Quantum field theory in spaces with closed timelike curves

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Gott spacetime has closed timelike curves, but no locally anomalous stress energy. A complete orthonormal set of eigenfunctions of the wave operator is found in the special case of a spacetime in which the total deficit angle is 2π. A scalar quantum field theory is constructed using these eigenfunctions. The resultant interacting quantum field theory is not unitary because the field operators can create real, on-shell, particles in the noncausal region. These particles propagate for finite proper time accumulating an arbitrary phase before being annihilated at the same spacetime point as that at which they were created. As a result, the effective potential within the noncausal region is complex, and probability is not conserved. The stress tensor of the scalar field is evaluated in the neighborhood of the Cauchy horizon; in the case of a sufficiently small Compton wavelength of the field, the stress tensor is regular and cannot prevent the formation of the Cauchy horizon.

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I. INTRODUCTION

Spacetimes with closed timelike lines have generally been considered unphysical [1] because of logical paradoxes, the lack of a well-posed Cauchy problem, or the sense that they are obviously wrong. Prompted by the work of Morris *et al.* [2, 3], there has recently been an extensive reexamination of the question. The conclusion of this reexamination is that it is not trivial to decide whether closed timelike curves are physically allowed. Indeed, spacetimes with closed timelike curves exist, and solutions to field equations on these spaces exist. These solutions are complete on some spacelike surfaces in at least some noncausal spacetimes [4, 5]. Although the causality properties of these spacetimes are unfamiliar, they do not appear to be self-contradictory, and, if one is prepared to consider them at all, one must address the question of their acceptability in other terms.

The original wormhole spacetimes of Morris *et al.* [2] require that the stress-energy which supports the wormhole fail to satisfy the positive-energy condition. Although the matrix elements of the stress energy of a quantum field do not in general satisfy the positive-energy condition, their volume integrals over distances large compared with the wavelengths associated with the field excitations are in general positive, and it is still somewhat problematic to reconcile the existence of the wormholes with the stability of matter. On the other hand, Gott[1] [6] has pointed out that spacetimes with two relatively moving infinitely long straight strings can possess closed timelike curves. These spacetimes are vacuum spacetimes except for conical singularities at the strings, and each string alone is a physically acceptable solution to Einstein's equations. Although Carroll *et al.* [8] and Deser *et al.* [9] have pointed out that such spacetimes cannot arise from a spacetime which initially contains no closed timelike curves and has a positive-definite total energy, the Gott spacetime itself does not have any local properties which are physically unacceptable. The existence of the acausal region in which future directed nonspacelike curves can intersect each other is the only peculiarity of the spacetime.

These considerations suggest that it is worth studying the properties of Gott space in more detail. A point mass in $2+1$ dimensions produces a spacetime which is everywhere flat. Coordinates may be chosen in which the metric is the flat Minkowski metric, but with a wedge removed from the space, and the points along the edges of the wedge identified. The resultant cone has a singularity at its tip where the point mass which produces the space is located. The circumference of a circle of radius r is $(2\pi - \theta)r$, where θ is the deficit angle of the cone and is a measure of the mass [7, 10–12]. The Gott spacetime is generated by two such point masses moving relative to one another. The scalar wave equation is particularly easy to analyze in the special limiting case in which the deficit angles of the two points are both π. (That space is open, whereas if the sum of the deficit angles were greater than 2π the space would be closed [13].) In particular, a scalar quantum field may be constructed on the space using a path integral to calculate the propagators. The resultant free field theory appears to be fully acceptable; however, an interacting field theory is not unitary.

The procedure for constructing the field theory is fairly straightforward. Coordinates may be chosen in which the metric is the Minkowski metric, but with the edges of the removed wedges identified with additional boosts. Because of the boosts, the point (in $2+1$ dimensions) or string (in 3+1 dimensions) singularities are moving. Despite the unusual boundary conditions, it is possible to solve the scalar wave equation on the spacetime. A complete orthonormal set of functions which are eigenfunctions of the wave operator is exhibited in Sec. IV. Using these eigenfunctions, the functional integral which

[1]See also Deser *et al.* [7].

defines, in causal spacetimes, the time-ordered matrix elements of the field operators in the vacuum will be evaluated in Sec. V. The resulting matrix elements appear as an infinite sum of terms corresponding to the various possible winding numbers of paths around the singularities. Alternatively, each term may be viewed as corresponding to a given image under (boosted) reflection in the boundaries. In this acausal spacetime the individual terms are precisely the functions which one would naively write down, however, with each term in the sum being separately time ordered according to whether the field point is in the past or future of the source point image. As a result, when both the points of the matrix element of a pair of field operators (or Green's function) are in the acausal region, the points may be connected by a future-directed timelike curve for some winding numbers and by a past-directed timelike curve for other winding numbers. Because of this, the functional integral result for $\langle(\phi(x)\phi(x'))_+\rangle$ cannot be regarded as a matrix element in which the field associated with the earlier time lies to the right of the field associated with the later time. When both points are in the acausal region, each point is both in the future and in the past of the other point. The functional integral defines the matrix element by having either field create positive energy excitations which travel along future directed timelike curves to be annihilated by the other field.

To put this another way, time-ordered products cannot be constructed because there is not a well-defined time ordering for pairs of points in the acausal region. However, the propagation of particles (or waves) is well ordered in that they propagate forward in time. (These spacetimes do have a well-defined direction of time.)

