Relaxation time approximation for relativistic dense matter

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In this article the relaxation time approximation for a system of spin-$\frac{1}{2}$ fermions is studied with a view to calculating those transport properties obeyed by relativistic dense matter such as viscosity coefficients, thermal conductivities, spin diffusion, etc. This is achieved *via* the use of covariant Wigner functions. The collision term is, of course, linear in the deviation of the Wigner function from equilibrium, and *a priori* involves arbitrary functions of the four-momentum. These functions are restricted from physical arguments and from the requirement of Lorentz invariance. The kinetic equation obeyed by the Wigner function is then split into a mass-shell constraint and "true" kinetic equations, whose solution is sought within the Chapman-Enskog approximation. It is also realized that, in a relativistic quantum framework, there exist *two* expansion parameters: the new parameter occurs because of the existence of a new length scale defined by the Compton wavelength; in some cases (e.g., when the effective mass of the fermions goes to zero), this last quantity can be of the order of the mean free path. From the first-order solutions and from the Landau-Lifshitz matching conditions, the main transport properties of the system are obtained as functions of the macroscopic quantities (temperature, density, polarization) *and* of various relaxation times to be determined elsewhere by a specific physical model. Finally, all the results obtained are discussed and suggestions for some extensions are given.

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I. INTRODUCTION

In the last few years many attempts have been made to calculate the transport coefficients of relativistic dense matter, whether for nuclear (symmetric or pure neutrons) or quark matter. Two main motivations are behind these attempts. First the recent developments of experiments involving heavy ions collisions demand a better understanding of an assumed hydrodynamical stage [1] of the resulting fireball, and also of directly connected questions such as the entropy production and a possible return to a state of local equilibrium. Second, the description of dense objects studied in relativistic astrophysics (neutron stars, strange stars) necessitates the knowledge of transport properties in many important problems (cooling of such stars, energy and momentum transfer from inner to outer parts of the star, etc.).

Although they can be evaluated *via* the use of Kubo's relations (or connected models) [2], the transport coefficients [3,4] (heat conduction, viscosities, diffusion coefficients, etc.) are generally calculated on the basis of kinetic theory [5], whether relativistic [6,7] or not, i.e., through the use of a Boltzmann equation (or of its quantum version, the so-called Uhlenbeck-Uehling equation [8]) or of any other kinetic equation. Such an equation generally involves the dynamics of dense matter under the form of a cross section. For quark matter, we believe that, at the present moment, it is probably illusory to obtain a credible cross section from QCD in the energy domain of interest (i.e., between $T\sim 100$ MeV and, say, $T\lesssim 500$ MeV, so as to have an order of magnitude in mind), a domain where precisely the deconfining transition is supposed to occur: this transition is neither completely understood from a theoretical point of view nor is it established experimentally. For relativistic nuclear matter, the transition amplitude is generally evaluated at second order in the coupling constants of the model (one boson exchange) although the perturbation expansion does not converge. Hence, the results obtained are quite problematic.

For these reasons, a more phenomenological approach has often been preferred, based on the use of a relaxation time approximation of the collision term, where all the dynamics are supposed to be included in a single parameter (*via* the relaxation time) that should be evaluated elsewhere with a specific model. Next, the calculation of the transport coefficient follows from a first-order Chapman-Enskog expansion of the kinetic equation. Owing to the above remarks, we believe that such an approach, somewhat more modest than a general one, is the most reasonable at the present moment. It presents the advantage of giving the general structure of the transport coefficients as functions of the temperature T, of the energy density ρ (or the chemical potential, or the particle density, etc.) *and* the relaxation time τ; furthermore, when this last quantity is roughly evaluated as

$$\tau=\frac{1}{\sigma n\langle v\rangle} \tag{1.1}$$

[where n is a particle (or possibly a hole or quasiparticle) density, where σ is a total cross section, and $\langle v\rangle$ is the average relative four-velocity of two colliding particles], then reasonable orders of magnitude can be obtained.

Following this line, most authors have used a relaxation time model for the (relativistic) kinetic equation, of the general form

$$p\cdot\partial f(x,p)=-\frac{f(x,p)-f_{\rm eq}(x,p)}{\tau(p)}\ , \tag{1.2}$$

where $f(x,p)$ is the distribution function of the particles (or quasiparticles), $f_{\rm eq}(x,p)$ is the (local) equilibrium distribution function [9], and $\tau(p)$ is *a priori* a four-momentum-dependent relaxation time. For instance, the choice $\tau(p)$=const has mainly been studied by Marle [10], while the choice [11]

$$\tau(p)=\frac{\tau}{u\cdot p} \tag{1.3}$$

is the one of Anderson and Witting [12] (where τ=const and where u^μ is the local average four-velocity of the medium). This last choice is the most popular one for the following reasons [12]: (i) τ has truly the meaning of a relaxation *time,* whether the system is at $T=0$ K or is ultrarelativistic $T\gg m$, whether the mass of the particles is zero or not; (ii) the so-called Landau and Lifschitz matching conditions [13], i.e.,

$$u_\mu J^\mu_{(1)}=0\ , \tag{1.4a}$$

$$u_\mu T^{\mu\nu}_{(1)}=0 \tag{1.4b}$$

where the index (1) refers to the deviation from equilibrium, is automatically satisfied as a consequence of the conservation laws for the four-current J^μ and the energy-momentum tensor $T^{\mu\nu}$. Moreover, as has been discussed by Danielewicz and Gyulassy [4], the Landau and Lifschitz form of relativistic hydrodynamics leads to more sensible results at low densities. For subsequent use, let us also notice the first-order Chapman-Enskog solution [12] of Eqs. (1.2) and (1.3),

$$f_{(1)}(x,p)=-\frac{\tau}{p\cdot u}p\cdot\partial f_{\rm eq}(x,p)+O(\tau^2)\ . \tag{1.5}$$

Nevertheless, although quite natural and valid more or less for dilute unpolarized systems, the Anderson-Witting equation possesses some obvious limitations which we briefly review. First, the concept of a distribution function does possess a well-known domain of validity and instead one should rather use a Wigner function, defined in a covariant way [7,14–16]. Next, nucleons or quarks are fermions obeying some sort of Dirac equation and spin is taken into account *via* a 4×4 covariant Wigner matrix (or a larger one when internal degrees of freedom are taken into account) while it is definitely not so in the Anderson-Witting equation. Another remark may also be in order. While the Anderson-Witting equation *must* be supplemented by a mass-shell–type constraint on p^μ, it is not so in the Wigner function approach (see below and Refs. [7] and [15]). Finally, polarized matter can be dealt with more completely in our Wigner function approach than in an extension of Eq. (1.2) (see Appendix A). Moreover, the Anderson-Witting equation cannot be used in its original form either to obtain spin waves, internal quantum number waves, or a possible coupling between them.

Unfortunately, neither the obtaining of a relaxation term for a relativistic kinetic equation satisfied by the covariant Wigner function nor its Chapman-Enskog expansion is a trivial problem. As to the collision term, it is indeed difficult to infer its general form due to the matrix character of the Wigner function. As to a Chapman-Enskog expansion, quantum theory *and* relativity do introduce a supplementary length (the usual Compton wave length) and, accordingly, one more expansion parameter. Besides the usual dimensionless parameter,

$$\epsilon\equiv\frac{\tau\langle v\rangle}{L}\sim\frac{\tau}{L} \tag{1.6}$$

(where L is a macroscopic length, $\langle v\rangle\sim c$ for a relativistic system), there also exists the following parameter

$$\eta\equiv\frac{1}{mL} \tag{1.7}$$

or, equivalently,

$$\bar\eta\equiv\frac{1}{m\tau}\ . \tag{1.8}$$

However, in ordinary cases, η is generally much smaller than ϵ and its contribution can perfectly be neglected [17]. This is the case for on-shell particles of dilute systems. Nevertheless, when one thinks of systems of quasifermions their (effective) mass depends on T, ρ, etc. and can, in principle, be arbitrarily small, leading thereby to arbitrarily large values of the parameters η and $\bar\eta$. A well-known example can be found in the Walecka model [17,18] for nuclear matter or in the so-called scalar plasma [19] where the effective mass of the nucleon (or of a quark) is given by

$$M=m-g\langle\phi\rangle\ , \tag{1.9}$$

where $\langle\phi\rangle$ is a scalar meson condensate and where g is a coupling constant.

In this paper, we would like to explore these problems and provide some solutions. We expect new qualitative phenomena, such as a self-diffusion in the medium due to the "two-fluid" character of its constituents (i.e., spin-up and spin-down particles and/or particles and antiparticles) or spin waves, besides some modifications on the structure of the transport coefficients themselves. We shall not, however, investigate here other interesting cases that also give rise to modifications of the latter and, more important, to new transport coefficients. For instance, neither collective effects [20] and/or external fields [21] are dealt with below.

In Sec. II we recall briefly the basic equations and definitions used throughout this paper while in Sec. III an introductory discussion of a particular relaxation term is given. In Sec. IV a class of general relaxation terms is studied (and also their Chapman-Enskog solution). Section V is devoted to the general properties of an acceptable collision term. In Sec. VI transport properties are studied while a discussion and some remarks are given in Sec. VII.

II. BASIC EQUATIONS AND DEFINITIONS

In this section the basic definitions and equations needed in what follows are briefly recalled and details can be found elsewhere [7,14,15]. The one-fermion Wigner function is defined as

$$F(x,p)=\frac{1}{(2\pi)^4}\int d^4R \times e^{-ip\cdot R}\left\langle \bar{\psi}\left[x+\frac{R}{2}\right]\otimes\psi\left[x-\frac{R}{2}\right]\right\rangle , \tag{2.1}$$

where ψ obeys a Dirac equation which does not need to be specified further here and where the angular brackets denote a quantum statistical average value. $F(x,p)$ is normalized through

$$J^\mu(x)=\mathrm{Tr}\int d^4p\,\gamma^\mu F(x,p) , \tag{2.2}$$

where $J^\mu(x)$ is the conserved four-current of the fermions, i.e., a charge, or baryonic-number, or leptonic-number, etc. four-current. $J^\mu(x)$ defines the average four-velocity u^μ of the system under consideration through

$$J^\mu(x)\equiv nu^\mu , \tag{2.3}$$

where n is the charge (or baryonic-number, etc.) density of the system, and with $u^2=1$.

The main physical quantities needed are, besides the four-current, the average energy-momentum tensor

$$T^{\mu\nu}(x)=\mathrm{Tr}\int d^4p\,p^\mu\gamma^\nu F(x,p) \tag{2.4}$$

and the spin-density tensor

$$S^{\mu\nu\lambda}(x)=\frac{i}{4}\mathrm{Tr}\int d^4p\{\gamma^\lambda\sigma^{\mu\nu}+\sigma^{\mu\nu}\gamma^\lambda\}F(x,p) , \tag{2.5}$$

which is necessary for the description of polarized media. Equation (2.4) is nothing but the average value of the *canonical* energy-momentum tensor and hence is not necessarily symmetric and is conserved in the absence of external fields or condensates, the only case which is considered in this paper. $S^{\mu\nu\lambda}$, a completely antisymmetric tensor, is not conserved since this is the case only for the *total* (spin *plus* kinetic momentum) angular-momentum density [7,22]. It can be rewritten as

$$S^{\mu\nu\lambda}(x)=\frac{1}{2}\mathrm{Tr}\int d^4p\,\epsilon^{\mu\nu\lambda\alpha}\gamma_5\gamma_\alpha F(x,p) . \tag{2.6}$$

Note that a symmetric form of $T^{\mu\nu}$ and a conserved version of $S^{\mu\nu\lambda}$ have been given in Ref. [7]; unfortunately, they are valid only for noninteracting systems and hence are generally of little use.

For free particles [23] the covariant Wigner distribution (2.1) obeys the following equations [14]

$$\{i\gamma\cdot\partial+2[\gamma\cdot p-m]\}F(x,p)=0 , \tag{2.7a}$$

$$F(x,p)\{i\gamma\cdot\overleftarrow{\partial}-2[\gamma\cdot p-m]\}=0 \tag{2.7b}$$

which can easily be derived from the Dirac equations obeyed by ψ and $\bar{\psi}$. The most general kinetic equations for $F(x,p)$, assuming as usual pointlike collisions [i.e., the scale of length (and time) in which the system is described is much larger than the interaction region (and the interaction time is much smaller than any other times)] and no collective effects [20], is obtained by adding a phenomenological collision term to Eqs. (2.7).

Equilibrium

In equilibrium the collision term must vanish whatever its form, the system is invariant under spacetime translations and $F_{\mathrm{eq}}(p)$ obeys the system (2.7), which reduces to

$$(\gamma\cdot p-m)F_{\mathrm{eq}}(p)=0=F_{\mathrm{eq}}(p)(\gamma\cdot p-m) . \tag{2.8}$$

Multiplying these equations on the appropriate side by $(\gamma\cdot p+m)$, one gets

$$(p^2-m^2)F_{\mathrm{eq}}(p)=0 \tag{2.9}$$

which indicates that $F_{\mathrm{eq}}(p)$ is necessarily proportional to $\delta(p^2-m^2)$. Next, $F_{\mathrm{eq}}(p)$ is decomposed on the Dirac algebra generated by the 16 matrices γ^A $(A=1,2,\ldots,16)$ as

$$F_{\mathrm{eq}}(p)=\tfrac{1}{4}\{f(p)I+f_\mu(p)\gamma^\mu+if_{\mu\nu}(p)\sigma^{\mu\nu} +f_{5\mu}(p)\gamma_5\gamma^\mu+if_5(p)\gamma^5\} \tag{2.10}$$

so that the various f^A's obey the following system [equivalent to Eqs. (2.8)]

$$p_\mu f^\mu-mf=0 , \tag{2.11}$$

$$p^\mu f-mf^\mu=0 , \tag{2.12}$$

$$2mf_{\mu\nu}+\epsilon_{\rho\lambda\mu\nu}p^\lambda f_5^\rho=0 , \tag{2.13}$$

$$f_5=0 , \tag{2.14}$$

$$\epsilon_{\lambda\rho\nu\mu}p^\lambda f^{\rho\nu}+mf_{5\mu}=0 , \tag{2.15}$$

$$p_\lambda f^{\lambda\mu}=0 , \tag{2.16}$$

$$p_{[\mu}f_{\nu]}=0 , \tag{2.17}$$

$$p_\mu f_5^\mu=0 , \tag{2.18}$$

$$p_\mu f_5=0 , \tag{2.19}$$

which are obtained after elementary manipulations [21]. Equation (2.12) and the mass-shell constraint $p^2=m^2$ imply Eq. (2.11). Equations (2.13) and (2.15) are equivalent owing to the conditions (2.16) and (2.18). Equations (2.14), (2.17), (2.18), and (2.19) then appear to be trivial and one finally obtains

$$F_{\mathrm{eq}}(p)=\frac{1}{4}\left[f_{\mathrm{eq}}(p)I+\frac{p\cdot\gamma}{m}f_{\mathrm{eq}}(p) -\frac{i}{2m}\epsilon_{\rho\lambda\mu\nu}p^\lambda f_{5\mathrm{eq}}^\rho(p)\sigma^{\mu\nu} +f_{5\mu\mathrm{eq}}(p)\gamma_5\gamma^\mu\right] . \tag{2.20}$$

Therefore, the most general equilibrium Wigner function does depend on two as-yet unknown functions $f_{\mathrm{eq}}(p)$ and $f_{5\mathrm{eq}}^\lambda(p)$ to be determined by a direct calculation from the definition (2.1) and the ordinary relativistic density operator. For $f_{\mathrm{eq}}(p)$, one finds [24]

$$f_{\rm eq}(p)=\frac{\varepsilon(p^0)}{(2\pi)^3}\delta(p^2-m^2)\frac{1}{\exp\beta(p_\mu u^\mu-\mu)+1} \qquad (2.21)$$

where $\varepsilon(p^0)$ is the sign of p^0 (with $p^0 \equiv p\cdot u$) and where μ is the chemical potential. When the system is unpolarized, $f_{5\rm eq}^\lambda \equiv 0$ and one has

$$F_{\rm eq}(p)=\frac{\gamma\cdot p+m}{4m}f_{\rm eq}(p) \; . \qquad (2.22)$$

For polarized systems $f_{5\rm eq}^\lambda(p)\neq 0$ and can always be rewritten as

$$f_{5\rm eq}^\lambda(p)\equiv S^\lambda(p)f_{\rm eq}(p) \; , \qquad (2.23)$$

where $S^\lambda(p)$ is a pseudo four-vector orthogonal to p_λ [Eq. (2.18)]

$$p_\lambda S^\lambda(p)=0 \; . \qquad (2.24)$$

In such a case, $F_{\rm eq}(p)$ possesses the general form

$$F_{\rm eq}(p)=\frac{\gamma\cdot p+m}{2m}\,\frac{1+\gamma_5\gamma^\mu S_\mu(p)}{2}f_{\rm eq}(p) \; , \qquad (2.25)$$

where use has been made of Eq. (2.24). Note that the first term of the right-hand side of this last equation is nothing but the usual projector over the positive-energy states while the second cannot be interpreted as a projection over spin-up states [25], since $S^\mu S_\mu=-S^2\neq 1$; note also that these two matrix operators commute due to the fact that $p_\lambda S^\lambda=0$.

At this point the (pseudo) four-vector $S^\mu(p)$ is completely arbitrary, and this for two reasons. First, it depends on the polarization of the system whatever its definition. Next, this arbitrariness reflects that of the Dirac spinors of the free particle. Therefore, a physical choice must be made as to $S^\mu(p)\equiv S(p)N^\mu(p)$, with $N^\mu(p)N_\mu(p)=-1$. As to $S(p)$, it can easily be shown to be directly connected to the polarization of the medium through the density operator, whose spin part reads

$$\frac{1+\gamma_5\gamma_\mu N^\mu(p)}{2}\xi(p)+\frac{1-\gamma_5\gamma_\mu N^\mu(p)}{2}[1-\xi(p)] =\frac{1+[2\xi(p)-1]\gamma_5\gamma_\mu N^\mu(p)}{2} \; , \qquad (2.26)$$

where $\xi(p)$ is the percentage of spin-up particles. Hence, as a result, one has

$$S(p)\equiv 2\xi(p)-1 \; . \qquad (2.27)$$

On the other hand, the simplest physical choice for $N^\mu(p)$ is the following

$$N^\mu(p)=\frac{u^{[\mu}n^{\nu]}p_\nu}{\{(u^\lambda p_\lambda)^2-(n^\lambda p_\lambda)^2\}^{1/2}} \; , \qquad (2.28)$$

where u^μ is the average four-velocity of the system and n^μ is a unit spacelike pseudo four-vector orthogonal to u^μ,

$$u^\mu n_\mu=0, \quad n^\mu n_\mu=-1 \; , \qquad (2.29)$$

whose physical meaning is that it represents a given global spin-quantization axis. The above choice (2.28) for $N^\mu(p)$ is, in fact, a consequence of a simple analysis of the way a system of charged particles is usually polarized. Suppose, indeed, that the system under consideration is placed in a magnetic field: the various spins align along its direction thus leading to a (more or less, according to the temperature) polarized system. If we now switch off the magnetic field, the system is then metastable and will depolarize more or less rapidly (for helium III, this relaxation time is of order of a couple of days). If we now look at the Wigner distribution f_5^μ *in the presence* of the magnetic field, one can see [21] that *it is proportional* to $N^\mu(p)$ where n^μ is the space direction of the magnetic field. The *choice* (2.28) is, accordingly, quite natural.

At this point, it should be remarked that, although spin does not commute with the Dirac Hamiltonian, our choice is quite sensible since we do not deal with a true equilibrium state but rather with a metastable one.

In order to gain more insights on $S(p)$, let us evaluate the various macroscopic quantities $S^{\mu\nu\lambda}$ (spin-density tensor) and M^μ (polarization four-vector). One has

$$M^\mu=\int d^4p\, f_5^\mu(p) =\int d^4p\frac{S(p)u^{[\mu}n^{\nu]}p_\nu}{\{(u\cdot p)^2-(n\cdot p)^2\}^{1/2}}f_{\rm eq}(p) \; . \qquad (2.30)$$

With the *choice*

$$S(p)=\mathcal{P}\{(u\cdot p)^2-(n\cdot p)^2\}^{1/2} \quad (\mathcal{P}={\rm const}) \qquad (2.31)$$

M^μ can easily be calculated as

$$M^\mu=-\mathcal{P}n^\mu n_{\rm eq}m \qquad (2.32)$$

so that

$$\mathcal{P}=\frac{1}{mn_{\rm eq}}n_\mu M^\mu \; , \quad M^\mu u_\mu=0 \; . \qquad (2.33)$$

M^μ then appears to be parallel to the spin-quantization axis and $\mathcal{P}$ is the polarization of the medium. The choice (2.31) has been used after previous results of the magnetic field case [21].

