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Conformally exact metric and dilaton in string theory on curved spacetime

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Using a Hamiltonian approach to gauged Wess-Zumino-Witten models, we present a general method for computing the conformally exact metric and dilaton, to all orders in the $1/k$ expansion, for any bosonic, heterotic, or type-II superstring model based on a coset G/H. We prove the following relations: (i) For type-II superstrings the conformally exact metric and dilaton are identical to those of the nonsupersymmetric *semiclassical* bosonic model except for an overall renormalization of the metric obtained by $k \to k-g$. (ii) The exact expressions for the heterotic superstring are derived from their exact bosonic string counterparts by shifting the central extension $k \to 2k-h$ [but an overall factor $(k-g)$ remains unshifted]. (iii) The combination $e^{\Phi}\sqrt{-G}$ is independent of k and therefore can be computed in lowest-order perturbation theory. The general formalism is applied to the coset models $\mathrm{SO}(d-1,2)_{-k}/\mathrm{SO}(d-1,1)_{-k}$ that are relevant for string theory on curved spacetime. Explicit expressions for the conformally exact metric and dilaton for the cases $d=2,3,4$ are given. In the semiclassical limit ($k \to \infty$) our results agree with those obtained with the Lagrangian method up to one loop in perturbation theory.

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I. INTRODUCTION

During the past year there has been extensive investigations of curved-spacetime string backgrounds generated by noncompact cosets G/H. All models with spacetime dimension $d \leq 4$ require the noncompact current algebra coset $\mathrm{SO}(d-1,2)_{-k}/\mathrm{SO}(d-1,1)_{-k}$ as part of, or as the full, conformal field theory [1]. The action is written in the form of a gauged Wess-Zumino-Witten (WZW) model [2]. For models involving more than four spacetime coordinates there are other possibilities which have been classified [3], but so far not investigated. The semiclassical analysis [4] for $k \to \infty$ has shown that these are useful models for learning more about string and particle propagation in gravitationally singular spaces such as black holes and more interesting singularities in various dimensions. By now essentially all models up to dimension four have been subjected to the semiclassical analysis [5–12]. A cosmological interpretation has also been found [13–15]. A group-theoretical method for the global analysis of these semiclassical geometries, including an explicit solution of the geodesics, has been formulated and explicitly applied to some cases [14].

As in [8], heterotic and type-II superstring actions can be constructed in exactly four dimensions in the form of $N=1$ superconformal gauged WZW models. We believe that a heterotic string model of this type, perhaps with some variations, taken with a cosmological interpretation, provides the kind of setting suitable for a discussion of the physics of the early Universe in the context of string theory.

The principal method of semiclassical investigation followed Ref. [4], which used a Lagrangian method. Quantum corrections, which were necessary to obtain the dilaton and satisfy the perturbative equations for conformal invariance [16], were limited to one loop. In practical terms one cannot carry out the quantum computation of the σ-model-like theory to all orders with this method. However, the main interest in these models stems from the fact that they are conformally exact current algebra theories, which are in principle exactly solvable quantum theories. In order to take advantage of this fact it is desirable to go back to the Hamiltonian method and use the algebraic properties of the current algebra. The model can then be investigated via the coset methods for noncompact current algebras [17,18].

In this paper we will show how to use the Hamiltonian approach to compute the gravitational metric and dilaton backgrounds to all orders in the quantum theory (all orders in the central extension k). These will then provide a more accurate representation of the conformally exact vacuum configuration of the string at the "classical" level (i.e., no string loops). We have managed to obtain these quantities for bosonic, type-II supersymmetric, and heterotic string theories in $d \leq 4$.[1]

The main idea is the following: The conformally exact Hamiltonian is the sum of left and right Virasoro generators $L_0^L + L_0^R$ that may be written purely in terms of Casimir operators of G and H at the tachyon level. The exact dependence on the central extension k is included in this form. If we investigate the exact quantum eigenstates in configuration space, then the Casimir operators become Laplacians constructed as differential operators in group parameter space ($\mathrm{dim}G$). If the state ψ is a singlet under the gauge group H (acting simultaneously on

[1]The corresponding results are also given for a particle theory whose WZW-like action was defined in [14]. The particle theory can be thought of as a string shrunk to a point which has no interactions with string excitations. For this case the semiclassical result is actually exact.

left and right movers), then gauge invariance requires that it is a function of singlet combinations of group parameters. There are exactly $\dim(G/H)$ such invariants which we choose as our string coordinates X^a. We have recently shown [14] that these invariants provide a global description of the geometry. In this way we can write the conformally exact Hamiltonian $L_0^L + L_0^R$ as a differential operator in the global curved-space-time manifold involving only the string coordinates X^a. By comparing to the expected general form

$$(L_0^L + L_0^R)\psi = \frac{-1}{e^{\Phi}\sqrt{-G}}\partial_a(e^{\Phi}\sqrt{-G}G^{ab}\partial_b\psi)$$

for the singlet ψ, we read off the exact global metric and dilaton.

We have applied this program to the general bosonic, heterotic, and type-II superstrings and derived relationships among the exact quantities of these theories as announced in the abstract of this paper. For the specific cosets of interest $SO(d-1,2)/SO(d-1,1)$ explicit expressions are given below. The large-k limit of our results agrees with the previous semiclassical computations. In the special case of two dimensions we also agree with another previous derivation of the exact metric and dilaton for the $SO(2,1)/SO(1,1)$ bosonic string [19].

II. ALGEBRAIC FORMALISM FOR COMPUTING THE EXACT METRIC AND DILATON

Let us consider a bosonic string theory for closed strings in d curved-space-time dimensions, based on a σ-model conformal field theory with string coordinates X^a, $a=0,1,\ldots,d-1$. The space-time metric and dilaton fields are $G_{ab}(X)$ and $\Phi(X)$, respectively. We begin with the effective action for the tachyon field $T(X)$ in d space-time dimensions. The most general form of this effective action is

$$S[T] = \int d^dX\sqrt{-G}\,e^{\Phi}(G^{ab}\partial_a T\partial_b T - V(T)) ,$$
$$V(T) = 2T^2 + O(T^3) , \qquad (2.1)$$

where $V(T)$ is the tachyon potential whose precise form is not necessary for the analysis that follows. From the point of view of conformal field theory the tachyon is completely defined through the action of the zero modes L_0^L and L_0^R of the stress tensors for the left and right movers. Therefore (2.1) must be equivalent to the following action:

$$S[T] = \int d^dX\sqrt{-G}\,e^{\Phi}(T(L_0^L + L_0^R)T - V(T)) . \qquad (2.2)$$

Comparison of (2.1) with (2.2) determines the form of $L_0^L + L_0^R$ as a differential operator in configuration space,

$$(L_0^L + L_0^R)T = -\frac{1}{e^{\Phi}\sqrt{-G}}\partial_a(G^{ab}e^{\Phi}\sqrt{-G}\,\partial_b T) . \qquad (2.3)$$

