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Periodic instantons and quantum-mechanical tunneling at high energy

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The tunneling process at high energy is investigated for a one-dimensional system with the double-well potential. The path-integral method is used to calculate the transition amplitude between excited states in the two wells, as well as the level splitting of the excited states by expanding the action about a periodic instanton solution. The solution of half a period between two turning points is treated like an instanton configuration and the singularity of the Feynman kernel between turning points for the finite Euclidean time interval is smoothened with the end-point integrals. The level splitting obtained is in exact agreement with the WKB result. For weak coupling and energies far below the barrier height, the transition amplitude grows with energy exponentially. For energies approaching the barrier height anharmonic contributions must be taken into account.

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I. INTRODUCTION

Instanton transitions in electroweak theory have attracted widespread attention recently. The possibility of baryon- and lepton-number violation is one of the interesting phenomena related to instanton transitions between neighboring vacua. Following the conclusions of Ringwald's work [1], that to leading order the two-particle scattering cross section in the semiclassical approximation around the instanton may not be suppressed at high energy, interest in the subject grew tremendously. Most of the existing calculations rely on a certain type of perturbation expansion about the ordinary zero-energy or vacuum instanton. However, it has been pointed out [2] that since the zero-energy instanton prescribes the vacuum boundary conditions, the calculation based on it alone is inadequate at high energies and new solutions to the field equations [2], satisfying manifestly nonvacuum boundary conditions, would have to be used. Periodic instantons characterized by nonzero energy have been proposed [2–4] by several authors as candidates of the new solutions. The periodic instanton moves between two turning points.

It is needless to say that a full investigation of the tunneling process dominated by the periodic instanton configuration for a one-dimensional system is of great interest. Here we extend the vacuum instanton method for quantum tunneling [5,6] to that of the periodic instanton configuration. Unlike the cases of the vacuum instanton [5] and the bounce [6], in which the two turning points can be reached only asymptotically, the Feynman kernel here is divergent between two turning points for a finite Euclidean time interval. The modification is not a trivial matter. We achieve this, however, by calculating the transition amplitude between excited states in two wells directly. The singularity of the Feynman kernel is thereby smoothed out by the end-point integrations. We then obtain a result which agrees with that of WKB calculations. We mention that the same subject, namely, the calculation of the transition amplitude, was treated in a recent publication [7] but was based on the use of the vacuum instanton configuration alone.

For the case of weak coupling and energies far below the barrier height the expansion of our result shows that the amplitude grows with energy exponentially, which is in agreement with results of other considerations in the literature [2,3] for all models possessing instantons. For energies approaching the barrier height, the situation is, however, quite different. The effect of anharmonic oscillations becomes important.

II. PERIODIC INSTANTONS FOR THE DOUBLE-WELL POTENTIAL

We consider a scalar field ϕ in one time and zero space dimensions. The Lagrangian is

$$L=\frac{1}{2}\left[\frac{d\phi}{dt}\right]^2-V(\phi) \tag{2.1}$$

(mass $m_0=1$ and $\hbar=1$ being used throughout), where

$$V(\phi)=\frac{\eta^2}{2}\left[\phi^2-\frac{m^2}{\eta^2}\right]^2$$

is a double-well potential (Fig. 1). Following Manton and Samols [8], a nontrivial classical solution ϕ_c which extremizes the Euclidean action and satisfies the following classical equation in terms of Euclidean time τ,

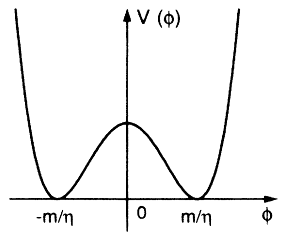

FIG. 1. The double-well potential.