From M^μ, one gets

$$S^{\mu\nu\lambda}=-\tfrac{1}{2}\epsilon^{\mu\nu\lambda\alpha}M_\alpha \; , \qquad (2.34)$$

so that, locally, the spin tensor $M^{\mu\nu}$ reads

$$M^{\mu\nu}\equiv S^{\mu\nu\lambda}u_\lambda=-\tfrac{1}{2}\epsilon^{\mu\nu\lambda\alpha}u_\lambda M_\alpha \; . \qquad (2.35)$$

In the rest frame of the system where $u^\mu\equiv(1,\mathbf{0})$ and taking n^μ along the third axis, the only nonvanishing component of $M^{\mu\nu}$ is M^{12}. Finally, our choice (2.28) and (2.31) for $S^\mu(p)$ appears to describe correctly a polarized medium in (metastable) thermodynamical equilibrium. Other choices are, of course, possible but they deal with systems prepared in particular ways: particles endowed with four-momentum p contain a prescribed p-dependent percentage of spin-up particles, etc.

III. AN ILLUSTRATIVE EXAMPLE

A few years ago, a simple relaxation time kinetic equation that generalizes the Anderson-Witting one [12] was suggested [26]. It reads

$$\{i\gamma\cdot\partial+2[\gamma\cdot p-m]\}F=-i\gamma\cdot u\frac{F-F_{\rm eq}}{\tau}\ , \tag{3.1a}$$

$$F\{i\gamma\cdot\overleftarrow{\partial}-2[\gamma\cdot p-m]\}=-i\frac{F-F_{\rm eq}}{\tau}\gamma\cdot u \tag{3.1b}$$

and it has the property that the Landau and Lifshitz matching conditions (1.4) are satisfied: this can easily be checked (i) by taking the trace of both equations, (ii) by integrating over p, (iii) by adding both results of each equation (3.1), and finally (iv) by taking account of the current conservation; also the same procedure must be repeated after multiplying by p^ν. Another property of this system is that it leads to the Anderson-Witting equation when the "magnetic polarization" is neglected: this can be seen through [26] the Gordon decomposition of $f^\mu(x,p)$ which yields [14,24] $f^\mu\propto p^\mu f$ or $f_{\mu\nu}\sim 0$, and $f_5\sim 0$, $f_5^\mu\sim 0$. Of course both Eqs. (3.1) are consistent with each other (see Sec. V for the case of a general collision term).

The interest of this kinetic equation not only lies in its simplicity but also in the fact that it raises a number of problems to be found in general quantum and relativistic kinetic equations.

Let us first try to solve the system (3.1) by a *naive* Chapman-Enskog expansion at order 1 in τ, with

$$F=F_{\rm eq}+\tau F_{(1)}+\cdots \tag{3.2}$$

where $F_{\rm eq}$ is now a *local* equilibrium Wigner function; i.e., it depends on x^μ *via* the macroscopic quantities (T,u^μ,μ) involved. Then, the right-hand side of Eq. (3.1a) should be considered at order zero in τ, and taking into account Eqs. (2.8), one gets (with $\gamma\cdot u\gamma\cdot u=1$)

$$F_{(1)}(x,p)=\gamma\cdot u\gamma\cdot\partial F_{\rm eq}(x,p) \tag{3.3}$$

$$=\gamma\cdot u\gamma^\lambda[\gamma\cdot p+m]\frac{\partial_\lambda f_{\rm eq}(x,p)}{4m}\ . \tag{3.4}$$

Similarly, from Eq. (3.1b) one obtains

$$F_{(1)}(x,p)=[\gamma\cdot p+m]\gamma^\lambda\gamma\cdot u\frac{\partial_\lambda f_{\rm eq}(x,p)}{4m} \tag{3.5}$$

which is different from (and in contradiction with) the result (3.4) whereas Eqs. (3.1) are consistent.

To see exactly what was wrong in this naive Chapman-Enskog expansion, the system (3.1) is rewritten as

$$\gamma\cdot\partial F=-\left[\frac{\gamma\cdot u}{\tau}-2i(\gamma p-m)\right](F-F_{\rm eq})\ , \tag{3.6a}$$

$$\partial F\cdot\gamma=-(F-F_{\rm eq})\left[\frac{\gamma\cdot u}{\tau}+2i(\gamma\cdot p-m)\right]\ , \tag{3.6b}$$

where use has been made of the properties (2.8) for $F_{\rm eq}$. Equivalently Eqs. (3.6) are transformed to

$$F-F_{\rm eq}=-\frac{\gamma\cdot\left[\frac{u}{\tau}-2ip\right]-2im}{\left[\frac{u}{\tau}-2ip\right]^2+4m^2}\gamma\cdot\partial F\ , \tag{3.7a}$$

$$F-F_{\rm eq}=-\partial F\cdot\gamma\frac{\gamma\cdot\left[\frac{u}{\tau}+2ip\right]+2im}{\left[\frac{u}{\tau}+2ip\right]^2+4m^2}\ . \tag{3.7b}$$

Setting now $p^\mu=m\xi^\mu$ and after some rearrangements, it then reads

$$F-F_{\rm eq}=\frac{\gamma\cdot\left[\frac{u}{m\tau}-2i\xi\right]-2i}{\left[\frac{u}{m\tau}+2i\xi\right]^2+4}\frac{\gamma\cdot\partial F}{m}\ , \tag{3.8a}$$

$$F-F_{eq}=-\frac{\partial F\cdot\gamma}{m}\frac{\gamma\cdot\left[\frac{u}{m\tau}+2i\xi\right]+2i}{\left[\frac{u}{m\tau}+2i\xi\right]^2+4}\ . \tag{3.8b}$$

In this last system there appears *two* expansion parameters: (i) $1/mL$, which comes from the term $\partial F/m$ and (ii) $1/m\tau$. Let us first expand these equations with respect to $\bar\eta\equiv 1/m\tau$; one obtains [from Eq. (3.8a)]

$$F_{(1)}\simeq-\frac{1}{2}\frac{\gamma\cdot\xi+1}{\eta\xi\cdot u}\frac{\gamma\cdot\partial F_{\rm eq}}{m} \tag{3.9}$$

for the dominant term (in $1/\bar\eta$); or, equivalently

$$F_{(1)}\simeq-\frac{\tau}{2p\cdot u}(\gamma\cdot p+m)\gamma\cdot\partial f_{\rm eq}\left[\frac{\gamma\cdot p+m}{4m}\right]\ , \tag{3.10}$$

$$F_{(1)}\sim-\frac{\tau}{p\cdot u}p\cdot\partial F_{\rm eq}\ . \tag{3.11}$$

The *same* results are obtained from Eq. (3.8b) achieving thereby a consistent solution of both Eqs. (3.1). In the passage from Eqs. (3.10) to Eq. (3.11) use was made of Eq. (B2).

Several remarks are now in order. It should first be noticed that Eq. (3.11) does not only constitute the dominant part of the solution (i.e., in $1/\bar\eta$) but also the correct first-order Chapman-Enskog solution since the term $\tau\partial$ is precisely of order τ/L. This circumstance is, in fact, fortuitous and does not appear with general collision terms (see Sec. V). Another remark is that the solution (3.11) leads exactly to the Anderson-Witting results. Let us briefly show this. What is needed to get the transport coefficients is $T^{\mu\nu}$ and J^ν which can be obtained from Eqs. (2.2) and (2.4). These last equations can also be rewritten as

$$J^\nu=\int d^4p\, f^\nu \quad\text{and}\quad T^{\mu\nu}=\int d^4p\, p^\mu f^\nu \tag{3.12}$$

indicating that only the function $f^\mu_{(1)}(x,p)$ is of interest in view of our goal. It is given by

$$f^{\nu}_{(1)} \equiv \mathrm{Tr}\gamma^{\nu}F_{(1)} = -\frac{\tau}{p\cdot u}\frac{p^{\nu}}{m}p\cdot\partial f_{\rm eq} \quad (3.13)$$

$$= \frac{p^{\nu}}{m}f_{(1)} \; . \quad (3.14)$$

In other words, it has exactly the same form as (and is identical with) the Anderson-Witting result [12]. This, of course, does not mean that the physical content of Eqs. (3.1) is identical with the one of the Anderson-Witting equation; for instance, higher-order approximations do differ; or while Eqs. (3.1) is suitable for polarized media (see below), this is not the case for the Anderson-Witting equation. However, it has been found that, for unpolarized media, most collision terms do possess the property that the first-order Chapman-Enskog solution coincides with the Anderson-Witting solution (see Sec. IV).

Let us now give a glance at polarized media, i.e., those whose equilibrium Wigner function is given by Eq. (2.25). It is not difficult to realize that Eq. (3.11) [in which Eq. (2.25) is inserted] is still valid and, more important, so is the case for Eqs. (3.13) and (3.14). It follows that the transport coefficients so obtained are identical with those given by Anderson and Witting [12]. However, from Eq. (3.11), one can obtain the relaxation of the average polarization four-vector $f^{\mu}_{5(1)}(x,p)$,

$$f^{\mu}_{5(1)}(x,p) \equiv \mathrm{Tr}\{\gamma_5\gamma^{\mu}F_{(1)}(x,p)\} \; , \quad (3.15)$$

as

$$M^{\mu}_{(1)}(x) = \int d^4p \, f^{\mu}_{5(1)}(x,p) \quad (3.16)$$

$$= -\tau\int d^4p \frac{p^{\lambda}\partial_{\lambda}}{p\cdot u}f^{\mu}_{5\rm eq}(x,p) \quad (3.17)$$

and the corresponding transport coefficients

$$M^{\mu}_{(1)}(x) = u^{\mu}\left\{-\tau\lambda\mathcal{P}n\cdot X + \frac{\tau}{3}i_{4-1}[\pi^{\mu\nu}\partial_{\mu}n_{\nu}\mathcal{P} + n\cdot\partial\mathcal{P}]\right\} + n^{\mu}\left\{\tau\left[i_{21}-\frac{1}{3}i_{4-1}\right]\dot{\mathcal{P}}\right\} + \pi^{\mu\alpha}\left\{\frac{\tau}{3}\mathcal{P}i_{4-1}n\cdot\partial u_{\alpha} + \tau\mathcal{P}i_{21}\dot{n}_{\alpha}\right\} , \quad (3.18)$$

where λ is the thermal conductivity, X^{α} is

$$X^{\alpha} \equiv \Delta^{\alpha\lambda}(u)[\partial_{\lambda}\beta + \beta\dot{u}_{\lambda}] \; , \quad (3.19)$$

and the i_{mn} are integrals defined in Appendix A.

It is finally clear that the system (3.1) does not allow—at least at first order in the Chapman-Enskog expansion—a coupling between polarization and 4-current.

IV. A GENERAL RELAXATION TIME MODEL

A general relaxation time model—although not the most general one—is now studied. It is chosen so that the collision term has the following form

$$C(F) = M\cdot(F-F_{\rm eq})\cdot N \; , \quad (4.1)$$

where M and N are 4×4 complex matrices. *A priori*, it depends on 2×16 complex parameters while the most general relaxation collision term assumes the form

$$C(F) = \chi^{abcd}(F_{cd}-F^{\rm eq}_{cd}) \; , \quad (4.2)$$

where (a,b,c,d) are spinor indices running from 1 to 4. Equation (4.2) depends on 4^4 complex parameters and much less when symmetries are taken into account. However, despite this lack of generality Eq. (4.1) possesses a sufficient degree of complexity to accommodate most useful physical cases, and our relativistic quantum kinetic system reads

$$\{i\gamma\cdot\partial + 2[\gamma\cdot p - m]\}F = C(F) \; , \quad (4.3a)$$

$$F\{i\gamma\cdot\overleftarrow{\partial} - 2[\gamma\cdot p - m]\} = \tilde{C}(F) \; , \quad (4.3b)$$

where $\tilde{C}$ is chosen in such a way that this system be consistent, i.e., so that

$$\tilde{C}(F) = -\gamma^0 C^{\dagger}(F)\gamma^0 \; , \quad (4.4)$$

where this last property results from the following one [7]

$$\gamma^0 F^{\dagger}\gamma^0 = F \quad (4.5)$$

and from the requirement of consistency.

In order to expand F in a many-parameter Chapman-Enskog series, the above system (4.3) is first written in terms of the dimensionless variables $\xi^{\mu}\equiv p^{\mu}/m$ and $\bar{x}^{\mu}\equiv x^{\mu}/L$ and thus reads

$$\frac{i\gamma\cdot\bar{\partial}}{L}F + \frac{2}{\ell_c}(\gamma\cdot\xi - 1)F = \frac{C(F)}{\ell_{mfp}} \; , \quad (4.6a)$$

$$iF\frac{\bar{\partial}\cdot\gamma}{L} - \frac{2F}{\ell_c}(\gamma\cdot\xi - 1) = \frac{\tilde{C}(F)}{\ell_{mfp}} \; , \quad (4.6b)$$

where ℓ_c is the Compton wavelength of the particles of the system and where ℓ_{mfp} is their mean free path. ℓ_{mfp} has been made apparent in Eqs. (4.6) *via* an implicit redefinition of $C(F)$: this has been made in accordance with the fact that a collision term is always supposed to be of order ℓ^{-1}_{mfp} (or τ^{-1} if one considers that, in a relativistic system, $\langle v\rangle \lesssim 1$). Note also that ℓ_{mfp} is supposed to be *any* of all possible collision lengths (such that the following developments make sense): for instance, there exists a collision length for spin flip, another one for the attenuation of polarization, etc. Another important point to be noticed is that there may exist the interesting situation where one of the relaxation times is much greater than all others. In such a case one has to face a Chapman-Enskog expansion whose first term is not necessarily a stable equilibrium state but rather a metastable one. Such a case can be found, in a nonrelativistic context, in the study of polarized ^{3}He: during experiments performed (e.g., spin-echo ones) the medium remains fully polarized [27].

Now, only two expansion parameters,

$$\eta \equiv \ell_c/L, \quad \epsilon = \ell_{mfp}/L \quad (4.7)$$

are considered although there exist possibly many others.

In general, η is much smaller than ϵ. There exist, however, physical situations where these parameters are of the same order of magnitude or where $\epsilon \ll \eta$. Among these last cases, one finds those where collective effects are important [18–20].

Setting now

$$\partial = \partial_{(0)} + \epsilon \partial_{(1)\epsilon} + \eta \partial_{(1)\eta} + \cdots \ , \tag{4.8}$$

$$F = F_{(0)} + \epsilon F_{(1)\epsilon} + \eta F_{(1)\eta} + \cdots \tag{4.9}$$

for the expansion of our various quantities in terms of powers of ϵ and η and separating the different orders, one gets

$$i\gamma \cdot \partial_{(0)} F_{(0)} + 2(\gamma \cdot p - m) F_{(1)\eta} = M F_{(1)\epsilon} N \ , \tag{4.10a}$$

$$i\partial_{(0)} \cdot \gamma F_{(0)} - 2F_{(1)\eta}(\gamma \cdot p - m) = -\bar{N} F_{(1)\epsilon} \bar{M} \ , \tag{4.10b}$$

$$(\gamma \cdot p - m) F_{(1)\epsilon} = F_{(1)\epsilon}(\gamma \cdot p - m) = 0 \ , \tag{4.11}$$

$$M F_{(1)\eta} N = 0 = \bar{N} F_{(1)\eta} \bar{M} \ , \tag{4.12}$$

where we have set $\bar{M} \equiv \gamma^0 M^\dagger \gamma^0$. Here, we limit ourselves to the "simple" case of an isotropic collision term, i.e., depending only on p^μ and u^μ so that the matrices M and N read

$$\begin{aligned} M &= \mu_1 I + \mu_2 \gamma \cdot p + \mu_3 \gamma \cdot u + u_4 \sigma \cdot u \cdot p \ , \\ N &= \nu_1 I + \nu_2 \gamma \cdot p + \nu_3 \gamma \cdot u + \nu_4 \sigma \cdot u \cdot p \ , \end{aligned} \tag{4.13}$$

with the following notation

$$\sigma \cdot u \cdot p \equiv \sigma^{\mu\nu} u_\mu p_\nu \ .$$

In the following, it appears quite helpful to use the decomposition of the various matrices $M, N, F_{(1)\epsilon}, F_{(1)\eta} \ldots$ obtained, with the projectors

$$P_\pm = \frac{m \pm \gamma \cdot p}{2m} \ ; \tag{4.14}$$

they are such that

$$\begin{aligned} &P_+ + P_- = I \ , \\ &P_+ P_- = P_- P_+ = 0 \end{aligned} \tag{4.15}$$

and for any 4×4 matrix X, one has

$$X = P_+ X P_+ + P_+ X P_- + P_- X P_+ + P_- X P_- \ . \tag{4.16}$$

General form of $F_{(1)\epsilon}$

From Eqs. (4.11), it is easily seen that $F_{(1)\epsilon}$ has necessarily the form

$$F_{(1)\epsilon} = P_+ F_{(1)\epsilon} P_+ \ . \tag{4.17}$$

Multiplying now both sides of Eq. (4.10a) by P_+, one obtains

$$i P_+ \gamma \cdot \partial F_{\text{eq}} P_+ = P_+ M P_+ F_{(1)\epsilon} P_+ N P_+ \ , \tag{4.18}$$

where use has been made of Eqs. (4.15) and (4.17). On the other hand, the matrices M and N of the form (4.13) do possess the property

$$P_+ M P_+ = \left[\mu_1 + m\mu_2 + \frac{p \cdot u}{m} \mu_3 \right] P_+ \tag{4.19}$$

so that one gets

$$F_{(1)\epsilon} = \frac{i P_+ \gamma \cdot \partial F_{\text{eq}} P_+}{\left[\mu_1 + m\mu_2 + \frac{p \cdot u}{m} \mu_3 \right] \left[\nu_1 + m\nu_2 + \frac{p \cdot u}{m} \nu_3 \right]} \tag{4.20}$$

which reduces to

$$F_{(1)\epsilon} = \frac{i \frac{p}{m} \cdot \partial F_{\text{eq}}}{\left[\mu_1 + m\mu_2 + \frac{p \cdot u}{m} \mu_3 \right] \left[\nu_1 + m\nu_2 + \frac{p \cdot u}{m} \nu_3 \right]} \tag{4.21}$$

where use was made of the properties (see Appendix B)

$$\begin{aligned} &P_+ \gamma^\mu P_+ = \frac{p^\mu}{m} P_+ \ , \\ &\left[P_+, \frac{1 + \gamma_5 \gamma^\mu S_\mu}{2} \right] = 0 \ . \end{aligned} \tag{4.22}$$

Had we used Eq. (4.10b) instead of Eq. (4.10a), we would have found

$$F_{(1)\epsilon} = \frac{-i \frac{p}{m} \cdot \partial F_{\text{eq}}}{\left[\mu_1^* + m\mu_2^* + \frac{p \cdot u}{m} \mu_3^* \right] \left[\nu_1^* + m\nu_2^* + \frac{p \cdot u}{m} \nu_3^* \right]} \ . \tag{4.23}$$

It follows that the consistency of the results (4.21) and (4.23) implies that the denominator should be purely imaginary

$$\left[\mu_1 + \mu_2 m + \frac{p \cdot u}{m} \mu_3 \right] \left[\nu_1 + \nu_2 m + \frac{p \cdot u}{m} \nu_3 \right] \equiv i A(p) \ , \tag{4.24}$$

where $A(p)$ is a *known* (but as-yet unspecified) function of p and, accordingly, one has

$$F_{(1)\epsilon} = -\frac{p \cdot \partial F_{\text{eq}}}{m A(p)} \ . \tag{4.25}$$

Note that the case $A(p) \equiv 0$ only leads to constraints on local equilibrium [see Eq. (4.18)] which implies $p \cdot \partial F_{\text{eq}} = 0$.

It should be emphasized that the general form (4.25), despite its apparent generality, is essentially similar to the one obtained in Sec. III and reduces to Anderson and Witting's when the denominator is chosen to be proportional to $p \cdot u$. Note also that it is valid whether the system is polarized or not.