Now let us consider the σ-model-like action which results from an exact conformal theory based on the gauged WZW action. Using the equivalent current algebra coset model G/H, we can write L_0^L in terms of the quadratic Casimir operators Δ_G^L and Δ_H^L for the group and the subgroup as follows:

$$L_0^L T = \left[\frac{\Delta_G^L}{k-g} - \frac{\Delta_H^L}{k-h}\right]T ,$$
$$\Delta_G^L \equiv \mathrm{Tr}(J_G^L)^2, \quad \Delta_H^L \equiv \mathrm{Tr}(J_H^L)^2 , \qquad (2.4)$$

where J_G^L and J_H^L are anti-Hermitian group and subgroup generators obeying the appropriate Lie algebras, and g,h are the Coxeter numbers for the group and the subgroup. For the cases of interest in this paper $g=d-1, h=d-2$ for $d\geq 3$, and $g=2, h=0$ for $d=2$.[2] An expression similar to (2.4) can also be written for L_0^R. As shown below, we construct the generators $J_G^L, J_G^R, J_H^L, J_H^R$ as first-order differential operators acting on group parameter space. Then the Casimir operators $\Delta_G^L, \Delta_H^L, \Delta_G^R, \Delta_H^R$ contain single and double derivatives with respect to all $\dim G$ parameters in G.[3]

For the purposes of this paper it is sufficient to concentrate on gauge-invariant tachyon level states T which satisfy[4]

$$(J_H^L + J_H^R)T = 0 . \qquad (2.5)$$

The number of conditions is $\dim H$ and therefore T can depend only on $d=\dim(G/H)$ parameters, X^a (string coordinates), which are H invariants. The fact that there are exactly $\dim(G/H)$ such independent invariants is not immediately obvious, but it should become apparent to the reader by considering a few specific examples. As discussed in [14] these are in fact the coordinates that globally describe the σ-model geometry. Consequently, using the chain rule, we reduce the derivatives in (2.4) to only derivatives with respect to the d string coordinates X^a. Moreover, using the fact that $\Delta_G^L = \Delta_G^R$ for any group,[5] together with the fact that the gauge-invariance condition (2.5) leads to $(\Delta_H^L - \Delta_H^R)T = 0$ [see (2.14) below], we ensure the physical condition for closed bosonic strings $(L_0^L - L_0^R)T = 0$. Then using (2.3) and (2.4) one can deduce uniquely the expression for the inverse metric G^{ab} by comparing the coefficients of the double derivatives $\partial_a\partial_b T$. Comparison of the single-derivative terms $\partial_a T$ will give a system of d coupled linear partial

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[2]For the particle theory of footnote 1 the Hamiltonian contains no Coxeter numbers since the higher string excitation are absent. Then $L_0^L = (\Delta_G^L - \Delta_H^L)/k$.

[3]Since we have defined our Casimir operators as the square of anti-Hermitian generators we differ by a minus sign from usual conventions. For example, for SU(2) we would get the eigenvalues $\Delta_G = -j(j+1)$ instead of $+j(j+1)$.

[4]If H contains an Abelian U(1) or $\mathbb{R}$ factor there is the alternative of imposing the axial gauging condition $(J_H^L - J_H^R)T = 0$ for the currents associated with the Abelian factor. For an application see [20].

[5]This follows from $J_G^R = -g^{-1}J_G^L g + (1/D)\mathrm{Tr}(g^{-1}J_G^L g)$, where D is the dimension of the matrix g. The second term is present because g and J_G^L do not commute as quantum operators and J_G^R has to be traceless. But despite its presence the relation $\Delta_G^L = \Delta_G^R$ is derived from it.

differential equations, whose solution determines the dilaton field Φ.

The general k dependence of the exact expressions takes a particular form that can be seen as follows. In the large-k limit (2.4) becomes proportional to $(1/k)(\Delta_G^L-\Delta_H^L)$, from which we can read off the semiclassical metric and dilaton according to the above procedure. Therefore, we may rewrite (2.4) in the form

$$L_0^L T=\frac{1}{k-g}\left[\Delta_{G/H}^L+\frac{g-h}{k-h}\Delta_H^L\right]T\ , \tag{2.6}$$

where $\Delta_{G/H}^L=\Delta_G^L-\Delta_H^L$. Then it is evident that, except for the overall factor $(k-g)$, all dependence on k has the form $(g-h)/(k-h)$. This applies to the bosonic string. For the heterotic and type-II superstrings the k dependence can be derived by the same technique as will be seen below. It is evident that for the particle theory of footnotes 1 and 2 there are no such corrections to the semiclassical result.

Let us specialize to the coset models $SO(d-1,2)_{-k}/SO(d-1,1)_{-k}$ with $d=2,3,4$ since these are the ones of interest for a theory in four dimensions. We want to find the currents appropriate for right or left transformations of the group elements of $SO(d-1,2)$ in an $SO(d-1,1)$ basis. It is convenient to parametrize the group element of $SO(d-1,2)$ as the product $g=ht$, where $h\in SO(d-1,1)$ and $t\in SO(d-1,2)/SO(d-1,1)$. The h,t are given by[6]

$$\begin{aligned} h&=\begin{bmatrix}1 & 0\\ 0 & h_\mu{}^\nu\end{bmatrix}\ ,\\ t&=\begin{bmatrix} b & (b+1)x^\nu\\ -(b+1)x_\mu & [\eta_\mu{}^\nu-(b+1)x_\mu x^\nu]\end{bmatrix}\ . \end{aligned} \tag{2.7}$$