$$\frac{1}{2}\left[\frac{d\phi_c}{d\tau}\right]^2 - V(\phi_c) = -E , \tag{2.2}$$

is obtained by imposing on ϕ_c the periodicity condition

$$\phi_c|_{\tau=-2T} = \phi_c|_{\tau=2T} = 0 , \tag{2.3}$$

as

$$\phi_c = \frac{\kappa b(\kappa)}{\eta}\mathrm{sn}(b(\kappa)\tau) . \tag{2.4}$$

The configuration can therefore be called a periodic instanton [2,3] although this terminology is not unambiguous since the word is also used for some specific field-theoretical configurations. The Jacobian elliptic function sn has real periods $4n\mathcal{K}(\kappa)$, n being an integer, and $\mathcal{K}(\kappa)$ is the quarter period given by the usual complete elliptic integral of the first kind. The periodic boundary condition (2.3) is satisfied if we let $b(\kappa)T = \mathcal{K}(\kappa)$, $n=1$. The parameters κ, $b(\kappa)$ are defined by

$$\kappa^2 \equiv \frac{1-u}{1+u}, \quad u = \frac{\eta}{m^2}\sqrt{2E} , \quad b(\kappa) = m\left[\frac{2}{1+\kappa^2}\right]^{1/2} . \tag{2.5}$$

The periodic instanton starts its motion from $\phi=0$ at $\tau=-2T$ and reaches the turning point $-a$ at time $\tau=-T$ with velocity $d\phi_c/d\tau|_{\tau=-T}=0$. It then returns to the origin $\phi=0$ at $\tau=0$ and then travels back and reaches the turning point $\phi=a$ at $\tau=T$, again with zero velocity. It then returns to the origin at $\tau=2T$. The motion of this periodic configuration is shown schematically in Fig. 2.

The topological charge for one period is zero but it is not zero for the half-period, i.e.,

$$Q = \frac{\eta}{2m}[\phi_c(T) - \phi_c(-T)] = \kappa\left[\frac{2}{1+\kappa^2}\right]^{1/2} . \tag{2.6}$$

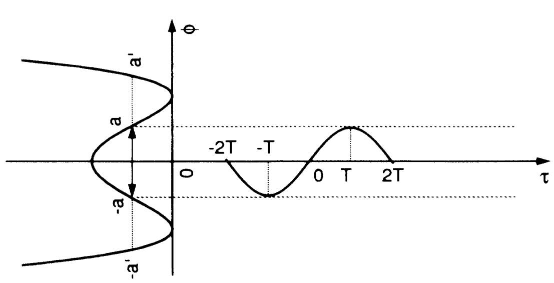

FIG. 2. The motion of the periodic instanton.

It tends to 1, namely, the topological charge of the vacuum instanton, for $\kappa \to 1$ and the solution (2.4) reduces to the usual vacuum instanton solution. The small fluctuation equation about the periodic instanton configuration (2.4) can be shown [9] to be Lame's equation

$$\frac{d^2\chi}{dz^2} + \{\lambda - n(n+1)\kappa^2 \mathrm{sn}^2 z\}\chi = 0 , \tag{2.7}$$

where $\lambda = (\omega^2 + 2m^2)/b^2(\kappa)$ and $n=2$ with the periodic boundary condition. The eigenvalues and eigenfunctions are given in Ref. [9]. In the literature [4] the configuration which we call a periodic instanton here is also referred to as a bounce configuration in view of the latter's similar motion for one period. However, the term "bounce" is not appropriate in our case, since in the limit $\kappa \to 1$ the negative fluctuation mode associated with the solution (2.4) merges into the zero mode of the small fluctuation equation. The one-period configuration should therefore be considered as an instanton–anti-instanton pair configuration, as pointed out by Manton and Samols [8]. In the following we calculate the amplitude for a transition between excited states in the two wells, and we then expand the path integral about the solution (2.4) in a half-period as for one instanton. In addition to the instanton, the infinite number of instanton–anti-instanton pairs will have to be taken into account.