Determination of $F_{(1)\eta}$

Let us now solve Eqs. (4.10) and (4.12) for $F_{(1)\eta}$ and, to this end, let us decompose this last matrix as in Eq. (4.16) and multiply Eq. (4.10a) from the right by P_- and from the left by P_+; one gets

$$P_+MP_+F_{(1)\epsilon}P_+NP_-=0 \tag{4.26}$$

which leads to a nontrivial solution only if

$$P_+NP_-=0\ , \tag{4.27}$$

similarly, Eq. (4.10b) yields $P_-\bar{N}P_+=0$. Multiplying now one of the equations (4.10) on both sides by P_-, one is led to

$$2P_-F_{(1)\eta}P_-=-P_-MP_+F_{(1)\epsilon}P_+NP_-=0\ . \tag{4.28}$$

The other components, i.e., $P_+F_{(1)\eta}P_-$ and $P_-F_{(1)\eta}P_+$, are obtained from P_- (4.10a) P_+ and P_+ (4.10b) P_-, respectively, as

$$P_-F_{(1)\eta}P_+=\frac{1}{4m}[iP_-\gamma\cdot\partial F_{\mathrm{eq}}P_+-P_-MF_{(1)\epsilon}NP_+]\ , \tag{4.29}$$

$$P_+F_{(1)\eta}P_-=-\frac{1}{4m}[iP_+\partial F_{\mathrm{eq}}\cdot\gamma P_++P_+\bar{N}F_{(1)\epsilon}\bar{M}P_+]\ . \tag{4.30}$$

The last term, $P_+F_{(1)\eta}P_+$, is expressed as a function of the other ones by multiplying Eq. (4.12) on both sides by P_+ *and* using Eq. (4.19)

$$P_+F_{(1)\eta}P_+=-\left\{\frac{P_+F_{(1)\eta}P_-NP_+}{\left[\nu_1+m\nu_2+\frac{p\cdot u}{m}\nu_3\right]}+\frac{P_+MP_-F_{(1)\eta}P_+}{\left[\mu_1+m\mu_2+\frac{p\cdot u}{m}\mu_3\right]}\right\}\ . \tag{4.31}$$

Gathering now all these results, $F_{(1)\eta}$ is given by

$F_{(1)\eta}$=Eq. (4.28)+ Eq. (4.29)+ Eq. (4.30)+ Eq. (4.31)

or

$$\begin{aligned}F_{(1)\eta}=&\left[1-\frac{P_+MP_-}{\left[\mu_1+m\mu_2+\frac{p\cdot u}{m}\mu_3\right]}\right]\left[\frac{P_-i\gamma\cdot\partial F_{\mathrm{eq}}}{4m}-\frac{P_-MP_+}{\left[\mu_1+m\mu_2+\frac{p\cdot u}{m}\mu_3\right]}\cdot\frac{P_+i\gamma\cdot\partial F_{\mathrm{eq}}}{4m}\right]\\&-\left[\frac{P_+\partial F_{\mathrm{eq}}i\gamma P_+}{4m}-\frac{P_+\partial F_{\mathrm{eq}}i\gamma P_+}{4m}\frac{P_+\bar{M}P_-}{\left[\mu_1+m\mu_2+\frac{p\cdot u}{m}\mu_3\right]^*}\right]\left[1-\frac{P_-NP_+}{\left[\nu_1+m\nu_2+\frac{p\cdot u}{m}\nu_3\right]}\right]\ .\end{aligned} \tag{4.32}$$

This part of the complete first-order solution constitutes the physically interesting one since it yields various polarization effects connected to charge and energy-momentum flows, leading thereby to a number of new transport properties such as spin diffusion.

Let us now give an explicit solution for $F_{(1)\eta}$. It is obtained by replacing M and N in Eq. (4.32) by their specific form (4.13). After a straightforward but quite long and tedious calculation, one finds

$$\begin{aligned}F_{(1)\eta}=I\cdot&\left[-\frac{i}{4}(\beta-\beta^*)\Delta^{\alpha\beta}(p)u_\alpha r_\beta+(\delta-\delta^*)\frac{ip\cdot r}{4m}-\frac{1}{4m}(\beta+\beta^*)\epsilon^{\lambda\rho\alpha\beta}p_\lambda u_\rho t_{\alpha\beta}\right]\\
&+\gamma_\mu\left[-\frac{i}{4m}(\beta-\beta^*)\Delta^{\alpha\beta}(p)u_\alpha r_\beta p^\mu+(\delta-\delta^*)\frac{ip\cdot r}{4m}\frac{p^\mu}{m}-i(\alpha-\alpha^*)\frac{p\cdot r}{4m}\Delta^{\mu\nu}(p)u_\nu\right.\\
&\left.+(\alpha+\alpha^*)\frac{p^\lambda t_{\lambda\rho}}{4m}\epsilon^{\rho\lambda\nu\mu}u_\lambda p_\nu+\frac{1}{4}(\beta+\beta^*)\Delta^\alpha{}_\lambda(p)\Delta^\beta{}_\nu(p)u_\alpha t_{\beta\rho}\epsilon^{\rho\lambda\nu\mu}-\epsilon^{\rho\lambda\nu\mu}\frac{p_\nu}{m}t_{\lambda\rho}\right]\\
&+\sigma^{\mu\nu}\left[\frac{i}{4m}r_{[\mu}p_{\nu]}-\frac{i}{4m}(\alpha+\alpha^*)\frac{p\cdot r}{2m}u_{[\mu}p_{\nu]}-\frac{i}{8}(\beta+\beta^*)\Delta^\alpha{}_\mu(p)\Delta^\beta{}_\nu(p)u_{[\alpha}r_{\beta]}\right.\\
&+\frac{(\alpha-\alpha^*)}{8m}p^\lambda t^\rho{}_\lambda\Delta^{\alpha\beta}(p)u_\alpha\epsilon_{\beta\rho\mu\nu}-\frac{1}{8m}(\beta-\beta^*)\epsilon_{\lambda\alpha\beta[\mu}p^\lambda u^\alpha t^\beta{}_{\nu]}\\
&\left.+\frac{(\beta-\beta^*)}{8m}\Delta^{\alpha\beta}(p)u_\alpha t^\rho{}_\beta p^\lambda\epsilon_{\lambda\rho\mu\nu}-\frac{(\delta-\delta^*)}{8m}\epsilon_{\alpha\beta\mu\nu}\frac{p^\alpha}{m}p_\lambda t^{\lambda\beta}\right]\end{aligned}$$

$$+\gamma_5\gamma^\mu\left[\frac{1}{4}(\beta+\beta^*)z_\mu+\frac{i}{4m}(\alpha-\alpha^*)\frac{p_\mu}{m}t_{\lambda\rho}p^\lambda u^\rho-\frac{i}{4}(\beta-\beta^*)\Delta^{\alpha\beta}(p)u_\alpha t_{\beta\mu}+\frac{i}{4m}(\delta-\delta^*)p^\lambda t_{\lambda\mu}\right.$$
$$\left.-\frac{i}{4}(\beta-\beta^*)\Delta_{\mu\alpha}(p)\Delta_{\nu\beta}(p)u^{[\alpha}t^{\beta]\nu}\right]+\gamma_5\left[\frac{i}{4m}(\alpha+\alpha^*)p^\lambda t_{\lambda\rho}u^\rho\right] \quad (4.33)$$

where we have set

$$r_\mu=\frac{1}{4m}\partial_\mu f_{\text{eq}} , \quad (4.34)$$

$$t_{\mu\nu}=\frac{1}{4m}\partial_\mu[S_\nu f_{\text{eq}}] , \quad (4.35)$$

$$z_\alpha=\epsilon_{\lambda\mu\nu\alpha}\frac{p^\lambda}{m}u^\mu r^\nu , \quad (4.36)$$

$$\Delta^{\alpha\beta}(p)=\eta^{\alpha\beta}-\frac{p^\alpha p^\beta}{m^2} , \quad (4.37)$$

$$\alpha=\frac{\mu_3+m\mu_4}{\mu_1+m\mu_2+\frac{p\cdot u}{m}\mu_3} , \quad (4.38)$$

$$\beta=\frac{\mu_3-m\mu_4}{\mu_1+m\mu_2+\frac{p\cdot u}{m}\mu_3} , \quad (4.39)$$

$$\delta=\alpha\beta\Delta^{\mu\nu}(p)u_\mu u_\nu . \quad (4.40)$$

The above solution (4.33) finally depends on four real parameters only, namely the real and imaginary parts of α and β. In the case of an unpolarized system, the above expression for $F_{(1)\eta}$ has a slightly simpler form obtained by setting

$$S^\mu(p)\equiv 0, \quad t^{\alpha\beta}(p)\equiv 0 . \quad (4.41)$$

Remarks

(1) So far, Eq. (4.12) has not been completely used and the consistency of $P_+F_{(1)\eta}P_+$ as obtained from either equation can be shown to require the following constraints on M and N

$$\frac{P_-\overline{M}P_+}{\left[\mu_1+m\mu_2+\frac{p\cdot u}{m}\mu_3\right]^*}=\frac{P_-NP_+}{\left[\nu_1+m\nu_2+\frac{p\cdot u}{m}\nu_3\right]} . \quad (4.42)$$

Furthermore, the following relations hold

$$P_-NP_-=0=P_-\overline{N}P_- , \quad (4.43)$$

$$P_-MP_-=\frac{P_-MP_+MP_-}{\left[\mu_1+m\mu_2+\frac{p\cdot u}{m}\mu_3\right]} \quad (4.44)$$

if one does not want any very particular constraint on local equilibrium. More explicitly, these constraints lead to the following relations

$$\nu_3=m\nu_4 , \quad (4.45)$$

$$\nu_1=m\nu_2+\frac{p\cdot u}{m}\nu_3 , \quad (4.46)$$

$$\frac{\nu_3+m\nu_4}{\nu_1+m\nu_2+\frac{p\cdot u}{m}\nu_3}=\frac{\nu_3}{\nu_1}=\left[\frac{\mu_3-m\mu_4}{\mu_1+m\mu_2+\frac{p\cdot u}{m}\mu_3}\right]^* , \quad (4.47)$$

$$\frac{(\mu_3+m\mu_4)(\mu_3-m\mu_4)\left[1-\frac{(p\cdot u)^2}{m^2}\right]}{\left[\mu_1+m\mu_2+\frac{p\cdot u}{m}\mu_3\right]\left[\mu_1-m\mu_2-\frac{p\cdot u}{m}\mu_3\right]}=1 . \quad (4.48)$$

Let us also recall that Eq. (4.24), that connects these various parameters, is purely imaginary.

From the constraints (4.45)–(4.47), one concludes that the most general admissible form for N is

$$N=\left[\nu_2+\frac{\nu_3}{m}\gamma\cdot u\right](\gamma\cdot p+m) . \quad (4.49)$$

(2) It can easily be verified that $F_{(1)\eta}$ does possess the property

$$f^\mu_{(1)\eta}p_\mu=mf_{(1)\eta} \quad (4.50)$$

but verifies neither $f_{(5)}\equiv 0$ nor $f^\mu_5\cdot p_\mu=0$, although these conditions are quite natural and indeed are true in most cases studied with a general collision term [see Sec. V]. These last conditions can be obtained by imposing either

$$u^\rho p\cdot\partial[S_\rho(p)f_{\text{eq}}]=0$$

and, in this case, we constrain the form of local equilibrium, or by setting $\alpha=0$. When this last condition is assumed then the matrix M has necessarily the general form

$$M=(\gamma\cdot p+m)\left[\mu_2+\frac{\mu_3}{m}\gamma\cdot u\right] . \quad (4.51)$$

(3) When the system is not polarized then $F_{(1)\epsilon}$ does not lead to polarization phenomena (in other words, $f_{5(1)\epsilon}\equiv 0$ as well as $f^{\mu\nu}_{(1)\epsilon}\equiv 0$). It is not so for $F_{(1)\eta}$ since $f^\mu_{5(1)\eta}\neq 0$. $F_{(1)\eta}$ implicitly contains the possibility of *polarized* perturbations. The average polarization four-vector of such a perturbation is given by

$$M^\mu_{(1)}(x)=\int d^4p\, f^\mu_{5(1)\eta}(p) \quad (4.52)$$

and, in the case of an unpolarized system [see Eqs. (4.41)], it reads

$$M^{\mu}_{(1)}(x)=\frac{\epsilon^{\lambda\alpha\beta\mu}}{4m}\int d^4p(\beta+\beta^*)p_\lambda u_\alpha\partial_\beta f_{\rm eq}(p)\ . \tag{4.53}$$

Moreover, one can also note that

$$M^{\mu}_{(1)}(x)u_\mu=0 \tag{4.54}$$

as it should be.

(4) It is desirable, although not mandatory, that the Landau and Lifshitz matching conditions (1.4) be satisfied, or

$$u_\mu\cdot\int d^4p\{\epsilon f^{\mu}_{(1)\epsilon}+\eta f^{\mu}_{(1)\eta}\}=0=\frac{u_\mu}{L}\int d^4p\left\{\tau f^{\mu}_{(1)\epsilon}+\frac{1}{m}f^{\mu}_{(1)\eta}\right\} \tag{4.55}$$

and a similar equation with p_μ included inside the integration symbol.

However, it should be realized that the integrand of Eq. (4.55) can also be considered as a first-order expansion of f^μ with respect to the (supposed) small parameter $(m\tau)^{-1}$. Accordingly, the following question raises itself in a natural manner: should the Landau and Lifshitz conditions be satisfied at order $O(1/m\tau)$ or at order $O(1/m^2\tau^2)$? When the first possibility is chosen—and it corresponds to the usual first-order Chapman-Enskog term—then it provides a condition on $f^{\mu}_{(1)\epsilon}$ only.

V. GENERAL PROPERTIES OF THE RELAXATION TIME APPROXIMATION

Let us now investigate the most general relativistic quantum Bhatnagar-Gross-Krook (BGK) [28] collision term and, to this end the system (4.3) reads

$$i\gamma\cdot\partial F+2[\gamma\cdot p-m]F=C(F)\ , \tag{5.1a}$$

$$i\partial F\cdot\gamma-2F[\gamma\cdot p-m]=\tilde{C}(F)\ , \tag{5.1b}$$

where the collision term $\{C\cdot\tilde{C}\}$ must satisfy the consistency condition (4.4): i.e., $\tilde{C}=-\gamma^0C^+\gamma^0$. It should be noticed that the following discussion is valid for whatever $\{C,\tilde{C}\}$, be it a collision term or an arbitrary interaction term. Only the results specialized to linear $\{C,\tilde{C}\}$ are not general: in a relaxation time approximation F appears linearly in $C(F)$ and, since $C(F_{\rm eq})=0$, $C(F)$ must depend on F through the combination

$$\delta F\equiv F-F_{\rm eq}\ . \tag{5.2}$$

Besides these simple properties $C(F)$ must also obey the general conditions which have to be satisfied by all collision terms: (i) $C(F_{\rm eq})=0$, (ii) conservation laws must be verified, and (iii) an H theorem should exist. Property (i) is satisfied by construction in our specific case while consequences of point (ii) are studied below. As to the existence of an H theorem, it is required in order to ensure the monotonic relaxation toward equilibrium. Actually, not only is the definition of an entropy for off-equilibrium states quite moot but it is also difficult to express the entropy itself in terms of the Wigner function the more so since it *a priori* includes polarization [29]. Therefore, we shall content ourselves by demanding (i) that all f^A ($A=1,2,\ldots,16$) relax towards $f^A_{\rm eq}$ and (ii) that the second principle of thermodynamics be satisfied.

Note that entropy density could be defined through the thermodynamic relation

$$\rho=-P+\mu n+Ts\ , \tag{5.3}$$

where n is the charge density, ρ is the energy density, P is the pressure, and μ the chemical potential. However, even though this would be more or less satisfactory for off-equilibrium states of spinless particles, it could hardly be so when spin is properly taken into account.

Note also that, since F is a 4×4 matrix, there exist *a priori* 16 possible relaxation times and also C^2_{16} possible couplings between the 16 components of F. It follows that, *a priori,* the general collision term C depends on 136 constant. Of course, this number is drastically reduced when symmetries are taken into account.

In this section, the system (5.1) is first set in a more suitable form, i.e., as a set of equations for the f^A's from which a system of coupled kinetic equations for the same quantities is deduced. The consistency of this system and some natural conditions then impose some constraints on C. Among these conditions is the demand that they are truly relaxation equations and that they are similar to ordinary ones (such as the Anderson-Witting equation) in this sense that they can be solved with a Chapman-Enskog expansion. In particular, since C generally depends on macroscopic quantities such as the equilibrium average four-velocity u^μ, it is demanded that no derivative enters into it: if τ is one of the various relaxation times, C should be of order τ^{-1}, a property which would no longer persist if C would contain, e.g., terms like $\partial_\mu u_\nu$ or $\partial_\mu T$, etc. Other consistency properties will also be imposed, e.g., as to the mass shell on which evolves the particles within the system.

Finally, C is decomposed on the Dirac algebra as

$$C=\tfrac{1}{4}\{cI+c_\mu\gamma^\mu+c_{\mu\nu}\sigma^{\mu\nu}+c_{5\mu}\gamma_5\gamma^\mu+c_5\gamma^5\} \tag{5.4}$$

and a similar decomposition for $\tilde{C}$ whose coefficients are denoted by $\tilde{c}_A$ ($A=1,2,\ldots,16$).

Kinetic and constraint equations for the f^A's

The general system (5.1) does not look like a relativistic kinetic equation [7,30]. This is due to the fact that it includes both transport properties and mass-shell constraints. They can be disentangled in several ways. To this end, the system (5.1) is rewritten as

$$2(\gamma\cdot\pi_+ - m)F = C\ , \tag{5.5a}$$

$$2F(\gamma\cdot\pi_- - m) = \tilde{C}\ , \tag{5.5b}$$

where

$$\pi_\pm = p \pm \frac{i}{2}\partial\ . \tag{5.6}$$

Multiplying Eq. (5.5a) from the left by $(\gamma\cdot\pi_+ + m)$ and Eq. (5.5b) by $(\gamma\cdot\pi_- + m)$ from the right, adding and substracting the resulting equations, one obtains

$$(p^2 - m^2 - \tfrac{1}{4}\Box)F = \tfrac{1}{2}\{m(C+\tilde{C}) + \gamma\cdot\pi_+ C + \tilde{C}\gamma\cdot\pi_-\}\ , \tag{5.7}$$

$$ip\cdot\partial F = \tfrac{1}{4}\{(\gamma\cdot\pi_+ + m)C - \tilde{C}(\gamma\cdot\pi_- + m)\}\ . \tag{5.8}$$

These last two equations have the unpleasant feature that they involve the derivatives of the collision term $\{C,\tilde{C}\}$ and also $\Box F$. In fact such terms can be eliminated by considering the following two quantities

$$i\gamma\cdot\partial \text{Eq. (5.5a)} \mp \text{Eq. (5.5b)} i\gamma\cdot\overleftarrow{\partial} \tag{5.9}$$

which lead to

$$\tfrac{1}{2}\{i\gamma\cdot\partial C - i\partial\tilde{C}\cdot\gamma\} = \{i\gamma\cdot\partial\,\gamma\cdot p\,F - i\partial F\gamma p\cdot\gamma\} - \Box F - mi[\gamma\cdot\partial f - \partial F\cdot\gamma] \tag{5.10}$$

and

$$\tfrac{1}{2}\{i\gamma\cdot\partial C + i\partial\tilde{C}\cdot\gamma\} = i\gamma\cdot\partial\,\gamma\cdot pF + i\partial F\gamma p\cdot\gamma - im[\gamma\cdot\partial F + \partial F\cdot\gamma]\ , \tag{5.11}$$

respectively. Introduced into Eqs. (5.7) and (5.8), they yield

$$(p^2 - m^2)F = \tfrac{1}{2}\{i\gamma\cdot\partial(\gamma\cdot p - m)F - F(\gamma\cdot p - m)i\gamma\cdot\overleftarrow{\partial}\} + \tfrac{1}{2}\{(\gamma\cdot p - m)C - \tilde{C}(\gamma p - m)\}\ , \tag{5.12}$$

$$ip\cdot\partial F = \tfrac{1}{4}\{(\gamma\cdot p + m)C - \tilde{C}(\gamma\cdot p + m) + i\gamma\cdot\partial\,\gamma\cdot pF + i\partial\,F\gamma p\cdot\gamma\} - im\{\gamma\cdot\partial F + \partial F\cdot\gamma\} \tag{5.13}$$

where the "unpleasant" features of Eqs. (5.7) and (5.8) have disappeared.

