{[n/2]} \frac{n!}{(2k)!\, (n-2k)!}\, |(2^{2k}-2) B_{2k}| \times (\pi b_0)^{2k-1}\, a_{n-2k+1} , \tag{2.14}$$

and $\tilde{c}_0 = a_1/(\pi b_0)$. Here $[x]$ denotes the integer part of x, and B_n are the Bernoulli numbers. Note that the dispersive coefficient at a given order is determined by the absorptive coefficient at one higher-order plus absorptive coefficients at lower orders.

The existence of zeros in the $\sin(\pi b_0 z)$ denominator in the relation between Borel transforms (2.13) implies that the Borel transform of the dispersive coefficients $C(z)$ *must* have singularities at integer multiples of $1/b_0$ unless the absorptive transform $A(z)$ has compensating zeros.[4] These singularities constrain the possible large-order behavior of the dispersive perturbative coefficients [7]. This is discussed in greater generality below.

B. General β functions and renormalization schemes

In the body of this paper, we show that the preceding one-term β function results do have a natural generalization to the case of a general β function. The exact form of the results depends on the coefficients (2.6) of the β function which in turn depend on the scheme used for defining the renormalized coupling. As is well known and will be reviewed at the beginning of Sec. V, the first two coefficients in the expansion of the β function are independent of the choice of scheme, while all higher coefficients may be arbitrarily adjusted by a redefinition of the renormalized coupling [2, 8]. We shall find it convenient to choose a definition of the renormalized coupling for which the *inverse* β function contains only two terms:

$$\frac{1}{\beta(g^2)} = -\frac{1}{b_0 g^4} + \frac{\lambda}{b_0 g^2} , \tag{2.15}$$

or

$$\beta(g^2) = -b_0 g^4 \;/\; (1 - \lambda g^2) , \tag{2.16}$$

where

$$\lambda \equiv b_1/b_0 . \tag{2.17}$$

With this choice, the solution of the renormalization group equation (2.5) is particularly simple:

$$\frac{q^2}{\mu^2} = \left(\frac{g^2(q^2)}{g^2(\mu^2)} \right)^{\lambda/b_0} \exp\left\{ \frac{1}{b_0 g^2(q^2)} - \frac{1}{b_0 g^2(\mu^2)} \right\} , \tag{2.18}$$

or

$$q^2 = \Lambda^2\, (g^2(q^2))^{\lambda/b_0} \exp\left\{ \frac{1}{b_0 g^2(q^2)} \right\} , \tag{2.19}$$

where

[4]From the definition (2.8), one sees that $A(z)$ does have a zero at the origin. Hence $z = 0$ is not a singularity of $C(z)$.

$$\Lambda^2 \equiv \mu^2 \left(g^2(\mu^2)\right)^{-\lambda/b_0} \exp\left\{-\frac{1}{b_0 g^2(\mu^2)}\right\} \tag{2.20}$$

is the physical, renormalization-group-invariant mass scale of the theory.

This choice of the β function will also greatly reduce the algebra in our work. Another simple alternative is to choose a different coupling $\bar{g}^2$ whose β function has only two terms [2]:

$$\bar{\beta}(\bar{g}^2) \equiv \mu^2\, d\bar{g}^2/d\mu^2 = -b_0 \bar{g}^4 \left(1 + \lambda \bar{g}^2\right). \tag{2.21}$$

As will be shown at the beginning of Sec. V, the two schemes are related by a shift in the inverse coupling, $\bar{g}^{-2} = g^{-2} - \lambda$. Moreover, as shown in Appendix B (and by an independent argument in Sec. V), such a redefinition only changes the Borel transforms in the two schemes by a simple exponential factor:

$$\bar{A}(z) = e^{-\lambda z} A(z)\,, \qquad \bar{C}(z) = e^{-\lambda z} C(z)\,. \tag{2.22}$$

Thus our results employing the inverse two-term β function may be easily converted to the other case where the β function itself has only two terms.

C. Modified Borel transforms

Remarkably, the simple relation (2.10) between Borel transforms with a one-term β function does generalize to a two-term (inverse) β function provided that one considers a suitably "modified" Borel transform. As shown in Sec. V, one is naturally led to introduce the following definition. Given an asymptotic series

$$f(z) \sim \sum_{n=0}^{\infty} f_n\, z^n\,, \tag{2.23}$$

the *modified* Borel transform is defined as

$$\mathcal{F}(z) \equiv \sum_{n=0}^{\infty} \frac{\Gamma(1+\lambda z)}{\Gamma(n+1+\lambda z)}\, f_n\, z^n\,. \tag{2.24}$$

Just like the ordinary Borel transform, this modified transform may be viewed as a generating function for the coefficients $\{f_n\}$. Provided that the coefficients $\{f_n\}$ grow no faster than $n!\,k^n$ (for some constant k), $\mathcal{F}(z)$ is analytic in a neighborhood of the origin. The coefficients $\{f_n\}$ may be extracted from the derivatives of the transform $\mathcal{F}(z)$ evaluated at the origin. Because of the presence of the shift by λz in the argument of the Γ functions, a given coefficient f_n is not simply proportional to the nth derivative of $\mathcal{F}$. Instead, f_n is given by a linear combination of the first n derivatives of $\mathcal{F}(z)$ evaluated at the origin. In Sec. V [cf. Eq. (5.46)] we derive the contour integral representation

$$f_n = \oint \frac{dz}{2\pi i z}\, \chi_n(1/z)\, \mathcal{F}(z)\,, \tag{2.25}$$

where $\chi_n(y)$ is the nth-order polynomial

$$\chi_n(y) = \delta_{n,0} + n\, y^n\, \frac{\Gamma(n+\lambda/y)}{\Gamma(1+\lambda/y)}\,, \tag{2.26}$$

and the contour circles the origin. The residue of the integrand in Eq. (2.25) generates the appropriate linear combination of derivatives.

The modified Borel transform is related to the ordinary Borel transform

$$F(z) \equiv \sum_{n=0}^{\infty} \frac{f_n}{n!}\, z^n \tag{2.27}$$

through the integral relation

$$F(z) = \oint \frac{dy}{2\pi i y} \left(1 + \frac{z/y}{(1-z/y)^{1+\lambda y}}\right) \mathcal{F}(y)\,, \tag{2.28}$$

where the contour wraps about the cut connecting the branch points of the integrand at $y = 0$ and $y = z$ and excludes any singularities of $\mathcal{F}(y)$. This result follows directly from Eqs. (2.25) and (2.26) by using the generalized binomial theorem. An alternative derivation is given in Sec. V [cf. Eq. (5.42)]. It will be shown in Sec. V [Eq. (5.35)] that the inverse of this relation also has a simple form

$$\mathcal{F}(z) = \lambda \int_0^z dy\, (1 - y/z)^{\lambda z - 1} F(y)\,. \tag{2.29}$$

The integral relation (2.28) shows that the domain of analyticity of the ordinary Borel transform includes that of the modified Borel transform[5] [since singularities in $F(z)$ only develop when the contour is pinched between the branch point at z and a singularity of $\mathcal{F}(z)$]. In Appendix C it is shown that a basic effect of the transformation (2.28) or its inverse (2.29) is to shift the exponents of algebraic singularities. If the modified transform has the singular behavior

$$\mathcal{F}(z) \sim [1 - (z/R)]^{-\alpha}\,, \tag{2.30}$$

then the standard Borel transform has the singular behavior

$$F(z) \sim [1 - (z/R)]^{-\alpha-\lambda R}\, \frac{\Gamma(\alpha+\lambda R)}{\Gamma(\alpha)\,\Gamma(1+\lambda R)} \tag{2.31}$$

as $z \to R$, and conversely. The subleading, nonanalytic terms in this correspondence are suppressed by a relative factor of $[1-(z/R)]\ln[1-(z/R)]$.

Just as for the usual Borel transform, the location and nature of the singularities of $\mathcal{F}(z)$ closest to the origin determine the leading asymptotic behavior of the coefficients $\{f_n\}$. If the modified transform $\mathcal{F}(z)$ has a radius of convergence larger than K, then a trivial bound of the

[5]Actually, $F(z)$ is analytic in a larger domain than $\mathcal{F}(z)$ since the overall factor of $\Gamma(1+\lambda z)$ in the definition of the modified Borel transform (2.24) causes $\mathcal{F}(z)$ to have simple poles at $z = -n/\lambda$, $n = 1, 2, \ldots$. These poles are removed by the integral transform (2.28) which yields the standard transform $F(z)$. The overall factor of $\Gamma(1+\lambda z)$ could, of course, be omitted in the definition of the modified transform, but it will prove convenient not to do so.

integral (2.25) using the asymptotic form of the Γ function shows that the large-order growth of the coefficients $\{f_n\}$ is bounded by

$$|f_n| \leq C\, n!\, K^{-n}\, n^{|\lambda| K} \tag{2.32}$$

for some constant C. If the nearest singularity to the origin has the form (2.30), then the generalized binomial expansion of the corresponding Borel transform (2.31) shows that the coefficients $\{f_n\}$ have the large-order behavior

$$f_n \sim R^{-n}\, \frac{\Gamma(n+\alpha+\lambda R)}{\Gamma(\alpha)\,\Gamma(1+\lambda R)}\,, \tag{2.33}$$

with corrections suppressed by $\ln n/n$, or, using the asymptotic behavior of the Γ function,

$$f_n \sim \frac{n!\, R^{-n}\, n^{\alpha+\lambda R-1}}{\Gamma(\alpha)\,\Gamma(1+\lambda R)}\,. \tag{2.34}$$

Given the Borel transform of an asymptotic series, one may generate a function whose asymptotic expansion coincides with the original series by performing a Laplace transform

$$f(y) \equiv \frac{1}{y} \int_0^\infty dz\, e^{-z/y}\, F(z)\,, \tag{2.35}$$

since expanding $F(z)$ yields

$$f(y) \sim \sum_{n=0}^{\infty} f_n\, y^n\,. \tag{2.36}$$

In terms of the modified Borel transform, the same construction reads

$$f(y) \equiv \frac{1}{y}(1-\lambda y) \int_0^\infty dz\, e^{-z/y}\, \frac{(z/y)^{\lambda z}}{\Gamma(1+\lambda z)}\, \mathcal{F}(z)\,. \tag{2.37}$$

This is derived in Appendix C. If f_0 vanishes, then the function

$$\bar{f}(y) \equiv \int_0^y dy'\, f(y')/y' \sim \sum_{n=1}^{\infty} f_n\, y^n/n \tag{2.38}$$

has a slightly simpler form. Inserting Eq. (2.37) into the integral in Eq. (2.38) and using

$$\frac{1}{y'^2}(1-\lambda y') e^{-z/y'}\, (z/y')^{\lambda z} = \frac{1}{z}\frac{\partial}{\partial y'} \left\{ e^{-z/y'}\, (z/y')^{\lambda z} \right\} \tag{2.39}$$

yields

$$\bar{f}(y) = \int_0^\infty \frac{dz}{z}\, e^{-z/y}\, \frac{(z/y)^{\lambda z}}{\Gamma(1+\lambda z)}\, \mathcal{F}(z)\,. \tag{2.40}$$

This will be the appropriate "inverse" to use if $\mathcal{F}(z)$ is the modified Borel transform of a series $\{n\, g_n\}$ where each coefficient is scaled by n, since in this case $\bar{f}(y) \sim \sum_{n=1}^{\infty} g_n\, y^n$. This form will be used below.

If the (ordinary or modified) Borel transform is analytic in a neighborhood of the real axis and is well behaved as $z \to \infty$, then the inverse transform (2.35) or (2.37) defines a unique "sum" of the asymptotic series which satisfies certain boundedness and analyticity conditions [9]. If the Borel transform has a singularity at some point z_0 on the positive real axis, then different contour prescriptions for the integrals (2.35) or (2.37) will produce "sums" of the asymptotic series $\sum_n f_n\, y^n$ which differ by exponentially small terms of order $\exp(-z_0/y)$. This will be discussed in more detail below.

D. General β function results

Let $\mathcal{A}(z)$ denote the modified Borel transform of the absorptive coefficients $\{n\, a_n\}$:

$$\mathcal{A}(z) = \sum_{n=1}^{\infty} \frac{\Gamma(1+\lambda z)}{\Gamma(n+1+\lambda z)}\, n\, a_n\, z^n\,. \tag{2.41}$$

Similarly, let $\mathcal{C}(z)$ denote the modified transform of the dispersive coefficients $\{n\, c_n\}$ with a suitably chosen constant piece:

$$\mathcal{C}(z) = (\tilde{c}_0 - \lambda \tilde{c}_{-1}) + \sum_{n=1}^{\infty} \frac{\Gamma(1+\lambda z)}{\Gamma(n+1+\lambda z)}\, n\, c_n\, z^n\,. \tag{2.42}$$

In Sec. V we show that these transforms obey the same simple relation as in the one-term β-function case:

$$\mathcal{A}(z) = \sin(\pi b_0 z)\, \mathcal{C}(z)\,. \tag{2.43}$$

When Eq. (2.25) is used to extract the original absorptive coefficients from this relation, one finds that[6]

$$a_n = (\tilde{c}_0 - \lambda \tilde{c}_{-1}) \sum_{k=0}^{[\frac{n-1}{2}]} (-)^k \frac{(\pi b_0)^{2k+1}}{(2k+1)!}\, \lambda^{n-2k-1}\, I^{n-1}_{0,2k} + \sum_{m=1}^{n-1} m\, c_m \sum_{k=0}^{[\frac{n-m-1}{2}]} (-)^k\, \frac{(\pi b_0)^{2k+1}}{(2k+1)!}\, \lambda^{n-m-2k-1} \times I^{n-1}_{m,2k}\,. \tag{2.44}$$

Here the $\{I^n_{m,l}\}$ are combinatorial factors defined by the generating function

$$\frac{\Gamma(n+1+x)}{\Gamma(m+1+x)} \equiv \sum_{l=0}^{n-m} x^{n-m-l} I^n_{m,l}\,. \tag{2.45}$$

Alternatively, applying Eq. (2.25) to

$$\mathcal{C}(z) = \mathcal{A}(z)/\sin(\pi b_0 z) \tag{2.46}$$

yields the inverse relations

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[6] Amusingly, the explicit relation (2.44) between the absorptive and dispersive coefficients, and its inverse (2.47), were derived before the appropriate definition of the modified Borel transform satisfying (2.43) was found. Direct (but tedious) methods for obtaining these results are described in Appendix D.

$$c_n = \frac{a_{n+1} - \lambda a_n}{\pi b_0\, n} + \sum_{m=1}^{n-1} m\, a_m \sum_{k=1}^{[\frac{n-m+1}{2}]} \frac{|(2^{2k}-2)B_{2k}|}{(2k)!} (\pi b_0)^{2k-1}\, \lambda^{n-m-2k+1}\, I^{n-1}_{m,2k-2}\,, \tag{2.47}$$

together with $\tilde{c}_0 = (a_1-\lambda a_0)/(\pi b_0)$ and $\tilde{c}_{-1} = -a_0/(\pi b_0)$. These results are derived in Sec. V. It is easy to verify that they reduce to the previous results (2.11), (2.12), and (2.14) for the case of the one-term β function when the limit $\lambda \to 0$ is taken. Note that, just as in the case of the one-term β function, the relations (2.44) and (2.47) have a "triangular structure": The right-hand side of Eq. (2.44) involves coefficients of lower order; the right-hand side of Eq. (2.47) involves a coefficient of one higher-order plus coefficients of lower order.

E. Large-order behavior of the dispersive coefficients

The existence of zeros in the $\sin(\pi b_0 z)$ denominator in the relation between the modified Borel transforms (2.46) implies that $\mathcal{C}(z)$ will have singularities at all nonzero integer values of $b_0\, z$ unless $\mathcal{A}(z)$ has compensating zeros. Hence, one of the following possibilities for the large-order behavior must occur.