III. TRANSITION AMPLITUDE FOR QUANTUM TUNNELING BETWEEN EXCITED STATES IN THE TWO WELLS

We let $|E\rangle_+$ and $|E\rangle_-$ be eigenstates of the same energy E in the right- and left-hand wells, respectively. The smaller contribution due to quantum tunneling leads to the effect of level separation or splitting ΔE, which removes the asymptotic degeneracy. The corresponding eigenstates of the Hamiltonian separate into odd and even states $|E\rangle_0$ and $|E\rangle_e$ which are superpositions of $|E\rangle_+, |E\rangle_-$ such that $|E\rangle_0 = 1/\sqrt{2}(|E\rangle_+ - |E\rangle_-)$, and $|E\rangle_e = 1/\sqrt{2}(|E\rangle_+ + E\rangle_-)$ with eigenvalues $E \pm \Delta E/2$, respectively. The amplitude for the transition from state $|E\rangle_-$ in the left-hand well to the state $|E\rangle_+$ in the right-hand well in the time interval $2T$ can be calculated as

$$\begin{aligned} A_{+,-} &= {}_+\langle E|e^{-2\hat{H}T}|E\rangle_- \\ &= -e^{-2ET}\sinh(T\Delta E) , \end{aligned} \tag{3.1}$$

where ΔE is the splitting of the energy levels due to tunneling. The amplitude can be calculated with the help of the path-integral method from

$$A_{+,-} = \int \psi_{E+}(\phi'')\psi_{E-}(\phi')K(\phi'',T;\phi',-T)d\phi' d\phi'' , \tag{3.2}$$

where the Feynman kernel is defined by

$$K(\phi'',T;\phi',-T) = \int \mathcal{D}\{\phi\}\exp(-S) , \tag{3.3}$$

where

$$S = \int_{-T}^{T}\left[\frac{1}{2}\left[\frac{d\phi}{d\tau}\right]^2 + V(\phi)\right]d\tau$$

is the classical Euclidean action and $\psi_{E+}(\phi'')=\langle \phi''|E\rangle_+$ and $\psi_{E-}(\phi')=\langle \phi'|E\rangle_-$ are eigenfunctions of the energy which dominate in the right- and left-hand wells, respectively.

IV. THE ONE-INSTANTON CONTRIBUTION

The Feynman kernel (3.3) can be evaluated with the help of the standard path-integral method. Expanding $\phi(\tau)$ about the periodic instanton (2.4) we set

$$\phi(\tau)=\phi_c(\tau)+\chi(\tau) \tag{4.1}$$

with the boundary condition $\chi(T)=\chi(-T)=0$. Substituting this for $\phi(\tau)$ in Eq. (3.3), we have

$$K=\exp[-S(\phi_c)]\int \mathcal{D}\{\chi\}\exp[-\delta S]$$
$$\equiv\exp[-S(\phi_c)]I \ . \tag{4.2}$$

With Eq. (2.2) and the derivative of the Jacobian elliptic function (2.4), i.e.,

$$\frac{d\phi_c}{d\tau}=\frac{\kappa b^2(\kappa)}{\eta}\mathrm{cn}(b(\kappa)\tau)\mathrm{dn}(b(\kappa)\tau) \tag{4.3}$$

the classical action can be found to be [using (2.2)]

$$S(\phi_c)=W(\phi_c(T),\phi_c(-T),E)+2ET \ , \tag{4.4}$$

where

$$W(\phi_c(T),\phi_c(-T),E)=\frac{4m^3}{3\eta^2}(1+u)^{1/2}[E(\kappa)-u\mathcal{K}(\kappa)] \ . \tag{4.5}$$

Here $E(\kappa)$ is the complete elliptic integral of the second kind. The integral containing the elliptic function $\mathrm{cn}^2\chi\,\mathrm{dn}^2\chi$ is evaluated with the help of tables of integrals [10]. For weak coupling [which will be defined more precisely later, (cf. (6.1)] the cubic and quartic terms in $\chi(\tau)$ of Eq. (4.2) can be dropped in the one-loop approximation, so that the fluctuation action is

$$\delta S=\int_{-T}^{T}d\tau\left[\frac{1}{2}\left[\frac{d\chi}{d\tau}\right]^2+\chi^2(3\eta^2\phi_c^2-m^2)\right] \ . \tag{4.6}$$