Furthermore, h can be written in the form $h_\mu{}^\nu=[(1+a)(1-a)^{-1}]_\mu{}^\nu$, with $a_{\mu\nu}=-a_{\nu\mu}$ when both indices are lowered. To ensure that t is a $SO(d-1,2)$ group element we take $b=(1-x^2)/(1+x^2)$. By considering the infinitesimal left transformations $\delta_L g=\epsilon_L g$ we can read off the form of the generators,

$$\begin{aligned} J_{\mu\nu}^L&=\tfrac{1}{2}(1+a)_{\mu\alpha}(1+a)_{\nu\beta}\frac{\partial}{\partial a_{\alpha\beta}}\ ,\\ J_\mu^L&=-\tfrac{1}{2}(1+x^2)\left[\frac{1+a}{1-a}\right]_\mu{}^\nu\frac{\partial}{\partial x^\nu}\\ &\quad+\tfrac{1}{2}(1+a)_{\mu\alpha}(1+a)_{\beta\gamma}x^\gamma\frac{\partial}{\partial a_{\alpha\beta}}\ . \end{aligned} \tag{2.8}$$

It can be shown that the above generators obey the commutation rules of the $SO(d-1,2)$ algebra. Namely,

$$\begin{aligned} [J_{\mu\nu}^L,J_{\alpha\beta}^L]&=J_{\mu\alpha}^L\eta_{\nu\beta}-J_{\nu\alpha}^L\eta_{\mu\beta}+J_{\nu\beta}^L\eta_{\mu\alpha}-J_{\mu\beta}^L\eta_{\nu\alpha}\ ,\\ [J_{\mu\nu}^L,J_\alpha^L]&=\eta_{\mu\alpha}J_\nu^L-\eta_{\nu\alpha}J_\mu^L\ ,\\ [J_\mu^L,J_\nu^L]&=J_{\mu\nu}^L\ . \end{aligned} \tag{2.9}$$

If we consider instead, the infinitesimal right transformations $\delta_R g=g\epsilon_R$ we find the expressions

$$\begin{aligned} J_{\mu\nu}^R&=-\tfrac{1}{2}(1-a)_{\mu\alpha}(1-a)_{\nu\beta}\frac{\partial}{\partial a_{\alpha\beta}}-x_{[\mu}\frac{\partial}{\partial x^{\nu]}}\ ,\\ J_\mu^R&=\tfrac{1}{2}(x^2-1)\frac{\partial}{\partial x^\mu}-x_\mu x^\nu\frac{\partial}{\partial x^\nu}\\ &\quad-\tfrac{1}{2}(1-a)_{\mu\alpha}(1-a)_{\gamma\beta}x^\gamma\frac{\partial}{\partial a_{\alpha\beta}}\ . \end{aligned} \tag{2.10}$$

These currents obey the same commutation rules as in (2.9) and moreover commute with the left currents $[J^L,J^R]=0$. Now we construct the quadratic Casimir operator associated with the left and right currents. We find

$$\begin{aligned} \Delta_G^L&=\tfrac{1}{2}(J^L)_{\mu\nu}(J^L)^{\mu\nu}+(J^L)_\mu(J^L)^\mu\\ &=\tfrac{1}{4}(1+x^2)^2\frac{\partial^2}{\partial x^\mu\partial x_\mu}-\frac{d-2}{2}(1+x^2)x_\mu\frac{\partial}{\partial x_\mu}+\tfrac{1}{4}(1+a)_{\mu\gamma}x^\gamma(1+a)_{\alpha\delta}x^\delta(1-a^2)_{\nu\beta}\frac{\partial^2}{\partial a_{\mu\nu}\partial a_{\alpha\beta}}\\ &\quad+\tfrac{1}{2}(1+a)_{\mu\alpha}x^\alpha(1-a^2)_{\nu\beta}x^\beta\frac{\partial}{\partial a_{\mu\nu}}+\tfrac{1}{2}(1+x^2)(1+a)_{\mu\alpha}x^\alpha(1+a)_{\nu\beta}\frac{\partial^2}{\partial a_{\mu\nu}\partial x_\beta}\\ &\quad+\tfrac{1}{8}(1-a^2)_{\mu\alpha}(1-a^2)_{\nu\beta}\frac{\partial^2}{\partial a_{\mu\nu}\partial a_{\alpha\beta}}+\tfrac{1}{4}(a-a^3)_{\mu\nu}\frac{\partial}{\partial a_{\mu\nu}}\ . \end{aligned} \tag{2.11}$$

The expression for Δ_G^R is identical for reasons explained in footnote 5. The quadratic Casimir operators corresponding to the subgroup $H=SO(d-1,1)$ are

$$\begin{aligned} \Delta_H^L&=\tfrac{1}{2}(J^L)_{\mu\nu}(J^L)^{\mu\nu}\\ &=\tfrac{1}{8}(1-a^2)_{\mu\alpha}(1-a^2)_{\nu\beta}\frac{\partial^2}{\partial a_{\mu\nu}\partial a_{\alpha\beta}}+\tfrac{1}{4}(a-a^3)_{\mu\nu}\frac{\partial}{\partial a_{\mu\nu}} \end{aligned} \tag{2.12}$$

and

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[6] We follow the notation of [14]. As explained there, to compare with [7,8] where another vector X^μ was used, one should set $X^\mu=2x^\mu/(x^2-1)$. These Lorentz vectors should not be confused with the Lorentz-invariant string coordinates X^a even though they have the same dimension $d=\dim(G/H)$.

$$\Delta_H^R=\tfrac{1}{2}(J^R)_{\mu\nu}(J^R)^{\mu\nu}$$
$$=\tfrac{1}{8}(1-a^2)_{\mu\alpha}(1-a^2)_{\nu\beta}\frac{\partial^2}{\partial a_{\mu\nu}\partial a_{\alpha\beta}}+\tfrac{1}{4}(a-a^3)_{\mu\nu}\frac{\partial}{\partial a_{\mu\nu}}$$
$$+(x^2\eta_{\mu\nu}-x_\mu x_\nu)\frac{\partial^2}{\partial x_\mu\partial x_\nu}-(d-1)x_\mu\frac{\partial}{\partial x_\mu}$$
$$+(1+a)_{\mu\alpha}x^\alpha(1+a)_{\nu\beta}\frac{\partial^2}{\partial a_{\mu\nu}\partial x_\beta}\ . \tag{2.13}$$

The two expressions differ by the last line in (2.13) which equals

$$\Delta_H^R-\Delta_H^L=\tfrac{1}{2}((J^R)^{\mu\nu}-(J^L)^{\mu\nu})((J^R)_{\mu\nu}+(J^L)_{\mu\nu})\ . \tag{2.14}$$

Using the expressions (2.8) and (2.10) the gauge-invariance conditions (2.5) take the form

$$\left[a_{[\mu\lambda}\frac{\partial}{\partial a_\lambda{}^{\nu]}}-x_{[\mu}\frac{\partial}{\partial x^{\nu]}}\right]T=0\ , \tag{2.15}$$

where the brackets indicate antisymmetrization of the μ,ν indices. This form is recognized as the global Lorentz generator and it requires that T be constructed only from Lorentz invariants that can be formed from x_μ and $a_{\mu\nu}$. Next we specialize to the cases $d=3$ and $d=4$. The $d=2$ case corresponding to the two-dimensional black hole is discussed in the Appendix.