Taking now the traces of these last two equations with the 16 matrices of the Dirac algebra, we finally arrive at the following "transport" equations

$$p\cdot\partial f = \frac{1}{2i}p^\mu(c_\mu + \tilde{c}_\mu)\ , \tag{5.14}$$

$$p\cdot\partial f_\nu - p_\mu\partial^\nu f^\mu + m\partial^\nu f = \frac{p^\mu}{i}(c_{\mu\nu} - \tilde{c}_{\mu\nu}) + \frac{m}{2i}(c^\nu + \tilde{c}^\nu)\ , \tag{5.15}$$

$$p\cdot\partial f^{\mu\nu} + \partial^{[\mu}f^{\nu]\lambda}p_\lambda = \frac{1}{4i}\epsilon^{\mu\nu\alpha\beta}p_\alpha(c_{5\beta} + \tilde{c}_{5\beta})\ , \tag{5.16}$$

$$p\cdot\partial f_5 = \frac{p^\mu}{2}(c_{5\mu} - \tilde{c}_{5\mu}) - \frac{m}{2}(c_5 + \tilde{c}_5)\ , \tag{5.17}$$

$$p\cdot\partial f_5^\lambda - p_\mu\partial^\lambda f_5^\mu = \frac{1}{2}\epsilon^{\mu\nu\alpha\lambda}p_\alpha(c_{\mu\nu} + \tilde{c}_{\mu\nu}) \tag{5.18}$$

and at the following constraint (mass-shell) equations

$$(p^2 - m^2)f = \frac{p^\mu}{4}(c_\mu - \tilde{c}_\mu) + \frac{m}{4}(c - \tilde{c}) + \partial^\mu p^\nu f_{\mu\nu}\ , \tag{5.19}$$

$$(p^2 - m^2)f_\nu = \frac{p^\mu}{2}(c_{\mu\nu} - \tilde{c}_{\mu\nu}) + \frac{m}{4}(c_\nu - \tilde{c}_\nu) + \frac{p_\nu}{4}(c - \tilde{c}) + m\partial^\mu f_{\mu\nu} - \frac{1}{2}p^\mu\partial^\lambda f_5^\rho\epsilon_{\rho\lambda\mu\nu}\ , \tag{5.20}$$

$$(p^2 - m^2)f_{\mu\nu} = -\frac{1}{8}\epsilon_{\rho\lambda\mu\nu}(c_5^\rho - \tilde{c}_5^\rho)p^\lambda + \frac{m}{4i}(c_{\mu\nu} - \tilde{c}_{\mu\nu}) + \frac{1}{8i}p_{[\mu}(c_{\nu]} + \tilde{c}_{\nu]}) + \frac{1}{4}\epsilon_{\rho\lambda\mu\nu}p^\lambda\partial^\rho f_5 - \frac{1}{4}p_{[\mu}\partial_{\nu]}f - \frac{m}{4}\partial_{[\mu}f_{\nu]}\ , \tag{5.21}$$

$$(p^2 - m^2)f_5 = -\frac{p^\mu}{4i}(c_{5\mu} + \tilde{c}_{5\mu}) + \frac{m}{4i}(c_5 - \tilde{c}_5) + \frac{m}{2}\partial_\mu f_5^\mu - \frac{1}{2}\epsilon_{\lambda\rho\nu\mu}p^\mu\partial^\lambda f^{\rho\nu}\ , \tag{5.22}$$

$$(p^2 - m^2)f_5^\lambda = -\frac{1}{4i}\epsilon^{\mu\nu\alpha\lambda}p_\alpha(c_{\mu\nu} - \tilde{c}_{\mu\nu}) + \frac{m}{4}(c_5^\lambda - \tilde{c}_5^\lambda) - \frac{1}{4}p^\lambda(c_5 + \tilde{c}_5) + \frac{1}{2}\epsilon^{\mu\nu\alpha\lambda}p_\alpha\partial_\mu f_\nu - \frac{m}{2}\partial^\lambda f_5\ . \tag{5.23}$$

Note that the "transport equations" (5.14)–(5.18) as well as the "constraint equations" (5.19)–(5.23) are generally valid whether $\{C,\tilde{C}\}$ is a collision term or an arbitrary interaction term; no assumption has yet been made. In particular, without any further hypothesis, it has not the form of a relaxation equation

$$p\cdot\partial f^A = K^A{}_B\cdot(f^B - f^B_{\rm eq})\quad (A,B = 1,2,\ldots,16) \tag{5.24}$$

which can be cast into the form

$$\frac{d}{d\tau}X = K\cdot(X - X_{\rm eq}) \tag{5.25}$$

with $X \equiv \|f^A\|$ and $K \equiv \|K^A{}_B\|$ so that each f^A relaxes monotonously towards its equilibrium value $f^A_{\rm eq}$. In order that the system (5.14)–(5.18) have the expected form (5.24) two general conditions must be obeyed: (i) $C(F)$ must be linear and must depend on F through $\delta F = F - F_{\rm eq}$ and (ii) the superfluous terms $-p_\mu\partial^\nu f^\mu + m\partial^\nu f$, $\partial^{[\mu}f^{\nu]\lambda}p_\lambda$, and $p_\mu\partial^\lambda f_5^\mu$ occurring in the left-hand side of Eqs. (5.15), (5.16), and (5.18) must not be present. As a matter of fact, in a first-order Chapman-Enskog expansion they disappear completely (see below).

The constraint equations (5.19)–(5.23) are now worth discussing. In the absence of any external field (or condensate) whatsoever, in the kinetic regime we are considering, collisions are pointlike and hence particles lie on the mass shell $p^2 = m^2$. This property can be seen in another way: when the solution of the transport equations are expanded in a convergent approximation whose zeroth order is such that $p^2=m^2$ (as is the case for an equilibrium Wigner function) then, owing to the linearity of $\{C(F),\tilde{C}(F)\}$, each order is on the mass shell, and so is the complete solution.

For future use, let us also give a system of equations [31] obeyed by the f^A's and obtained directly from Eqs. (5.1) by taking the various traces, summing and adding:

$$p_\mu f^\mu - mf = \tfrac{1}{4}(c-\tilde{c}) , \tag{5.26a}$$

$$\partial_\mu f^\mu = \frac{1}{2i}(c+\tilde{c}) , \tag{5.26b}$$

$$\partial_\nu f^{\mu\nu} + p^\mu f - mf^\mu = \frac{1}{2i}(c^\mu - \tilde{c}^\mu) , \tag{5.27a}$$

$$\partial_\mu f + 4p^\lambda f_{\lambda\mu} = \frac{1}{2i}(c_\mu + \tilde{c}_\mu) , \tag{5.27b}$$

$$\tfrac{1}{2}\partial_{[\mu} f_{\nu]} - 2mf_{\mu\nu} - p^\lambda \epsilon_{\rho\lambda\mu\nu} f_5^\rho = \frac{1}{2i}(c_{\mu\nu} - \tilde{c}_{\mu\nu}) , \tag{5.28a}$$

$$\tfrac{1}{2}\partial^\lambda f_5^\rho \epsilon_{\lambda\rho\nu\mu} + p_{[\mu} f_{\nu]} = \tfrac{1}{2}(c_{\mu\nu} + \tilde{c}_{\mu\nu}) , \tag{5.28b}$$

$$\partial_\mu f_5 - 2p^\lambda \epsilon_{\lambda\rho\nu\mu} f^{\rho\nu} - 2mf_{5\mu} = \tfrac{1}{2}(c_{5\mu} - \tilde{c}_{5\mu}) , \tag{5.29a}$$

$$\partial_\lambda f^{\rho\nu} \epsilon_{\lambda\rho\nu\mu} + 2p_\mu f_5 = -\frac{1}{2i}(c_{5\mu} + \tilde{c}_{5\mu}) , \tag{5.29b}$$

$$p_\mu f_5^\mu = -\tfrac{1}{4}(c_5 + \tilde{c}_5) , \tag{5.30a}$$

$$\partial_\mu f_5^\mu + 2mf_5 = -\frac{1}{2i}(c_5 - \tilde{c}_5) . \tag{5.30b}$$

Constraints on the collision term

Besides the linearity of $C(F)$, the form of the collision term has to be such that the usual conservation laws are satisfied. Furthermore, the possibility of a Chapman-Enskog expansion of the solution [or possibly another type of expansion such as that implied by the 14-moment method (i.e., the relativistic grad moment method); however, here we limit ourselves to the Chapman-Enskog expansion] should also be allowed.

Let us first investigate the conservation laws and, to this end, let us integrate Eq. (5.26b) over p, after it has been successively multiplied by 1 or by p^λ. One gets [see Eqs. (2.2) and (2.4)]

$$\partial_\mu J^\mu = \frac{1}{2i}\int d^4p(c+\tilde{c}) = 0 , \tag{5.31}$$

$$\partial_\mu T^{\mu\lambda} = \frac{1}{2i}\int d^4p\, p^\lambda (c+\tilde{c}) = 0 . \tag{5.32}$$

These last two equations are, in fact, weak constraints that lead to matching conditions [5–7,10] to be satisfied, whether Landau-Lifshitz ones or others. For instance, it is clear that, when

$$(c+\tilde{c}) = \text{const} \times (f - f_{\text{eq}}) ,$$

then the Marle matching conditions are obeyed, while for

$$(c+\tilde{c}) = \text{const} \times u_\mu (f^\mu - f^\mu_{\text{eq}}) ,$$

this is the case for the Landau and Lifshitz ones. However, these last expressions for $(c+\tilde{c})$ are by no means unique.

Besides the conservation laws (5.31) and (5.32), there also exists the conservation of the total angular momentum, which includes some kind of a balance equation for the spin. It reads

$$\partial_\lambda S^{\mu\nu\lambda} - (T^{\mu\nu} - T^{\nu\mu}) = 0 \tag{5.33}$$

which, in terms of Wigner functions, can be written as

$$\tfrac{1}{2}\partial_\lambda \int d^4p\, \epsilon^{\rho\lambda\mu\nu} f_{5\rho} - \int d^4p\, p^{[\mu} f^{\nu]} = 0 . \tag{5.34}$$

Comparison of Eq. (5.34) and Eq. (5.28b) yields

$$\int d^4p [c_{\mu\nu} + \tilde{c}_{\mu\nu}] = 0 \tag{5.35}$$

which also constitutes a weak constraint.

Let us now look at the possibility of a Chapman-Enskog approximation. Performing the same expansion of Eq. (5.1a) as in Sec. IV, we arrive at quite similar expressions that read [see Eqs. (4.10)–(4.12)]

$$i\gamma\cdot\partial F_{\text{eq}} + 2(\gamma\cdot p - m)F_{(1)\eta} = C[F_{(1)\epsilon}] , \tag{5.36a}$$

$$(\gamma\cdot p - m)F_{(1)\epsilon} = 0 , \tag{5.36b}$$

$$C[F_{(1)\eta}] = 0 \tag{5.36c}$$

and similar equations resulting from the expansion of Eq. (5.1b). Multiplying Eq. (5.36a) from the right by P_- and from the left by P_+, one finds the constraint

$$P_+ C P_- = 0 \tag{5.37}$$

(and also the equivalent constraint $P_- \tilde{C} P_+ = 0$). Similarly, Eq. (5.36a) multiplied from the left by P_+ yields

$$ip\cdot\partial F_{\text{eq}} = mP_+ C \tag{5.38a}$$

while the unwritten analogous equation provides

$$ip\cdot\partial F_{\text{eq}} = m\tilde{C} P_+ , \tag{5.38b}$$

$$P_+ C - \tilde{C} P_+ = 0 \tag{5.39}$$

which implies Eq. (5.37). A comparison between Eq. (5.13) and Eq. (5.38) shows that the consistency of the first-order Chapman-Enskog solution gives rise to

$$\tfrac{1}{8}[i\gamma\cdot\partial C + i\partial\tilde{C}\cdot\gamma] = O(\epsilon) . \tag{5.40}$$

At this point, it must be strongly emphasized that $F_{(1)}$ appearing implicitly in C and $\tilde{C}$ is always $F_{(1)\epsilon}$ [see Eq. (5.36a)] and hence all the above constraints refer only to $f^A_{(1)\epsilon}$ *and not* to $F_{(1)\eta}$. We come back to this point below.

Let us now examine the consequences of the mass-shell constraint $p^2=m^2$ *and* of the Chapman-Enskog expansion

$$p^2 = m^2, \quad \partial F \to \partial F_{\text{eq}} . \tag{5.41}$$

From Eq. (5.12) and the fact that $P_- F_{\text{eq}} = F_{\text{eq}} P_- = 0$, one gets

$$CP_- + P_- \tilde{C} = 0 . \tag{5.42}$$

Conditions (5.39) and (5.42) can now be decomposed on the Dirac matrices and yield 20 equations which do not present much interest except in their main consequence:

the various coefficients $\{c^A, \tilde{c}^A\}$ can be expressed in terms of $\{c, \tilde{c}\}$ and $\{c_5^\lambda \cdot \tilde{c}_5^\lambda\}$ *only*. Introduced into the transport equations (5.14)–(5.18) they provide

$$p\cdot\partial f_{\rm eq} = \frac{m}{2i}(c+\tilde{c}) , \tag{5.43}$$

$$p\cdot\partial f^{\mu}_{\rm eq} = \frac{p^\mu}{2i}(c+\tilde{c}) , \tag{5.44}$$

$$p\cdot\partial f^{\mu\nu}_{\rm eq} = \frac{1}{4i}\epsilon^{\mu\nu\alpha\beta}p_\alpha(c_{5\beta}+\tilde{c}_{5\beta}) , \tag{5.45}$$

$$p\cdot\partial f^{\mu}_{5\rm eq} = \frac{im}{2}\Delta^{\mu\nu}(p)(c_{5\nu}+\tilde{c}_{5\nu}) , \tag{5.46}$$

$$p\cdot\partial f_{5\rm eq} = 0 . \tag{5.47}$$

We still insist, at this point, that while the left-hand side of these last equations refers to equilibrium functions, their right-hand side concerns $F_{(1)\epsilon}$ only while $F_{(1)\eta}$ has to be determined from Eqs. (5.36a)–(5.36c). Accordingly, the various $f^A_{(1)\epsilon}$'s obey the following relations, as well as their equilibrium analogues,

$$p_\mu f^{\mu}_{(1)\epsilon} = m f_{(1)\epsilon} , \tag{5.48a}$$

$$p_\mu f^{\mu\nu}_{(1)\epsilon} = 0 , \tag{5.48b}$$

$$p_\mu f^{\mu}_{5(1)\epsilon} = 0 , \tag{5.48c}$$

$$f_{5\epsilon} = 0 \tag{5.48d}$$

which result from Eq. (5.36b), i.e., from $P_- F_{(1)\epsilon} = F_{(1)\epsilon} P_- = 0$.

The collision term

Let us now turn to admissible collision terms. To this end, it is necessary to construct the most general scalars and pseudovectors from those at our disposal, i.e., from

$$f, f^\mu, f^{\mu\nu}, f_5^\mu, f_5; p^\mu, u^\mu, n^\mu; \epsilon^{\mu\nu\alpha\beta} . \tag{5.49}$$

First, it should be recalled that the various $f^A_{(1)\epsilon}$'s [for simplicity, the indices (1) and ϵ are suppressed; they are reestablished whenever necessary] obey the following relations similar to the equilibrium ones Eqs. (2.11)–(2.19)

$$f^\mu = \frac{p^\mu}{m} f , \quad f_5 \equiv 0 ,$$
$$f^{\mu\nu} = \frac{1}{2m}\epsilon^{\mu\nu\alpha\beta}p_\alpha f_{5\beta} , \tag{5.50}$$
$$f_5^\mu = \frac{1}{m}\epsilon^{\mu\nu\alpha\beta}p_\nu f_{\alpha\beta} .$$

It follows that, for instance, a scalar such as $u_\mu f^\mu$ is proportional to f and hence should not be considered as essentially different from f. Finally, with all these constraints in mind, one can write

$$p\cdot\partial f_{\rm eq} = \frac{m}{2i}(c+\tilde{c}) = a f_{(1)\epsilon} + b_\lambda f^{\lambda}_{5(1)\epsilon} , \tag{5.51}$$

$$p\cdot\partial f_5^\mu = -\frac{im}{2}\Delta^{\mu\nu}_{(p)}(c_{5\mu}+\tilde{c}_{5\mu}) = c^\mu f_{(1)\epsilon} + d^{\mu\lambda} f_{5\lambda(1)\epsilon} . \tag{5.52}$$

Moreover, because of the relation $p_\mu f^\mu_{5\rm eq} = 0 = p_\mu f^\mu_{5(1)\epsilon}$, the tensors $b^\mu, c^\mu, d^{\mu\lambda}$ should be chosen as being orthogonal to p^μ, so that Eqs. (5.51) and (5.52) contain $1+3+3+3\times 3=16$ independent parameters. The $a, b^\mu, c^\mu, d^{\mu\lambda}$ are functions of the various components of p^μ and of constant phenomenological relaxation times.

Let us now discuss the system (5.51) and (5.52).

1. The fact that $f^\mu \propto p^\mu f$ has the interesting consequence that the energy-momentum tensor is now symmetric. As a consequence, the local polarization tensor is conserved at order ϵ:

$$\partial_\lambda S^{\mu\nu\lambda} = -\tfrac{1}{2}\epsilon^{\mu\nu\lambda\rho}\partial_\lambda \int d^4p\, f_{5\rho} = O(\epsilon) . \tag{5.53}$$

2. Unlike the results of the preceding section where the first-order solution, in the parameter ϵ, implied a complete decoupling between f and f_5^λ, here there exists a possible coupling *via* the functions b^μ and c^μ. As a consequence, spin diffusion and other transport phenomena connected with polarization do appear at this order and not only at order η.

3. When the medium is not polarized, then $n^\mu \equiv 0$ and Eqs. (5.51) and (5.52) decouple still *at order* ϵ.

4. Equations (5.44) and (5.45) for f^μ and $f^{\mu\nu}$ do not contain any new information, but these functions can be obtained from relations (5.50) once the system (5.51)–(5.52) has been solved.

5. One could be tempted to take as the most general form of the relativistic quantum relaxation time approximation Eqs. (5.51)–(5.52) with the left-hand side now containing the f^A's instead of the $f^A_{\rm eq}$'s and on the right-hand side the $(f^A - f^A_{\rm eq})$ instead of the $f^A_{(1)\epsilon}$. In fact, this would be equivalent to an equation (or rather a system) in which $F_{(1)\eta} \equiv 0$; indeed, this would imply that $F_{1)\eta}$ obeys both Eqs. (5.36b) and $C(F_{(1)\eta}) = 0$, and it is not difficult to realize that their only solution is precisely $F_{(1)\eta} \equiv 0$. This choice would also mean that the nontrivial part of the solution would be at least $O(\eta^2)$.

6. The system (5.51) and (5.52) is linear in the unknown functions $f_{(1)\epsilon}$ and $f^{\lambda}_{5(1)\epsilon}$ and hence can be solved without any particular difficulty (see Sec. VI) and, consequently, allows the calculation of transport coefficients at $O(\epsilon)$ and $O(\eta)$. On the other hand, the explicit calculation of $F_{(1)\eta}$—although straightforward since all our equations are linear—is much more involved.

VI. TRANSPORT PROPERTIES

In the preceding section, it has been shown that a great variety of possible collision terms (in the relaxation time approximation) can actually be constructed. In this section we would like to obtain a specific collision term, as simple as possible but nontrivial in the sense that it should also contain effects occurring because of spin contributions (e.g., spin precession, spin diffusion, etc.). A similar collision term is also well needed in other problems such as the evaluation of transport coefficients of a quantum relativistic plasma embedded in a magnetic field.

The only function that can *a priori* be determined is the function $a(p)$ occurring in Eq. (5.51) since, when the system is unpolarized, the equation for f should reduce to the Anderson-Witting one. Accordingly, $a(p)$ can be chosen to be

$$a(p)=-\frac{p\cdot u}{\tau}\ ; \tag{6.1}$$

note, however, that we could have added an arbitrary term proportional to $\mathcal{P}$, the polarization of the system.