Next we introduce the mapping [11]

$$y(\tau)=\chi(\tau)-\int_{-T}^{\tau}\frac{\dot N(\tau')}{N(\tau')}\chi(\tau')d\tau' \tag{4.7}$$

such that $N(\tau)$ satisfies the following equation derivable from (4.6):

$$\ddot N=2[3\eta^2\phi_c^2(\tau)-m^2]N \tag{4.8}$$

which has a solution $N(\tau)=d\phi_c/d\tau=N(-\tau)$. The integral I in Eq. (4.2) then becomes

$$I=\int \mathcal{D}\{y(\tau)\}d\alpha\left|\frac{\mathcal{D}\chi}{\mathcal{D}y}\right|\exp\left\{-\left[\int_{-T}^{T}d\tau\frac{1}{2}\left[\frac{dy}{d\tau}\right]^2+\alpha\left[y(T)+N(T)\int_{-T}^{T}\frac{\dot N(\tau')}{N^2(\tau')}y(\tau')d\tau'\right]\right]\right\} , \tag{4.9}$$

where α is the Lagrange multiplier, which inserts as a constraint on $y(\tau)$ the boundary condition $\chi(\pm T)=0$, and $|\mathcal{D}\chi/\mathcal{D}y|$ is the functional Jacobian associated with the mapping (4.7). The integral can be carried out by direct integration [11] and yields

$$I=\left[\frac{1}{2\pi}\right]^{1/2}\left[\frac{1}{N(T)N(-T)}\right]^{1/2}\left[\int_{-T}^{T}\frac{d\tau}{N^2(\tau)}\right]^{-1/2} . \tag{4.10}$$

The kerneel (4.10) between the two turning points at $\phi=\pm a$ (i.e., $\tau=\pm T$) is divergent, in view of the zero velocity at turning points, i.e., $d\phi_c(T)/d\tau=d\phi_c(-T)/d\tau=0$ (in agreement with (4.3) in which $\mathrm{cn}(b(\kappa)T)=\mathrm{cn}(\mathcal{K})=0$). This is unlike the cases of vacuum instantons [5] or vacuum bounces [6] in which the turning points can be reached only asymptotically and so there this difficulty is avoided in considerations of a large but finite T. The transition amplitude, however, must be finite and hence the singularity of kernel (4.10) has to be smoothed out by the end-point integrations $d\phi'$ and $d\phi''$. To this end we introduce a formula established in the Appendix, i.e., that

$$\frac{\partial^2 S}{\partial\phi^2(T)}=\left[\frac{1}{N(T)N-T)}\right]\left[\int_{-T}^{T}\frac{d\tau}{N(\tau)^2}\right]^{-1} . \tag{4.11}$$

The transition amplitude (3.2) thus becomes

$$A_{+,-}=\left[\frac{1}{2\pi}\right]^{1/2}\int\left[\frac{\partial^2 S}{\partial\phi^2(T)}\right]^{1/2}\psi_{E+}(\phi'')\psi_{E-}(\phi')e^{-S(\phi'',\phi',T)}d\phi'd\phi'' \ . \tag{4.12}$$

In order to be able to evaluate the integral we need the proper eigenfunctions ψ_E. A natural choice is to take ψ_E of WKB type. Inside the central barrier $(-a<\phi<a)$, these wave functions are [12]

$$\psi_{E+}(\phi'')=\frac{C_+}{\sqrt{\dot\phi''}}e^{-\Omega(\phi'')} ,$$
$$\psi_{E-}(\phi')=\frac{C-}{\sqrt{\dot\phi'}}e^{-\Omega(\phi')} , \tag{4.13}$$

where

$$\Omega(\phi'')=\int_{\phi''}^{\phi(T)}\dot{\phi}\,d\phi,\quad \Omega(\phi')=\int_{\phi(-T)}^{\phi'}\dot{\phi}\,d\phi\;. \tag{4.14}$$