III. 3d BOSONIC STRING

The $3d$ model based on the coset model SO(2,2)/SO(2,1) was discussed semiclassically from the gauged WZW model Lagrangian point of view in [5,7,9]. This model may be viewed as the $3d$ submanifold of a four-dimensional model which is constructed by adjoining a factor of U(1) or $\mathbb{R}$ to the coset. The global structure of the $3d$ manifold was analyzed in [14] by finding the global coordinates and examining the particle trajectories. In particular, it was found that the space consists of two topologically distinct sectors. There is a curvature singularity with the topology of "pinched double trousers" in one sector and that of a "double saddle" in the other. In this case the antisymmetric matrix $a_{\mu\nu}$ has three parameters; therefore, it is possible to reparameterize it in terms of a three-dimensional vector y^μ, as $a_{\mu\nu}=\epsilon_{\mu\nu\lambda}y^\lambda$. Then the gauge condition (2.15) takes the form

$$\left[y_{[\mu}\frac{\partial}{\partial y^{\nu]}}+x_{[\mu}\frac{\partial}{\partial x^{\nu]}}\right]T=0\ . \tag{3.1}$$

The constraint (3.1) requires that T depends only on the three Lorentz invariants $x^2,y^2,x\cdot y$, or their combinations. In fact, in order to make correspondence with previous results we choose the same invariants as in [14] with $T(v,u,b)$ where

$$b=\frac{1-x^2}{1+x^2},\quad v=\frac{2}{1+y^2},\quad u=-2\frac{(x\cdot y)^2}{x^2(1+y^2)}\ . \tag{3.2}$$

Using the chain rule, we transform the derivatives with respect to the vectors x^μ and $y^\mu=\frac{1}{2}\epsilon^{\mu\nu\lambda}a_{\nu\lambda}$ in (2.11) and (2.12) to derivatives with respect to the H invariants v,u,b, e.g.,

$$\frac{\partial}{\partial x_\mu}T=\{2v(x\cdot y)/x^4[(x\cdot y)x^\mu-x^2y^\mu]\}\frac{\partial}{\partial u}T$$
$$-(b+1)^2x^\mu\frac{\partial}{\partial b}T\ ,$$

etc. Finally, with the dot products in the Laplacians, L_0 is written only in terms of (v,u,b) when acting on T. Then comparison of the double-derivative terms in (2.3) and (2.4) gives the nonzero elements of the inverse of the metric [we omit an overall factor of $1/[2(k-2)]$ which will be restored later in (3.5)],

$$\begin{aligned}&G^{bb}=4(b^2-1)\ ,\\&G^{vv}=-4\frac{b-1}{b+1}v(v-u-2)+\frac{4}{k-1}v(v-2)\ ,\\&G^{uu}=4\frac{b+1}{b-1}u(v-u-2)+\frac{4}{k-1}u(u+2)\ ,\\&G^{vu}=\frac{4}{k-1}vu\ ,\end{aligned} \tag{3.3}$$

and comparison of the single-derivative terms yields a system of linear partial differential equations which determine the dilaton,

$$\begin{aligned}&\frac{\partial}{\partial b}\ln(\sqrt{-G}e^\Phi)=\frac{b}{b^2-1}\ ,\\&\frac{\partial}{\partial v}(G^{vv}\sqrt{-G}e^\Phi)+\frac{\partial}{\partial u}(G^{uv}\sqrt{-G}e^\Phi)\\&\qquad=2\sqrt{-G}e^\Phi\left[\frac{b-1}{b+1}(u+2-3v)+\frac{4v-2}{k-1}\right]\ ,\\&\frac{\partial}{\partial u}(G^{uu}\sqrt{-G}e^\Phi)+\frac{\partial}{\partial v}(G^{vu}\sqrt{-G}e^\Phi)\\&\qquad=2\sqrt{-G}e^\Phi\left[\frac{b+1}{b-1}(v-2-3u)+\frac{4u+2}{k-1}\right]\ .\end{aligned} \tag{3.4}$$

Note that without a dilaton these equations have no solutions. Therefore, even without the hindsight of general string arguments, a dilaton must be introduced in our approach in order to have a solution to these equations. If we invert the inverse metric we get for the line element the following expression [we also restore the overall factor $2(k-2)$]:

$$ds^2=2(k-2)(G_{bb}db^2+G_{vv}dv^2+G_{uu}du^2+2G_{vu}dvdu)\ , \tag{3.5}$$

with

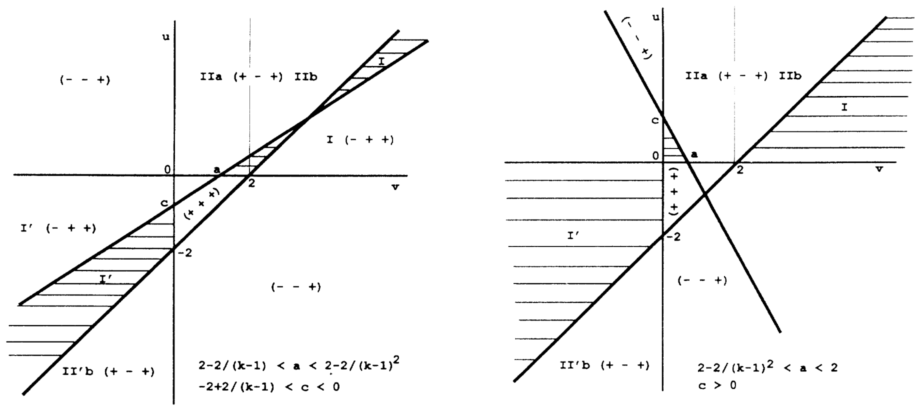

FIG. 1. Allowed regions of the $3d$ manifold for the coset model SO(2,2)/SO(2,1). (a) $b>k/(k-2)$. (b) $1<b<k/(k-2)$.