In order to say something about the remaining undetermined phenomenological functions $b^\mu, c^\mu, d^{\mu\lambda}$, one can still resort to the matching conditions. If one wishes, for example, to implement the Landau and Lifshitz matching conditions, as was the case in the nonpolarized case, the deviations to the baryonic current and energy-momentum tensor to first order in ϵ should obey

$$u_\mu J^\mu_{(1)}=0,\quad u_\nu T^{\mu\nu}_{(1)}=0\ . \tag{6.2}$$

In this section it is shown how these conditions do restrict the range of the arbitrary functions that occur in the collision term. Next, the explicit form of the conservation relations are studied since they are repeatedly used in the calculation of $J^\mu_{(1)}$ and $T^{\mu\nu}_{(1)}$. Then a simple admissible form for δf is proposed and it obeys the Landau and Lifshitz matching conditions. The results obtained for $J^\mu_{(1)}$ and $T^{\mu\nu}_{(1)}$ are given while the corresponding transport coefficients are derived and discussed. Finally, δf^μ_5 is dealt with in another subsection.

The conservation relations

The equilibrium conservation relations read

$$\partial_\mu J^\mu_{\text{eq}}=O(\epsilon)\ , \tag{6.3}$$

$$\partial_\mu T^{\mu\nu}_{\text{eq}}=O(\epsilon)\ , \tag{6.4}$$

$$\partial_\lambda S^{\mu\nu\lambda}_{\text{eq}}=O(\epsilon) \tag{6.5}$$

with

$$J^\mu_{\text{eq}}=nu^\mu\ , \tag{6.6}$$

$$T^{\mu\nu}_{\text{eq}}=\rho u^\mu u^\nu-P\Delta^{\mu\nu}\ , \tag{6.7}$$

$$S^{\mu\nu\lambda}_{\text{eq}}=\tfrac{1}{2}\epsilon^{\mu\nu\lambda\rho}\mathcal{P}_n m n_\rho \tag{6.8}$$

so that Eqs. (6.3)–(6.5) lead to

$$\dot n+n\theta=0\ , \tag{6.9}$$

$$\dot\rho+(\rho+P)\theta=0\ , \tag{6.10}$$

$$\Delta^{\mu\nu}\partial_\nu(P)=(\rho+P)\dot u^\mu\ , \tag{6.11}$$

$$\dot{\mathcal{P}}=\mathcal{P}(\theta+u'\cdot n)=\mathcal{P}\pi^{\mu\nu}\partial_\mu u_\nu\ , \tag{6.12}$$

$$\mathcal{P}\pi^{\mu\nu}\partial_\lambda(n)=-n\pi^{\mu\nu}[\partial_\lambda(\mathcal{P})+\mathcal{P}n'_\lambda]\ , \tag{6.13}$$

$$\epsilon^{\mu\nu\lambda\rho}n_\nu\partial_\lambda(n_\rho)=0\ . \tag{6.14}$$

Relation (6.14) will also prove useful when cast into the form

$$\pi^{\mu\nu}u_\alpha\partial_\lambda n^\alpha=\pi^{\mu\lambda}\dot n_\lambda\ , \tag{6.15}$$

$$\epsilon^{\mu\nu\alpha\beta}u_\mu n_\nu\partial_\alpha(n_\beta)=F^{\mu\nu}\partial_\mu(n_\nu)=0\ . \tag{6.16}$$

In Eqs. (6.9)–(6.16), use was made of the notations

$$\dot a\equiv u^\mu\partial_\mu(a)\ , \tag{6.17}$$

$$a'\equiv n^\mu\partial_\mu(a)\ , \tag{6.18}$$

$$\Delta^{\mu\nu}\equiv\eta^{\mu\nu}-u^\mu u^\nu\ , \tag{6.19}$$

$$\pi^{\mu\nu}\equiv\eta^{\mu\nu}-u^\mu u^\nu+n^\mu n^\nu\ , \tag{6.20}$$

$$F^{\mu\nu}\equiv\epsilon^{\mu\nu\alpha\beta}u_\alpha n_\beta\ , \tag{6.21}$$

$$\theta\equiv\partial_\mu u^\mu \tag{6.22}$$

with the properties

$$\pi^{\mu\nu}u_\nu=\pi^{\mu\nu}n_\nu=F^{\mu\nu}u_\nu=F^{\mu\nu}n_\nu=0\ , \tag{6.23}$$

$$F^\mu{}_\lambda\pi^{\lambda\nu}=F^{\mu\nu}\ , \tag{6.24}$$

$$F^{\mu\lambda}F_\lambda{}^\nu=-\pi^{\mu\nu}\ . \tag{6.25}$$

Another useful tensor is

$$^*F^{\mu\nu}=\tfrac{1}{2}\epsilon^{\mu\nu\alpha\beta}F_{\alpha\beta}=-u^{[\mu}n^{\nu]}\ . \tag{6.26}$$

In terms of the derivatives of the chemical potential μ and of the inverse temperature β, or rather of the parameters $\alpha=\beta\mu$ and $\gamma=m\beta$, Eqs. (6.9)–(6.11) may be rewritten as

$$\dot\alpha I_{21}-\dot\gamma I_{22}+\gamma\frac{I_{40}}{3}\theta=0\ , \tag{6.27}$$

$$\dot\alpha I_{22}-\dot\gamma I_{23}+\gamma\frac{I_{41}}{3}\theta=0\ , \tag{6.28}$$

$$\Delta^{\mu\lambda}\partial_\lambda(\alpha)=\frac{I_{41}}{I_{40}}\Delta^{\mu\lambda}[\partial_\lambda(\gamma)+\gamma\dot u_\lambda]\ , \tag{6.29}$$

where the I_{nm} are defined in Appendix A.

Let us also recall here the relations provided by the normalization and orthogonality of u^μ and n^μ:

$$u_\alpha\partial_\lambda u^\alpha=0=n_\alpha\partial_\lambda n^\alpha\ , \tag{6.30}$$

$$u_\alpha\partial_\lambda n^\alpha=-n_\alpha\partial_\lambda u^\alpha\ . \tag{6.31}$$

Determination of $\delta f\equiv f_{(1)\epsilon}$

The first-order correction δf is used when evaluating the transport coefficients of the system *via* the calculation of $J^\mu_{(1)}$ and $T^{\mu\nu}_{(1)}$. The starting point of our analysis is the most general possible form for δf. It reads

$$\delta f=Ap\cdot\partial f_{\text{eq}}+B^\lambda p\cdot\partial f_{5\lambda\text{eq}}\ , \tag{6.32}$$

where A is an arbitrary scalar function of p^μ and B^λ is the most general pseudovector orthogonal to p^λ that can be constructed from $p^\mu, n^\mu, u^\mu, \epsilon^{\mu\nu\alpha\beta}$. Accordingly, δf can be written as

$$\begin{aligned}\delta f=&\,a(p)p\cdot\partial f_{\text{eq}}+b(p)n_\lambda p\cdot\partial f^\lambda_{5\text{eq}}\\&+c(p)p\cdot n\,\pi^{\mu\nu}p_\mu p\cdot\partial f_{5\nu\text{eq}}\\&+d(p)p\cdot n\,\epsilon^{\mu\nu\rho\lambda}p_\mu u_\nu n_\rho p\cdot\partial f_{5\lambda\text{eq}}\ ,\end{aligned} \tag{6.33}$$

where the functions $a(p), b(p), c(p)$, and $d(p)$ are *a priori* arbitrary *scalar* functions [32] of p.

First, one must notice that, in the absence of polarization (i.e., when $\mathcal{P}\equiv 0$), δf should reduce to the Anderson-Witting form. Consequently, $a(p)$ can be chosen to be

$$a(p)=-\frac{\tau}{p\cdot u}\ . \tag{6.34}$$

A second remark is the following. In the other terms of Eq. (6.33), the expression $p\cdot\partial f_{5\mathrm{eq}}$ appears and one has

$$\begin{aligned} p\cdot\partial f_5^\lambda &= p\cdot\partial[-u^{[\lambda}(n^{\alpha]}/m)p_\alpha \mathcal{P}f] \\ &= -u^{[\lambda}n^{\alpha]}\frac{p_\alpha}{m}p\cdot\partial(\mathcal{P}f)-\mathcal{P}fp\cdot\partial\left[u^{[\lambda}n^{\alpha]}\frac{p_\alpha}{m}\right]\ . \end{aligned} \tag{6.35}$$

The first term in this last expression, once contracted either with $\pi^{\mu\nu}p_\nu$ or with $\epsilon^{\mu\nu\rho\lambda}p_\mu u_\nu n_\rho$ [which occur in the third and last term of Eq. (6.33), respectively] vanishes. Therefore, the peculiar effects due to the existence of *spatial gradients of polarization* occur only when the second term is present, i.e., when $b(p)\neq 0$.

The second term also contains the gradient $-(p\cdot u/m)\mathcal{P}b(p)p\cdot\partial f$. An explicit calculation of $J^\mu_{(1)}$ and $T^{\mu\nu}_{(1)}$ shows that this expression contributes terms proportional to u^μ and $u^\mu u^\nu$, respectively, besides normal terms orthogonal to u^μ. These terms are quite undesirable in view of the Landau and Lifshitz matching conditions (6.2) but are the only one of this structure. The only possibility left to eliminate these undesirable contributions is by way of choosing $b(p)$ so that the same structure as the first term in (6.33) be reproduced; this occurs when

$$b(p)=-\tau_s\frac{m}{(p\cdot u)^2}\ . \tag{6.36}$$

Other undesirable terms, proportional to u^μ or $u^\mu u^\nu$, come from the remaining parts of the second term, i.e., from

(i) $\dfrac{\tau_s}{p\cdot u}p\cdot\partial(\mathcal{P})f_{\mathrm{eq}}$

and

(ii) $\tau_s\dfrac{n_\lambda}{(p\cdot u)^2}p\cdot\partial\{u^{[\lambda}n^{\alpha]}p_\alpha\}\mathcal{P}f_{\mathrm{eq}}$.

From (i) one finds

$$\tau_s\frac{m^3}{2\pi^2}i_{21}\dot{\mathcal{P}}u^\mu \quad (\text{in } J^\mu_{(1)})\ , \tag{6.37a}$$

$$\tau_2\frac{m^4}{2\pi^2}i_{22}\dot{\mathcal{P}}u^\mu u^\nu \quad (\text{in } T^{\mu\nu}_{(1)}) \tag{6.37b}$$

while from (ii) one gets

$$-\tau_s\frac{m^3}{6\pi^2}i_{4-1}\mathcal{P}(\theta+n\cdot u')u^\mu \quad (\text{in } J^\mu_{(1)})\ , \tag{6.38a}$$

$$-\tau_s\frac{m^4}{6\pi^2}i_{40}\mathcal{P}(\theta+n\cdot u')u^\mu u^\nu \quad (\text{in } T^{\mu\nu}_{(1)})\ . \tag{6.38b}$$

From Eq. (6.12), which stems from the conservation of angular momentum, it is found that, in fact, these terms have the same general structure and can be grouped as

$$\tau_s\frac{m^3}{2\pi^2}\left[i_{21}-\frac{i_{4-1}}{3}\right]\dot{\mathcal{P}}u^\mu \quad (\text{in } J^\mu_{(1)}) \tag{6.39}$$

and a similar expression in $T^{\mu\nu}_{(1)}$.

Let us now consider the third term of Eq. (6.33), i.e., the $c(p)$ term. An explicit calculation shows that its contribution along u^μ has exactly the same structure as the undesirable extra term (6.39). Accordingly, both terms can be chosen so as to cancel each other and, to this end, it appears necessary that

$$\frac{1}{15m^2}\int d^4p\begin{Bmatrix}(p\cdot u)\\(p\cdot u)^2\end{Bmatrix}[\Delta(u)\cdot p\cdot p]^2c(p) = \begin{cases}\tau_s\dfrac{m^3}{2\pi^2}\left[i_{21}-\dfrac{i_{4-1}}{3}\right] & (\text{for } J^\mu_{(1)})\ ,\\[2ex] \tau_s\dfrac{m^4}{2\pi^2}\left[i_{22}-\dfrac{i_{40}}{3}\right] & (\text{for } T^{\mu\nu}_{(1)})\ .\end{cases} \tag{6.40}$$

A simple *choice* for $c(p)$ that allows these conditions to be obeyed is given by

$$c(p)=-15\tau_s\frac{\Omega^{\mu\nu}p_\mu p_\nu}{(\Delta^{\mu\nu}(u)p_\mu p_\nu)^2}\frac{m}{(p\cdot u)^2}\ , \tag{6.41}$$

where

$$\Omega^{\mu\nu}\equiv u^\mu u^\nu-n^\mu n^\nu\ .$$

The fourth term also brings contributions in u^μ (for the four-current) or in $u^\mu u^\nu$ (for the energy-momentum tensor) although with a very different structure; in the four-current, for instance, it is proportional to

$$\mathcal{P}\epsilon^{\lambda\nu\alpha\beta}u_\lambda n_\nu\partial_\alpha u_\beta\cdot u^\mu\ .$$

This has no counterpart elsewhere. It seems therefore that one should choose $d(p)\equiv 0$.

In fact, in the local rest frame of the system such a term is proportional to

$$\mathcal{P}(\mathbf{\Delta}\wedge\mathbf{u})\cdot\mathbf{n}\ ,$$

so that it vanishes when the local equilibrium three-velocity is *irrotational.* In such a case, it is not necessary to impose $d(p)=0$. More generally, it seems that the condition $(\mathbf{\nabla}\wedge\mathbf{u})\cdot\mathbf{n}=0$ can be obeyed in a great number of physical cases.

Finally, the *simplest* possible form for δf can be taken as

$$\delta f=-\tau\frac{p\cdot\partial}{p\cdot u}f_{\rm eq}-\tau_s\frac{m}{(p\cdot u)^2}\left[n_\lambda p\cdot\partial f^\lambda_{5\rm eq}+15\frac{\Omega^{\mu\nu}p_\mu p_\nu}{(\Delta\cdot p\cdot p)^2}p\cdot n\,\pi^{\mu\nu}p_\mu p\cdot\partial f^{\rm eq}_{5\nu}\right], \tag{6.42}$$

and it depends on two relaxation times τ and τ_s.

Transport coefficients

The calculation of the first-order correction $J^\mu_{(1)}$ for the four-current is straightforward and results from the integration of Eq. (6.42)

$$J^\mu_{(1)}=\int d^4p\frac{p^\mu}{m}\delta f \tag{6.43}$$

and from the use of the conservation relations for local equilibrium. The basic manipulations can be found, e.g., in Ref. [12] or in Ref. [20] and are therefore not repeated here. One finds

$$\begin{aligned}J^\mu_{(1)}=&[\tau-\tau_s\mathcal{P}]\frac{m^4}{6\pi^2}\left[\frac{I_{41}}{I_{40}}I_{4-1}-I_{40}\right]\Delta^{\mu\nu}[\partial_\nu(\beta)+\beta\dot u_\nu]\\&+\tau_s\frac{m^3}{2\pi^2}\left\{-\frac{i_{4-1}}{3}[\Delta^{\mu\nu}\partial_\nu(\mathcal{P})+\mathcal{P}\pi^{\mu\nu}\dot u_\nu]-\left[i_{21}-\frac{i_{4-1}}{3}\right]\mathcal{P}[\pi^{\mu\nu}\dot u_\nu+\pi^{\mu\nu}n'_\nu+n^\mu\pi^{\alpha\beta}\partial_\alpha n_\beta]\right\}.\end{aligned} \tag{6.44}$$

This expression obeys the first Landau-Lifshitz condition, and reduces to the expression first given by Anderson and Witting when the system is unpolarized, $\mathcal{P}\equiv 0$, as expected. The first term is the usual heat conduction term with, however, a slight modification: one now has an "effective" relaxation time $[\tau\to\tau-\tau_s\mathcal{P}]$ that depends on the polarization of the system.

Among the polarization-dependent terms one also notices the expression $\Delta^{\mu\nu}\partial_\nu(\mathcal{P})$, which is easily identified with the gradient of parallel polarization $\nabla(M_3)$ present in the classical derivation performed by, e.g., Lhuillier and Laloë [29]

$$\Delta^{\mu\nu}\partial_\nu(\mathcal{P})=-n_\alpha\Delta^{\mu\nu}\partial_\nu(\mathcal{P}n^\alpha). \tag{6.45}$$

The other terms do not appear in the nonrelativistic description. There is some ambiguity about the proper way of grouping them together.

The decomposition chosen in (6.44) has the advantage of reducing the number of transport coefficients to three, the first being the usual thermal conductivity, the second being the coefficient of the gradient of parallel polarization plus another term quite similar to Eckart's [33]. The third contribution to $J^\mu_{(1)}$ may be rewritten as

$$-\tau_s\frac{m^3}{2\pi^2}\left[i_{21}-\frac{i_{4-1}}{3}\right][n^{(\mu}\pi^{\alpha)\beta}\partial_\alpha(\mathcal{P}n_\beta)+\mathcal{P}\pi^{\mu\alpha}\dot u_\alpha] \tag{6.46}$$

and interpreted as a cross effect between transverse and parallel gradients of the polarization, plus another Eckart term [33].

Let us now turn to the energy-momentum tensor; from the integration of Eq. (6.42) one finds

$$\begin{aligned}T^{\mu\nu}_{(1)}=&\int d^4p\frac{p^\mu p^\nu}{m}\delta f\\=&\{\eta_0W_0^{\mu\nu}+\eta_1W_1^{\mu\nu}+\eta_2W_2^{\mu\nu}+\bar\eta\bar W^{\mu\nu}\}+\left\{\zeta\frac{\theta}{3}\Delta^{\mu\nu}\right\}+\{vV^{\mu\nu}\}\\&+u^{(\mu}\{c_1[\Delta^{\nu)\lambda}\partial_\lambda(\mathcal{P})+\mathcal{P}\pi^{\nu)\lambda}\dot u_\lambda]+c_2[n^{\nu)}\pi^{\alpha\beta}\partial_\alpha(\mathcal{P}n_\beta)+\pi^{\nu)\alpha}\dot u_\alpha]\}.\end{aligned} \tag{6.47}$$

The first line contains the shear viscosity effects. The shear viscosity is here split into three components η_0,η_1,η_2 and a cross effect between shear and bulk viscosity $\bar\eta$. The viscous stress tensor has been decomposed on a basis of orthogonal tensors $W_i^{\mu\nu}$ which are the covariant generalization [34] of Braginskii's [35] tensors, derived elsewhere

$$W_0^{\mu\nu}=\tfrac{3}{2}[n^\mu n^\nu+\tfrac13\Delta^{\mu\nu}][n^\alpha n^\beta+\tfrac13\Delta^{\alpha\beta}]2\sigma_{\alpha\beta} \tag{6.48a}$$

$$=[2n^\mu n^\nu+\pi^{\mu\nu}][n\cdot u'+\tfrac13\theta], \tag{6.48b}$$

$$W_1^{\mu\nu}=[\pi^{\mu\alpha}\pi^{\nu\beta}-\tfrac12\pi^{\mu\nu}n^\alpha n^\beta]2\sigma_{\alpha\beta}, \tag{6.49}$$

$$W_2^{\mu\nu} = -[\pi^{\mu\alpha} n^\nu n^\beta + \pi^{\nu\beta} n^\mu n^\alpha] 2\sigma_{\alpha\beta} \tag{6.50a}$$

$$= \pi^{(\mu\alpha} \dot{n}_\alpha n^{\nu)} - \pi^{(\mu\alpha} u'_\alpha n^{\nu)} , \tag{6.50b}$$

$$\overline{W}^{\mu\nu} = [\pi^{\mu\nu} + 2n^\mu n^\nu]\theta . \tag{6.51}$$

$2\sigma^{\alpha\beta}$ is the symmetric traceless shear tensor:

$$2\sigma^{\alpha\beta} = \partial^\alpha u^\beta + \partial^\beta u^\alpha - \dot{u}^\alpha u^\beta - u^\alpha \dot{u}^\beta - \tfrac{2}{3}\theta \Delta^{\alpha\beta} \tag{6.52}$$

with the property

$$2\sigma^{\mu\nu} = W_0^{\mu\nu} + W_1^{\mu\nu} + W_2^{\mu\nu} . \tag{6.53}$$

The shear viscosities then read

$$\eta_0 = (\tau - \tau_s \mathcal{P}) \frac{m^4}{2\pi^2} \gamma \frac{I_{6-1}}{15} + \tau_s \frac{m^4}{2\pi^2} \mathcal{P} \frac{4i_{6-2} - 5i_{40}}{105} , \tag{6.54a}$$

$$\eta_1 = (\tau - \tau_s \mathcal{P}) \frac{m^4}{2\pi^2} \gamma \frac{I_{6-1}}{15} + \tau_s \frac{m^4}{2\pi^2} \frac{15i_{40} + 2i_{6-2}}{105} , \tag{6.54b}$$

$$\eta_2 = (\tau - \tau_s \mathcal{P}) \frac{m^4}{2\pi^2} \gamma \frac{I_{6-1}}{15} + \tau_s \frac{m^4}{2\pi^2} \mathcal{P} \frac{80i_{40} - 105i_{22} - 8i_{6-2}}{210} , \tag{6.54c}$$

$$\overline{\eta} = \tau_s \frac{m^4}{2\pi^2} \mathcal{P} \frac{8i_{6-2} - 10i_{40}}{105} . \tag{6.54d}$$

The bulk viscosity can be picked up in the second line of $T_{(1)}^{\mu\nu}$

$$\zeta = (\tau - \tau_s \mathcal{P}) \frac{m^4}{6\pi^2} \gamma \left[\frac{I_{23} I_{40}^2 + I_{21} I_{41}^2 - 2I_{22} I_{40} I_{41}}{I_{22}^2 - I_{21} I_{23}} + I_{6-1} \right] \tag{6.55}$$

and has the same structure as in the nonpolarized case if one replaces the relaxation time by a polarization-dependent one, $\tau \rightarrow \tau_s \mathcal{P}$. From the vorticity tensor $V^{\mu\nu}$,

$$V^{\mu\nu} = \pi^{\alpha[\mu} n^{\nu]} n^\beta [\partial_\alpha(u_\beta) - \partial_\beta(u_\alpha)] , \tag{6.56}$$

one obtains the new coefficient

$$\upsilon = \tau_s \frac{m^4}{2\pi^2} \frac{105i_{22} + 10i_{40} - 8i_{6-2}}{210} \tag{6.57}$$

and this term may constitute a new dissipative effect.