(1) If $\mathcal{A}(z)$ has a radius of convergence greater than $1/b_0$ [so that the absorptive coefficients $\{a_n\}$ grow slower than $n!\, K^n$ for some $K < 1/b_0$], then $\mathcal{C}(z)$ will have simple poles at $z = \pm 1/b_0$. If the residues $\mathcal{A}_\pm \equiv \mathcal{A}(\pm 1/b_0)$ are not both zero, Eqs. (2.30) and (2.34) (for $\alpha = 1$) show that the dispersive coefficients will have large-order behavior which is completely determined by these residues:

$$c_n \sim (n-1)! \left\{ \frac{\mathcal{A}_+\, b_0^n\, n^{\lambda/b_0}}{\pi\, \Gamma(1+\lambda/b_0)} - \frac{\mathcal{A}_-\, (-b_0)^n\, n^{-\lambda/b_0}}{\pi\, \Gamma(1-\lambda/b_0)} \right\} \tag{2.48}$$

as $n \to \infty$.

(2) If $\mathcal{A}(z)$ has a radius of convergence equal to $1/b_0$, so will $\mathcal{C}(z)$. If the singularity nearest to the origin lies on the real axis, then the dispersive coefficients will grow faster than the absorptive coefficients by a single power of n.[7] For example, if the absorptive coefficients behave for large n as

$$a_n \sim (n-1)! \left\{ \frac{\mathcal{A}_+\, b_0^n\, n^{\lambda/b_0+\gamma_+}}{\Gamma(1+\lambda/b_0)} + \frac{\mathcal{A}_-\, (-b_0)^n\, n^{-\lambda/b_0+\gamma_-}}{\Gamma(1-\lambda/b_0)} \right\} \tag{2.49}$$

for some constants $\mathcal{A}_\pm$ and $\gamma_\pm > -1$, then from Eq. (2.34) the modified Borel transform $\mathcal{A}(z)$ will have the behavior $\mathcal{A}(z) \sim \mathcal{A}_\pm\, (1 \mp b_0 z)^{-1-\gamma_\pm}\, \Gamma(1+\gamma_\pm)$ as $b_0 z \to \pm 1$. Dividing by $\sin(\pi b_0 z)$ gives the behavior of the modified dispersive transform, $\mathcal{C}(z) \sim \pm \mathcal{A}_\pm\, (1 \mp b_0 z)^{-2-\gamma_\pm}\, \Gamma(1+\gamma_\pm)/\pi$. Thus the dispersive coefficients will grow like

$$c_n \sim n! \left\{ \frac{\mathcal{A}_+\, b_0^n\, n^{\lambda/b_0+\gamma_+}}{\pi\, (1+\gamma_+)\, \Gamma(1+\lambda/b_0)} - \frac{\mathcal{A}_-\, (-b_0)^n\, n^{-\lambda/b_0+\gamma_-}}{\pi\, (1+\gamma_-)\, \Gamma(1-\lambda/b_0)} \right\} \tag{2.50}$$

as $n \to \infty$.

(3) If $\mathcal{A}(z)$ has a radius of convergence less than $1/b_0$, so will $\mathcal{C}(z)$. In this case, the dispersive coefficients will have the same large-order behavior (within an overall constant factor) as the absorptive coefficients, with both growing faster than $b_0^n\, n!$ as $n \to \infty$.

These conditions on the possible large-order behavior are independent of the specific dynamics of the asymptotically free theory and follow solely from the existence of renormalized perturbation theory. Each of the possible behaviors above is fully consistent with the constraints of analyticity and the renormalization group, contradicting the claim of unique large-order behavior asserted in [1]. [The asymptotic behavior given in this reference also does not obey the constraint given in point (3) above, as shown in Appendix A.] However, in view of Eqs. (2.48) and (2.50) and the asymptotic form of the Γ function, large-order behavior of the form

$$c_n \sim \mathcal{C}_+\, b_0^n\, \Gamma(n + \gamma_+) - \mathcal{C}_-\, (-b_0)^n\, \Gamma(n - \gamma_-) \tag{2.51}$$

for the dispersive coefficients is, in some measure, a *generic* possibility. This behavior (for some values of $\mathcal{C}_\pm$ and $\gamma_\pm$) will occur *unless* either (i) the absorptive transform $\mathcal{A}(z)$ is singular within $b_0|z| \le 1$ [in which case both absorptive and dispersive coefficients grow more rapidly than (2.51)], or (ii) the absorptive transform has zeros at both $z = \pm 1/b_0$ and a radius of convergence greater than $1/b_0$ [in which case the coefficients grow more slowly than (2.51)].

In a different renormalization scheme, for which the inverse β function does not have the simple two-term form (2.15), the exact relation between the dispersive and absorptive modified transforms will not have the simple form $\mathcal{C}(z) = \mathcal{A}(z)/\sin(\pi b_0 z)$. However, as shown in Appendix B, a coupling redefinition corresponding to a change in renormalization scheme cannot change the location or nature of any singularities in a Borel transform which are within the radius of convergence of the Borel transform of redefined beta function. Thus, the above constraints on the possible large-order behavior of the dispersive coefficients holds without modification in any

[7] If $\mathcal{A}(z)$ has both a nonzero value and a branch point at $b_0 z = \pm 1$ with a vanishing discontinuity at the branch point [e.g., $\mathcal{A}(z) \sim \mathcal{A}_+ + (1 - b_0 z)^\alpha$, with $\alpha > 0$], then the dominant singularity in $\mathcal{C}(z)$ is a simple pole at $b_0 = \pm 1$, and the large-order behavior of the dispersive coefficients is governed by Eq. (2.48). In this case, the dispersive coefficients grow faster than the absorptive coefficients by more than one power of n.

renormalization scheme in which the β function coefficients grow slower than $n!\,\kappa^n$ for some $\kappa < b_0$.[8]

F. Renormalons and operator-product expansions

Explicit studies of perturbation theory in QCD show the following [2–5].

(i) The ultraviolet behavior of individual multiloop diagrams can generate contributions behaving as $c_{m+1} \sim (-b_0/k)^m\, m!$, for $k = 1, 2, \ldots$, leading to singularities in the Borel transform $C(z)$ at the points $z = -k/b_0$ on the negative real axis. Near the first singularity [4], $C(z) \sim (b_0\, z+1)^{-1+\gamma}$, where γ is related to the anomalous dimension of local operators of dimension six. These contributions are referred to as *ultraviolet renormalons.*

(ii) The infrared behavior of multiloop diagrams can generate contributions behaving as $c_{m+1} \sim (b_0/k)^m\, m!$, for $k = 2, 3, \ldots$, corresponding to singularities in $\mathcal{C}(z)$ at the points $z = k/b_0$ on the positive real axis. Near the first singularity [5], $C(z) \sim (b_0\, z-2)^{-1-2\lambda/b_0}$, or equivalently [using (2.29)] the modified transform $\mathcal{C}(z)$ has a simple pole, with a subleading logarithmic branch cut.[9] These contributions are referred to as *infrared renormalons.*

(iii) Instanton–anti-instanton pairs generate singularities in the Borel transform on the positive real axis (starting at $z = 16\pi^2$) to the right of the leading infrared renormalon singularity.

(iv) No other sources of singularities in the Borel transform are known.

The presence of infrared renormalon singularities on the positive real axis is directly related to the existence of nonperturbative vacuum expectation values of composite operators [11]. The operator-product expansion of two electromagnetic currents reads

$$i\,\mathcal{T}(j_\mu(y+\tfrac{1}{2}x)\,j_\nu(y-\tfrac{1}{2}x)) \sim \sum_i C^i_{\mu\nu}(x)\; O_i(y)\,, \tag{2.52}$$

where $\{O_i\}$ denotes the appropriate set of local operators. The Fourier transform of the vacuum expectation value of this expansion gives

$$K(-q^2) \sim \sum_i \tilde{C}^i(q)\, \langle 0|\,O_i|0\rangle\,, \tag{2.53}$$

where

$$\tilde{C}^i(q) \equiv \frac{g^{\mu\nu}}{3q^2}\int d^4x\, e^{-iqx}\, C^i_{\mu\nu}(x)\,. \tag{2.54}$$

The coefficient functions for the scalar operators $\{O_i\}$ with nonvanishing vacuum expectation values have the form

$$\tilde{C}^i(q) = \left(q^2\right)^{-d_i/2} D^i(q^2/\mu^2,\, g^2(\mu^2))\,, \tag{2.55}$$

where the d_i are the physical dimensions of the operators O_i and the D^i are dimensionless functions which have a perturbative expansion in powers of $g^2(\mu^2)$. For QCD, the lowest dimension gauge-invariant composite operators with nonvanishing vacuum expectation values are the unit operator $\hat{1}$, with dimension $d = 0$, $[F^{\mu\nu}F_{\mu\nu}]$ and $[m\,\bar{\psi}\psi]$, both of dimension $d = 4$, followed by various operators of dimension 6, 8, ...[10] The entire asymptotic expansion of the Euclidean correlation function (2.4) is contained in the coefficient function of the unit operator. The dimensionless functions which are not associated with the unit operator obey a homogeneous renormalization group equation

$$\left\{\mu^2\frac{\partial}{\partial\mu^2} + \beta(g^2)\frac{\partial}{\partial g^2} - \gamma^i(g^2)\right\} D^i(q^2/\mu^2,\, g^2(\mu^2)) = 0\,, \tag{2.56}$$

where the γ^i are the anomalous dimensions of the operators O_i. The renormalization group may be used to transfer the momentum dependence into the running coupling:

$$D^i(q^2/\mu^2,\, g^2(\mu^2)) = \exp\left\{-\int_{g^2(\mu^2)}^{g^2(q^2)} dg^2\,\frac{\gamma^i(g^2)}{\beta(g^2)}\right\} D^i(1,\, g^2(q^2))\,. \tag{2.57}$$

By using the expansion (2.6) for the β function and writing $\gamma^i(g^2) = \gamma^i_0\, g^2 + \gamma^i_1\, g^4 + \cdots$ one finds that

$$-\int_{g^2(\mu^2)}^{g^2(q^2)} dg^2\,\frac{\gamma^i(g^2)}{\beta(g^2)} = \frac{\gamma^i_0}{b_0}\ln\frac{g^2(q^2)}{g^2(\mu^2)} + \cdots\,, \tag{2.58}$$

where the ellipsis stands for a power series in $g^2(q^2)$ minus the same series in $g^2(\mu^2)$. The terms involving $g^2(\mu^2)$ may be absorbed by making a suitable multiplicative redefinition of the renormalized operator O_i to yield a renormalization group invariant which is independent of μ^2. The vacuum expectation value of this invariant produces a dimensionless numerical constant times Λ^{d_i}, where Λ is the renormalization-group-invariant mass parameter defined in Eq. (2.20). The power series in $g^2(q^2)$ can be absorbed into a redefinition of the coefficient function $D^i(1,\, g^2(q^2))$. Thus, a given term in the operator-product expansion produces a contribution to the current correlation function of the form

$$K^i(-q^2) \sim \left(\frac{\Lambda^2}{q^2}\right)^{d_i/2} [g^2(q^2)]^{\gamma^i_0/b_0}\bar{D}^i(g^2(q^2))\,, \tag{2.59}$$

where $\bar{D}^i(g^2(q^2))$ admits an asymptotic expansion in

[8] Minimal subtraction schemes, in which the β function is believed not to contain "renormalon" singularities, should satisfy this condition. See, for example, Refs. [3] and [10].

[9] This entails a branch point in $\mathcal{A}(z)$ with a vanishing discontinuity as discussed in the previous footnote.

[10] Chiral symmetry requires that nonchirally invariant operators like $[\bar{\psi}\psi]$ be accompanied by a factor of the quark mass. Hence, only even dimension composite operators appear in the expansion.

powers of $g^2(q^2)$. Expressing q^2/Λ^2 in terms of the running coupling defined by the two-term inverse β function [Eq. (2.18)] shows that these contributions give essential singularities at the origin in the $g^2(q^2)$ plane. The leading behavior at $g^2(q^2) = 0$ for each term in the operator-product expansion is given by

$$K^i(-q^2) \sim \left[g^2(q^2)\right]^{(\gamma_0^i - \lambda d_i/2)/b_0} \exp\left\{-\frac{d_i}{2b_0 g^2(q^2)}\right\} \tilde{D}^i , \tag{2.60}$$

where $\tilde{D}^i \equiv \bar{D}^i(0)$ is a constant.

The contribution from a composite operator is inherently scheme dependent; one can always redefine the operator by adding a constant multiple of the unit operator. This changes the vacuum expectation value of the operator, at the cost of moving (part of) the nonperturbative contribution (2.60) into the coefficient function of the unit operator. Consequently, in any method to "resum" the perturbation series, one should expect to find ambiguities of precisely the form (2.60). Infrared renormalon singularities in the Borel transform are precisely the reflection of these ambiguities. As mentioned earlier, each singularity on the positive real axis generates a nonperturbative ambiguity in the inverse Borel transform. It is instructive to work out a specific example. We consider the case where the modified Borel transform $\mathcal{C}(z)$ has a branch point at $b_0 z = d$ near which $\mathcal{C}(z) \sim (d - b_0 z)^{\sigma-1}$. A singularity of this form in the inverse transform (2.40) creates an ambiguity resulting from different possible choices for routing the contour about the branch cut which starts at $z = d/b_0$. The discontinuity across this cut gives a measure of this ambiguous contribution. Retaining only the leading term near $g^2(q^2) = 0$ we obtain

$$\begin{aligned} \Delta K(-q^2) &\sim \text{const} \times \int_{d/b_0}^{\infty} dz \exp\left\{-\frac{z}{g^2(q^2)}\right\} \left[g^2(q^2)\right]^{-\lambda z} \\ &\qquad \times (b_0 z - d)^{\sigma-1} \\ &\sim \text{const}' \times \left[g^2(q^2)\right]^{\sigma - \lambda d/b_0} \exp\left\{-\frac{d}{b_0 g^2(q^2)}\right\} . \end{aligned} \tag{2.61}$$

This is precisely the structure of a nonperturbative operator-product contribution (2.60) if we identify $d = d_i/2$ and $\sigma = \gamma_0^i/b_0$.