$$G_{bb}=\frac{1}{4(b^2-1)}\ ,$$

$$G_{vv}=-\frac{\beta(v,u,b)}{4v(v-u-2)}\left[\frac{b+1}{b-1}+\frac{1}{k-1}\frac{u+2}{v-u-2}\right]\ , \tag{3.6}$$

$$G_{uu}=\frac{\beta(v,u,b)}{4u(v-u-2)}\left[\frac{b-1}{b+1}-\frac{1}{k-1}\frac{v-2}{v-u-2}\right]\ ,$$

$$G_{vu}=\frac{1}{4(k-1)}\frac{\beta(v,u,b)}{(v-u-2)^2}\ ,$$

and where the function $\beta(v,u,b)$ is defined as

$$\beta^{-1}(v,u,b)=1+\frac{1}{k-1}\frac{1}{v-u-2}\times\left[\frac{b-1}{b+1}(u+2)-\frac{b+1}{b-1}(v-2)-\frac{2}{k-1}\right]\ . \tag{3.7}$$

It remains to solve the system of differential equations (3.4). Although the solution to those equations is straightforward, it is illuminating to guess the solution by recalling some of the results of [7] and [8] (see also [21]). There it was found that various factors in the measure, including the group Haar measure, the Faddeev-Popov determinant in a unitary gauge, and the determinant produced by integrating out the gauge fields, combine together to give the square root of the determinant of the metric. That is

$$\text{Haar}\times\text{Faddev}-\text{Popov}/\text{determinant}=\sqrt{-G}\ . \tag{3.8}$$

Moreover, the dilation Φ which solves the conformal conditions [16] was identified as $e^{\Phi}=$determinant. These observations led to the concluison that the purely group-theoretical Haar$\times$Faddeev$-$Popov can always be written in the form $e^{\Phi}\sqrt{-G}$. This result was true in the semiclassical limit $k\to\infty$. Because of the group-theoretic nature of the derivation it was conjectured that the combination $e^{\Phi}\sqrt{-G}$ is k independent and equal to the one-loop result, although individually the metric and the dilaton can receive $1/k$ corrections. In the notation of [14] the one-loop semiclassical result is

$$e^{\Phi}\sqrt{-G}\,|_{k\to\infty}=\left[\frac{b^2-1}{vu}\right]^{1/2}\ . \tag{3.9}$$

One can now check that this expression indeed satisfies the system of differential equations (3.4) for all values of k. Therefore, the conjecture is correct as we expected on the basis of the group-theoretical argument above. So we have proven the theorem

$$e^{\Phi}\sqrt{-G}=e^{\Phi}\sqrt{-G}\,|_{k\to\infty}\quad\text{for all } k\ . \tag{3.10}$$

This is also true in all string and superstring models we consider in the present paper. We are convinced that this is a general feature of gauged WZW models. After calculating $\sqrt{-G}$ from (3.3) or (3.6) the result for the dilaton is

$$\Phi=\ln\left[\frac{(b^2-1)(v-u-2)}{\sqrt{\beta(v,u,b)}}\right]+\Phi_0\ , \tag{3.11}$$

where Φ_0 is the constant of integration. In the limit $k\to\infty,\beta\to 1$ both the metric and the dilaton tend to their semiclassical expressions of [14].

One might ask the question: How does the finite value of k modify the manifold? In Figs. 1–3 the allowed regions in the v-u plane are indicated at fixed values of b. As in [14], the three signs inside the parentheses in the various regions are the signs of the coefficients of dv^2,du^2, and db^2 in the semiclassical metric. A minus (plus) sign corresponds to a time (space) coordinate, thus indicating the signature of the region. The regions with one time coordinate correspond to the SO(2,2)/SO(2,1) coset. The remaining regions correspond to the analytic

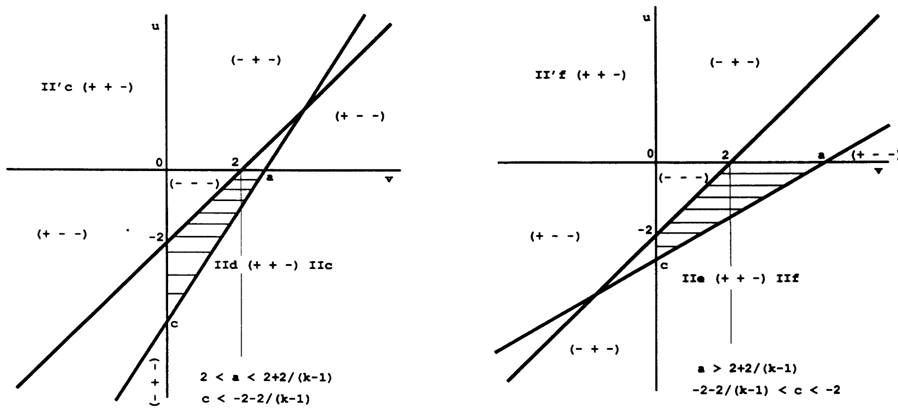

FIG. 2. Same as Fig. 1 but with (a) $0<b<1$ and (b) $-1<b<0$.

continuations SO(3,1)/SO(3) $(+++)$, SO(4)/SO(3) $(---)$, and SO(3,1)/SO(2,1) (one plus). Thus, by specializing to each one of these regions our exact metric and dilaton describe those cosets as well. The 45° line $u=v-2$ is a curvature singularity in the semiclassical limit, the other two being at $b=\pm 1$. The way the b-fixed planes are sliced up by the lines at $u=0, v=0, v-u-2=0$ into regions of various signatures is a purely group-theoretical result about the coset manifolds that are listed above. That is, the coset manifold SO(2,2)/SO(2,1) lives in the ranges of (v,u,b) parameters indicated in the figures, independently of any metric (similarly for the other manifolds). As it turns out, the full region coincides with the properties of the semiclassical metric. However, quantum corrections may require additional constraints on the acceptable regions in order to maintain the signature. This is indeed what happens, and how the k dependence of the exact metric shows up. The second line, with varying slope (which depends on b) is a singularity of the function $\beta(v,u,b)$. For the case of the coset SO(2,2)/SO(2,1) and $k>2$,[7] one must demand that $\beta(v,u,b)>0$ so that the exact metric has the correct signature, as seen from the determinant of the metric (one time coordinate requires $\det G<0$). This leads to further restrictions for the allowed regions. The result is shown as the shaded areas in the figures: they have switched signature due to the quantum corrections. Therefore, although they were previously allowed, they are now off limits since a classical geodesic in the SO(2,2)/SO(2,1) unshaded regions cannot enter the shaded areas. Therefore quantum amplitudes are expected to decay off and tunnel in these regions. This implies that quantum effects have created a screening of the classical singularity, although not everywhere.