Outside the barrier, i.e., for $|\phi|>|a|$ the product of the wave functions ψ_{E+},ψ_{E-} which dominate in the two wells vanishes sufficiently fast. The normalization constants can be calculated to be [12]

$$C_+=\left[\frac{1/2}{\int_a^{a'}\frac{d\phi}{\sqrt{2(E-V)}}}\right]^{1/2},\qquad C_-=\left[\frac{1/2}{\int_{-a'}^{-a}\frac{d\phi}{\sqrt{2(E-V)}}}\right]^{1/2}, \tag{4.15}$$

where $\pm a$ and $\pm a'$ are the four turning points (see Fig. 2) with

$$a=\phi(T)=\left[\frac{m^2}{\eta^2}-\frac{\sqrt{2E}}{\eta}\right]^{1/2}$$

and

$$a'=\left[\frac{m^2}{\eta^2}+\frac{\sqrt{2E}}{\eta}\right]^{1/2}.$$

Evaluating the elliptic integrals [10] in Eq. (4.15) the result is

$$C_+=C_-=\left[\frac{m\sqrt{1+u}}{2\mathcal{K}(\kappa')}\right]^{1/2}, \tag{4.16}$$

where $\kappa'^2=2u/(1+u)$, $\mathcal{K}(\kappa')$ being again the complete elliptic integral. Next, we expand $S(\phi'',\phi',T)$ and $\Omega(\phi'')$ [and correspondingly $\Omega(\phi')$] as a power series of $[\phi''-\phi_c(\tau)]$ and keep terms up to the second order for S and only the zero-order term for Ω for our one-loop approximation. The integration of $d\phi''$ becomes a Gaussian integral, the factor $\partial^2 S/\partial\phi^2(\tau)$ in (4.12) being canceled out by the corresponding factor of the Gaussian integration $d\phi''$. Now

$$S(\phi'',\phi',T)=S(\phi_c(T),\phi_c(-T),T)+\frac{1}{2}\frac{\partial^2 S}{\partial\phi(T)^2}(\phi''-\phi_c(T))^2+\cdots\;. \tag{4.17}$$

Thus the final result for the one-instanton contribution is

$$A_{+,-}=\frac{Tm\sqrt{1+u}}{\mathcal{K}(\kappa')}\exp\left[-\frac{4m^3}{3\eta^2}(1+u)^{1/2}[E(\kappa)-u\mathcal{K}(\kappa)]\right]e^{-2ET}\;. \tag{4.18}$$

The factor $2T$ in the numerator arises from the starting point integration $d\phi'$ in the leading approximation by writing this $\dot{\phi}'d\tau$.

V. SUM OVER INSTANTON AND INSTANTON-ANTI-INSTANTON PAIRS

The path integral requires a sum over all possible paths. Hence in addition to the contribution of one instanton, the contribution of the infinite number of instanton-anti-instanton pairs must be taken into account; i.e., the transition amplitudes becomes

$$A_{+,-}=\sum_{n=0}^{\infty}A_{+,-}^{(2n+1)}\;, \tag{5.1}$$

where $A_{+,-}^{(2n+1)}$ denotes the amplitude calculated for one instanton plus n instanton-anti-instanton pairs. As an example we consider the one-instanton-plus-one-pair amplitude, i.e.,

$$A_{+,-}^{(3)}=A_{+,-}\left[T,\frac{T}{3}\right]A_{+,-}\left[\frac{T}{3},-\frac{T}{3}\right]\times A_{+,-}\left[-\frac{T}{3},-T\right]. \tag{5.2}$$