With one exception, the singularities in the modified transform $\mathcal{C}(z)$ generated by the zeros of the $\sin(\pi b_0 z)$ denominator in Eq. (2.46) are at precisely those locations which correspond to ultraviolet and infrared renormalons. The exception is the zero at $b_0 z = 1$. A singularity at this location will generate a nonperturbative ambiguity of order $1/q^2$ in the inverse Borel transform which, because there is no gauge-invariant local scalar operator of dimension 2, cannot be attributed to any physical vacuum expectation value. The absence of this singularity is possible only if the modified absorptive Borel transform $\mathcal{A}(z)$ has a zero at $b_0\, z = 1$. This constraint has not been previously noted. Alternatively, if a singularity at $b_0 z = 1$ does exist, leading to $O(\Lambda^2/q^2)$ contributions to the inverse Borel transform, then this would have to be interpreted as an unexpected nonperturbative correction to the coefficient function of the unit operator in the operator-product expansion.[11] This would be a major problem for phenomenological applications of the operator-product expansion (e.g., QCD sum rules [12]) which are based on the assumption that there exists a range of momenta where nonperturbative $\langle[F^2]\rangle/q^4$ and $\langle[m\bar{\psi}\psi]\rangle/q^4$ contributions are significant and correctly parametrize the leading nonperturbative effects, while simultaneously all coefficient functions may be well approximated by the first term or two of their perturbative expansions. No convincing argument demonstrating either the presence or absence of such $1/q^2$ corrections is known to the authors.[12]

III. RENORMALIZATION GROUP AND ANALYTICITY

We turn at last to the details of our work, beginning with a review of the renormalization group equation for the current-current correlation function and the origin of the large momentum asymptotic expansion (2.4). Using a mass-independent renormalization scheme, the dimensionless function $K(t)$ depends on the renormalization point μ^2, the renormalized coupling $g^2(\mu^2)$, and any mass parameters $m(\mu^2)$ in the form

$$K(t) = K(t/\mu^2, g^2(\mu^2), m^2(\mu^2)/\mu^2) . \tag{3.1}$$

The function $K(t)$ has a perturbative expansion in powers of $g^2(\mu^2)$:

$$K(t) \sim \sum_{n=0}^{\infty} k_n(t/\mu^2, m^2/\mu^2)\, \left[g^2(\mu^2)\right]^n . \tag{3.2}$$

Since all momentum dependence is hidden in the coefficients k_n, this expansion is not directly useful for examining the large momentum behavior of $K(t)$. However, Eq. (3.1) implies that a variation in t is equivalent to a variation in μ^2 combined with a suitable compensating change in the coupling $g^2(\mu^2)$ and mass $m(\mu^2)$. The dependence of $K(t)$ on the renormalization point μ is described by the inhomogeneous renormalization group equation

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[11] In massless QCD, any $1/q^2$ correction must have a nonperturbative origin. However, with nonvanishing quark masses in the Lagrangian, $O(m^2/q^2)$ terms, calculable in perturbation theory, also appear in the operator-product expansion.

[12] Conceivably, similar $1/q^2$ nonperturbative contributions might appear in many other correlation functions. Such contributions may already have been seen in Wilson loop expectation values, where available numerical data suggests the presence of corrections proportional to the area of the loop in the limit of *small* loops (i.e., for loops large compared to the lattice spacing but small compared to $1/\Lambda$) [13].

$$\mu^2 \frac{d}{d\mu^2} K(t/\mu^2, g^2(\mu^2), m^2(\mu^2)/\mu^2) = \left[\mu^2 \frac{\partial}{\partial \mu^2} + \beta(g^2)\frac{\partial}{\partial g^2} + \delta(g^2)\, m^2 \frac{\partial}{\partial m^2}\right] K(t/\mu^2, g^2, m^2/\mu^2) = D(g^2)\,, \tag{3.3}$$

where the β function $\beta(g^2)$ describes the μ dependence of the renormalized coupling,

$$\mu^2 \frac{d}{d\mu^2}\, g^2(\mu^2) = \beta(g^2(\mu^2))\,, \tag{3.4}$$

and the anomalous dimension $\delta(g^2)$ characterizes the variation of the running mass:

$$\mu^2 \frac{d}{d\mu^2}\, m^2(\mu^2) = m^2(\mu^2)\, \delta(g^2(\mu^2))\,. \tag{3.5}$$

Because the electromagnetic currents are conserved they acquire no anomalous dimension. However, since the product of two current operators is singular, one subtraction proportional to the unit operator is required for the proper definition of the time-ordered product in Eq. (2.1); the inhomogeneous term $D(g^2)$ characterizes the dependence of this subtraction term on the renormalization scale. The functions $D(g^2)$, $\beta(g^2)$, and $\delta(g^2)$ have perturbative expansions of the form

$$D(g^2) \sim d_0 + d_1\, g^2 + d_2\, g^4 + \cdots\,, \tag{3.6}$$

$$\beta(g^2) \sim -b_0\, g^4 - b_1\, g^6 - \cdots\,, \tag{3.7}$$

$$\delta(g^2) \sim \delta_0\, g^2 + \delta_1\, g^4 + \cdots\,, \tag{3.8}$$

with $b_0 > 0$ in an asymptotically free theory such as QCD.

We will first consider the current-correlation function $K(t)$ in the Euclidean region where $-t$ is real and positive. To solve the renormalization group equation (3.3), one first introduces a running coupling $g^2(-t)$ defined by

$$\int_{g^2(\mu^2)}^{g^2(-t)} \frac{dg^2}{\beta(g^2)} \equiv \ln\left(\frac{-t}{\mu^2}\right)\,. \tag{3.9}$$

The coupling $g^2(-t)$ is independent of the renormalization point μ but obeys Eq. (3.4) with μ^2 replaced by $-t$. Having defined the running coupling, we may now define the momentum-dependent mass parameter

$$m^2(-t) = m^2(\mu^2) \exp\left\{\int_{g^2(\mu^2)}^{g^2(-t)} dg^2\, \frac{\delta(g^2)}{\beta(g^2)}\right\}\,. \tag{3.10}$$

With these definitions in hand, the general solution of the full renormalization group equation (3.3) may be written as

$$\begin{aligned} K(t/\mu^2, g^2(\mu^2), m^2(\mu^2)/\mu^2) &\\ = K(-1, g^2(-t), m^2(-t)/(-t)) &\\ - \int_{g^2(\mu^2)}^{g^2(-t)} dg^2\, \frac{D(g^2)}{\beta(g^2)}\,. \end{aligned} \tag{3.11}$$

This result may now be expanded in powers of $g^2(-t)$. However, the presence of the inhomogeneous term involving $D(g^2)$ alters the perturbative expansion of $K(t)$. To see this, we note that the expansions (3.6) and (3.7) imply that

$$\begin{aligned} -\int_{g^2(\mu^2)}^{g^2(-t)} dg^2 \frac{D(g^2)}{\beta(g^2)} = {}& \frac{d_0}{b_0}\left[\frac{1}{g^2(\mu^2)} - \frac{1}{g^2(-t)}\right] \\ &+ \left(\frac{d_1}{b_0} - \frac{d_0 b_1}{b_0^2}\right) \ln\left[\frac{g^2(-t)}{g^2(\mu^2)}\right] \\ &+ \cdots\,, \end{aligned} \tag{3.12}$$

where the ellipsis stands for a power series in $g^2(-t)$ minus the same series in $g^2(\mu^2)$. All the terms involving $g^2(\mu^2)$ may be absorbed in a single μ^2-dependent parameter $\kappa(\mu^2)$. The series in $g^2(-t)$ combines with the perturbative expansion (3.2) of $K(-1, g^2(-t), m^2(-t)/(-t))$ to yield a modified expansion in powers of $g^2(-t)$. Hence $K(t)$ has a large-t asymptotic expansion of the form

$$\begin{aligned} K(t) \sim {}& \kappa(\mu^2) + \tilde{c}_{-1}\, g^2(-t)^{-1} + \tilde{c}_0\, \ln g^2(-t) \\ &+ \sum_{n=1}^{\infty} c_n\, g^2(-t)^n\,, \end{aligned} \tag{3.13}$$

as quoted earlier in Eq. (2.4). The coefficients $\{\tilde{c}_{-1}, \tilde{c}_0, c_n\}$ are independent of μ^2 and all renormalized masses. The mass independence follows since $K(t)$ is finite in the massless limit and $m^2(-t)/t$ vanishes faster[13] than any power of $g^2(-t)$ as $t \to \infty$. All remaining μ dependence is contained in the momentum-independent term $\kappa(\mu^2)$. The presence of the $1/g^2(-t)$ and $\ln g^2(-t)$ terms in this result may, at first glance, appear odd. However, since the expansion of Eq. (3.9) gives

$$\frac{1}{g^2(-t)} = \frac{1}{g^2(\mu^2)} + b_0 \ln\left(\frac{-t}{\mu^2}\right) - \frac{b_1}{b_0} \ln\left[\frac{g^2(-t)}{g^2(\mu^2)}\right] + \cdots\,, \tag{3.14}$$

the $1/g^2(-t)$ term is precisely what is required to generate the $\ln(q^2/\mu^2)$ behavior of the free-field correlation function. Similarly, the $\ln g^2(-t)$ term reflects the pres-

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[13] Inserting the expansions (3.7) and (3.8) into the definition (3.10) of the momentum-dependent mass shows that, in the large $-t$ limit where $g^2(-t)$ tends to zero,

$$m^2(-t) \sim \left[g^2(-t)\right]^{-\delta_0/b_0} m^2\,,$$

where

$$m^2 \equiv \left[g^2(\mu^2)\right]^{\delta_0/b_0} \exp\left\{-\int_0^{g^2(\mu^2)} dg^2 \left[\frac{\delta(g^2)}{\beta(g^2)} + \frac{\delta_0}{b_0 g^2}\right]\right\} \times m^2(\mu^2)\,,$$

is independent of the scale mass μ. In QCD $\delta_0 < 0$, and thus $m^2(-t)$ vanishes as a positive power of the coupling when $-t \to \infty$. The additional suppression by $1/t$ makes such mass terms asymptotically insignificant.

ence of $\ln[\ln(q^2/\mu^2)]$ terms in the large momentum behavior of $K(t)$.

It is worth noting that using different renormalization prescriptions may result in the addition of a finite, q^2-independent term $P(g^2(\mu^2))$ to the renormalized current-correlation function $K(t/\mu^2, g^2(\mu^2))$, with $P(g^2(\mu^2))$ having a power series expansion

$$P(g^2(\mu^2)) \sim \sum_{n=0}^{\infty} p_n\, g^{2n}(\mu^2)\,. \tag{3.15}$$

Such a change will add a contribution

$$\mu^2 \frac{dP(g^2(\mu^2))}{d\mu^2} = \beta(g^2(\mu^2)) \frac{dP(g^2)}{dg^2} \tag{3.16}$$

to the inhomogeneous term $D(g^2(\mu^2))$ in the renormalization group equation (3.3). Since this shift in $D(g^2(\mu^2))$ has an expansion in powers of $g^2(\mu^2)$ starting at order $g^4(\mu^2)$, we learn that, by appropriately choosing the renormalization scheme, one can remove from $D(g^2)$ all but the first two terms in its perturbative expansion, and obtain

$$D(g^2) = d_0 + d_1 g^2\,. \tag{3.17}$$

The remaining coefficients d_0 and d_1 cannot be altered by this redefinition. It is these terms in the function $D(g^2)$ that produce the $1/g^2(-t)$ and $\ln g^2(-t)$ pieces in the asymptotic expansion (3.13) of the correlation function $K(t)$.

The inhomogeneous term in the renormalization group equation for $K(t/\mu^2, g^2(\mu^2))$ can be avoided altogether if one studies instead the differentiated function[14] $t\,\partial K(t/\mu^2, g^2(\mu^2))/\partial t$. Here, to keep the discussion simple, we consider only the massless case. Since this function satisfies a homogeneous renormalization group equation, we may choose $\mu^2 = -t$ to obtain a power series expansion in the running coupling $g^2(-t)$:

$$t\frac{\partial}{\partial t} K(t/\mu^2, g^2(\mu^2)) \sim \sum_{n=0}^{\infty} k'_n\, g^{2n}(-t)\,. \tag{3.18}$$

The expansion of the original correlation function may be recovered by integrating over t and using $dt/t = dg^2(-t)/\beta(g^2(-t))$:

$$K(t/\mu^2, g^2(\mu^2)) \sim K(1, g^2(\mu^2)) + \int_{g^2(\mu^2)}^{g^2(-t)} \frac{dg^2}{\beta(g^2)} \sum_{n=0}^{\infty} k'_n\, g^{2n}\,. \tag{3.19}$$

Comparing this expansion with the previous results (3.12) and (3.13) shows that the first two terms of the differentiated current correlation function [and of the inhomogeneous term $D(g^2)$] are given by

$$\begin{aligned} k'_0 &= -d_0 = b_0 \tilde{c}_{-1}\,, \\ k'_1 &= -d_1 = b_1 \tilde{c}_{-1} - b_0 \tilde{c}_0\,. \end{aligned} \tag{3.20}$$

Since the correlation function $K(t)$ is analytic throughout the cut t plane, in addition to the Euclidean space asymptotic expansion (3.13), we may examine the asymptotic behavior of $K(te^{i\theta})$ as $t \to -\infty$ for an arbitrary phase θ. The same steps as before yield

$$K(te^{i\theta}/\mu^2, g^2(\mu^2), m^2(\mu^2)/\mu^2) = K(-e^{i\theta}, g^2(-t), m^2(-t)/(-t)) - \int_{g^2(\mu^2)}^{g^2(-t)} dg^2\, \frac{D(g^2)}{\beta(g^2)}\,, \tag{3.21}$$

and the large-t asymptotic expansion in powers of $g^2(-t)$ now produces θ-dependent (but m and μ independent) coefficients:

$$K(te^{i\theta}) \sim \kappa(\mu^2) + \tilde{c}_{-1}\, ib_0\theta + \tilde{c}_{-1}\, g^2(-t)^{-1} + \tilde{c}_0 \ln g^2(-t) + \sum_{n=1}^{\infty} c_n(\theta)\, g^2(-t)^n\,. \tag{3.22}$$

Here, so as to simplify the later notation, we anticipate the result that the phase dependence of the g^2 independent term has the simple linear form $\tilde{c}_{-1}\, ib_0\theta$.

Because variations in the phase of t are equivalent to variations in θ, the θ dependence of the coefficients $\{c_n(\theta)\}$ is controlled by the β function:

$$\begin{aligned} 0 &= \left(i\frac{\partial}{\partial\theta} + t\frac{\partial}{\partial t}\right) K(te^{i\theta}) \\ &\sim \left[-\frac{\tilde{c}_{-1}}{g^2(-t)} + \tilde{c}_0\right] \frac{\beta(g^2(-t))}{g^2(-t)} \\ &\quad -\tilde{c}_{-1} b_0 + \sum_{n=1}^{\infty} \left[i\frac{\partial}{\partial\theta} + n\frac{\beta(g^2(-t))}{g^2(-t)}\right] c_n(\theta)\, g^2(-t)^n\,. \end{aligned} \tag{3.23}$$

Hence the coefficients $\{c_n(\theta)\}$ satisfy the recursion relation

$$i\frac{d}{d\theta} c_{n+1}(\theta) = -b_{n+1}\, \tilde{c}_{-1} + b_n\, \tilde{c}_0 + \sum_{m=1}^{n} m\, b_{n-m}\, c_m(\theta)\,. \tag{3.24}$$

Our goal is to solve this recursion relation explicitly. This will enable us to relate the asymptotic behavior in the Euclidean region where $-t$ is real and positive, to the corresponding behavior in the Lorentzian domain where $t = s + i0^+$, with s positive. Taking the imaginary part of the expansion (3.22) when $\theta = -\pi$, and comparing with the expansion of the discontinuity,

[14]The occurrence of the inhomogeneous term in the renormalization group is related to the need for a subtraction in the Lehmann (dispersion relation) representation for $K(t/\mu^2, g^2(\mu^2))$. The Lehmann representation for $t\,\partial K(t/\mu^2, g^2(\mu^2))/\partial t$ dispenses with this additional subtraction constant.

$$\operatorname{Im} K(s+i0^+) \sim \sum_{n=0}^{\infty} a_n\, g(s)^{2n}\,, \tag{3.25}$$

shows that the absorptive coefficients $\{a_n\}$ are given by

$$a_n = \operatorname{Im} c_n(-\pi) \tag{3.26}$$

for $n \geq 1$, while

$$a_0 = -\pi b_0\, \tilde{c}_{-1}\,. \tag{3.27}$$

IV. BOREL SUMS AND COHERENT STATES

To illustrate our method in simple terms, we first consider the solution to the recursion relation (3.24) for the special case of a β function which contains only one term, $\beta(g^2) = -b_0 g^4$. The recursion relation (3.24) simplifies to

$$i\frac{d}{d\theta}\, c_{n+1}(\theta) = n b_0\, c_n(\theta) \tag{4.1}$$

for $n \geq 1$, plus

$$i\frac{d}{d\theta}\, c_1(\theta) = b_0\, \tilde{c}_0\,. \tag{4.2}$$

To solve these equations, it is convenient to regard the coefficients as defining an abstract vector which may be represented as a state of a simple harmonic oscillator:

$$|C_\theta\rangle \equiv |0\rangle\, \tilde{c}_0 + \sum_{n=1}^{\infty} |n\rangle \frac{n}{\sqrt{n!}}\, c_n(\theta)\,. \tag{4.3}$$

Here $|0\rangle$ is the usual ground state defined by

$$a|0\rangle \equiv 0\,, \tag{4.4}$$

and the basis states

$$|n\rangle \equiv \frac{(a^\dagger)^n}{\sqrt{n!}}|0\rangle \tag{4.5}$$

are eigenstates of the number operator $N = a^\dagger a$, where $a^\dagger$ and a are standard creation and annihilation operators obeying

$$[a, a^\dagger] = 1\,. \tag{4.6}$$

Since

$$a^\dagger|n\rangle = |n+1\rangle\sqrt{n+1}\,, \tag{4.7}$$

the recursion relations (4.1) and (4.2) are equivalent to a simple operator equation,

$$i\frac{d}{d\theta}|C_\theta\rangle = b_0\, a^\dagger|C_\theta\rangle\,, \tag{4.8}$$

which has the immediate solution

$$|C_\theta\rangle = e^{-ib_0\theta a^\dagger}|C\rangle\,, \tag{4.9}$$

where $|C\rangle$ is the initial vector with $\theta = 0$.