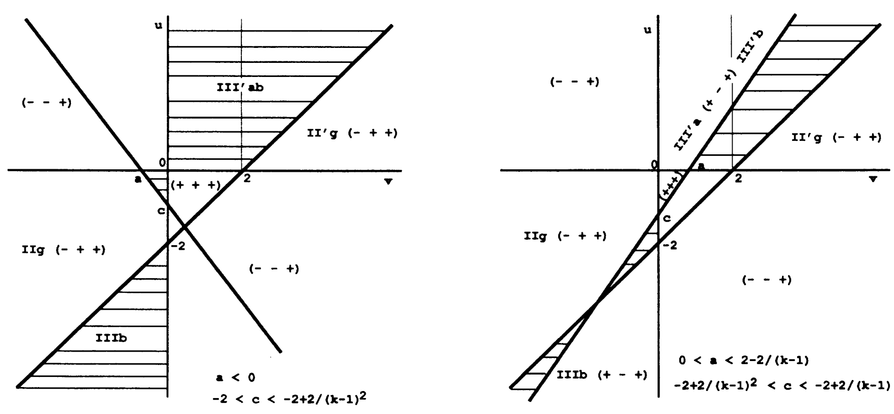

FIG. 3. Same as Fig. 1 but with (a) $-k/(k-2)<b<-1$ and (b) $b<-k/(k-2)$.

[7]The signature depends crucially on the sign of $k-2$ as seen from (3.5). Demanding $c=26$ for the bosonic string gives $k\simeq 2.48$ or $k\simeq 0.91$, and $c=15$ for the superstring gives $k=20/7$. It is believed that a consistent quantum theory requires $k>2$ [18]. Nevertheless, one could perform a similar analysis even when $k<1$.

IV. 4d BOSONIC STRING

The 4*d* model based on the coset SO(3,2)/SO(3,1) was analyzed in [8] where expressions for the perturbative metric and dilaton were given in some patches of the manifold. We will see that the method we have been following in the present paper leads to the discovery of the global coordinates as well. The four invariants one can construct and which satisfy the gauge condition (2.15) are

$$x^2,\quad z_1=\tfrac{1}{4}\,\mathrm{Tr}(a^2),\quad z_2=\tfrac{1}{4}\,\mathrm{Tr}(a^*a),\quad z_3=xa^2x/x^2\ , \tag{4.1}$$

where $a^*_{\mu\nu}=\frac{1}{2}\epsilon_{\mu\nu\alpha\beta}a^{\alpha\beta}$ is the dual of $a_{\mu\nu}$. However, the metric written in these coordinates is nondiagonal and very complicated. Instead, we use a different set of four invariants b,u,v,w for which the semiclassical metric is diagonal,

$$\begin{aligned} b&=\frac{1-x^2}{1+x^2},\quad u=\frac{1+z_2^2+2(z_1-z_3)}{1-2z_1-z_2^2}\ ,\\ v&=\frac{1+z_1+(z_1^2+z_2^2)^{1/2}}{1-z_1-(z_1^2+z_2^2)^{1/2}},\quad w=\frac{1+z_1-(z_1^2+z_2^2)^{1/2}}{1-z_1+(z_1^2+z_2^2)^{1/2}}\ .\end{aligned} \tag{4.2}$$

To find the ranges in which the above global coordinates take their values we consider a Lorentz frame that can cover all possibilities without loss of generality. First, we notice that by Lorentz transformations the antisymmetric matrix $a_{\mu\nu}$ can always be transformed, as in [8], to a block-diagonal matrix, with the nonzero elements

$$a_{01}=\tanh t\quad\text{or}\ \coth t,\quad a_{23}=\tan\phi\ . \tag{4.3}$$

Then using (4.2) one can deduce the form of the global variables: $v=\pm\cosh 2t$, $w=\cos 2\phi$, and

$$u=\frac{1}{x^2}[w(x_0^2-x_1^2)-v(x_2^2+x_3^2)]\ .$$

When $x^2<0$, apart from an overall factor we may give its components by $(0,\cos\theta,0,\sin\theta)$ or by $(\sinh\psi,\cosh\psi,0)$, on the other hand, when it is timelike $x^2>0$ we may write $(\cosh\psi,\psi,0,\sinh\psi,0)$. These three possibilities give the following expressions for u: (i) $u=w\cos^2\theta+v\sin^2\theta$, (ii) $u=-w\sinh^2\psi+v\cosh^2\psi$, (iii) $u=w\cosh^2\psi-v\sinh^2\psi$. Then the string variables take values in the following regions with the signature indicated in the (b,u,v,w) basis

$$\begin{aligned} &(++-+):\ b^2>1,\ \{-1<w<u<1<v\ \text{or}\ v<-1<u<w<1\ \text{or}\ -1<w<1<u<v\}\\ &(+-++):\ b^2>1,\ \{-1<w<1<v<u\ \text{or}\ u<v<-1<w<1\}\\ &(-+++):\ b^2<1,\ \{u<w<1<v\ \text{or}\ v<-1<w<u\ \text{or}\ v<u<-1<w<1\}\ .\end{aligned} \tag{4.4}$$

Then by considering states of the type $T=T(b,u,v,w)$ and following a procedure similar to the 3*d* case we find the line element

$$ds^2=2(k-3)(G_{bb}db^2+G_{uu}du^2+G_{vv}dv^2+G_{ww}dw^2+2G_{uv}dudv+2G_{uw}dudw+2G_{vw}dvdw)\ , \tag{4.5}$$