Finally, in the third line of Eq. (6.47) is a contribution to the heat current whose structure is similar to that of the polarization part of $u^{(\mu} J_{(1)}^{\nu)}$. The coefficients c_1 and c_2 are given by

$$c_1 = -\tau_s \frac{m^4}{6\pi^2} \mathcal{P} i_{40} , \tag{6.58a}$$

$$c_2 = -\tau_s \frac{m^4}{2\pi^2} \mathcal{P} \left[i_{22} - \frac{i_{40}}{3} \right] . \tag{6.58b}$$

The result (6.44) and (6.47), schematically rewritten as

$$J^\mu = n u^\mu + K X^\mu + q_1^\mu , \tag{6.59a}$$

$$T^{\mu\nu} = (\rho + p) u^\mu u^\nu - P \eta^{\mu\nu} + u^{(\mu} q_2^{\nu)} + \Sigma^{\mu\nu} + \zeta \frac{\theta}{3} \Delta^{\mu\nu} \tag{6.59b}$$

(with

$$X^\mu = \Delta^{\mu\lambda} \left[\partial_\lambda \left[\frac{1}{\beta} \right] - \frac{\dot{u}_\lambda}{\beta} \right] , \tag{6.60}$$

q_1^μ and q_2^ν are the polarization contributions to the heat current, $\Sigma^{\mu\nu}$ is the viscous stress tensor), may be put in either Eckart form

$$J_E^\mu = n U_E^\mu , \tag{6.61a}$$

$$T_E^{\mu\nu} = (\rho + P) U_E^\mu U_E^\nu - P \eta^{\mu\nu} + U_E^{(\nu} Q_E^{\nu)} + \Sigma^{\mu\nu} + \zeta \frac{\theta}{3} \Delta^{\mu\nu} \tag{6.61b}$$

or in Landau and Lifshitz form

$$J_L^\mu = n U_L^\mu + Q_L^\mu , \tag{6.62a}$$

$$T_L^{\mu\nu} = (\rho + P) U_L^\mu U_L^\nu - P \eta^{\mu\nu} + \Sigma^{\mu\nu} + \zeta \frac{\theta}{3} \Delta^{\mu\nu} \tag{6.62b}$$

with a $O(\epsilon)$ redefinition of u^μ, respectively,

$$U_E^\mu = u^\mu + \frac{K}{n} X^\mu + \frac{q_1^\mu}{n} , \tag{6.63a}$$

$$U_L^\mu = u^\mu + \frac{1}{\rho + P} q_2^\mu . \tag{6.63b}$$

Determination of $\delta f_5^\mu = f_{5(1)\epsilon}^\mu$

The most general possible form for δf_5^μ is

$$\delta f_5^\mu = C^\mu p \cdot \partial f + D^{\mu\lambda} p \cdot \partial f_{5\lambda} \tag{6.64}$$

where C^μ and $D^{\mu\lambda}$ *a priori* involve twelve arbitrary functions of p. In order to reduce this number and obtain a nontrivial minimal form, no such compelling conditions as the Landau and Lifshitz conditions in the case of δf can be imposed: however, three such conditions on $\mathcal{P}$ and n^μ could exist; this question is addressed at the end of this subsection.

It is known from classical [29] calculations that the heat current and parallel component of the spin current undergo a coupled relaxation, while the transverse components of the spin current precess around the polarization axis and, of course, relax toward their equilibrium value.

These last features, when written in a noncovariant form, then lead to the following system for the various components of δf_5^μ:

$$\delta f_5^1 = -\frac{\tau_\perp}{p \cdot u} p \cdot \partial f_5^1 + \frac{\tau_\times}{p \cdot u} p \cdot \partial f_5^2 , \tag{6.65a}$$

$$\delta f_5^2 = -\frac{\tau_\times}{p \cdot u} p \cdot \partial f_5^1 - \frac{\tau_\perp}{p \cdot u} p \cdot \partial f_5^2 , \tag{6.65b}$$

$$\delta f_5 = -\frac{\tilde{\tau}}{p \cdot u} p \cdot \partial f - \frac{\tilde{\tau}_s}{p \cdot u} p \cdot \partial f_5^3 \tag{6.65c}$$

or, covariantly (and keeping in mind that $p_\mu \delta f_5^\mu = 0 = p_\mu p\cdot\partial f_5^\mu$ has to be verified)

$$\delta f_5^\mu = -\frac{\tau_\perp}{p\cdot u}\frac{\eta^{\mu\rho}p\cdot u - p^\rho u^\mu}{p\cdot u}\pi_{\rho\nu}p\cdot\partial f_{5\mathrm{eq}}^\nu - \frac{\tau_\times}{p\cdot u}\frac{\eta^{\mu\rho}p\cdot u - p^\rho u^\mu}{p\cdot u}\epsilon_{\rho\nu\alpha\beta}u^\alpha\eta^\beta p\cdot\partial f_{5\mathrm{eq}}^\nu - \tilde{\tau}\frac{u^{[\mu}n^{\nu]}p_\nu}{(p\cdot u)^2}p\cdot\partial f_{\mathrm{eq}} - \tilde{\tau}_s\frac{u^{[\mu}n^{\nu]}p_\nu}{(p\cdot u)^2}n_\lambda p\cdot\partial f_5^\lambda \; . \tag{6.66}$$

It should be noticed that the expression chosen in (6.66) allows for different relaxation times for the parallel and transverse components, as has been pointed out by Jeon and Mullin [36] or Dominguez Tenreiro and Hakim [34].

Furthermore, it is not very difficult to implement one matching condition for the polarization. Requiring that the definition of the polarization $\mathcal{P}$ should not be modified at first order in ϵ, i.e., that $\mathcal{P}_{(1)} \equiv 0$, demands that $n_\mu \int d^4p\, \partial f_5^\mu \equiv 0$. From a derivation similar to the one previously used, it turns out that this condition is obeyed provided δf_5^μ is of the following form

$$\delta f_5^\mu = \left[-\tilde{\tau}\frac{p\cdot u}{m^2}p\cdot\partial f_{\mathrm{eq}} - \frac{\tilde{\tau}_s}{m}\left[n_\lambda p\cdot\partial f_{5\mathrm{eq}}^\lambda + 15\frac{\Omega pp}{(\Delta pp)^2}p\cdot n\,\pi^{\alpha\beta}p_\alpha p\cdot\partial f_{5\beta}^{\mathrm{eq}} \right] \right] \frac{u^{[\mu}n^{\nu]}p_\nu}{p\cdot u} - \frac{\tau_\perp}{p\cdot u}\frac{\eta^{\mu\rho}p\cdot u - p^\rho u^\mu}{p\cdot u}\pi_{\rho\nu}p\cdot\partial f_{5\mathrm{eq}}^\nu - \frac{\tau_\times}{p\cdot u}\frac{\eta^{\mu\rho}p\cdot u - p^\rho u^\mu}{p\cdot u}\epsilon_{\rho\nu\alpha\beta}u^\alpha n^\beta p\cdot\partial f_{5\mathrm{eq}}^\nu \; . \tag{6.67}$$

With the choices (6.66) or (6.67), one finds, respectively,

$$\begin{aligned}\int d^4p\;\delta f_5^\mu = & -\tau_\perp\frac{m^3}{2\pi^2}\left[i_{21}\mathcal{P}\pi^{\mu\nu}\dot{n}_\nu + \frac{i_{4-1}}{3}\mathcal{P}(\pi^{\mu\nu}u'_\nu + \pi^{\alpha\beta}\partial_\alpha n_\beta u^\mu)\right] \\ & -\tau_\times\frac{m^3}{2\pi^2}\left[i_{21}\mathcal{P}\epsilon^{\mu\nu\alpha\beta}u_\nu n_\alpha \dot{n}_\beta + \frac{i_{4-1}}{3}\mathcal{P}\epsilon^{\mu\nu\alpha\beta}u_\nu n_\alpha u'_\beta\right] \\ & +\tilde{\tau}\frac{m^3}{2\pi^2}\left[\left[I_{20}\dot{\alpha} - I_{21}\dot{\gamma} + \gamma\frac{I_{4-1}}{3}\right] n^\mu + \left[I_{4-2}\frac{I_{41}}{I_{40}} - I_{4-1}\right]\frac{1}{3}n^\lambda[\partial_\lambda(\gamma) + \gamma\dot{u}_\lambda]u^\mu\right] \\ & +\tilde{\tau}_s\frac{m^3}{2\pi^2}\left[\left[\frac{i_{4-1}}{3} - i_{21}\right]\dot{\mathcal{P}}n^\mu - \frac{i_{4-1}}{3}u^\mu n^\lambda\partial_\lambda(\mathcal{P}) \right. \\ & \left. - \left[I_{4-1}\frac{I_{41}}{I_{40}} - I_{40}\right]\frac{1}{3}n^\lambda[\partial_\lambda(\gamma) + \gamma\dot{u}_\lambda]u^\mu \right] \quad \text{[from Eq. (6.66)]}\end{aligned} \tag{6.68}$$

or

$$\begin{aligned}\int d^4p\;\delta f_5^\mu = & -\tau_\perp\frac{m^3}{2\pi^2}\left[i_{21}\mathcal{P}\pi^{\mu\nu}\dot{n}_\nu + \frac{i_{4-1}}{3}\mathcal{P}(\pi^{\mu\nu}u'_\nu + \pi^{\alpha\beta}\partial_\alpha n_\beta u^\mu)\right] \\ & -\tau_\times\frac{m^3}{2\pi^2}\left[i_{21}\mathcal{P}\epsilon^{\mu\nu\alpha\beta}u_\nu n_\alpha \dot{n}_\beta + \frac{i_{4-1}}{3}\mathcal{P}\epsilon^{\mu\nu\alpha\beta}u_\nu n_\alpha u'_\beta\right] \\ & +\tilde{\tau}_s\frac{m^3}{2\pi^2}\left[-\frac{i_{40}}{3}\mathcal{P}'u^\mu + \left[i_{22} - \frac{i_{40}}{3}\right]\mathcal{P}\pi^{\alpha\beta}\partial_\alpha n_\beta u^\mu\right] \quad \text{[from Eq. (6.67)]} \; .\end{aligned} \tag{6.69}$$

It follows that the off-equilibrium part of the spin tensor reads

$$\delta S^{\mu\nu\lambda} = -\frac{1}{2}\epsilon^{\mu\nu\lambda\rho}\int d^4p\;\delta f_{5\rho}$$

or, from Eq. (6.69),

$$\delta S^{\mu\nu\lambda}=\tau_\perp\frac{m^3}{4\pi^2}\left[i_{21}\mathcal{P}u^{[\mu}n^{\nu}F^{\lambda]\alpha}\dot{n}_\alpha+\frac{i_{4-1}}{3}\mathcal{P}u^{[\mu}n^{\nu}F^{\lambda]\alpha}u'_\alpha-\frac{i_{4-1}}{3}\mathcal{P}\pi^{\alpha\beta}\partial_\alpha n_\beta F^{[\mu\nu}n^{\lambda]}\right]$$

$$+\tau_\times\frac{m^3}{4\pi^2}\left[i_{21}\mathcal{P}u^{[\mu}n^{\nu}\pi^{\lambda]\alpha}\dot{n}_\alpha+\frac{i_{4-1}}{3}\mathcal{P}u^{[\mu}n^{\nu}\pi^{\lambda]\alpha}u'_\alpha\right]$$

$$+\tilde{\tau}\frac{m^3}{4\pi^2}\left[\left[I_{20}\dot{\alpha}-I_{21}\dot{\gamma}+\gamma\frac{I_{4-1}}{3}\theta\right]F^{[\mu\nu}u^{\lambda]}+\frac{1}{3}\left[\frac{I_{41}}{I_{40}}I_{4-2}-I_{4-1}\right]n^\alpha[\partial_\alpha(\gamma)+\gamma\dot{u}_\alpha]F^{[\mu\nu}n^{\lambda]}\right]$$

$$+\tilde{\tau}_s\frac{m^3}{4\pi^2}\left[\left[\frac{i_{4-1}}{3}-i_{21}\right]\dot{\mathcal{P}}F^{[\mu\nu}u^{\lambda]}-\frac{i_{4-1}}{3}\mathcal{P}'F^{[\mu\nu}n^{\lambda]}-\frac{1}{3}\left[\frac{I_{41}}{I_{40}}I_{4-1}-I_{40}\right]n^\alpha[\partial_\alpha(\gamma)+\gamma\dot{u}_\alpha]F^{[\mu\nu}n^{\lambda]}\right] \tag{6.70}$$

and from Eq. (6.67)

$$\delta S^{\mu\nu\lambda}=\tau_\perp[\text{same as in Eq. (6.70)}]+\tau_\times[\text{same as in Eq. (6.70)}]$$

$$+\tilde{\tau}_s\frac{m^3}{4\pi^2}\left[-\frac{i_{40}}{3}\mathcal{P}F^{[\mu\nu}n^{\lambda]}+\left[i_{22}-\frac{i_{40}}{3}\right]\mathcal{P}\pi^{\alpha\beta}\partial_\alpha n_\beta F^{[\mu\nu}n^{\lambda]}\right] \tag{6.71}$$

where $a^{[\alpha}b^{\beta}c^{\delta]}$ represents the antisymmetrized combination

$$a^{[\alpha}b^{\beta}c^{\delta]}=a^\alpha b^\beta c^\delta-a^\alpha b^\delta c^\beta+a^\delta b^\alpha c^\beta-a^\delta b^\beta c^\alpha+a^\beta b^\delta c^\alpha-a^\beta b^\alpha c^\delta\ .$$

$\delta S^{\mu\nu\lambda}$ may also be rewritten in a more transparent form which is here exemplified for Eq. (6.71)

$$\delta S^{\mu\nu\lambda}=-\frac{1}{2}\int d^4p(p^{[\lambda}/m)\delta f^{\mu\nu]}\ , \tag{6.72}$$

where relations (5.50) were used to transform δf_5^μ into $\delta f^{\mu\nu}$. This last form is more like what one would expect for the current of a quantity described by the distribution function $f^{\mu\nu}$.

$\delta S^{\mu\nu\delta}$ is rewritten in terms of the gradients of a polarization tensor $\mathcal{P}^*F^{\mu\nu}$ as [from Eq. (6.67)]

$$\delta S^{\mu\nu\lambda}=-\tau_\perp\frac{m^3}{4\pi^2}\left[i_{21}F^{[\mu\alpha}{}^*F^{\nu\beta}u^{\lambda]}u^\gamma\partial_\gamma(\mathcal{P}^*F_{\alpha\beta})+\frac{i_{4-1}}{3}[F^{[\mu\alpha}{}^*F^{\nu\beta}\Delta^{\lambda]\gamma}\partial_\gamma(\mathcal{P}^*F_{\alpha\beta})-F^{[\mu\nu}u^{\lambda]}n^\alpha\partial_\beta(\mathcal{P}^*F_{\alpha\beta})]\right]$$

$$-\tau_\times\frac{m^3}{4\pi^2}\left[i_{21}\pi^{[\mu\alpha}{}^*F^{\nu\beta}u^{\lambda]}u^\gamma\partial_\gamma(\mathcal{P}^*F_{\alpha\beta})+\frac{i_{4-1}}{3}[\pi^{[\mu\alpha}{}^*F^{\nu\beta}\Delta^{\lambda]\gamma}\partial_\gamma(\mathcal{P}^*F_{\alpha\beta})+F^{[\mu\nu}u^{\lambda]}n^\alpha F^{\beta\gamma}\partial_\gamma(\mathcal{P}^*F_{\alpha\beta})]\right]$$

$$+\tilde{\tau}_s\frac{m^3}{4\pi^2}\left[\frac{i_{22}}{2}F^{[\mu\nu}{}^*F^{\alpha\beta}\Delta^{\lambda]\gamma}\partial_\gamma(\mathcal{P}^*F_{\alpha\beta})+\left[i_{22}-\frac{1}{3}i_{40}\right]F^{[\mu\nu}n^{\lambda]}u^\alpha\partial^\beta(\mathcal{P}^*F_{\alpha\beta})\right]\ . \tag{6.73}$$

A final remark

Finally, when δf and δf_5^μ are known, one can go back to the transport equations by solving a linear system. For instance, if one chooses δf as given by Eq. (6.42) and δf_5^μ as in Eq. (6.70), then the transport equations turn out to be

$$p\cdot\partial f=-p\cdot u\left[\frac{\tilde{\tau}_s}{\tau\tilde{\tau}_s-\tilde{\tau}\tau_s}\right]\delta f+\frac{m}{(p\cdot u)^2}\left[\frac{\tau_s}{\tau\tilde{\tau}_s-\tilde{\tau}\tau_s}\right]n_\mu\delta f_5^\mu\ , \tag{6.74}$$

$$p\cdot\partial f_5^\mu=[\eta^\mu{}_\rho p\cdot u-p_\rho u^\mu]\left[-\frac{\tau_\perp}{\tau_\perp^2+\tau_\times^2}\pi^{\rho\nu}\delta f_{5\nu}+\frac{\tau_\times}{\tau_\perp^2+\tau_\times^2}F^{\rho\nu}\delta f_{5\nu}\right]$$

$$+u^{[\mu}n^{\alpha]}p_\alpha\left[\frac{pu}{m}\frac{\tilde{\tau}}{\tau\tilde{\tau}_s-\tilde{\tau}\tau_s}\delta f-\frac{m}{pu}\frac{\tau}{\tau\tilde{\tau}_s-\tilde{\tau}\tau_s}n_\lambda\delta f_5^\lambda\right.$$

$$\left.+15\frac{\Omega\cdot p\cdot p}{(\Delta\cdot p\cdot p)^2}p\cdot n\frac{\tau_\perp}{\tau_\perp^2+\tau_\times^2}\pi^{\alpha\beta}p_\alpha\delta f_{5\beta}-15\frac{\Omega\cdot p\cdot p}{(\Delta\cdot p\cdot p)^2}p\cdot n\frac{\tau_\times}{\tau_\perp^2+\tau_\times^2}F^{\alpha\beta}p_\alpha\delta f_{5\beta}\right] \tag{6.75}$$

with $F^{\alpha\beta} \equiv \epsilon^{\alpha\beta\mu\nu} u_\mu n_\nu$. Note that these transport equations are valid at order $O(\epsilon)$ only. They may, however, be studied *per se* as if they were true at any order.