To express this solution in an explicit form, we utilize coherent states defined by

$$\langle z| \equiv \langle 0|e^{az} = \sum_{n=0}^{\infty} \frac{z^n}{\sqrt{n!}}\langle n|\,, \tag{4.10}$$

which are (left) eigenvectors of the creation operator:

$$\langle z|a^\dagger = z\langle z|\,. \tag{4.11}$$

Thus the coherent state representative, defined as

$$C_\theta(z) = \langle z|C_\theta\rangle\,, \tag{4.12}$$

obeys

$$C_\theta(z) = e^{-ib_0\theta z}C(z)\,. \tag{4.13}$$

This coherent state representation

$$C_\theta(z) = \tilde{c}_0 + \sum_{n=1}^{\infty} \frac{n\, c_n(\theta)}{n!}\, z^n \tag{4.14}$$

is precisely the Borel transform of the perturbative coefficients $\{\tilde{c}_0,\, n\, c_n(\theta)\}$. For $\theta = 0$ this is the same transform of the dispersive coefficients introduced earlier in Eq. (2.9).

The absorptive coefficients may also be assembled to form an abstract vector,

$$|A\rangle = \sum_{n=1}^{\infty} |n\rangle \frac{n}{\sqrt{n!}}\, a_n\,, \tag{4.15}$$

whose coherent state projection

$$A(z) = \langle z|A\rangle \tag{4.16}$$

gives the Borel transform (2.8) of the absorptive coefficients $\{n\, a_n\}$:

$$A(z) = \sum_{n=1}^{\infty} \frac{n\, a_n}{n!}\, z^n\,. \tag{4.17}$$

Using Eq. (4.13) to rotate from $\theta = 0$ to $\theta = -\pi$ (from the negative real t axis back to the positive axis) and taking the imaginary part gives

$$A(z) = \sin(\pi b_0 z)\, C(z)\,. \tag{4.18}$$

This is the result of [7] quoted in Eq. (2.10). We have obtained it in a very simple fashion which demonstrates an interesting and useful connection: The Borel transform is the coherent state representation of the abstract vector which describes the original perturbative series.

V. GENERAL β FUNCTION

As was discussed earlier, the first two terms of the perturbative expansion of the β function are scheme independent and uniquely determined. On the other hand, the higher-order terms are scheme dependent. They may be altered by using different renormalization schemes corresponding to coupling redefinitions of the form

$$\bar{g}^2 = g^2 + d_4\, g^4 + d_6\, g^6 + \cdots \tag{5.1}$$

which leave the lowest, order-g^2, term unchanged. This is straightforward to verify directly by inserting Eq. (5.1) into

$$\mu^2 \frac{d\bar{g}^2}{d\mu^2} \equiv \bar{\beta}(\bar{g}^2)\,, \tag{5.2}$$

and comparing with

$$\mu^2 \frac{dg^2}{d\mu^2} \equiv \beta(g^2)\,. \tag{5.3}$$

As we shall see, it will prove convenient to exploit this freedom and employ a "two-term inverse β function"

$$\frac{1}{\beta(g^2)} = -\frac{1}{b_0 g^4} + \frac{\lambda}{b_0 g^2}\,. \tag{5.4}$$

Before proceeding to extend the results of the preceding section to this case of an essentially general β function, it is worth pausing to describe the relation of this coupling definition to that where the β function itself contains only two terms:

$$\bar{\beta}(\bar{g}^2) = -b_0\,\bar{g}^4 - b_0\lambda\,\bar{g}^6. \tag{5.5}$$

It is easy to check from Eqs. (5.2) and (5.3) that

$$\frac{1}{g^2} = \frac{1}{\bar{g}^2} + \lambda \tag{5.6}$$

converts $\beta(g^2)$ into $\bar{\beta}(\bar{g}^2)$. The relationship of the perturbation series for the current-current correlation function for these two choices of the β function is, in fact, a simple application of the mathematical techniques developed in the preceding section. In view of Eq. (3.14), with a one-term β function a change of scale from μ_1^2 to μ_2^2 induces a change in the coupling of

$$\frac{1}{g^2(\mu_2^2)} = \frac{1}{g^2(\mu_1^2)} + b_0\,\ln(\mu_2^2/\mu_1^2)\,. \tag{5.7}$$

This is precisely the coupling redefinition (5.6) if we identify $g^2 = g^2(\mu_2^2)$, $\bar{g}^2 = g^2(\mu_1^2)$, and $\lambda = b_0 \ln(\mu_2^2/\mu_1^2)$. Thus, if we replace the phase rotation $e^{i\theta}$ used in the previous section by the scale factor $\mu_2^2/\mu_1^2 = e^{\lambda/b_0}$, then the previous solution (4.13) of the one-term renormalization group relations (4.1), (4.2) implies that

$$C(z) = e^{\lambda z}\bar{C}(z)\,, \tag{5.8}$$

where $C(z)$ and $\bar{C}(z)$ are the Borel transforms, defined by Eq. (2.9), for the two different schemes. This is the result quoted earlier in Eq. (2.22). Since the relation between these two schemes has this simple, explicit form, it suffices to work out the consequences of the more convenient, two-term inverse β function.

We turn now to solve the recursion relation (3.24) for the two-term inverse β function (5.4). This we shall do by developing an operator technique which generalizes that introduced in the previous section. Inserting the two-term inverse β function (5.4) into the recursion relation (3.24) gives

$$i\frac{d}{d\theta}c_{n+1}(\theta) = b_0(\tilde{c}_0 - \lambda\tilde{c}_{-1})\lambda^n + \sum_{m=1}^{n} m b_0 \lambda^{n-m} c_m(\theta)\,. \tag{5.9}$$

By subtracting successive equations, this can be rewritten as

$$i\frac{d}{d\theta}\{c_{n+1}(\theta) - \lambda c_n(\theta)\} = n b_0 c_n(\theta) \tag{5.10}$$

for $n \geq 1$, while

$$i\frac{d}{d\theta}c_1(\theta) = b_0(\tilde{c}_0 - \lambda\tilde{c}_{-1})\,. \tag{5.11}$$

The generalization of the previous abstract vector definition (4.3),

$$|C_\theta\rangle = |0\rangle(\tilde{c}_0 - \lambda\tilde{c}_{-1}) + \sum_{n=1}^{\infty} |n\rangle \frac{n}{\sqrt{n!}}\, c_n(\theta)\,, \tag{5.12}$$

transcribes relations (5.10) and (5.11) into an operator equation

$$\left(1 - \lambda a^\dagger \frac{1}{N}\right) i\frac{d}{d\theta}|C_\theta\rangle = b_0 a^\dagger |C_\theta\rangle\,, \tag{5.13}$$

or

$$i\frac{d}{d\theta}|C_\theta\rangle = b_0 S|C_\theta\rangle\,, \tag{5.14}$$

where

$$S = \left(1 - \lambda a^\dagger \frac{1}{N}\right)^{-1} a^\dagger\,. \tag{5.15}$$

Using the commutation relation $[N, a^\dagger] = a^\dagger$, this operator may be rewritten as

$$\begin{aligned} S &= a^\dagger \left(1 - \lambda a^\dagger \frac{1}{N+1}\right)^{-1} \\ &= a^\dagger (N+1) \frac{1}{N+1-\lambda a^\dagger} \\ &= a^\dagger + \lambda a^\dagger a^\dagger \frac{1}{N - \lambda a^\dagger + 1}\,. \end{aligned} \tag{5.16}$$

Just as in the work of the previous section, Eq. (5.14) has the straightforward operator solution:

$$|C_\theta\rangle = e^{-ib_0\theta S}|C\rangle, \tag{5.17}$$

where $|C\rangle$ is the initial vector at $\theta = 0$.

Continuing to work in the spirit of the previous section, we introduce left eigenstates of the operator S,

$$\zeta\langle\zeta| = \langle\zeta|S\,, \tag{5.18}$$

which reduce to coherent states at $\lambda = 0$. The projection of the abstract vector $|C_\theta\rangle$ onto these states defines a generalization of the coherent state representation:

$$C_\theta(\zeta) = \langle\zeta|C_\theta\rangle\,. \tag{5.19}$$

In this representation, the abstract operator relation (5.17) becomes a concrete relation between ordinary functions:

$$C_\theta(\zeta) = e^{-ib_0\theta\zeta}\, C(\zeta)\,. \tag{5.20}$$

We again assemble the absorptive coefficients into an abstract vector,

$$|A\rangle = \sum_{n=1}^{\infty} |n\rangle \frac{n}{\sqrt{n!}}\, a_n\,, \tag{5.21}$$

which has the generalized coherent state representation

$$\mathcal{A}(\zeta) = \langle \zeta | A \rangle \,. \tag{5.22}$$

Using Eq. (5.20) to rotate from the negative real t axis to the positive axis and taking the discontinuity gives

$$\mathcal{A}(\zeta) = \sin(\pi b_0 \zeta)\, \mathcal{C}(\zeta) \,. \tag{5.23}$$

This is the relation (2.43) quoted in Sec. II. The functions $\mathcal{A}(\zeta)$ and $\mathcal{C}(\zeta)$ will be seen to be precisely the modified Borel transforms of the absorptive and dispersive coefficients.

To use this result, we need the explicit form of the generalized coherent state representation. To derive it, we first note that the explicit construction of the $\langle\zeta|$ states may be obtained from their coherent state representative $\langle \zeta | z^* \rangle$. To obtain this representative, we multiply Eq. (5.18) from the right by $(N - \lambda a^\dagger + 1)|z^*\rangle$ and use

$$\frac{d}{dz^*} |z^*\rangle = a^\dagger |z^*\rangle \,, \tag{5.24}$$

which follows from the definition (4.10), to arrive at the differential equation

$$\zeta \left[z^* \frac{d}{dz^*} - \lambda \frac{d}{dz^*} + 1 \right] \langle \zeta | z^* \rangle = \left(1 + z^* \frac{d}{dz^*} \right) \frac{d}{dz^*} \langle \zeta | z^* \rangle \,, \tag{5.25}$$

which is a standard confluent hypergeometric equation in the argument ζz^*. The solution we need is, however, easy to obtain directly. Since the differential equation (5.25) is linear in z^*, it is solved by a Laplace transform involving the kernel e^{pz^*} which converts the derivative d/dz^* into a multiplication by p and replaces z^* by the derivative d/dp. This method produces a first-order differential equation, which leads to the solution

$$\langle \zeta | z^* \rangle = \lambda \zeta \int_0^1 du\, (1-u)^{\lambda\zeta - 1} e^{\zeta u z^*} \,. \tag{5.26}$$

Because $z^* = 0$ is a regular-singular point of the differential equation (5.25), the other linearly independent solution is singular at the origin. The scalar product of a coherent state $|z^*\rangle$ with a vector of finite norm produces an entire analytic function of z^*. Hence the solution (5.26) is the proper solution to the differential equation (5.25) since it defines an entire analytic function of z^*. The state $|z^* = 0\rangle$ is the ground state $|0\rangle$. The solution (5.26) is normalized so that $\langle \zeta | 0 \rangle = 1$.

Expanding Eq. (5.26) in powers of z^*, using the standard integral representation of Euler's beta function, and using

$$\langle n | z^* \rangle = \frac{z^{*\,n}}{\sqrt{n!}} \tag{5.27}$$

gives the number state representation of $\langle \zeta |$:

$$\begin{aligned} \phi_n(\zeta) &\equiv \langle \zeta | n \rangle / \sqrt{n!} \\ &= \frac{\Gamma(1+\lambda\zeta)}{\Gamma(n+1+\lambda\zeta)}\, \zeta^n \,. \end{aligned} \tag{5.28}$$

Near the origin $\zeta = 0$, $\phi_n(\zeta)$ has a power series expansion which starts from ζ^n, so the functions $\{\phi_n(\zeta)\}$ ($n = 0, 1, 2, \ldots$) are linearly independent and form a complete set of analytic functions in the region $|\zeta| < 1/|\lambda|$. Making use of these functions gives

$$\mathcal{A}(\zeta) = \sum_{n=0}^{\infty} \langle \zeta | n \rangle \langle n | A \rangle = \sum_{n=1}^{\infty} \phi_n(\zeta)\, n a_n \,, \tag{5.29}$$

and

$$\mathcal{C}(\zeta) = (\tilde{c}_0 - \lambda \tilde{c}_{-1}) + \sum_{n=1}^{\infty} \phi_n(\zeta)\, n c_n \,, \tag{5.30}$$

which are just the results (2.41) and (2.42) quoted in Sec. II.

We turn now to investigate the relationship between the new transform and the standard Borel transform, and to also put our previous results in a more general setting. The power series expansion coefficients $\{f_n\}$ of some function $f(x)$ may be used to define an abstract vector according to

$$|F\rangle = \sum_{n=0}^{\infty} |n\rangle \frac{f_n}{\sqrt{n!}} \,. \tag{5.31}$$

The coherent state representation produces the Borel transform

$$F(z) = \langle z | F \rangle = \sum_{n=0}^{\infty} f_n \frac{z^n}{n!} \,, \tag{5.32}$$

while the new representation produces the modified Borel transform

$$\mathcal{F}(\zeta) = \langle \zeta | F \rangle = \sum_{n=0}^{\infty} f_n\, \phi_n(\zeta) \,. \tag{5.33}$$

The relation between the modified Borel transform and the Borel transform can be derived by using the transformation function $\langle \zeta | z^* \rangle$ given in Eq. (5.26). We first note that, according to Eq. (4.10),

$$\begin{aligned} \left| \frac{d}{dz} \right\rangle \left\langle z \right|_{z=0} &= \sum_{n=0}^{\infty} \frac{1}{n!}\, a^{\dagger\, n} |0\rangle\langle 0| a^n \\ &= \sum_{n=0}^{\infty} |n\rangle\langle n| = 1 \,. \end{aligned} \tag{5.34}$$

Utilizing this identity, the relation between the two transforms can be easily obtained,

$$\begin{aligned} \mathcal{F}(\zeta) = \langle \zeta | F \rangle &= \left\langle \zeta \left| \frac{d}{dz} \right\rangle \langle z | F \rangle \right|_{z=0} \\ &= \lambda\zeta \int_0^1 du\ (1-u)^{\lambda\zeta-1} e^{\zeta u \frac{d}{dz}} \langle z | F \rangle \Big|_{z=0} \\ &= \lambda\zeta \int_0^1 du\, (1-u)^{\lambda\zeta - 1} F(\zeta u) \,. \end{aligned} \tag{5.35}$$

This expression was previously quoted in Eq. (2.29) of Sec. II.