Evaluating each Feynman kernel and the end-point integrals, one obtains

$$A_{+,-}^{(3)}=\int_{-T}^{T}d\tau_1\int_{-T}^{\tau_1}d\tau_2\int_{-T}^{\tau_2}d\tau\left[\frac{m\sqrt{1+u}}{2\mathcal{K}(\kappa')}\right]^3 e^{-3W}e^{-2ET}=\frac{(2T)^3}{3!}\left[\frac{m\sqrt{1+u}}{2\mathcal{K}(\kappa')}\right]^3 e^{-3W}e^{-2ET}\,, \tag{5.3}$$

where W is given by (4.5). The expression for $A^{(2n+1)}$ is obtained by a straightforward generalization. Finally the transition amplitude is found to be

$$A_{+,-}=e^{-2ET}\sinh\left\{\frac{Tm\sqrt{1+u}}{\mathcal{K}(\kappa')}\exp\left[-\left[\frac{4m^3}{3\eta^2}(1+u)^{1/2}(E(\kappa)-u\mathcal{K}(\kappa))\right]\right]\right\}. \tag{5.4}$$

Comparing this expression with Eq. (3.1), the level splitting is seen to be

$$\Delta E=\frac{m\sqrt{1+u}}{\mathcal{K}(\kappa')}\exp\left[-\left[\frac{4m^3}{3\eta^2}(1+u)^{1/2}[E(\kappa)-u\mathcal{K}(\kappa)]\right]\right]. \tag{5.5}$$

This is the WKB result [12,13] as expected. Our result indicates explicitly that the periodic instantons are indeed responsible for the quantum tunneling at excited states and therefore confirms the validity of the recent proposal to calculate the two-particle scattering cross section in the semiclassical approximation around the periodic instanton instead of the vacuum instanton [2–4].

VI. EXPANSION OF THE LEVEL SPLITTING ΔE

Following Ref. [7], we introduce a dimensionless coupling constant $g^2=\eta^2/m^3$. The potential in the neighborhood of the minima at $\phi'=\phi\pm m/\eta$ is $V(\phi')\approx -2m^2\phi'^2$ so that the frequency of small oscillations in each well is $\omega=2m$. In the limit $g\rightarrow 0$ in which the two minima of the potential are infinitely separated, while the central barrier becomes infinitely high, the system reduces to a pair of independent harmonic oscillators. In this case we have $E\rightarrow E_n=(n+\frac{1}{2})\omega, n=0,1,2,\ldots$. We consider first the weak coupling case with $g\ll 1$ and energy E far below the barrier height $m^4/2\eta^2$ (the sphaleron mass) such that u of (2.5) becomes

$$u=\frac{\eta\sqrt{2E_n}}{m^2}=2g\sqrt{(n+\frac{1}{2})}\ll 1 \quad \text{and } \kappa\approx 1 \ . \tag{6.1}$$

Under these conditions the complete elliptic integrals can be expanded as [10]

$$E(\kappa)=1+\frac{1}{2}\left[\ln\left[\frac{4}{\kappa'}\right]-\frac{1}{2}\right]\kappa'^2 +\frac{3}{16}\left[\ln\left[\frac{4}{\kappa'}\right]-\frac{13}{12}\right]\kappa'^4+\cdots \ , \tag{6.2}$$

$$\mathcal{K}(\kappa')=\ln\left[\frac{4}{\kappa'}\right]+\frac{1}{4}\left[\ln\left[\frac{4}{\kappa'}\right]-1\right]\kappa'^2+\cdots \ , \tag{6.3}$$

where $\kappa'=\sqrt{1-\kappa^2}\approx 0$. For W of (4.5) we then obtain

$$W=\frac{4}{3g^2}+2\left[n+\frac{1}{2}\right]\ln\left[\frac{g}{4}\right]+\left[n+\frac{1}{2}\right]\ln\left[n+\frac{1}{2}\right] -\left[n+\frac{1}{2}\right] . \tag{6.4}$$