where

$$\begin{aligned} G_{bb}&=\frac{1}{4(b^2-1)}\ ,\\ G_{uu}&=\frac{\beta(b,u,v,w)}{4(u-w)(v-u)}\left\{\frac{b-1}{b+1}-\frac{1}{k-2}\frac{(v-w)^2}{(v-u)(u-w)}\left[1-\frac{1}{k-2}\frac{b+1}{b-1}\right]\right\}\ ,\\ G_{vv}&=-\frac{(v-w)\beta(b,u,v,w)}{4(v^2-1)(v-u)}\left\{\frac{b+1}{b-1}-\frac{1}{k-2}\frac{1}{(v-u)(u-w)}\right.\\ &\qquad\left.\times\left[1-u^2+\left[\frac{b+1}{b-1}\right]^2(v-u)(v-w)+\frac{1}{k-2}\frac{b+1}{b-1}\frac{(1+v^2)(u+w)-2v(1+uw)}{v-w}\right]\right\}\ ,\\ G_{ww}&=\frac{(v-w)\beta(b,u,v,w)}{4(1-w^2)(u-w)}\left\{\frac{b+1}{b-1}-\frac{1}{k-2}\frac{1}{(v-u)(u-w)}\right.\\ &\qquad\left.\times\left[1-u^2+\left[\frac{b+1}{b-1}\right]^2(u-w)(v-w)-\frac{1}{k-2}\frac{b+1}{b-1}\frac{(1+w^2)(u+v)-2w(1+uv)}{v-w}\right]\right\}\ ,\\ G_{uv}&=\frac{\beta(b,u,v,w)}{4(k-2)(v-u)^2}\left\{1-\frac{1}{k-2}\frac{b+1}{b-1}\frac{v-w}{u-w}\right\}\ ,\\ G_{uw}&=\frac{\beta(b,u,v,w)}{4(k-2)(u-w)^2}\left\{1-\frac{1}{k-2}\frac{b+1}{b-1}\frac{v-w}{v-u}\right\}\ ,\end{aligned} \tag{4.6}$$

$$G_{vw} = \frac{1}{(k-2)^2} \frac{b+1}{b-1} \frac{\beta(b,u,v,w)}{4(v-u)(u-w)} ,$$

where the function $\beta(b,u,v,w)$ is defined by

$$\beta^{-1} = 1 + \frac{1}{k-2} \frac{(v-w)^2}{(v-u)(w-u)} \left\{ \frac{b+1}{b-1} + \frac{b-1}{b+1} \frac{1-u^2}{(v-w)^2} + \frac{1}{k-2} \left[\frac{vw+u(v+w)-3}{(v-w)^2} - \left[\frac{b+1}{b-1} \right]^2 \right] \right\}$$
$$+ \frac{2}{(k-2)^3} \frac{b+1}{b-1} \frac{vw-1}{(v-u)(u-w)} . \tag{4.7}$$

The dilaton field is

$$\Phi = \ln \left[\frac{(b^2-1)(b-1)(v-u)(w-u)}{\sqrt{\beta(b,u,v,w)}} \right] + \Phi_0 . \tag{4.8}$$

It would be instructive to write the expression for the metric in the semiclassical limit and verify that the range for the string parameters (4.4) is such that there is only one time coordinate in every region. From (4.6) we obtain

$$\left. \frac{ds^2}{2(k-2)} \right|_{k\to\infty} = \frac{db^2}{4(b^2-1)} + \frac{b-1}{b+1} \frac{du^2}{4(v-u)(u-w)} + \frac{b+1}{b-1}(v-w) \left[\frac{dw^2}{4(1-w^2)(u-w)} - \frac{dv^2}{4(v^2-1)(v-u)} \right] . \tag{4.9}$$

This metric was derived in [8] in region $(++-+)$. The presence of a finite k modifies the manifold in a similar way to the 3d case, as considered in the preceding section.

Again, we can check that $e^{\Phi}\sqrt{-G}$ is independent of k, which proves the theorem for $d=4$.

V. TYPE-II AND HETEROTIC SUPERSTRINGS

In this section we consider superconformal extensions of the bosonic models we have been considering in the preceding sections. The general type-II and heterotic superstring model in curved space-time was defined in [8] in the form of a supersymmetric $N=1$ gauged WZW model. This corresponds to a Kazama-Suzuki model with a noncompact group SO$(1-1,2)$ [1] and therefore can be analyzed with current-algebra techniques [17,18]. The 4d case was worked out explicitly, to leading order in perturbation theory, using the Lagrangian method. As we shall see, the exact metric and dilaton for these superstring models are closely related to the corresponding expressions for the bosonic strings. It was pointed out in footnote 5 that for any WZW model $\Delta_G^L = \Delta_G^R \equiv \Delta_G$. Restricting to H-invariant tachyon states, as we were instructed to do by (2.5), gives another condition $\Delta_H^L T = \Delta_H^R T \equiv \Delta_H T$, or equivalently $\Delta_{G/H}^L T = \Delta_{G/H}^R T \equiv \Delta_{G/H} T$. For the bosonic string we saw that these remarks led to the k dependence exhibited in (2.6). Now we turn to the superstrings.

For the type-II superstring the coset model with $N=1$ superconformal symmetry ($N=2$ if G/H is Kählerian) is described by [1]

$$\frac{\mathrm{SO}(d-1,2)_{-k} \otimes \mathrm{SO}(d-1,1)_1}{\mathrm{SO}(d-1,1)_{-k+g-h}} , \tag{5.1}$$

for both the left and the right movers. For the tachyon the fermionic factor SO$(d-1,1)_1$ makes no contribution. However, the shifting of the level in the denominator in (5.1), i.e., $(-k+g-h) = -k+1$ instead of $-k$ for the bosonic case, has a profound effect and we obtain the quantum Hamiltonian with the exact dependence on k

$$(L_0^L + L_0^R) T_{\mathrm{II}} = \left[\frac{\Delta_G^L}{k-g} - \frac{\Delta_H^L}{k-g} + \frac{\Delta_G^R}{k-g} - \frac{\Delta_H^R}{k-g} \right] T_{\mathrm{II}}$$
$$= \frac{2}{k-g} \Delta_{G/H} T_{\mathrm{II}} . \tag{5.2}$$

Except for an overall renormalization of $k \to k-g$, this is exactly the expression in the semiclassical limit. Therefore, almost trivially we have proven a theorem: For any type-II superstring based on a Kazama-Suzuki coset, as in (5.1), the exact metric and dilaton are given by the one-loop perturbative result except for the overall normalization of the metric. In the special case of $d=2$ available field-theoretic perturbative computations verify this result up to five loops [22].

For a heterotic superstring only the left sector is supersymmetric whereas the right sector is not. Therefore, for the tachyon we can write

$$(L_0^L + L_0^R) T_{\mathrm{het}} = \left[\frac{\Delta_G^L}{k-g} - \frac{\Delta_H^L}{k-g} + \frac{\Delta_G^R}{k-g} - \frac{\Delta_H^R}{k-h} \right] T_{\mathrm{het}}$$
$$+ L_0^R(\mathrm{int}) T_{\mathrm{het}}$$
$$= \frac{2}{k-g} \left[\Delta_{G/H} + \frac{g-h}{2(k-h)} \Delta_H \right] T_{\mathrm{het}}$$
$$+ L_0^R(\mathrm{int}) T_{\mathrm{het}} , \tag{5.3}$$

where $L_0^R(\mathrm{int})$ is the internal part which does not contribute to the spacetime metric and dilaton. Comparing (2.6) to (5.3), we see that we can obtain the exact metric and dilaton for the heterotic superstring if we replace k in the bosonic expressions by $2k-h$ (except for the overall factor in the line element ds^2 which is the same in both cases). In the $k \to \infty$ limit these fields tend to their bosonic counterparts, as in [8], in agreement with the general arguments of [16].