To invert this relation, so as to express the ordinary Borel transform in terms of the modified Borel transform, we note that by Cauchy's formula any analytic function can be expressed as a superposition of simple poles. Hence the inverse relation to (5.35) may be found by studying how to invert a simple pole. The state $|\beta\rangle$ defined by the coherent state representation

$$B(z) = \langle z|\beta\rangle = 1 + \frac{\beta z}{(1-\beta z)^{1+\lambda/\beta}}\,, \tag{5.36}$$

produces a simple pole in the new representation. This can be proven from the observation that

$$\begin{aligned}\lambda\zeta \int_0^1 du\,(1-u)^{\lambda\zeta-1}\left[1+\frac{\beta\zeta u}{(1-\beta\zeta u)^{1+\lambda/\beta}}\right] &= 1-\frac{\beta\zeta}{1-\beta\zeta}\int_0^1 d\left[\frac{(1-u)^{\lambda\zeta}}{(1-\beta\zeta u)^{\lambda/\beta}}\right]\\ &= \frac{1}{1-\beta\zeta}\,.\end{aligned} \tag{5.37}$$

Hence, in view of Eq. (5.35),

$$\mathcal{B}(\zeta) = \langle\zeta|\beta\rangle = \frac{1}{1-\beta\zeta}\,. \tag{5.38}$$

Having found how to invert the relation (5.35) for a simple pole, we can now treat the general case. In the neighborhood of the origin where the modified Borel transform (5.33) is assumed to define an analytic function, Cauchy's formula may be applied:

$$\langle\zeta|F\rangle = \oint \frac{d\zeta'}{2\pi i}\,\frac{\langle\zeta'|F\rangle}{\zeta'-\zeta}\,, \tag{5.39}$$

where the contour circles about the origin with $|\zeta'| > |\zeta|$. In view of the transformation function (5.38), this Cauchy formula may be written as[15]

$$\langle\zeta|F\rangle = \oint \frac{d\zeta'}{2\pi i\zeta'}\,\langle\zeta|\,\beta = 1/\zeta'\,\rangle\langle\zeta'|F\rangle\,. \tag{5.40}$$

This implies the formal completeness relation[16]

$$\oint \frac{d\zeta}{2\pi i\zeta}\,|\,\beta = 1/\zeta\,\rangle\langle\zeta| = 1\,. \tag{5.41}$$

This expression of the identity holds when it is inserted in matrix elements and the contour chosen appropriately so as to enclose the relevant singularities. Inserting the completeness relation between $\langle z|$ and $|F\rangle$ yields the inverse relation to Eq. (5.35):

$$F(z) = \langle z|F\rangle = \oint \frac{d\zeta'}{2\pi i\zeta'}\,\langle z|\,\beta = 1/\zeta'\,\rangle\langle\zeta'|F\rangle\,. \tag{5.42}$$

This result was presented in Eq. (2.28) in Sec. II, where it was derived in a different fashion.

Number state matrix elements of the completeness relation (5.41) give

$$\oint \frac{d\zeta}{2\pi i\zeta}\,\chi_m(1/\zeta)\phi_n(\zeta) = \delta_{m,n}\,, \tag{5.43}$$

where

$$\chi_n(\beta) \equiv \langle n|\beta\rangle\sqrt{n!}\,. \tag{5.44}$$

Here the contour must encircle the origin with a radius constrained by $|\zeta| < 1/\lambda$ so as to avoid the singularities of $\phi_n(\zeta)$. With this restriction, Eq. (5.43) describes the way in which $\{\phi_n(\zeta)\}$ and $\{\chi_n(\beta)\}$ form reciprocal sets of basis functions. Expanding the coherent state representative $\langle z|\beta\rangle$ given by Eq. (5.36) in powers of z identifies the number state components $\langle n|\beta\rangle$ and yields the explicit form

$$\chi_n(\beta) = \delta_{n,0} + n\beta^n\,\frac{\Gamma(n+\lambda/\beta)}{\Gamma(1+\lambda/\beta)}\,, \tag{5.45}$$

which are polynomials in β. This is the result quoted in Eq. (2.26) of Sec. II.

The completeness relation (5.41) can also be exploited to extract the coefficients $\{f_n\}$ of an asymptotic series from the modified Borel transform:

$$\begin{aligned}f_n = \langle n|F\rangle\sqrt{n!} &= \oint \frac{d\zeta}{2\pi i\zeta}\,\langle n|\,\beta = 1/\zeta\,\rangle\langle\zeta|F\rangle\sqrt{n!}\\ &= \oint \frac{d\zeta}{2\pi i\zeta}\,\chi_n(1/\zeta)\langle\zeta|F\rangle\,.\end{aligned} \tag{5.46}$$

Since $\langle\zeta|F\rangle = \mathcal{F}(\zeta)$, this is the formula (2.25) stated in Sec. II.

With these results in hand, we may now derive the explicit relation between the dispersive coefficients $\{c_n\}$ and the absorptive coefficients $\{a_n\}$. Applying (5.46) to the absorptive transform and using the simple relation (5.23) between their modified Borel transforms, $\mathcal{A}(\zeta) = \langle\zeta|A\rangle = \sin(\pi b_0\zeta)\langle\zeta|C\rangle$, gives

$$\begin{aligned}\langle n|A\rangle\sqrt{n!} &= \oint \frac{d\zeta}{2\pi i\zeta}\,\chi_n(1/\zeta)\sin(\pi b_0\zeta)\langle\zeta|C\rangle\\ &= \oint \frac{d\zeta}{2\pi i\zeta}\,\chi_n(1/\zeta)\left\{\sin(\pi b_0\zeta)\sum_{m=0}^{\infty}\phi_m(\zeta)\langle m|C\rangle\sqrt{m!}\right\}\,,\end{aligned} \tag{5.47}$$

where in the second line Eq. (5.28) has been used. Conversely,

$$\langle n|C\rangle\sqrt{n!} = \oint \frac{d\zeta}{2\pi i\zeta}\,\chi_n(1/\zeta)\left\{\frac{1}{\sin(\pi b_0\zeta)}\sum_{m=0}^{\infty}\phi_m(\zeta)\langle m|A\rangle\sqrt{m!}\right\}\,. \tag{5.48}$$

[15]So as to keep the notation uncluttered, we use the symbol $|\beta = 1/\zeta'\rangle$ to denote the state $|\beta\rangle$ defined by Eq. (5.36), but evaluated at $\beta = 1/\zeta'$.

[16]This representation of the identity as a contour integral is a generalization of an idea of Schwinger [14].

Using the definitions (5.12) and (5.21) of the states $|C\rangle$ and $|A\rangle$, and the explicit forms (5.28) and (5.45) for the functions $\phi_n(\zeta)$ and $\chi_n(\zeta)$ gives, for $n \geq 1$,

$$\begin{aligned} na_n &= \oint \frac{d\zeta}{2\pi i\zeta}\,\frac{n\Gamma(n+\lambda\zeta)}{\Gamma(1+\lambda\zeta)}\,\zeta^{-n}\sin(\pi b_0\zeta)\left\{(\tilde c_0-\lambda\tilde c_{-1})+\sum_{m=1}^{\infty}\phi_m(\zeta)\,mc_m\right\} \\ &= \sum_{m=0}^{\infty}\oint\frac{d\zeta}{2\pi i\zeta}\,\frac{n\Gamma(n+\lambda\zeta)}{\Gamma(m+1+\lambda\zeta)}\,\zeta^{m-n}\sin(\pi b_0\zeta)\,\{(\tilde c_0-\lambda\tilde c_{-1})\delta_{m,0}+mc_m\}\,. \end{aligned} \tag{5.49}$$

The terms in the summation in Eq. (5.49) with $m \geq n$ vanish since the integrand is then regular at $\zeta = 0$. To evaluate this expression, we define combinatorial factors $I^n_{m,l}$ (where n, m, l are non-negative integers which satisfy $n \geq m+l$) by the generating function

$$\frac{\Gamma(n+1+x)}{\Gamma(m+1+x)} = \sum_{l=0}^{n-m} x^{n-m-l} I^n_{m,l}\,. \tag{5.50}$$

Inserting this expansion in Eq. (5.49) and expanding $\sin(\pi b_0\zeta)$ in powers of ζ now yields

$$\begin{aligned} a_n &= \sum_{m=0}^{n-1}\oint\frac{d\zeta}{2\pi i\zeta}\left(\sum_{k=0}^{\infty}(-)^k\frac{(\pi b_0\zeta)^{2k+1}}{(2k+1)!}\right)\sum_{l=0}^{n-m-1}(\lambda\zeta)^{n-m-l-1}\zeta^{m-n}I^{n-1}_{m,l}\,\{(\tilde c_0-\lambda\tilde c_{-1})\delta_{m,0}+mc_m\} \\ &= (\tilde c_0-\lambda\tilde c_{-1})\left\{\sum_{k=0}^{[\frac{n-1}{2}]}(-)^k\frac{(\pi b_0)^{2k+1}}{(2k+1)!}\,\lambda^{n-2k-1}I^{n-1}_{0,2k}\right\}+\sum_{m=1}^{n-1}mc_m\left\{\sum_{k=0}^{[\frac{n-m-1}{2}]}(-)^k\frac{(\pi b_0)^{2k+1}}{(2k+1)!}\,\lambda^{n-m-2k-1}I^{n-1}_{m,2k}\right\}. \end{aligned} \tag{5.51}$$

Here $[x]$ denotes the integer part of x. This result provides an explicit evaluation of the absorptive coefficients a_n in terms of the dispersive coefficients $\{c_m\}$ with smaller indices, $m < n$. It is the result (2.44) of Sec. II. In the limit $\lambda \to 0$, using

$$I^n_{n-k,k} = \frac{\Gamma(n+1)}{\Gamma(n-k+1)} = \frac{n!}{(n-k)!}\,, \tag{5.52}$$

we, find that, for $n > 0$,

$$\begin{aligned} \frac{a_n}{(n-1)!} &= \delta_{n,\text{odd}}\,(-)^{\frac{n-1}{2}}\frac{(\pi b_0)^n}{n!}\,\tilde c_0 \\ &\quad+\sum_{k=0}^{[\frac{n}{2}]-1}(-)^k\frac{(\pi b_0)^{2k+1}}{(2k+1)!}\,\frac{c_{n-2k-1}}{(n-2k-2)!}\,, \end{aligned} \tag{5.53}$$

which is precisely the previous result in [7] as quoted in Eqs. (2.11) and (2.12) of Sec. II.

Making use of the expansion

$$\frac{1}{\sin z} = \sum_{k=0}^{\infty}\frac{|(2^{2k}-2)B_{2k}|}{(2k)!}\,z^{2k-1}\,, \tag{5.54}$$

where B_n are the Bernoulli numbers, and going through similar steps, one may express the dispersive coefficients $\{c_n\}$ as a sum of the absorptive coefficients $\{a_n\}$. For $n > 1$, one has

$$\begin{aligned} c_n &= \sum_{m=1}^{n+1} ma_m\left\{\oint\frac{d\zeta}{2\pi i\zeta}\,\frac{\Gamma(n+\lambda\zeta)}{\Gamma(m+1+\lambda\zeta)}\,\frac{1}{\sin\pi b_0\zeta}\,\zeta^{m-n}\right\} \\ &= \frac{a_{n+1}-\lambda a_n}{n\pi b_0}+\sum_{m=1}^{n-1}ma_m\oint\frac{d\zeta}{2\pi i\zeta}\,\frac{1}{\sin\pi b_0\zeta}\sum_{l=0}^{n-m-1}I^{n-1}_{m,l}\,(\lambda\zeta)^{n-m-l-1}\zeta^{m-n} \\ &= \frac{a_{n+1}-\lambda a_n}{\pi b_0\,n}+\sum_{m=1}^{n-1}ma_m\sum_{k=1}^{[\frac{n-m+1}{2}]}\frac{|(2^{2k}-2)B_{2k}|}{(2k)!}(\pi b_0)^{2k-1}\lambda^{n-m-2k+1}I^{n-1}_{m,2k-2}\,, \end{aligned} \tag{5.55}$$

while

$$\tilde c_{-1} = -\frac{a_0}{\pi b_0}\,,\qquad \tilde c_0 = \frac{a_1-\lambda a_0}{\pi b_0}\,. \tag{5.56}$$

This is the result displayed in Eq. (2.47) of Sec. II. Again the dispersive coefficient c_n involves only the absorptive coefficients $\{a_m\}$ with $m \leq n+1$ and the $\lambda \to 0$ limit reproduces the previous result in [7] quoted in Eq. (2.14).

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APPENDIX A: CRITIQUE OF WEST'S PAPER

West [1] considers the differentiated correlation function $t\,dK(t)/dt$ which he terms $D(t/\mu^2, g^2(\mu^2))$ [defined in his Eq. (8)].[17] This is a renormalization group invariant which has a perturbative expansion in powers of the running coupling:

$$D(t/\mu^2, g^2(\mu^2)) \sim \sum_{n=0}^{\infty} (-)^n d_n(t/\mu^2)\, g(\mu^2)^{2n}\,. \quad \text{(A1)}$$

[To facilitate comparison with West's paper, we use his notation for the coefficients $\{d_n\}$ which, for $\mu^2 = -t$, differ from our $\{k_n'\}$ in Eq. (3.18) by a factor $(-)^n$.] West asserts [in his Eq. (20)] that these coefficients have the large-order behavior

$$d_n(t/\mu^2) \sim \left[\frac{2}{\pi\phi(k_1)}\right]^{1/2} k_1^{n-1} D(t/\mu^2, 1/k_1)\cos n\pi\,, \quad \text{(A2)}$$

where, in our notation, $k_1 \sim b_0(n-1)+\lambda$, and $\phi(k_1) \sim (b_0 k_1)^{-1}$. By using the free field limit $D(t/\mu^2, 0) = -\sum_f Q_f^2/4\pi^2$, where $\{Q_f\}$ are the quark charges (which follows from Eq. (8) in [1]), the claimed large-order behavior of the coefficients $d_n(t/\mu^2)$ can be written explicitly as

$$d_n(t/\mu^2) \sim -\frac{\sqrt{2\pi}}{4\pi^3} e^{-1+\lambda/b_0}\left(\sum\nolimits_f Q_f^2\right)(-b_0)^n\, n^{n-1/2}\,. \quad \text{(A3)}$$

We shall show that this result is both inconsistent and contradicts our exact relations.

Because $D(t/\mu^2, g^2(\mu^2))$ satisfies the homogeneous renormalization group equation

$$\mu^2 \frac{d}{d\mu^2} D(t/\mu^2, g^2(\mu^2)) = 0\,, \quad \text{(A4)}$$

the coefficients in the expansion (A1) must obey

$$\sum_{n=0}^{\infty} (-)^n g^{2n}(\mu^2)\left\{\mu^2\frac{\partial}{\partial\mu^2} + \frac{n\beta(g^2(\mu^2))}{g^2(\mu^2)}\right\} d_n(t/\mu^2) = 0\,. \quad \text{(A5)}$$

Using the asymptotic expansion of the β function, $\beta(g^2) \sim -b_0 g^4 - b_0\lambda g^6 - \cdots$, and identifying the coefficients of each power of the coupling yields

$$\mu^2\frac{\partial}{\partial\mu^2} d_n(t/\mu^2) = -b_0\{\,(n-1)d_{n-1}(t/\mu^2) - \lambda(n-2)d_{n-2}(t/\mu^2) + \cdots\}\,. \quad \text{(A6)}$$

Using the explicit large-order behavior (A3) on the right-hand side of Eq. (A6) implies that

$$\mu^2\frac{\partial}{\partial\mu^2} d_n(t/\mu^2) \sim \text{const} \times (-b_0)^n n^{n-1/2}\,, \quad \text{(A7)}$$

which states that the logarithmic derivative $(\mu^2 d/d\mu^2) d_n(t/\mu^2)$ has the same large-order behavior as does $d_n(t/\mu^2)$ itself. However, this contradicts Eq. (A3) which asserts that the leading-order behavior of $d_n(t/\mu^2)$ is independent of μ^2.