This result was also obtained in Ref. [7] however by a consideration of vacuum instanton transitions. Inserting (6.4) for W [cf. (4.5)] into (5.4), one can inquire about the behavior of the transition amplitude with energy in the weak coupling case. In this case the hyperbolic sine of (5.4) can be replaced by its argument. The energy dependence is demonstrated by the n dependence. The dominant n-dependent contribution of W is the term containing $\ln(g/4)$ and in the amplitude this competes with the argument of the exponential prefactor, i.e., in the weak coupling domain defined by (6.1) we have

$$\left|\ln\frac{g}{4}\right| \gg 2mT\simeq 2\mathcal{K}(\kappa=1) \ .$$

Thus the transition amplitude, although small, indeed grows with energy, in agreement with results of models possessing instantons. Inserting (6.2) and (6.3) into (5.5), the level splitting is found to be

$$\Delta E_n=\frac{2m}{\pi}\left[\frac{2^4 e}{g^2(n+\frac{1}{2})}\right]^{n+1/2} e^{-4/3g^2} \ . \tag{6.5}$$

Since $\kappa'^2=2u/(1+u)\approx 0$ we have neglected higher-order corrections in the expansion of $\mathcal{K}(\kappa')$ and have taken $\mathcal{K}(\kappa')\approx\pi/2$ in the denominator of (5.5). Using Stirling's formula $n!\approx\sqrt{2\pi}e^{-(n+1)}(n+1)^{n+1/2}$ the level splitting (6.5) can be rewritten as

$$\Delta E_n=\frac{2\sqrt{2}m}{\sqrt{\pi}n!}\left[\frac{2^4}{g^2}\right]^{n+1/2} e^{-4/3g^2} \ . \tag{6.6}$$

To our knowledge, the level splitting for excited states has been given previously only in Ref. [14]. Comparing (6.6) with formula (290) of Ref. [14], the result here differs by a factor of 2^{2n}. Since the level splitting formula (6.5) here is obtained with the low-energy expansion up to lowest nonzero order of Eqs. (6.2) and (6.3), we can expect agreement only for the ground state, and, in fact, this is seen to be the case. This result also agrees with that of vacuum instanton calculations [5]. At higher energies the exact formula (5.5) applies.

It should be emphasized that the result of Eqs. (6.4) and (6.5) is valid only under the condition (6.1), in which case the anharmonic corrections become negligible [7] in the prefactor $m\sqrt{1+u}/\mathcal{K}(\kappa')$ of (5.5). This result cannot be extended to very high energies.

It may be interesting to see the expansion of ΔE in the extremal case in which the energy tends to the top of the barrier such that $E\rightarrow m^4/2\eta^2$ and $u\rightarrow 1$, $\kappa\rightarrow 0$. The expansions of the complete elliptic integrals are, in this case, [10]

$$E(\kappa)=\frac{\pi}{2}\left[1-\frac{1}{4}\kappa^2+\cdots\right] , \tag{6.7}$$

$$\mathcal{K}(\kappa)=\frac{\pi}{2}\left[1+\frac{1}{4}\kappa^2+\cdots\right] . \tag{6.8}$$

Then [cf. (4.5)]

$$W\approx\frac{\sqrt{2}\pi}{g^2}\kappa^2 \quad \text{with } \kappa^2\rightarrow 0 \ . \tag{6.9}$$

Looking at (5.4) it is seen that the transition amplitude is not suppressed by the factor $e^{-4/3g^2}$ in this case. The situation is, however, not so simple. Since $\kappa'=2u/(1+u)\rightarrow 1$, the complete elliptic integral $\mathcal{K}(\kappa')$ has to be expanded as in (6.3), and we have (neglecting higher-order terms)

$$\frac{m\sqrt{1+u}}{\mathcal{K}(\kappa')}\approx\frac{\sqrt{2}m}{\ln(4/\kappa)} =\frac{\sqrt{2}m}{\ln(4/\kappa)}\rightarrow 0 \quad \text{with } \kappa\rightarrow 0 \ . \tag{6.10}$$

The transition amplitude as well as the level splitting are suppressed by the factor (6.10). The explanation is clear. When the energy is very high, the effect of anharmonic oscillations becomes important and the effective frequency (6.10), namely, the number of impacts per unit time at the turning poitns, approaches zero. However, in approaching the barrier height, our method is any case no longer valid. In that case different methods, such as expansions about the sphaleron configuration, should be used.