VI. CONCLUDING REMARKS

We found a general method for obtaining the conformally exact metric and dilaton fields for any theory which can be formulated as a coset model. Since the value of k that yields $c=26$ (or $c=15$) is actually small, our results represent substantial deviations from the semiclassical computations. We have applied our method to the coset models $SO(d-1,2)/SO(d-1,1)$. Our expressions hold true also for all the models that are obtained from this coset by appropriate analytic continuations, e.g., $SO(d,1)/SO(d)$,$SO(d+1)/SO(d)$, etc., by simply specializing to the appropriate region of our global space. For the cases $d=2,3,4$ we gave explicit results. We have also derived results that apply to the general gauged WZW model with or without supersymmetry. For any type-II superstring the perturbative one-loop results were shown to be also exact except for an overall factor in the metric. We have also shown that for a heterotic superstring the conformally exact metric and dilaton fields can be obtained by a simple shifting of k in the corresponding bosonic expressions. Finally, we have shown that the combination $e^{\Phi}\sqrt{-G}$ is indeed k independent as conjectured in earlier work.

Our exact results may be verified in perturbation theory, as was the case in $d=2$ (the $2d$ heterotic case has not yet been verified). However, this should be regarded as a challenge for perturbation theory which is beset with uncertainties over renormalization schemes. It is generally believed, but only tentatively proved that coset models and gauged WZW models are equivalent. This is certainly the case at the classical level, as can be seen from the equations of motion in the axial gauge [3,7,8,14]. Furthermore, our Hamiltonian approach should leave no doubt that semiclassically these two theories are equivalent. The Hamiltonian versus the path integral can be regarded as two possible approaches to quantization which may differ in higher orders of $\hbar$, unless one ensures that they are equivalent by choosing the correct measure of integration. If only to strengthen the relationship between these two formulations, it may be of interest to study the perturbative approach to verify our results for the more challenging cases in $d=3,4$. This should also be helpful in pinning down the appropriate renormalization scheme which may be useful for other computations in these models.

We have pointed out the screening effects of the quantum corrections in the neighborhoods of the singularities. It is not yet clear how to use these conformally exact results in physical applications. One needs to know how this "classical" string vacuum configuration competes with higher-genus string loop effects in specific physical situations. It may be that, together with the small-large duality properties pointed out in [14], one may arrive at believable physical conclusions in some regimes even from the zero-genus exact computation presented in this paper.

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APPENDIX

The reader who is familiar with the exact results for the metric and dilaton of the $2d$ Minkowski or Euclidean black hole, obtained in [19], may wonder how these can be deduced in our formalism. This appendix serves exactly this purpose. In the $2d$ case, for the coset model SO(2,1)/SO(1,1), the Casimir operators (2.11) and (2.12) take the form

$$\begin{aligned}\Delta_G^L &= \tfrac{1}{4}(1+x^2)^2\frac{\partial^2}{\partial x^\mu \partial x_\mu} - \tfrac{1}{4}(1+x^2)\frac{\partial^2}{\partial t^2} \\ &\quad + \tfrac{1}{2}(1+x^2)\frac{\partial}{\partial t}\epsilon_{\mu\nu}x^\mu \frac{\partial}{\partial x_\nu} , \\ \Delta_H^L &= -\frac{1}{4}\frac{\partial^2}{\partial t^2} ,\end{aligned} \tag{A1}$$

where we have used $a_{\mu\nu}=a\epsilon_{\mu\nu}$, and $(1-a^2)(\partial/\partial a)=\partial/\partial t$ for $a=\cosh t$ or $\sinh t$. The gauge condition (2.15) is

$$\epsilon_{\mu\nu}x^\mu \frac{\partial}{\partial x_\nu}T=0 . \tag{A2}$$

For gauge-invariant tachyon states of the form $T=T(b,t)$, where b is defined[8] by $b=(x^2-1)/(x^2+1)$, the Casimir operators reduce to the simpler equations

$$\begin{aligned}\Delta_G^L T &= \left[-(b^2-1)\partial_b^2 - 2b\partial_b + \frac{1}{2(b-1)}\partial_t^2\right]T , \\ \Delta_H^L T &= -\tfrac{1}{4}\partial_t^2 T .\end{aligned} \tag{A3}$$

Proceeding as before we obtain the line element

$$\begin{aligned} ds^2 &= 2(k-2)\left[\frac{db^2}{4(b^2-1)} - \beta(b)\frac{b-1}{b+1}dt^2\right] , \\ \beta^{-1}(b) &= 1-\frac{2}{k}\frac{b-1}{b+1} .\end{aligned} \tag{A4}$$

For the dilaton the corresponding expression is

$$\Phi = \ln\left[\frac{b+1}{\sqrt{\beta(b)}}\right]+\Phi_0 . \tag{A5}$$

The scalar curvature reads as

$$R=\frac{2k}{k-2}\frac{(k-2)b+k-4}{[(k-2)b+k+2]^2} . \tag{A6}$$

The scalar curvature is singular at $b=-(k+2)/(k-2)$ which is exactly the point of singularity for the function $\beta(b)$. To make contact with the results of [19] one needs to reparametrize b in the various patches. For instance, in the black-hole region outside the horizons at $b=1$ one

[8] We have here a sign difference with (3.2) which we used in higher dimensions. This allows us to agree with Fig. 4 in a previous paper [14] for $d=2$.

can set $b = \cosh 2r$, in the naked-singularity region $b = -\cosh 2r'$, and in the inside-the-horizons regions $b = \cos 2r''$. The corresponding expressions for the Euclidean black hole follow from (A4) and (A5) if we analytically continue $t \rightarrow i\theta$, where θ is compact, $0 < \theta < 2\pi$. The cigar ($b = \cosh 2r$) and trumpet ($b = -\cosh 2r'$) correspond to the SO(2,1)/SO(2) coset, whereas the cymbal region ($b = \cos 2r''$) to the SO(3)/SO(2) coset.

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