We shall now show that formula (A2) also contradicts our results. The assertion (A3) may be recast as a prediction for the dispersive coefficients $\{c_n\}$. Comparing the integrated expansion of $t\,dK/dt$, Eq. (3.19), to the original Euclidean expansion (3.13) and using our two-term inverse β function (2.15) yields

$$\begin{aligned} nc_n &= -(k_{n+1}' - \lambda k_n')/b_0 \\ &= (-)^n\{d_{n+1}(-1) + \lambda d_n(-1)\}/b_0\,. \end{aligned} \quad \text{(A8)}$$

This implies that the perturbative coefficients $\{k_n'\}$ of the differentiated correlation function have the same large-order behavior as that of the original coefficients $\{c_n\}$ (up to an overall constant factor), provided that the coefficients have $n!$ growth. If the differentiated coefficients have the asymptotic behavior (A3), then from Eq. (A8), the dispersive coefficients will satisfy

$$\begin{aligned} c_n &\sim \frac{(-)^n}{nb_0} d_{n+1}(-1) \times [1 + O(1/n)] \\ &\sim \frac{\sqrt{2\pi}}{4\pi^3} e^{\lambda/b_0}\left(\sum\nolimits_f Q_f^2\right) b_0^n\, n^{n-1/2} \\ &\sim \frac{e^{\lambda/b_0}}{4\pi^3}\left(\sum\nolimits_f Q_f^2\right)(eb_0)^n\,\Gamma(n)\,. \end{aligned} \quad \text{(A9)}$$

Such large-order behavior (A9) for the dispersive coefficients creates a singularity in the Borel transform

$$C(z) = \tilde{c}_0 - \lambda\tilde{c}_{-1} + \sum_{n=1}^{\infty} \frac{nc_n}{n!} z^n \quad \text{(A10)}$$

at $z = 1/eb_0$. (No such singularity, closer to the origin than the first ultraviolet renormalon at $b_0 z = -1$, has been found in any investigation of individual Feynman diagrams [5].) Inserting the asymptotic form (A9) into Eq. (A10) yields the leading behavior of $C(z)$ near $z = 1/eb_0$:

$$C(z) \sim \frac{e^{\lambda/b_0}}{4\pi^3}\left(\sum\nolimits_f Q_f^2\right)(1 - eb_0 z)^{-1}\,. \quad \text{(A11)}$$

We now go through similar steps for the absorptive coefficients a_n. Since Eq. (A8) is the special case of the general relation between the phase-dependent coefficients $\{c_n(\theta)\}$ and the momentum-dependent coefficients $\{d_n(t/|t|)\}$, with $t = -|t|e^{i\theta}$, the analytic continuation of Eq. (A8) produces

[17] Some of the properties of this function are discussed in Sec. III, Eqs. (3.18)–(3.20).

$$na_n = (-)^n \{\operatorname{Im} d_{n+1}(1) + \lambda \operatorname{Im} d_n(1)\}/b_0 \,. \quad \text{(A12)}$$

Equation (21) in [1] gives the predicted large-order behavior[18] of $\operatorname{Im} d_n(1)$:

$$\operatorname{Im} d_n(1) \sim (-)^{n+1} \frac{b_0}{8\pi^3} \left(\sum\nolimits_f Q_f^2\right) \frac{k_1^{n-3}}{\sqrt{2\pi\phi(k_1)}}$$
$$\sim (-)^{n+1} \frac{e^{-1+\lambda/b_0}}{8\pi^3\sqrt{2\pi}} \left(\sum\nolimits_f Q_f^2\right) b_0^{n-1} n^{n-5/2} \,. \quad \text{(A13)}$$

Hence the large-order behavior of the absorptive coefficients a_n is given by

$$a_n \sim \frac{e^{\lambda/b_0}}{8\pi^3\sqrt{2\pi}} \left(\sum\nolimits_f Q_f^2\right) b_0^{n-1}\, n^{n-5/2} \,,$$
$$\sim \frac{e^{\lambda/b_0}}{16\pi^4 b_0} \left(\sum\nolimits_f Q_f^2\right) (eb_0)^n\, \Gamma(n-2) \,. \quad \text{(A14)}$$

Comparing the large-order behavior of c_n given in Eq. (A9) with that of a_n given in Eq. (A14), we find that $a_n/c_n \sim 1/n^2$, which contradicts the constraint, described in Sec. II E, that a_n and c_n must have the same large-order behavior if the radius of convergence of the Borel transform is less than $1/b_0$. Hence (A2) cannot be correct.

Incidentally, we also find that substituting Eq. (21) into Eq. (12) in [1] does not give the large-order behavior for $r_n(1)$ stated in Eq. (22). Using $\alpha_s = g^2(s)/4\pi$ and

$$R(s) \sim 3\left(\sum\nolimits_f Q_f^2\right) \sum_{n=0}^{\infty} r_n(1) \left(\frac{\alpha_s(s)}{\pi}\right)^n \,, \quad \text{(A15)}$$

we find from Eq. (A14) that

$$r_n(1) \sim \frac{e^{\lambda/b_0}}{4\pi^3 b_0} (4\pi^2 e b_0)^n\, \Gamma(n-2) \,, \quad \text{(A16)}$$

which differs from Eq. (22) in [1] by the factor $-n^{\lambda/b_0}$. However, the corrected result (A16) must still be in error as it arises from the inconsistent result (A2).

APPENDIX B: COUPLING REDEFINITIONS AND BOREL TRANSFORM SINGULARITIES

The analysis in the text was simplified by choosing a definition of the renormalized coupling for which the inverse β function contains only two terms. One may wonder if the earlier results critically depend on this choice. We consider here the effects of a coupling redefinition of the form

$$\bar{g}^2 \sim g^2 + \sum_{n=2}^{\infty} d_n\, g^{2n} \,, \quad \text{(B1)}$$

as would result from using a different renormalization scheme. Provided that the coefficients $\{d_n\}$ grow sufficiently slowly, we shall show that such a redefinition does not change the position or the nature of the leading singularities in the Borel transform

$$F(z) = \sum_n f_n\, z^n/(n-1)! \quad \text{(B2)}$$

of a perturbative series. [This requires that the Borel transform of the coefficients $\{d_n\}$ have a larger radius of convergence than does $F(z)$.] Hence the results of the text concerning the possible singularities in the Borel transform of the current-current correlation function have a general validity.

It is instructive to first consider the special case of the coupling redefinition

$$\frac{1}{g^2} = \frac{1}{\bar{g}^2} + \lambda \,, \quad \text{(B3)}$$

or

$$\bar{g}^2 = g^2 + \lambda\, g^4 + \lambda^2\, g^6 + \cdots \,, \quad \text{(B4)}$$

on the Borel representation (or inverse Borel transform)

$$f(g^2) = \int_0^\infty \frac{dz}{z}\, e^{-z/g^2}\, F(z) \,. \quad \text{(B5)}$$

Substituting the redefinition (B3) into the representation (B5) immediately yields the altered Borel transform

$$\bar{F}(z) = e^{-\lambda z} F(z) \quad \text{(B6)}$$

which produces the redefined function $\bar{f}(\bar{g}^2) \equiv f(g^2(\bar{g}^2))$. This is just the result (2.22) of the text. If $F(z)$ has a singularity at $z = z_s$, then $\bar{F}(z)$ has the same singularity multiplied by a factor of $\exp\{-\lambda z_s\}$, plus lower-order singularities generated by the power series expansion of $\exp\{-\lambda(z - z_s)\}$.

The effect of an arbitrary coupling redefinition can be found by compounding the effects of infinitesimal redefinitions

$$g^2 = \bar{g}^2 - \delta g^2 \,, \quad \text{(B7)}$$

where

$$\delta g^2 \sim \delta_2\, g^4 + \delta_3\, g^6 + \delta_4\, g^8 + \cdots \,. \quad \text{(B8)}$$

Placing such an infinitesimal redefinition on the right-hand side of

$$\int_0^\infty \frac{dz}{z}\, e^{-z/\bar{g}^2}\, \bar{F}(z) = \int_0^\infty \frac{dz}{z}\, e^{-z/g^2} F(z) \quad \text{(B9)}$$

and expanding to first order in δg^2 produces

$$\bar{F}(z) = F(z) + \delta F(z) \,, \quad \text{(B10)}$$

with

$$\int_0^\infty \frac{dz}{z}\, e^{-z/g^2}\, \delta F(z) = -\frac{\delta g^2}{g^4} \int_0^\infty dz\, e^{-z/g^2} F(z) \,. \quad \text{(B11)}$$

Using

[18] This differs from the original form of Eq. (21) in [1] by a factor of $3(\sum_f Q_f^2)$, which we believe was missing there.

$$\Delta(z) \equiv \sum_{n=2}^{\infty} \delta_n \, z^n/(n-1)! \qquad \text{(B12)}$$

to denote the Borel transform of the series (B8), δg^2 has the Borel representation

$$\delta g^2 = \int_0^\infty \frac{dz}{z} \, e^{-z/g^2} \, \Delta(z) \, . \qquad \text{(B13)}$$

Since the series expansion (B8) for δg^2 starts at order g^4, both $\Delta(z)$ and its derivative $\Delta'(z)$ vanish at $z = 0$. Using this representation, the effect of the infinitesimal coupling redefinition on the original Borel transform $F(z)$ may be expressed as

$$\delta F(z) = -\delta_2 \, z \, F(z) - z \int_0^z dz' \, F(z') \, \frac{\partial^2}{\partial z^2} \left[\frac{\Delta(z - z')}{z - z'} \right] . \qquad \text{(B14)}$$

The validity of this formula is easily verified by inserting it into the integral on the left-hand side of Eq. (B11), interchanging the z and z' integrals, and performing the resulting z integration. The first part of this expression, $-\delta_2 z \, F(z)$, is just the infinitesimal version of the previous result (B6): The order λ term in the transformation (B4) is generated by an infinitesimal redefinition (B8) with only $\delta_2 = \lambda$ nonvanishing, in which case the convolution integral in Eq. (B14) vanishes. For a general redefinition, the convolution integral will become singular when an end point of the integral pinches a singularity of $F(z)$ or $\Delta(z)$. Because the kernel $(\partial^2/\partial z^2)[\Delta(z - z')/(z - z')]$ is regular at $z = z'$, the convolution weakens singularities in $F(z)$; if $F(z)$ has a singularity at $z = z_s$, then $\delta F(z) + \delta_2 z F(z)$ is less singular at this point by at least one power of $(z - z_s)$. Consequently, iterating this infinitesimal coupling redefinition cannot change the location or leading behavior of any singularity in $F(z)$ which is within the domain of analyticity of $\Delta(z)$. Therefore, except for an overall multiplicative factor, arbitrary coupling redefinitions cannot change the nature of any singularity in a Borel transform $F(z)$ which is within the radius of convergence of the Borel transformed coupling redefinition $\Delta(z)$.

APPENDIX C: BOREL TRANSFORM DETAILS

We show first the equivalence of the singularities of the modified Borel transform and the standard Borel transform presented in Eqs. (2.30) and (2.31). Suppose that the Borel transform $F(z)$ has a singularity at $z = R$ of the form

$$F(z) = (1 - z/R)^{-\alpha - \lambda R} \, . \qquad \text{(C1)}$$

Inserting Eq. (C1) into Eq. (2.29) gives the modified Borel transform

$$\mathcal{F}(\zeta) = \lambda\zeta \int_0^1 du \, (1 - u)^{\lambda\zeta - 1} (1 - \zeta u/R)^{-\alpha - \lambda R} \, , \qquad \text{(C2)}$$

which is Euler's integral representation of the hypergeometric function [15]:

$$\mathcal{F}(\zeta) = F(\alpha + \lambda R, 1; \lambda\zeta + 1; \zeta/R) \, . \qquad \text{(C3)}$$

Since the hypergeometric function $F(a, b; c; z)$ is analytic in the domain where $|z| < 1$, to examine the behavior of $\mathcal{F}(\zeta)$ near $\zeta = R$, we make use of the analytic continuation

$$F(a, b; c; z) = A_1 \, F(a, b; a + b - c + 1; 1 - z) + A_2 \, (1 - z)^{c-a-b} F(c - a, c - b; c - a - b + 1; 1 - z) \, , \qquad \text{(C4)}$$

where

$$A_1 = \frac{\Gamma(c)\Gamma(c - a - b)}{\Gamma(c - a)\Gamma(c - b)} \, , \qquad A_2 = \frac{\Gamma(c)\Gamma(a + b - c)}{\Gamma(a)\Gamma(b)} \, . \qquad \text{(C5)}$$

Noting that

$$F(a, b; a; z) = (1 - z)^{-b} \, , \qquad \text{(C6)}$$

we now find that the modified Borel transform $\mathcal{F}(\zeta)$ may be expressed as

$$\mathcal{F}(\zeta) = \frac{\lambda\zeta}{\eta} \, F(\lambda R + \alpha, 1; 1 - \eta; 1 - \zeta/R) + \frac{\Gamma(\lambda\zeta + 1)\Gamma(-\eta)}{\Gamma(\lambda R + \alpha)} \, (1 - \zeta/R)^{\eta} (\zeta/R)^{-\lambda\zeta} \, , \qquad \text{(C7)}$$

where $\eta = \lambda\zeta - \lambda R - \alpha$. Since $F(a, b; c; 0) = 1$, Eq. (C7) gives the leading singular behavior of $\mathcal{F}(\zeta)$ as $\zeta \to R$ as well as the first correction,

$$\mathcal{F}(\zeta) \sim \frac{\Gamma(\lambda R + 1)\Gamma(\alpha)}{\Gamma(\lambda R + \alpha)} \, (1 - \zeta/R)^{-\alpha} \times [1 - \lambda R \, (1 - \zeta/R) \ln(1 - \zeta/R) + O(1 - \zeta/R)] \, . \qquad \text{(C8)}$$

This shows that if the Borel transform $F(z)$ is exactly a simple power-law singularity, then the modified Borel transform $\mathcal{F}(\zeta)$ will contain a singularity at the same position but with a shifted power, and with subleading corrections suppressed by $(1 - \zeta/R) \ln(1 - \zeta/R)$. It is not difficult to see that the converse also holds: If the modified Borel transform $\mathcal{F}(\zeta)$ has only the first power-law term shown in Eq. (C8), then the Borel transform $F(z)$ will have the singularity shown in Eq. (C1) with a subleading correction suppressed by $(1 - z/R) \ln(1 - z/R)$. The equivalence we have demonstrated is just that between Eqs. (2.30) and (2.31) stated in the text.