VII. REMARKS AND DISCUSSION

Let us now summarize and discuss further the previous results as to our relativistic quantum kinetic equation and the subsequent transport properties of the system it describes. This kinetic equation is the generalization of the well-known Bhatnagar-Gross-Krook equation [28], i.e., of the relaxation time approximation, written for spin-$\frac{1}{2}$ fermions. The collision term involves (i) an equilibrium Wigner function for a *polarized* system and (ii) arbitrary functions of p occurring *via* the possible scalars, four-vectors, antisymmetric tensors, pseudo-four-vectors and pseudoscalars that can be constructed from those at our disposal. Next a solution of the kinetic equation was obtained in the Chapman-Enskog approximation and it was realized that *two* expansion parameters were necessary to this end; the first one, ϵ, is the usual parameter while the new one, η, occurs as a consequence of the advent of a new length scale determined by the Compton wavelength of the fermions. Then the Landau-Lifshitz matching conditions were used to simplify further the first-order solution and, as a consequence, the main transport properties (thermal conductivity, viscosities) of the system were derived as functions of the usual macroscopic parameters (temperature, density) *and* of various relaxation times, and of the polarization. On the other hand, what might be called for brevity the "spin part" or the "polarization part" of the solution could not be constrained on the basis of matching conditions and was determined only through physical arguments involving, e.g., spin precession, spin diffusion, cross effects, etc. All these questions have to be taken up and discussed.

(1) The "equilibrium" Wigner function of a polarized system of spin-$\frac{1}{2}$ fermions was shown to be of the general form (2.25)

$$F_{\text{eq}}(p) = \frac{\gamma \cdot p + m}{2m} \frac{1 + \gamma_5 \gamma^\mu S_\mu(p)}{2} f_{\text{eq}}(p) ,$$

where $S_\mu(p)$ is such that $p \cdot S(p) = 0$ and was determined from the density operator. However, there exists some flexibility in the choice of $S_\mu(p)$. The *simplest* possibility (2.28) can be replaced by the more general one

$$N^\mu(p) = \frac{v^{[\mu} n^{\nu]} p_\nu}{\{v^{[\alpha} n^{\beta]} p_\beta v_{[\alpha} n_{\lambda]} p^\lambda\}^{1/2}} ,$$

where v^μ is a unit timelike four vector, orthogonal to n^μ but *a priori* different from u^μ. This more general choice shows that $N^\mu(p)$ depends on *five* arbitrary constants (the five independent components of v^μ and n^μ) and hence $S_\mu(p)$ does depend on *six*. This is, in fact, quite satisfactory since the *macroscopic* spin tensor $M^{\mu\nu}$ *does* indeed possess six independent components. Note that expressions similar to Eqs. (2.30)–(2.35) can be obtained quite easily in this general case. However, such a choice would have made all our equations quite involved.

It should also be noticed that the complexity of, and the arbitrariness in, the description of a polarized system is specific to relativity: the nonuniqueness of timelike vectors (unlike the Newtonian case where all timelike "four-vectors" are parallel) leaves a freedom in the choice of v^μ while the orthogonality of p^μ and $S^\mu(p)$, the spin four-vector, gives rise to a more complex description than in the nonrelativistic case.

Also, in the absence of a magnetic field, polarized matter can only be in a *metastable* state so that it was implicitly assumed that the system relaxes towards a true equilibrium state in a time much longer than all other times under consideration in the problem. Had this assumption not been used then, in most cases, the Anderson-Witting results would have essentially been obtained: it is thus necessary to go over to *nonlinear* terms (and thus beyond the relaxation time approximation) in order to get interesting spin effects (see below).

(2) From the tensors at our disposal, a quite general linear collision term was written and was somewhat reduced from the fact that F, the covariant Wigner function, obeys a more general equation that the kinetic one [i.e., Eqs. (5.1)]. This general equation involves, as remarked by many authors, both the mass-shell constraint and the kinetic equation. In particular, the collision term was greatly simplified on the following basis: between collisions particles are free; this is a quite natural assumption valid as long as the system is not too dense and also in the absence of external fields whose action would modify the free motion between collisions. It should also be stressed, at this point, that *collective* effects are not considered in this paper; for instance, the motion of quasifermions between collisions is not "free" but occurs according to a dispersion equation of the general form [23]

$$\text{Det}\{\gamma \cdot p - \Sigma(p)\} = 0 .$$

It is clear that our assumption is an oversimplification which should be relaxed in the study of many interesting physical cases. Note, however, that such a simplification is common to all works presently published on the subject.

(3) Among the various conditions imposed on our collision term, it was demanded that it be so that a consistent Chapman-Enskog expansion be possible. Of course, other possible approximation schemes could have been used but such an expansion recommends itself. Other possibilities, such as the fourteen moments method or variational methods, could have also been imposed; however, it is not sure at all that new constraints would have been obtained and this way has not been explored.

The relativistic quantum Chapman-Enskog expansion was performed under the assumption that all relaxation times occurring in the collision term are of the same order of magnitude except that, as was noted above, the relaxation time of polarization was taken to be infinite. First-order formal expressions for the Wigner functions were obtained, in particular for $f_{(1)\epsilon}$, which was shown to be sufficient to derive the off-equilibrium part of the four-current and of the energy-momentum tensor. These expressions are quite sufficient when the parameter η is negligible, which property is valid whenever the effective mass of the fermions is large enough as $\eta \ll \epsilon$; this occurs, e.g., when one considers a nucleon on its mass shell, its effective mass being very close to its usual mass. The expression for $f_{(1)\eta}$ was obtained only in the "simple" case where the collision term is of the form $MF_{(1)}N$ (Sec. IV). When $\eta \ll \epsilon$ the Wigner function $f^{\mu}_{(1)\epsilon}$ had the customary form $(p^{\mu}/m)f_{(1)\epsilon}$ and the energy-momentum tensor was thus symmetric. As a consequence, the spin-density tensor $S^{\mu\nu\lambda}$, a nonconserved quantity, appeared to be conserved at order $O(\epsilon)$.

(4) The Landau-Lifshitz matching conditions were used to get a more specific form for $f_{(1)\epsilon}$. While $f_{(1)\epsilon}$ was completely determined from these conditions (up to three arbitrary relaxation times to be provided by a dynamical and/or statistical model), it should be noticed that this has been obtained by very strong demands and, in fact, a less stringent way to implement them would certainly lead to a more arbitrary $f_{(1)\epsilon}$. It should also be remarked that the usual (five) Landau-Lifshitz conditions

$$u_{\mu}J^{\mu}_{(1)\epsilon}=0, \quad u_{\mu}T^{\mu\nu}_{(1)\epsilon}=0 ,$$

have not all been used. Only the two following ones

$$u_{\mu}J^{\mu}_{(1)\epsilon}=0, \quad u_{\mu}u_{\nu}T^{\mu\nu}_{(1)\epsilon}=0 ,$$

have been incorporated in the solution, to which $\mathcal{P}_{(1)\epsilon}=0$ has been added. This means that the equilibrium quantities $n_{\text{eq}},\rho_{\text{eq}},\mathcal{P}_{\text{eq}}$ have been preserved. However, u^{μ} is left completely free and was fixed in the final result only *via* a change of the four-velocity of the form

$$u^{\mu} \rightarrow \tilde{u}^{\mu}+\xi^{\mu} ,$$

where ξ^{μ} is $O(\epsilon)$.

Several remarks are now in order as to the matching conditions. First the equilibrium Wigner function depends on five $[\beta,n_{\text{eq}},u^{\mu}]$ plus six $[\mathcal{P},v^{\mu},n^{\mu}]$ quantities and it seems that, *a priori,* eleven matching conditions must be imposed. In order to discuss this point let us consider a system of charged spin-$\frac{1}{2}$ fermions embedded in a strong magnetic field. Due to this magnetic field the system is polarized and the direction of the polarization vector is that of the magnetic field, say n^{μ}, in four-dimensional notation [21]. The conserved quantities (besides the charge four-current) are mainly the energy and the momentum along the magnetic-field axis. In the case of Eq. (3.1), this gave rise to the following matching conditions [26]

$$u_{\mu}J^{\mu}_{(1)}=0, u_{\mu}u_{\nu}T^{\mu\nu}_{(1)}=0, n_{\mu}u_{\nu}T^{\mu\nu}_{(1)}=0$$

(which are the analogues of the Landau-Lifshitz conditions), and no others. In particular, the polarization $\mathcal{P}$ is a function of the thermodynamic parameters n_{eq} and T and, of course, of the *external* magnetic field. Therefore, nothing particular had to be imposed on $\mathcal{P}$, which is not an independent state variable. Here, however, the situation is a little different. (i) The equilibrium Wigner function at hand does not correspond to a true thermodynamical equilibrium and hence the eleven (or eight if one identifies u^{μ} and v^{μ}) macroscopic quantities involved $[n_{\text{eq}},T,\mathcal{P},u^{\mu},n^{\mu}]$ should lead to eleven matching conditions expressing, in particular, the conservation of the general form of the equations of state. (ii) Unlike the strong magnetic field case, all three components of the three-momentum are conserved in collisions. Consequently, eleven (or eight) matching conditions have actually to be imposed or, besides the five usual ones, six (or three) more relations have to be satisfied. Unfortunately, there is no compelling reason why such or such condition should be imposed corresponding to the six new quantities connected with the macroscopic spin tensor. For instance, imposing the same form for the equation of state of the system even though slightly off equilibrium does not provide anything new. The equation of state $P=P(n_{\text{eq}},T)$ is the usual free gas Fermi equation and does not depend on the polarization $\mathcal{P}$. Similarly, $\mathcal{P}$ does not depend on n_{eq} and T. Thus there exists a large number of possibilities for the remaining conditions. For instance, the following six relations

$$v^{\mu}=u^{\mu}, \quad \Delta^{\lambda}_{\mu}(u)M^{\mu}_{(1)\epsilon}\equiv\Delta^{\lambda}_{\mu}(u)(\mathcal{P}n^{\mu})_{(1)}$$

could tentatively be imposed. It would remain to check *a posteriori* that no contradiction appears or that no interesting physical effect is eliminated. For instance, it would be desirable that the precession of the polarization four-vector would not explicitly disappear.

(5) From the solution $f_{(1)\epsilon}$ the main physical quantities, i.e., the off-equilibrium parts of the four-current and of the energy-momentum tensor were calculated thus leading thereby to the transport coefficients we were looking for. As in the magnetic-field case studied elsewhere new terms were obtained which have to be discussed.

A glance at Eq. (6.44) for $J^{\mu}_{(1)}$ indicates that this quantity involves *three* terms instead of the first one only, in the nonpolarized case [12,13]. The first term is the heat conduction term with this only difference that it now involved an *effective* relaxation time which is polarization dependent. The second one contains the space gradient of the polarization and hence represents a spin diffusion effect. However the third term is more difficult to interpret and is connected with the spatial variation of the quantization axis. Nevertheless, the decomposition used for $J^{\mu}_{(1)}$ is by no means unique and we could use another one involving spatial gradients of the density instead of the $\dot{u}$ terms, using a relation of the general form

$$\pi^{\mu\nu}\dot{u}_{\nu}=A\pi^{\mu\nu}\partial_{\nu}n_{\text{eq}}+B\pi^{\mu\nu}(\partial_{\nu}\beta+\beta\dot{u}_{\nu}) ,$$

where A and B are known expressions. Doing so it turns out that the off-equilibrium part of the four-current reads

$$J^{\mu}_{(1)}=\lambda_{\perp}\pi^{\mu\lambda}[\partial_{\lambda}\beta+\beta\dot{u}_{\lambda}]+\lambda_{\parallel}n^{\mu}n^{\lambda}[\partial_{\lambda}\beta+\dot{u}_{\lambda}] +D\pi^{\mu\lambda}\partial_{\lambda}n_{\text{eq}}+\sigma_{1}\Delta^{\mu\lambda}(u)\partial_{\lambda}\mathcal{P}+\sigma_{2}n^{(\mu}\pi^{\alpha)\beta}\partial_{\alpha}(\mathcal{P}n_{\beta}) ,$$

where we have set

$$D = -\frac{1}{3}\tau_s \mathcal{P} \frac{I_{40}}{I_{22}} ,$$

$$\lambda_\perp = (\tau - \tau_s \mathcal{P}) \frac{m^4}{6\pi^2} \left[\frac{I_{41}}{I_{40}} I_{4-1} - I_{40} \right] + \tau_s \frac{\mathcal{P} m^4}{6\pi^2} \left[\frac{I_{21} I_{41}}{I_{22}} - I_{40} \right] ,$$

$$\lambda_\parallel = (\tau - \tau_s \mathcal{P}) \frac{m^4}{6\pi^2} \left[\frac{I_{41}}{I_{40}} I_{4-1} - I_{40} \right]$$

and where the coefficients σ_1 and σ_2 are still those given in Sec. VI. This last expression for $J^\mu_{(1)}$ is more suited for interpretation than the form (6.44): D is a diffusion coefficient while $\lambda_\perp$ and $\lambda_\parallel$ are obviously perpendicular and parallel heat conductivity coefficients, respectively: similarly, σ_1 and σ_2 are directly connected to polarization and its spatial direction.

In fact, the main source of ambiguity in defining the above transport coefficients ultimately lies in the difficulty to get a satisfactory expression for the off-equilibrium entropy of systems of particles endowed with spin. Indeed, the knowledge of such an expression would allow a clear-cut definition of transport coefficients (see, e.g., in the relativistic case, Ref. [7]). Unfortunately, this problem is not specifically relativistic. In the nonrelativistic case, an approximate form has been suggested by Lhuillier and Laloë [29]; it is however not completely satisfactory when used in our case.

(6) The determination of $f^\mu_{5(1)}$ is somewhat more ambiguous, in the absence of any specific physical arguments and/or model. A precise form for this Wigner function is, in fact, useful, as noted earlier, in order to get a BGK collision term. Here, use was made of a "minimal" form, in the sense that the precession of polarization and a few spin effects (such as spin diffusion) were taken into account. Unfortunately, a full specification of $f^\mu_{5(1)}$ was not generally possible the more so since the matching conditions involving the polarization are not yet well established. It follows that the general BGK collision term has not been fully determined.

However, in the particular case where the collision term has the form $M\delta FN$, the precise form of the matrices M and V has been practically specified (Sec. IV) up to a few constants. Furthermore, its general solution has been obtained leading to a sufficiently rich variety of effects as to accommodate most physical situations.

(7) Several extensions of the above results can be obtained without any particular difficulty. For instance, global internal symmetries can be dealt with by including new indices in the Wigner function and in the relaxation "times." A simple example is provided by isospin $\frac{1}{2}$. Its treatment is completely similar to that of spin in the nonrelativistic case; this is so because of the complete decoupling of isospin (or other global internal symmetries) from the spacetime degrees of freedom. As to gauge invariances, the situation is more involved and can be handled in two different ways. First, one can use a gauge-covariant Wigner function [15] and a BGK collision term of the kind studied in the present article. The result is a gauge-covariant kinetic equation which is not very simple. However, the delicate point lies in the use of a (manifestly) gauge-covariant approximation method. Second [16], one can use a nonmanifestly-gauge-invariant formalism and next show that the final results are actually gauge invariant.

Another possible extension leading to new physical phenomena can be obtained by looking at *nonlinear* effects of spins *only*. To do that in a qualitative (and also semiquantitative) manner, it is sufficient to write down a matrix collision term which is quadratic [29] in the covariant Wigner matrix and is merely algebraic (i.e., it does not contain any momentum integration), the various coefficients involved being function of p. All these extensions are presently being studied.

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APPENDIX A: TWO FLUID MODEL

Let us take as a generalization of the "unpolarized" Anderson and Witting equation,

$$p\partial f = -\frac{p\cdot u}{\tau}\delta f$$

the following coupled transport equations

$$p\partial f_+ = -\frac{p\cdot u}{\tau_{++}}\delta f_+ - \frac{p\cdot u}{\tau_{+-}}\delta f_- , \tag{A1a}$$

$$p\cdot\partial f_- = -\frac{p\cdot u}{\tau_{-+}}\delta f_+ - \frac{pu}{\tau_{--}}\delta f_- , \tag{A1b}$$

where f_+ and f_- represent, respectively, the distribution function for spin-up and spin-down particles.

It is further assumed that the system (A1) can be inverted as

$$\begin{bmatrix} \delta f_+ \\ \delta f_- \end{bmatrix} = -\frac{\tau_{++}\tau_{--}\tau_{+-}\tau_{-+}}{\tau_{-+}\tau_{+-} - \tau_{++}\tau_{--}} \begin{bmatrix} \frac{1}{\tau_{--}} & -\frac{1}{\tau_{+-}} \\ -\frac{1}{\tau_{-+}} & \frac{1}{\tau_{++}} \end{bmatrix} \times \frac{p\cdot\partial}{p\cdot u} \begin{bmatrix} f_+ \\ f_- \end{bmatrix} . \tag{A2}$$

Next we perform a Chapman-Enskog approximation in the case

$$\frac{\tau_{++}\tau_{--}\tau_{+-}\tau_{-+}}{\tau_{-+}\tau_{+-} - \tau_{++}\tau_{--}} \frac{1}{\tau_{\rm macr}} \ll 1$$

and three other similar equations, where $\tau_{\rm macr}$ is a macroscopic time scale associated with the hydrodynamical gradient ∂. The first-order Chapman-Enskog solution is obtained by replacing $p\cdot\partial[{}^{f_+}_{f_-}]$ by $p\cdot\partial[{}^{f_{+\rm eq}}_{f_{-\rm eq}}]$ on the right-hand side of Eq. (A2). $f_{+\rm eq}$ and $f_{-\rm eq}$ are the following local equilibrium distribution functions

$$f_{+\mathrm{eq}}=\frac{\xi_+}{e^{\beta(p\cdot u-\mu)}+1}=\xi_+f_{\mathrm{eq}}\ ;$$
$$[\xi_+=\xi_+(p,x);\beta=\beta(x);\mu=\mu(x)]\ ,\tag{A3}$$
$$f_{-\mathrm{eq}}=\frac{\xi_-}{e^{\beta(p\cdot u-\mu)}+1}=\xi_-f_{\mathrm{ex}};\ \ [\xi_-=\xi_-(p,x)]$$

ξ_+ and ξ_- are the fractions of spin-up or spin-down particles at a given energy and momentum. In the following, for the sake of simplicity, it is assumed that the latter quantities are only x dependent: $\xi_+(p,x)=\xi_+(x)$ and $\xi_-(p,x)=\xi_-(x)$. We shall make frequent use of the property $\xi_++\xi_-=1$.