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APPENDIX

As in Eq. (4.4),

$$S(\phi(T),\phi(-T),T)=W(\phi(T),\phi(-T),E)+2ET \ , \tag{A1}$$

where

$$W=\int_{-T}^{T}\left[\frac{d\phi}{d\tau}\right]^2 d\tau=\int_{\phi(-T)}^{\phi(T)}\sqrt{2(V-E)}d\phi \ ,$$

$$E=E(\phi(T),\phi(-T)) \ .$$

Thus

$$\frac{\partial S}{\partial\phi(\pm T)}=\frac{\partial W}{\partial\phi(\pm T)}+\frac{\partial W}{\partial E}\frac{\partial E}{\partial\phi(\pm T)}+2T\frac{\partial E}{\partial\phi(\pm T)}=\frac{\partial W}{\partial\phi(\pm T)}, \ \text{since } 0=\frac{\partial S}{\partial E}=0 \ \text{so that } \frac{\partial W}{\partial E}+2T=0 \ . \tag{A2}$$

Then

$$\frac{\partial^2 S}{\partial\phi(T)\partial\phi(-T)}=\frac{\partial^2 W}{\partial\phi(T)\partial\phi(-T)}+\frac{\partial^2 W}{\partial\phi(T)\partial E}\frac{\partial E}{\partial\phi(-T)} \ . \tag{A3}$$

From (A1) we have $E=\partial S/\partial(2T)$ so that

$$\begin{aligned}\frac{\partial E}{\partial\phi(-T)}&=\frac{\partial^2 S}{\partial(2T)\partial\phi(-T)}\\&=\frac{\partial^2 W}{\partial\phi(-T)\partial E\partial(2T)/\partial E}\\&=-\frac{\partial^2 W}{\partial E\partial\phi(-T)}\Big/\frac{\partial^2 W}{\partial E^2} \ .\end{aligned} \tag{A4}$$

Substitution of Eq. (A4) into (A3) yields

$$\begin{aligned}\frac{\partial^2 S}{\partial\phi(T)\partial\phi(-T)}=&\frac{\partial^2 W}{\partial\phi(T)\partial\phi(-T)}\\&-\frac{\partial^2 W}{\partial\phi(T)\partial E}\frac{\partial^2 W}{\partial\phi(T)\partial E}\Big/\frac{\partial^2 W}{\partial E^2} \ .\end{aligned} \tag{A5}$$

In our case at the turning points $\partial W/\partial\phi(\pm T)=0$ and so $\partial^2 W/\partial\phi(T)\partial\phi(-T)\to 0$ in approaching $\pm T$. From the above expression for W we obtain

$$\frac{\partial^2 W}{\partial\phi(T)\partial E}=-\frac{1}{\dot\phi(T)}\to\infty \ ,$$

$$\frac{\partial^2 W}{\partial E\partial\phi(-T)}=-\frac{1}{\dot\phi(-T)}\to\infty \ ,$$

and similarly in approaching $\pm T$:

$$\frac{\partial^2 W}{\partial E^2}=-\int_{-T}^{T}\frac{d\tau}{\dot\phi^2(\tau)} \ .$$

Hence from (A5), with $N(\pm T)=\dot\phi(\pm T)$,

$$\frac{\partial^2 S}{\partial\phi(T)\partial\phi(-T)}=\left[\frac{1}{N(T)N(-T)}\right]\frac{1}{\int_{-T}^{T}\frac{d\tau}{N^2(\tau)}}$$

since $N(T)=N(-T)$,

$$\frac{\partial^2 S}{\partial\phi^2(T)}=\frac{1}{[N(T)]^2}\frac{1}{\int_{-T}^{T}\frac{d\tau}{N^2(\tau)}} \ .$$

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