We now turn to prove that the transformation (2.37) constructs from the modified Borel transform $\mathcal{F}(z)$ the same "inverse Borel transform" $f(y)$ as is produced by the Laplace transform (2.35) of the standard Borel transform. Substituting the expression (2.28) for the Borel transform $F(z)$ in terms of the modified Borel transform $\mathcal{F}(z)$ into the Laplace transform (2.35) gives

$$f(y) = \frac{1}{y}\int_0^\infty dz\, e^{-z/y} \oint \frac{d\zeta}{2\pi i\zeta}\left(1+\frac{z/\zeta}{(1-z/\zeta)^{1+\lambda\zeta}}\right) \times \mathcal{F}(\zeta) . \tag{C9}$$

Interchanging the order of the integrals[19] yields

$$f(y) = \frac{1}{y}\oint_C \frac{d\zeta}{2\pi i\zeta}\mathcal{F}(\zeta) \times \int_0^\infty dz\, e^{-z/y}\left(1+\frac{z/\zeta}{(1-z/\zeta)^{\lambda\zeta+1}}\right), \tag{C10}$$

where the contour C wraps counterclockwise about the entire path of the z integral. Separating the integrand into two pieces by writing

$$1+\frac{z/\zeta}{(1-z/\zeta)^{1+\lambda\zeta}} = \frac{1}{(1-z/\zeta)^{1+\lambda\zeta}} + \left\{1-\frac{1}{(1-z/\zeta)^{\lambda\zeta}}\right\}, \tag{C11}$$

and then, for the term in braces, integrating by parts in z yields

$$f(y) = \frac{1-\lambda y}{y}\oint_C \frac{d\zeta}{2\pi i\zeta}\mathcal{F}(\zeta) \times \int_0^\infty dz\, e^{-z/y}\left\{\frac{1}{(1-z/\zeta)^{1+\lambda\zeta}}\right\}. \tag{C12}$$

The ζ-contour integral can be written as a line integral of $\mathcal{F}(\zeta)$ times the discontinuity (in ζ) of the function $\int_0^\infty dz\, e^{-z/y}(1-z/\zeta)^{-1-\lambda\zeta}$. Hence

$$f(y) = -\frac{1-\lambda y}{y}\int_0^\infty d\zeta\, \mathcal{F}(\zeta) \times \oint_C \frac{dz}{2\pi i\zeta}\, e^{-z/y}(1-z/\zeta)^{-1-\lambda\zeta}, \tag{C13}$$

where the path of the ζ integral is now the same as the original path of z. The change of variable $z = yt+\zeta$ converts the inner integral into the standard representation of an inverse Γ function:

$$\begin{aligned} f(y) &= \frac{1-\lambda y}{y}\int_0^\infty d\zeta\, e^{-\zeta/y}(\zeta/y)^{\lambda\zeta}\mathcal{F}(\zeta) \times \frac{i}{2\pi}\int_C dt\, e^{-t}\,(-t)^{-\lambda\zeta-1} \\ &= \frac{1-\lambda y}{y}\int_0^\infty d\zeta\, e^{-\zeta/y}\frac{(\zeta/y)^{\lambda\zeta}}{\Gamma(1+\lambda\zeta)}\mathcal{F}(\zeta), \end{aligned} \tag{C14}$$

which is the result (2.37).

We can also show directly that the transformation (C14) defines a function which has the desired expansion. After rescaling ζ by a factor of y, the representation (C14) immediately shows that $f(y)$ has an asymptotic expansion in powers of y, $f(y) \sim \sum_{n=0}^\infty c_n\, y^n$. The coefficients may be computed by inserting the definition (2.24) of $\mathcal{F}$ into Eq. (C14), substituting $\zeta = sy$,

$$f(y) = (1-\lambda y)\int_0^\infty ds\, e^{-s}\, s^{\lambda sy}\sum_{m=0}^\infty f_m \frac{(sy)^m}{\Gamma(m+1+\lambda sy)}, \tag{C15}$$

and noting that only the first n terms in the sum can contribute terms of order y^n. Hence the coefficient c_n may be extracted by a contour integral encircling the origin:

$$\begin{aligned} c_n &= \oint \frac{dy}{2\pi i y}\frac{1-\lambda y}{y^n}\int_0^\infty ds\, e^{-s}\, s^{\lambda sy}\sum_{m=0}^n f_m \frac{(sy)^m}{\Gamma(m+1+\lambda sy)} \\ &= \sum_{m=0}^n f_m \oint \frac{dz}{2\pi i z}\frac{(z/\lambda)^{m-n}}{\Gamma(m+1+z)}\int_0^\infty ds\, e^{-s}\, s^{z+n}(1-z/s) = \sum_{m=0}^n f_m \oint \frac{dz}{2\pi i z}(z/\lambda)^{m-n}\frac{n\,\Gamma(n+z)}{\Gamma(1+m+z)}. \end{aligned} \tag{C16}$$

The terms with $m < n$ give no contribution since, for these terms, the ratio of Γ functions is a polynomial in z of order $n-m-1$ and thus when divided by z^{n-m+1} produces a vanishing residue at the origin. The final term gives

$$c_n = f_n \oint \frac{dz}{2\pi i z}\frac{n}{z+n} = f_n , \tag{C17}$$

so that, as expected, $f(y) \sim \sum_{n=0}^\infty f_n\, y^n$.

[19]This is valid under the condition that the integrals converge absolutely. For $\lambda \le 0$ it is sufficient that the modified Borel transform $\mathcal{F}(\zeta)$ be bounded and analytic within a neighborhood of the contour of the z integral (assumed to lie in the right half plane). For $\lambda > 0$ sufficient conditions are that $\mathcal{F}(\zeta)$ be bounded and analytic within some wedge enclosing the contour of the z integral (so that z/ζ may remain bounded away from one as both z and $\zeta \to \infty$). For sufficiently small values of y the integrand is then exponentially bounded.

APPENDIX D: ALTERNATIVE DERIVATIONS

Several different approaches may be used to derive the relations between the absorptive and dispersive perturbative expansion coefficients. This appendix sketches two complementary methods which do not involve the abstract vector representations employed in the main text.

By examining the asymptotic behavior of the dispersion relation satisfied by the correlation function $K(t)$, one may derive the result (2.47) expressing the dispersive coefficients in terms of the absorptive coefficients. The scalar correlation function $K(t)$ satisfies the once-subtracted dispersion relation

$$\Delta K(t) \equiv K(t) - K(0) = \frac{t}{\pi} \int_0^\infty \frac{ds}{s} \frac{\rho(s)}{(s-t)}, \qquad \text{(D1)}$$

where the spectral density $\rho(s)$ is the discontinuity of $K(t)$ across the positive real axis:

$$\rho(s) \equiv \mathrm{Im} K(s + i0^+) . \qquad \text{(D2)}$$

Given the asymptotic expansion of the spectral density,

$$\rho(s) \sim \sum_{n=0}^{\infty} a_n \, (g^2(s))^n , \qquad \text{(D3)}$$

this dispersion relation may be used to derive the asymptotic behavior of the correlation function $K(t)$ as $|t| \to \infty$ along any ray in the complex t plane. To carry this out, it is convenient to separate the dispersion integral (D1) into two pieces, $0 < s < |t|$, and $|t| < s < \infty$. In the low-momentum piece, one may expand the integrand in powers of s/t, and for the high-momentum piece, expand in powers of t/s. Hence

$$\begin{aligned} \pi \Delta K(t) &= -\int_0^{|t|} \frac{ds}{s} \rho(s) \left(\sum_{n=0}^{\infty} (s/t)^n \right) + \int_{|t|}^{\infty} \frac{ds}{s} \rho(s) \left(\sum_{n=1}^{\infty} (t/s)^n \right) \\ &\equiv -I_0^-(t) + \sum_{n=1}^{\infty} [I_n^+(t) - I_n^-(t)] , \end{aligned} \qquad \text{(D4)}$$

where

$$I_n^-(t) \equiv \int_0^{|t|} \frac{ds}{s} \rho(s) \, (s/t)^n , \qquad \text{(D5)}$$

and

$$I_n^+(t) \equiv \int_{|t|}^{\infty} \frac{ds}{s} \rho(s) \, (t/s)^n . \qquad \text{(D6)}$$

The asymptotic expansion of $I_n^\pm(t)$ may be computed by inserting the expansion (D3) of the spectral density, changing variables from s to $g^2(s)$, and performing the resulting integrations term by term. The final expressions are simplified if one uses the rescaled inverse coupling

$$z \equiv 1 \,/\, [b_0 \, g^2(s)] \qquad \text{(D7)}$$

as the variable rather than $g^2(s)$ itself. The relation (2.19) between s and $g^2(s)$ may then be expressed as

$$s = \Lambda^2 \, z^\gamma e^z , \qquad \text{(D8)}$$

where we have defined $\gamma \equiv -\lambda/b_0 = -b_1/b_0^2$. Similarly, let $y \equiv 1/[b_0 \, g^2(|t|)]$, and $t = |t| e^{i\phi}$. Using $ds/dz = s(1+\gamma/z)$, one finds for the high-momentum contribution

$$\begin{aligned} I_n^+(t) &\sim \int_y^\infty dz \, (1 + \gamma/z) \left(\sum_{m=0}^{\infty} a_m \, (b_0 z)^{-m} \right) (t/\Lambda^2)^n \, z^{-n\gamma} \, e^{-nz} \\ &\sim \sum_{m=0}^{\infty} (a_m + \gamma b_0 \, a_{m-1}) \, b_0^{-m} \, (t/\Lambda^2)^n \, n^{m+n\gamma-1} \, \Gamma(1-m-n\gamma, ny) \\ &\sim \sum_{m=0}^{\infty} (a_m + \gamma b_0 \, a_{m-1}) \, b_0^{-m} \sum_{k=0}^{\infty} (-)^k \frac{\Gamma(m+n\gamma+k)}{\Gamma(m+n\gamma)} \, n^{-k-1} y^{-m-k} e^{in\phi} , \end{aligned} \qquad \text{(D9)}$$

where $a_{-1} \equiv 0$, and the asymptotic expansion of the incomplete Γ function has been used,

$$\Gamma(1-\alpha, x) \equiv \int_x^\infty dz \, z^{-\alpha} e^{-z} \sim x^{-\alpha} e^{-x} \sum_{k=0}^{\infty} \frac{(-)^k}{x^k} \frac{\Gamma(\alpha+k)}{\Gamma(\alpha)} , \qquad \text{(D10)}$$

which is valid for $|x| \to \infty$ with $\arg(x) < 3\pi/2$.

When evaluating the low-momentum contribution, the asymptotic expansion of the spectral density can only be used when $s \gg \Lambda^2$. However, for $n > 0$, the integrand of $I_n^-(t)$ is strongly peaked about the upper limit $s = |t|$. Hence, one may ignore the contribution to the integral from low momenta, $s < \kappa$, for some cutoff κ chosen to scale with t so that $\kappa/|t|$ vanishes faster than any power of $g^2(t)$ while simultaneously $g^2(\kappa) \to 0$ as $|t| \to \infty$. Thus, for $n > 0$,

$$
\begin{aligned}
I_n^-(t) &= \int_\kappa^{|t|} \frac{ds}{s}\,\rho(s)\,(s/t)^n + O(\kappa/t)^n \\
&\sim \int_{w(\kappa)}^{y} dz\,(1+\gamma/z)\left(\sum_{m=0}^{\infty} a_m\,(b_0 z)^{-m}\right)(t/\Lambda^2)^{-n}\,z^{n\gamma}\,e^{nz} \\
&\sim \sum_{m=0}^{\infty} (a_m + \gamma b_0\, a_{m-1})\,(-b_0)^{-m}\,(t/\Lambda^2)^{-n}\,e^{i\pi n\gamma}\,n^{m-n\gamma-1}\left[\Gamma(1-m+n\gamma, e^{-i\pi}ny) - \Gamma(1-m+n\gamma, e^{-i\pi}nw)\right] \\
&\sim \sum_{m=0}^{\infty} (a_m + \gamma b_0\, a_{m-1})\,b_0^{-m} \sum_{k=0}^{\infty} \frac{\Gamma(m-n\gamma+k)}{\Gamma(m-n\gamma)}\, n^{-k-1} y^{-m-k} e^{-in\phi}\,,
\end{aligned}
\tag{D11}
$$

where $w(\kappa) \equiv 1/[b_0\, g^2(\kappa)]$.

To evaluate the final term $I_0^-(t)$, define a "subtracted" spectral density which is integrable from 0 to ∞,

$$
\bar\rho(s) \equiv \rho(s) - \theta(s-\mu^2)\,[a_0 + a_1 g^2(s)]\,,
\tag{D12}
$$

where $\theta(x)$ is the unit step function, and let $C(\mu^2) = \int_0^\infty (ds/s)\,\bar\rho(s)$. Then,

$$
\begin{aligned}
I_0^-(t) &\equiv C(\mu^2) + \int_{\mu^2}^{|t|} \frac{ds}{s}\,[a_0 + a_1 g^2(s)] - \int_{|t|}^{\infty} \frac{ds}{s}\,\bar\rho(s) \\
&\sim C(\mu^2) + a_0 \ln(|t|/\mu^2) + a_1 \int_{w(\mu^2)}^{y} dz\,(1+\gamma/z)\,(b_0 z)^{-1} - \sum_{m=2}^{\infty} a_m \int_y^\infty dz\,(1+\gamma/z)\,(b_0 z)^{-m} \\
&\sim C + a_0 y + (a_1 + \gamma b_0\, a_0)\, b_0^{-1} \ln y - \sum_{m=1}^{\infty} (a_{m+1} + \gamma b_0\, a_m)\, b_0^{-m-1}\, y^{-m}/m\,,
\end{aligned}
\tag{D13}
$$

where $C \equiv C(\mu^2) - a_0 \ln(\mu^2/\bar\Lambda^2) + a_1 b_0^{-1}\,\{\ln[b_0 g^2(\mu^2)] + \gamma b_0 g^2(\mu^2)\}$ is (despite appearances) a μ-independent constant.

Finally, putting these results together, we find that

$$
\Delta K(te^{i\phi}) \sim \tilde c_{-1}/g^2(t) + \tilde c_0\, \ln[g^2(t)] + \sum_{n=0}^{\infty} c_n(\phi)\, g^2(t)^n\,,
\tag{D14}
$$

where

$$
\tilde c_{-1} = (\pi b_0)^{-1}\,(-a_0)\,,
\tag{D15}
$$

$$
\tilde c_0 = (\pi b_0)^{-1}(a_1 - \lambda\, a_0)\,,
\tag{D16}
$$

$$
c_0(\phi) = (\pi b_0)^{-1}(a_1 - \lambda\, a_0)\,\ln(b_0) - \frac{1}{\pi}[a_0\, \Delta_{0,0}(-\phi) + C]\,,
\tag{D17}
$$

$$
\begin{aligned}
c_n(\phi) = {}& (\pi b_0)^{-1}\frac{1}{n}(a_{n+1} - \lambda\, a_n) \\
& - \frac{1}{\pi}\sum_{k=0}^{n} (a_{n-k} - \lambda\, a_{n-k-1})\, b_0^k\, \Delta_{k,n-k}(-\phi)\,,
\end{aligned}
\tag{D18}
$$

and

$$
\Delta_{k,m}(\phi) \equiv \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} e^{in\phi}\, n^{-k-1}\, \frac{\Gamma(m + k + n\lambda/b_0)}{\Gamma(m + n\lambda/b_0)}\,.
\tag{D19}
$$

For $k \geq 0$, $\Delta_{k,m}(\phi)$ is a polynomial in ϕ of order $k+1$. Inserting the expansion (2.45) of the ratio of Γ functions, noting that

$$
\sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} e^{in\phi}\, n^{-k-1} = -\frac{(2\pi i)^{k+1}}{(k+1)!}\, B_{k+1}(\phi/2\pi)\,,
\tag{D20}
$$

where $B_k(x)$ is a Bernoulli polynomial, and evaluating the result at $\phi = \pi$ yields the same result (2.47) found earlier.