We are now able to calculate the first-order deviations to the baryonic current and energy-momentum tensor

$$\delta J^\mu=\int d^4p\frac{p^\mu}{m}(\delta f_++\delta f_-)\ ,\tag{A4}$$

$$\delta T^{\mu\nu}=\int d^4p\frac{p^\mu p^\nu}{m}(\delta f_++\delta f_-)\ .\tag{A5}$$

Let us also define a spin current through

$$J^\mu_\xi=\int d^4p\frac{p^\mu}{m}(f_+-f_-)\ ,$$
$$\delta J^\mu_\xi=\int d^4p\frac{p^\mu}{m}(\delta f_+-\delta f_-)\ .\tag{A6}$$

Replacing δf_+ and δf_- by their approximations into Eqs. (A4)–(A6), one finds

$$\delta J^\mu=\int d^4p\frac{p^\mu}{m}\left[-\tau\frac{p\cdot\partial}{p\cdot u}f_{\mathrm{eq}}-\tilde{\tau}\frac{p\cdot\partial}{p\cdot u}(\xi_+-\xi_-)f_{\mathrm{eq}}\right]\ ,\tag{A7}$$

$$\delta T^{\mu\nu}=\int d^4p\frac{p^\mu p^\nu}{m}\left[-\tau\frac{p\cdot\partial}{p\cdot u}f_{\mathrm{eq}}-\tilde{\tau}\frac{p\cdot\partial}{p\cdot u}(\xi_+-\xi_-)f_{\mathrm{eq}}\right]\ ,\tag{A8}$$

$$\delta J^\mu_\xi=\int d^4p\frac{p^\mu}{m}\left[-\tau_s\frac{p\cdot\partial}{p\cdot u}f_{\mathrm{eq}}-\tilde{\tau}_s\frac{p\cdot\partial}{p\cdot u}(\xi_+-\xi_-)f_{\mathrm{eq}}\right]\tag{A9}$$

with

$$\tau=\frac{1}{2}\frac{\tau_{++}\tau_{--}(\tau_{+-}+\tau_{-+})-\tau_{-+}\tau_{+-}(\tau_{++}+\tau_{--})}{\tau_{++}\tau_{--}-\tau_{+-}\tau_{-+}}\ ,\tag{A10a}$$

$$\tilde{\tau}=\frac{1}{2}\frac{\tau_{++}\tau_{--}(\tau_{+-}-\tau_{-+})-\tau_{-+}\tau_{+-}(\tau_{++}-\tau_{--})}{\tau_{++}\tau_{--}-\tau_{+-}\tau_{-+}}\ ,\tag{A10b}$$

$$\tau_s=-\frac{1}{2}\frac{\tau_{++}\tau_{--}(\tau_{+-}-\tau_{-+})+\tau_{-+}\tau_{+-}(\tau_{++}-\tau_{--})}{\tau_{++}\tau_{--}-\tau_{+-}\tau_{-+}}\ ,\tag{A10c}$$

$$\tilde{\tau}_2=-\frac{1}{2}\frac{\tau_{++}\tau_{--}(\tau_{+-}+\tau_{-+})+\tau_{-+}\tau_{+-}(\tau_{++}+\tau_{--})}{\tau_{++}\tau_{--}-\tau_{+-}\tau_{-+}}\ .\tag{A10d}$$

As a matter of fact, the four τ's are actually not independent since one should have

$$\sigma_{++}=\sigma_{--},\ \ \sigma_{+-}=\sigma_{-+}\ ,$$

as a consequence of charge conjugation invariance. Furthermore, when the τ's are evaluated (roughly) by expressions of the usual form

$$\frac{1}{\tau}=n\sigma\langle v\rangle\ ,$$

n must necessarily involve both ξ_+ and ξ_- (besides n_{eq}) since only the density of *colliding* particles are to be taken into account. For instance, one has

$$\frac{1}{\tau_{++}}=\xi_+n_{\mathrm{eq}}\sigma_{++}\langle v\rangle\ .$$

After some straightforward manipulations, the deviations to equilibrium quantities $\delta J^\mu,\delta T^{\mu\nu},\delta J^\mu_\xi$ are found to be:

$$\delta J^\mu=-[\tau-\tilde{\tau}\mathcal{P}]4\pi m^3\left[\left[I_{21}\dot{\alpha}-I_{22}\dot{\gamma}+\gamma\frac{I_{40}}{3}\theta\right]u^\mu-\frac{I_{4-1}}{3}\Delta^{\mu\lambda}\partial_\lambda(\alpha)+\frac{I_{40}}{3}\Delta^{\mu\lambda}[\partial_\lambda(\gamma)+\gamma\dot{u}_\lambda]\right]$$
$$-\tilde{\tau}4\pi m^3\left[i_{21}\dot{\mathcal{P}}u^\mu-\frac{i_{4-1}}{3}\Delta^{\mu\lambda}\partial_\lambda(\mathcal{P})\right]\ ,\tag{A11}$$

$$\delta T^{\mu\nu}=-[\tau-\bar{\tau}\mathcal{P}]4\pi m^4\left[\left[I_{22}\dot{\alpha}-I_{23}\dot{\gamma}+\gamma\frac{I_{41}}{3}\theta\right]u^\mu u^\nu-\gamma\frac{I_{6-1}}{15}2\sigma^{\mu\nu}-\frac{\theta}{3}\Delta^{\mu\nu}\left[\gamma\frac{I_{6-1}}{3}+\frac{\dot{\alpha}}{\theta}I_{40}-\frac{\dot{\gamma}}{\theta}I_{41}\right]\right.$$
$$\left.+\left\{\frac{I_{41}}{3}[\partial_\lambda(\gamma)+\gamma\dot{u}_\lambda]-\frac{I_{40}}{3}\partial_\lambda(\alpha)\right\}\{u^\mu\Delta^{\nu\lambda}+u^\nu\Delta^{\mu\lambda}\}\right]$$
$$-\bar{\tau}4\pi m^4\left[i_{22}\dot{\mathcal{P}}u^\mu u^\nu-\frac{i_{40}}{3}\dot{\mathcal{P}}\Delta^{\mu\nu}-\frac{i_{40}}{3}(u^\mu\Delta^{\nu\lambda}+u^\nu\Delta^{\mu\lambda})\partial_\lambda(\mathcal{P})\right], \tag{A12}$$

$$\delta J^\mu_S=-[\tau_s-\bar{\tau}_s\mathcal{P}]4\pi m^3\left[\left[I_{21}\dot{\alpha}-I_{22}\dot{\gamma}+\gamma\frac{I_{40}}{3}\theta\right]u^\mu-\frac{I_{4-1}}{3}\Delta^{\mu\lambda}\partial_\lambda(\alpha)+\frac{I_{40}}{3}\Delta^{\mu\lambda}[\partial_\lambda(\gamma)+\gamma\dot{u}_\lambda]\right]$$
$$-\bar{\tau}_s4\pi m^3\left[i_{21}\dot{\mathcal{P}}u^\mu-\frac{i_{4-1}}{3}\Delta^{\mu\lambda}\partial_\lambda(\mathcal{P})\right]. \tag{A13}$$

In Eqs. (A11)–(A13) the following notations were employed

$$\gamma=m\beta\ ,$$
$$\alpha=\beta\mu\ ,$$
$$\mathcal{P}=\xi_+-\xi_-\ ,$$
$$I_{nm}=\int_0^\infty \sinh^n x\,\cosh^m x\frac{e^{\gamma\cosh x-\alpha}}{[e^{\cosh x-\alpha}+1]^2}dx\ ,$$
$$i_{nm}=\int_0^\infty \sinh^n x\,\cosh^m x\frac{1}{e^{\gamma\cosh x-\alpha}+1}dx$$

(we used the parametrization: $p^0=m\cosh x, p^1=m\sinh x\sin\theta\cos\varphi, p^2=m\sinh x\sin\theta\sin\varphi, p^3=m\sinh x\cos\theta$).

There remains to make use of the conservation relations

$$\partial_\mu J^\mu_{\rm eq}=0,\quad J^\mu_{\rm eq}=\int d^4p\frac{p^\mu}{m}(f^{\rm eq}_+ + f^{\rm eq}_-)=n_{\rm eq}u^\mu\ , \tag{A14}$$

$$\partial_\mu T^{\mu\nu}_{\rm eq}=0,\quad T^{\mu\nu}_{\rm eq}=\int d^4p\,p^\mu p^\nu(f^{\rm eq}_+ + f^{\rm eq}_-)=\rho u^\mu u^\nu-P\Delta^{\mu\nu}\ . \tag{A15}$$

These equations, respectively, lead to

$$\dot{n}_{\rm eq}+n\theta=0\ , \tag{A16}$$

$$\dot{\rho}+(\rho+P)\theta=0\ , \tag{A17}$$

$$\Delta^{\mu\nu}\partial_\lambda(P)=(\rho+P)\dot{u}^\mu\ , \tag{A18}$$

or in terms of $\dot{\alpha},\dot{\gamma},\dot{u}^\mu$

$$I_{21}\dot{\alpha}-I_{22}\dot{\gamma}+\frac{I_{40}}{3}\gamma\theta=0\ , \tag{A19}$$

$$I_{22}\dot{\alpha}-I_{23}\dot{\gamma}+\frac{I_{41}}{3}\gamma\theta=0\ , \tag{A20}$$

$$\Delta^{\mu\nu}\partial_\lambda(\alpha)=\frac{I_{41}}{I_{40}}\Delta^{\mu\lambda}[\partial_\lambda(\gamma)+\gamma\dot{u}_\lambda]\ . \tag{A21}$$

Inserting Eqs. (A19)–(A21) into Eqs. (A11)–(A13) simplifies the latter expressions to

$$\delta J^\mu=[\tau-\bar{\tau}\mathcal{P}]\frac{4\pi m^3}{3}\left[\frac{I_{41}I_{4-1}}{I_{40}}-I_{40}\right]\Delta^{\mu\lambda}[\partial_\lambda(\gamma)+\gamma\dot{u}_\lambda]-\bar{\tau}4\pi m^3\left\{i_{21}\dot{\mathcal{P}}u^\mu-\frac{i_{4-1}}{3}\Delta^{\mu\lambda}\partial_\lambda(\mathcal{P})\right\},$$

$$\delta T^{\mu\nu}=[\tau-\bar{\tau}\mathcal{P}]4\pi m^4\left[\gamma\frac{I_{6-1}}{15}2\sigma^{\mu\nu}+\theta\Delta^{\mu\nu}\left[\gamma\frac{I_{6-1}}{3}-\frac{\gamma}{3}\frac{2I_{22}I_{41}I_{40}-I_{23}I_{40}^2-I_{21}I_{41}^2}{I_{22}^2-I_{21}I_{23}}\right]\right]$$
$$-\bar{\tau}4\pi m^4\left[i_{22}\dot{\mathcal{P}}u^\mu u^\nu-\frac{i_{40}}{3}\dot{\mathcal{P}}\Delta^{\mu\nu}-\frac{i_{40}}{3}(u^\mu\Delta^{\nu\lambda}+u^\nu\Delta^{\mu\lambda})\partial_\lambda(\mathcal{P})\right],$$

$$\delta J^\mu_\xi=[\tau_s-\tilde{\tau}_s\mathcal{P}]\frac{4\pi m^3}{3}\left[\frac{I_{41}I_{4-1}}{I_{40}}-I_{40}\right]\Delta^{\mu\lambda}[\partial_\lambda(\gamma)+\gamma\dot{u}_\lambda]-\tilde{\tau}_s 4\pi m^3\left[i_{21}\dot{\mathcal{P}}u^\mu-\frac{i_{4-1}}{3}\Delta^{\mu\lambda}\partial_\lambda(\mathcal{P})\right].$$

Let us consider now a system in a state of quasistatic equilibrium where the polarization lifetime is supposed to be much longer than the kinetic time scale (such is indeed the case in ^{3}He where the polarization has been observed to last for a few days). This is equivalent to saying that the temporal gradient of the polarization can be taken to be zero to a very good approximation. On the right hand, the parameter $\xi_+-\xi_-=\mathcal{P}$ in our equations is readily identified as being the polarization, so that taking $\dot{\mathcal{P}}=0$ appears to be a natural choice. It should also be noticed that taking $\dot{\mathcal{P}}=0$ is equivalent to assuming the conservation of our spin current

$$J^\mu_{\xi\mathrm{eq}}=\mathcal{P}n_{\mathrm{eq}}u^\mu,\quad \partial_\mu J^\mu_\xi=n_{\mathrm{eq}}\dot{\mathcal{P}}\simeq 0\ . \tag{A22}$$

The deviations to equilibrium now have the structure:

$$\delta J^\mu=K\Delta^{\mu\lambda}[\partial_\lambda(\gamma)+\gamma\dot{u}_\lambda]+K_{\mathcal{P}}\Delta^{\mu\lambda}\partial_\lambda(\mathcal{P})\ , \tag{A23}$$

$$\delta T^{\mu\nu}=2\eta\sigma^{\mu\nu}+\xi\frac{\theta}{3}\Delta^{\mu\nu}+Q_{\mathcal{P}}(u^\mu\Delta^{\nu\lambda}+u^\nu\Delta^{\mu\lambda})\partial_\lambda(\mathcal{P})\ , \tag{A24}$$

$$\delta J^\mu_\xi=\tilde{K}\Delta^{\mu\lambda}[\partial_\lambda(\gamma)+\gamma\dot{u}_\lambda]+\tilde{K}_{\mathcal{P}}\Delta^{\mu\lambda}\partial_\lambda(\mathcal{P})\ . \tag{A25}$$

This is neither the Landau nor Eckart form. We can put the result into the Eckart form by redefining u^μ and adding to it a small quantity of order 2 in the relaxation times

$$U^\mu=u^\mu+\xi^\mu,\quad \xi^\mu=\frac{K}{n_{\mathrm{eq}}}\Delta^{\mu\lambda}[\partial_\lambda(\gamma)+\gamma\dot{u}_\lambda]+\frac{K_{\mathcal{P}}}{n_{\mathrm{eq}}}\Delta^{\mu\lambda}\partial_\lambda\mathcal{P}\ ,$$

$$\delta J^\mu_{(E)}=n_{\mathrm{eq}}U^\mu\ ; \tag{A26}$$

$$\delta T^{\mu\nu}_{(E)}=2\eta\sigma^{\mu\nu}+\xi\frac{\theta}{3}\Delta^{\mu\nu}+\frac{\rho+P}{mn_{\mathrm{eq}}}K\gamma^2(U^\mu\Delta^{\nu\lambda}+U^\nu\Delta^{\mu\lambda})\left[\partial_\lambda\left[\frac{1}{\beta}\right]-\frac{\dot{u}_\lambda}{\beta}\right]$$
$$+\left[Q_{\mathcal{P}}-\frac{\rho+P}{n_{\mathrm{eq}}}K_{\mathcal{P}}\right](U^\mu\Delta^{\nu\lambda}+U^\nu\Delta^{\mu\lambda})\partial_\lambda(\mathcal{P})\ , \tag{A27}$$

$$\delta J^\mu_{\xi_{(E)}}=(\mathcal{P}K-\tilde{K})\frac{\gamma^2}{m}\Delta^{\mu\lambda}\left[\partial_\lambda\left[\frac{1}{\beta}\right]-\frac{\dot{u}_\lambda}{\beta}\right]+(\tilde{K}_{\mathcal{P}}-\mathcal{P}K_{\mathcal{P}})\Delta^{\mu\lambda}\partial_\lambda(\mathcal{P})\ . \tag{A28}$$

From these equations the expressions of the transport coefficients can easily be read as

$$\lambda=\frac{\rho+P}{mn_{\mathrm{eq}}}K\gamma^2=[\tau-\tilde{\tau}\mathcal{P}]\frac{4\pi m^3}{3}\gamma^2\frac{I_{41}}{I_{40}}\left[\frac{I_{41}I_{4-1}}{I_{40}}-I_{40}\right]\quad\text{(thermal conductivity)}\ , \tag{A29}$$

$$\eta=[\tau-\tilde{\tau}\mathcal{P}]4\pi m^4\gamma\frac{I_{6-1}}{15}\quad\text{(shear viscosity)}\ , \tag{A30}$$

$$\xi=[\tau-\tilde{\tau}\mathcal{P}]\frac{4\pi m^4}{3}\gamma\left[\frac{I_{40}^2I_{23}+I_{41}^2I_{21}-2I_{22}I_{41}I_{40}}{I_{22}^2-I_{21}I_{23}}+I_{6-1}\right]\quad\text{(bulk viscosity)}\ . \tag{A31}$$

These three usual transport coefficients λ,η,ξ are only slightly modified with respect to the nonpolarized Anderson-Witting [10] case and, as a matter of fact, by changing the collision time τ into a polarization-dependent one $\tau-\tilde{\tau}\mathcal{P}$.

In addition, new transport coefficients associated with the polarization do appear, as resulting from coupling between heat current and polarization gradient

$$\lambda_{\mathcal{P}}=Q_{\mathcal{P}}-\frac{\rho+P}{n_{\mathrm{eq}}}K_{\mathcal{P}}=\tilde{\tau}\frac{4\pi m^4}{3}\left[i_{40}-\frac{I_{41}}{I_{40}}i_{4-1}\right] \tag{A32}$$

or from a coupling between spin current and temperature gradient

$$\sigma_H=(\mathcal{P}K-\tilde{K})\frac{\gamma^2}{m}=[\tau\mathcal{P}-\tilde{\tau}\mathcal{P}^2-\tau_s+\tau_s\mathcal{P}]\frac{4\pi m^2}{3}\gamma^2\left[\frac{I_{41}I_{4-1}}{I_{40}}-I_{40}\right] \tag{A33}$$

and also from a coupling between spin current and polarization gradient

$$\sigma=(\tilde{K}_{\mathcal{P}}-\mathcal{P}K_{\mathcal{P}})=[\tilde{\tau}_s-\mathcal{P}\tilde{\tau}]\frac{4\pi m^3}{3}i_{4-1}\ . \tag{A34}$$

APPENDIX B

In this appendix some useful formulas involving Dirac matrices, γ^{μ}, are given. They are based on the definitions and formulas summarized in Ref. [21], Appendix A. [Note that in Eqs. (A11) and (A12) of Ref. [21], in the last term of their right-hand sides, the indices μ and ν must be interchanged.]

Let p^{μ} and u^{μ} be two timelike four-vectors such that

$$p^2=m^2, \quad u^2=+1 , \tag{B1}$$

then the following formulas hold:

$$(\gamma\cdot p\pm m)\gamma^{\mu}(\gamma\cdot p\pm m)=2p^{\mu}(\gamma\cdot p\pm m) , \tag{B2}$$

$$(\gamma\cdot p+m)\sigma^{\alpha\beta}u_{\alpha}p_{\beta}(\gamma\cdot p+m)=0 , \tag{B3}$$

$$(\gamma\cdot p\pm m)\sigma^{\mu\nu}(\gamma\cdot p\pm m)=2\{p^{[\mu}\sigma^{\nu]\lambda}p_{\lambda}+m^2\sigma^{\mu\nu}\pm im\,\epsilon^{\mu\nu\alpha\beta}p_{\alpha}\gamma_5\gamma_{\beta}\} , \tag{B4}$$

$$\frac{\gamma\cdot p\pm m}{2m}\gamma_5\frac{\gamma\cdot p\mp m}{2m}=\gamma_5\frac{\gamma\cdot p\mp m}{2m} , \tag{B5}$$

$$\frac{\gamma\cdot p+m}{2m}\gamma_{\mu}\frac{\gamma\cdot p-m}{2m}=\frac{1}{2m^2}\{p_{\mu}\gamma\cdot p+m\sigma_{\mu\nu}p^{\nu}-m^2\gamma_{\mu}\} , \tag{B6}$$

$$\frac{\gamma\cdot p+m}{2m}\gamma_5\gamma_{\mu}\frac{\gamma\cdot p-m}{2m}=-\frac{p_{\mu}}{m}\gamma_5\frac{\gamma\cdot p-m}{2m} , \tag{B7}$$

$$\gamma\cdot p\,\sigma^{\mu\nu}\gamma\cdot p=m^2\sigma^{\mu\nu}+2p^{[\mu}\sigma^{\nu]\lambda}p_{\lambda} , \tag{B8}$$

$$\frac{\gamma\cdot p\pm m}{2m}\gamma^{\mu}\gamma^5\frac{\gamma\cdot\pm m}{2m}=\frac{1}{2}\gamma^{\mu}\gamma^5\pm\frac{i}{4m}\epsilon^{\mu\nu\alpha\beta}p_{\nu}\sigma_{\alpha\beta}-\frac{p^{\mu}}{2m^2}\gamma\cdot p\gamma^5 , \tag{B9}$$

$$\frac{\gamma\cdot p-m}{2m}\gamma\cdot u\frac{\gamma\cdot p+m}{2m}=\left\{\frac{p\cdot u}{m}-\gamma\cdot u\right\}\frac{\gamma\cdot p+m}{2m} , \tag{B10}$$

$$\frac{\gamma\cdot p+m}{2m}\sigma^{\mu\nu}\frac{\gamma\cdot p-m}{2m}=\frac{1}{2m^2}\{mp^{[\mu}\gamma^{\nu]}+p^{[\mu}\sigma^{\nu]\alpha}p_{\alpha}\} , \tag{B11}$$

$$(\sigma^{\mu\nu}u_{\mu}p_{\nu})^2=-\Delta^{\mu\nu}(u)p_{\mu}p_{\nu} , \tag{B12}$$

$$[\gamma\cdot p,\gamma\cdot u]_+=2p\cdot u , \tag{B13}$$

$$[\gamma\cdot p,\sigma^{\mu\nu}u_{\mu}p_{\nu}]_+=0 , \tag{B14}$$

$$[\gamma\cdot u,\sigma^{\mu\nu}u_{\mu}p_{\nu}]_+=0 . \tag{B15}$$

To these very useful formulas, used repeatedly throughout our calculations, we should add

$$[\gamma_5,\sigma_{\mu\nu}]_-=0 , \tag{B16}$$

$$\gamma_5\sigma^{\mu\nu}=-\frac{i}{2}\epsilon^{\mu\nu\alpha\beta}\sigma_{\alpha\beta} . \tag{B17}$$

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