The inverse to these relations, which will express the absorptive coefficients in terms of the dispersive coefficients, may be derived in a brute-force fashion by integrating the recursion relations

$$
i\frac{d}{d\phi} c_{n+1}(\phi) = -b_{n+1}\, \tilde c_{-1} + b_n\, \tilde c_0 + \sum_{m=1}^{n} m\, b_{n-m}\, c_m(\phi)
\tag{D21}
$$

previously derived in Sec. III [Eq. (3.24) with $\theta = \phi - \pi$]. Since there is no explicit ϕ dependence in these relations, one easily sees that $c_n(\phi)$ must be an nth-order polynomial in ϕ. If the initial conditions are known values of the Euclidean coefficients, $\{c_n \equiv c_n(\phi = \pi)\}$, then a general form for the ϕ dependence which reproduces the initial conditions is

$$
c_n(\pi - \phi) = c_n + \sum_{l=1}^{n} K_{n,l}\, \frac{(i b_0 \phi)^l}{l!}\,.
\tag{D22}
$$

Substituting this form into the recursion relations (D21) completely determines the coefficients $K_{n,l}$. For our β function with $b_n = b_0\,\lambda^n$, one finds that

$$K_{n+1,l+1} = (\tilde{c}_0 - \lambda\tilde{c}_{-1})\,\lambda^{n-l}\,I^n_{0,l} + \sum_{m=1}^{n-l} m\,c_m\,\lambda^{n-l-m}\,I^n_{m,l}\,, \qquad \text{(D23)}$$

where $\{I^n_{m,l}\}$ denotes the same combinatoric factors

$$I^n_{m,l} \equiv \sum_{k_1=m+1}^{n-l+1}\ \sum_{k_2=k_1+1}^{n-l+2} \cdots \sum_{k_l=k_{l-1}+1}^{n} k_1 k_2 \cdots k_l \qquad \text{(D24)}$$

whose generating function was introduced in (2.45). Evaluating this result at $\phi = \pi$ and taking the imaginary part immediately yields the expression for the absorptive coefficients quoted in Eq. (2.44).

APPENDIX E: KNOWN RESULTS

Here we collect previous perturbative results and present them in terms of our notation. In the modified-minimal subtraction ($\overline{\text{MS}}$) scheme which uses a coupling that we now denote by $\bar{g}^2$, the known QCD corrections to the R ratio have the form

$$R(s) = 3\Big(\sum\nolimits_f Q_f^2\Big)\left[1 + \frac{\bar{g}^2(s)}{4\pi^2} + r_2\left(\frac{\bar{g}^2(s)}{4\pi^2}\right)^2 + r_3\left(\frac{\bar{g}^2(s)}{4\pi^2}\right)^3 + O(\bar{g}^8)\right], \qquad \text{(E1)}$$

where Q_f is the charge of the quark of flavor f. The first two terms are not difficult to calculate (see, for example, [16]). The coefficient r_2 for the third term has been obtained independently by several groups [17–19],

$$r_2 = 1.9857 - 0.1153\,N_f\,, \qquad \text{(E2)}$$

where N_f is the total number of quark flavors. Recently, two groups [20, 21] have calculated the fourth coefficient

$$r_3 = -6.6368 - 1.2001\,N_f - 0.0052\,N_f^2 - 1.2395\,\frac{(\sum_f Q_f)^2}{(3\sum_f Q_f^2)}\,. \qquad \text{(E3)}$$

The β function $\bar{\beta}(\bar{g}^2) \equiv \mu^2 d\bar{g}^2/d\mu^2$ in the $\overline{\text{MS}}$ scheme has the asymptotic expansion

$$\bar{\beta}(\bar{g}^2) = -b_0\bar{g}^4 - b_1\bar{g}^6 - b_2\bar{g}^8 - O(\bar{g}^{10})\,, \qquad \text{(E4)}$$

where

$$4\pi^2 b_0 = \frac{1}{4}\left(11 - \frac{2}{3}N_f\right), \qquad \text{(E5)}$$

while [22, 23]

$$(4\pi^2)^2 b_1 = \frac{1}{16}\left(102 - \frac{38}{3}N_f\right), \qquad \text{(E6)}$$

and [24]

$$(4\pi^2)^3 b_2 = \frac{1}{64}\left(\frac{2857}{2} - \frac{5033}{18}N_f + \frac{325}{54}N_f^2\right). \qquad \text{(E7)}$$

To apply our results, we need first to express the $\overline{\text{MS}}$ expansion in terms of our coupling g^2 which has a two-term inverse β function. The two couplings are related by

$$\bar{g}^2 = g^2 + \alpha_2 g^4 + \alpha_3 g^6 + O(g^8)\,, \qquad \text{(E8)}$$

and

$$\bar{\beta}(\bar{g}^2(g^2)) = \beta(g^2)\frac{d\bar{g}^2(g^2)}{dg^2}\,. \qquad \text{(E9)}$$

Requiring that

$$\mu^2\frac{dg^2}{d\mu^2} \equiv \beta(g^2) = -\frac{b_0 g^4}{1-\lambda g^2}\,, \qquad \text{(E10)}$$

and inserting the expansions (E4) and (E8) into the relation (E9) yields

$$\alpha_3 = \frac{b_2}{b_0} - \lambda^2 + \alpha_2(\alpha_2 + \lambda)\,, \qquad \text{(E11)}$$

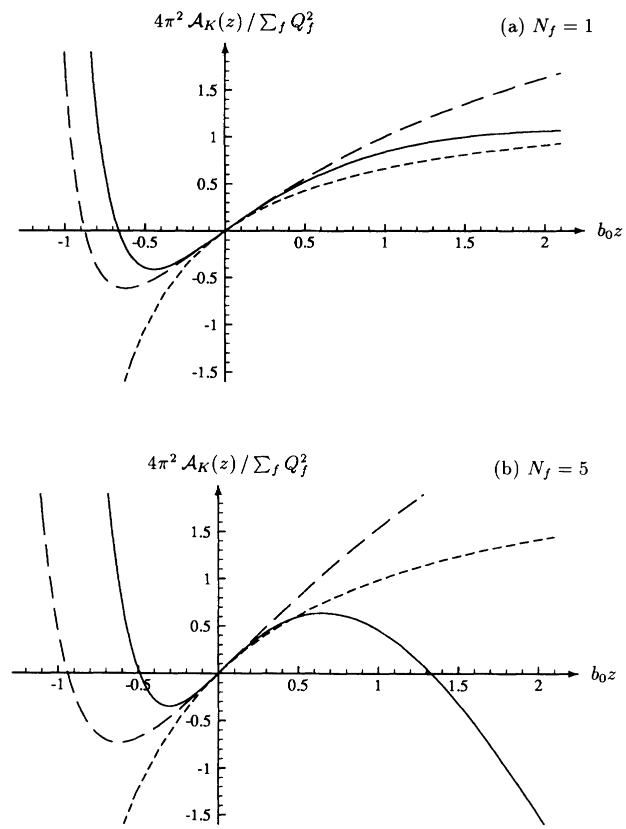

FIG. 1. First three partial sums for the absorptive modified Borel transform $\mathcal{A}(z)$ for $N_f = 1$ and $N_f = 5$. Plotted is $4\pi^2\mathcal{A}_K(z)/\sum_f Q_f^2$ versus $b_0 z$. The short dash, long dash, and solid lines denote the first, second, and third partial sums, respectively.

with no constraint on the first parameter α_2. Since we have not yet specified the renormalization point for the coupling g^2, there is a one-parameter family of couplings with a two-term inverse β function.

We shall require that α_2 vanishes so as to keep g^2 as close to $\bar{g}^2$ as possible. This redefinition then shifts only the fourth term in the expansion of $R(s)$:

$$R(s) = 3\left(\sum_f Q_f^2\right)\left[1 + \frac{g^2}{4\pi^2} + r_2\left(\frac{g^2}{4\pi^2}\right)^2 + (r_3 + 16\pi^4\alpha_3)\left(\frac{g^2}{4\pi^2}\right)^3 + O(g^8)\right]. \tag{E12}$$

Since $R(s) = 12\pi\,\mathrm{Im}K(s + i0^+)$, the first four absorptive coefficients are

$$\begin{aligned} a_0 &= \frac{1}{4\pi}\sum_f Q_f^2, \\ 4\pi^2\, a_1 &= a_0, \\ (4\pi^2)^2\, a_2 &= r_2\, a_0, \\ (4\pi^2)^3\, a_3 &= (r_3 + 16\pi^4\alpha_3)a_0. \end{aligned} \tag{E13}$$

Relations (5.55) and (5.56) determine the first four dispersive coefficients in terms of the first four absorptive coefficients:

$$\begin{aligned} \tilde{c}_{-1} &= -\frac{a_0}{\pi b_0}, \\ \tilde{c}_0 &= \frac{a_1 - \lambda a_0}{\pi b_0}, \\ c_1 &= \frac{a_2 - \lambda a_1}{\pi b_0}, \\ c_2 &= \frac{a_3 - \lambda a_2}{2\pi b_0} + \frac{\pi b_0 a_1}{6}. \end{aligned} \tag{E14}$$

For five flavors, $N_f = 5$, the numerical values of the parameters are

$$\begin{aligned} 4\pi^2 b_0 &= 1.917, \qquad \lambda = b_1/b_0 = 0.03194, \\ r_2 &= 1.409, \qquad r_3 = -12.81, \\ (4\pi^2)^2\alpha_3 &= -0.12. \end{aligned} \tag{E15}$$

The values of the first four absorptive coefficients become

$$\begin{aligned} a_0 &= 0.09726, \\ 4\pi^2 a_1 &= a_0, \\ (4\pi^2)^2 a_2 &= 1.409\, a_0, \\ (4\pi^2)^3 a_3 &= -12.92\, a_0, \end{aligned} \tag{E16}$$

and the first four dispersive coefficients are given by

$$\begin{aligned} \tilde{c}_{-1} &= -6.556\, a_0, \\ \tilde{c}_0 &= -0.04333\, a_0, \\ 4\pi^2 c_1 &= 0.02463\, a_0, \\ (4\pi^2)^2 c_2 &= -0.2168\, a_0. \end{aligned} \tag{E17}$$

The presence of the term $(4\pi^2)^2\,\alpha_3$, caused by the difference between our coupling g^2 and the more conventional $\overline{\mathrm{MS}}$ coupling $\bar{g}^2$, produces only a small (1%) change in the fourth expansion coefficient of the R ratio (E12).

Given these results, one may compute the first three partial sums in the modified Borel transform of the absorptive coefficients:

$$\mathcal{A}_K(z) \equiv \sum_{n=1}^{K} \frac{\Gamma(1+\lambda z)}{\Gamma(n+1+\lambda z)}\, n\, a_n\, z^n. \tag{E18}$$

The results are plotted in Fig. 1 for two different numbers of quark flavors. The series of partial sums is obviously highly unstable when $b_0 z$ is less than about $-1/2$. Based on these graphical results, the presence of an ultraviolet renormalon singularity at $b_0 z = -1$ is certainly not surprising. Most intriguing, however, is the behavior of these partial sums near $b_0 z = +1$. As discussed in Sec. II, unless the absorptive transform $\mathcal{A}(z)$ vanishes at $b_0 z = 1$ the dispersive Borel transform will be singular at $b_0 z = 1$, leading to previously unknown nonperturbative $1/q^2$ corrections in the operator-product expansion. If the complete $\mathcal{A}(z)$ does have a zero at $b_0 z = 1$, then in the sequence of partial sums one might hope to see a zero on the positive axis whose position converges to $b_0 z = 1$. As Fig. 1 shows, the differences between the partial sums grow as z increases from zero. However, for one quark flavor, at $b_0 z = 1$ the first two partial sums differ from the third by only -21% and $+19\%$, respectively. While hardly conclusive, the data for one flavor appear to support the simplest hypothesis: that further partial sums will converge to a nonzero value at $b_0 z = 1$, leading one to predict the existence of nonperturbative $1/q^2$ effects. As the number of quark flavors increase, the partial sums (for fixed values of $b_0 z$) become increasingly unstable. For $N_f = 5$, the last partial sum does have a zero near $b_0 z = 1.3$, but the differences between the different partial sums are clearly too large to draw any meaningful conclusion about the true value at $b_0 z = 1$.

[1] G. B. West, Phys. Rev. Lett. **67**, 1388 (1991); **67**, 3732(E) (1991).

[2] G. 't Hooft, in *The Whys of Subnuclear Physics*, Proceedings of the International School, Erice, Italy, 1977, edited by A. Zichichi, Subnuclear Series Vol. 15 (Plenum, New York, 1979).

[3] See, for example, *Large Order Behavior of Perturbation Theory*, edited by J. C. Le Guillou and J. Zinn-Justin (North-Holland, Amsterdam, 1990), and references therein.

[4] G. Parisi, Phys. Lett. **76B**, 65 (1978).

[5] A. H. Mueller, Nucl. Phys. **B250**, 327 (1985).

[6] J. D. Bjorken, in *Particle Physics: Cargèse 1989*, edited by M. Lévy *et al.* (Plenum, New York, 1990).
[7] L. S. Brown and L. G. Yaffe, Phys. Rev. D **45**, R398 (1992).
[8] See, for example, D. J. Gross, in *Methods in Field Theory*, Proceedings of the Les Houches Summer School, Les Houches, France, 1975, edited by R. Balian and J Zinn-Justin, Les Houches Summer School Proceedings Vol. XXVIII (North-Holland, Amsterdam, 1977).
[9] A standard reference is G. N. Hardy, *Divergent Series* (Oxford University Press, Oxford, 1949). A sharper version of the uniqueness of the Borel transform appears in F. Nevanlinna, Ann. Acad. Sci. Fenn. Ser. **A12**, No. 3 (1916-18) and A. Sokal, J. Math. Phys. **21**, 261 (1980).
[10] F. David, Nucl. Phys. **B209**, 433 (1982).
[11] See, for example, F. David, Nucl. Phys. **B234**, 237 (1984).
[12] See, for example, M. Shifman, Proceedings of the 11th Autumn School of Physics, Lisbon, Portugal, 1989 (unpublished); V. Novikov, M. Shifman, A. Vainshtein, M. Voloshin, and V. Zacharov, Nucl. Phys. **B237**, 525 (1984), and references therein.
[13] G. P. Lepage and P. B. Mackenzie, in *Lattice '90*, Proceedings of the International Workshop, Tallahassee, Florida, 1990, edited by U. M. Heller, A. D. Kennedy, and S. Sanielevici [Nucl. Phys. B (Proc. Suppl.) **20**, 173 (1991)].
[14] Julian Schwinger, *Quantum Kinematics and Dynamics* (Benjamin, New York, 1970), Chap. IV, pp. 133–140.
[15] For the properties of hypergeometric functions, see, for example, *Higher Transcendental Functions (Bateman Manuscript Project)*, edited by A. Erdélyi (McGraw-Hill, New York, 1953), Vol. 1. In particular, the formulas we need are (10), p. 59, (1), p. 108, (5), p. 109, and (2), p. 105.
[16] Lowell S. Brown, *Quantum Field Theory* (Cambridge University Press, Cambridge, England, 1992).
[17] K. G. Chetyrkin, A. L. Kataev, and F. V. Tkachov, Phys. Lett. **85B**, 277 (1979).
[18] M. Dine and J. Sapirstein, Phys. Rev. Lett. **43**, 668 (1979).
[19] W. Celmaster and J. Sapirstein, Phys. Rev. Lett. **44**, 560 (1980).
[20] S. G. Gorishny, A. L. Kataev, and S. A. Larin, Phys. Lett. B **259**, 144 (1991).
[21] L. R. Surguladze and M. A. Samuel, Phys. Rev. Lett. **66**, 560 (1991).
[22] W. Caswell, Phys. Rev. Lett. **33**, 244 (1974).
[23] D. R. T. Jones, Nucl. Phys. **B75**, 531 (1974).
[24] O. V. Tarasov, A. A. Vladimirov, and A. Yu. Zharkov, Phys. Lett. **93B**, 429 (1980).