The complete set of eigenfunctions of the wave operator then provides a complete set of solutions of the wave equation. These are complete on a given spacelike surface in the causal region, and that completeness constitutes a special case of the theorem proven by Morris and Friedman [5]: It is possible to arbitrarily specify the positive-frequency field on an initial spacelike surface which lies entirely outside the acausal region; this uniquely determines the positive frequency field throughout the spacetime.

If a Hermitian product of field operators acts at the same spacetime point in a causal spacetime, its vacuum matrix element must be real because it can only create virtual particles which are reabsorbed at the same point. If the point is in the causal region of an acausal spacetime, the same result holds, but in the acausal region one field can create an excitation which propagates forward in time, accumulating an arbitrary positive phase, to be annihilated by another field acting at the same spacetime point. Since no negative phases can be produced (particles always propagate forward in time), the matrix element of the product of the field operators in the vacuum will not in general be real. The fact that matrix elements of "physical" operators such as current densities and stress-energy tensors no longer possess the expected reality properties in the acausal region can be ignored in a noninteracting theory because these operators are uncoupled in that theory. The lack of Hermiticity appears only in multiplicative factors of the matrix elements which are canceled in the renormalization process.

The situation is quite different for an interacting field. The equation for the field itself contains products of more than one field. In a causal spacetime and in the causal region of an acausal spacetime, the vacuum expectation value of the interaction term becomes a real effective potential in which the excitations propagate. In the acausal region of an acausal spacetime, the vacuum expectation value gives an effective potential which is complex due to the phases of the particles which are created and reabsorbed. This potential does not yield unitary propagation, and the lack of unitarity is not associated with the creation of particles which appear at future infinity I^+. It is associated with the real creation of particles which are reabsorbed by the same interaction. That is, it is associated with what happens within the acausal region. In the case of an interacting field, complete information about what happens there is not available on spacelike surfaces restricted to the causal region, and the data which establishes the state of the system in the acausal region cannot be given on such surfaces. The first-order corrections due to the complex potential produce probabilities which may be greater than one, thereby violating unitarity in any sense.

There has been some discussion of the response of the metric to the stress tensor induced by the existence of an acausal region [14, 15]. In the model discussed here, since the propagator is calculated exactly, it is straightforward to calculate the matrix element of the stress-energy tensor. In the causal region it has the correct Hermiticity, and, after it has been renormalized, it is regular everywhere in the causal region but becomes singular as the Cauchy horizon is approached. This singularity is of order $1/[x^\pm \ln^2(x^+x^-)]$, where $x^\pm = 0$ defines the Cauchy horizon. When this is inserted into Einstein's equations, the resultant metric is of the form

$$ds^2 = e^{2\psi(-x^+x^-)}dx^+dx^- + e^{2\phi(-x^+x^-)}dy^2 \tag{1.1}$$

where $e^{2\psi} \to (-x^+x^-)^{2C\ln\ln[Y_0^2/(-x^+x^-)]}$ as $-x^+x^- \to 0^+$. For zero mass fields, this singularity is fairly weak, and it is not clear what a simultaneous solution of Einstein's equations and the field equations would yield as a self-consistent solution. Nor is it clear how this relates to Hawking's chronology protection conjecture [15]. However, for nonzero mass fields and for sufficiently small relative rapidity of the point masses relative to their separation, there is no singularity of the stress energy and, therefore, no mechanism for chronology protection.

Hartle [20] has studied the problem of unitarity in acausal spacetimes using the decoherence approach of Gell-Mann and Hartle [21]. He too finds that there is a lack of unitarity, but for cases where there is an acausal region to the future of the spacelike surfaces on which measurements are made, in addition to the case where the acausal region lies between the surfaces. The procedure followed in this work exhibits a lack of unitarity only when the acausal region lies between the surfaces on which measurements are made. This seems more reasonable in that our ability to make consistent measurements now should not be compromised by the behavior of the system at points which are in the future of the entire region in which the measurements are made.

The order of presentation is as follows. The following section consists of a brief outline of Gott space, establishing the conventions used here. The causal properties of the space are discussed in Sec. III, followed by, in Sec. IV, a derivation of the complete orthonormal set of eigenfunctions of the wave operator on the space. The Green's function and operator products derived in Sec. V are used in Sec. VI to derive the properties of the matrix elements of the stress tensor and the operator ϕ^2. These results are used in Sec. VII to discuss the properties of the interacting field, and exhibit the lack of unitarity of the interacting field. The effect of the quantum field on the metric is discussed in the last section.

II. GOTT SPACE

The general matter-free solution to Einstein's equations in $2+1$ dimensions is everywhere flat, and the solution with point masses is flat except for conical singularities at the locations of the masses [7]. The masses are proportional to the deficit angles at the singularities. A $(3+1)$-dimensional spacetime with infinitely long straight strings running parallel to the z axis is also flat with conical singularities along the strings.

Gott [6] pointed out that two relatively moving point masses in $2+1$ dimensions (or two strings in $3+1$ dimensions) produce a spacetime which has closed timelike lines. He considered the physically realistic case in which the deficit angles of the masses are small, and, as a consequence, the relative velocity of the masses must be large in order to produce an acausal spacetime. The space is open only if the sum of the deficit angles associated with the masses is less than or equal to 2π [7]. Only the limiting case of two equal masses with deficit angle π will be considered in this work; in that case it is easy to obtain explicit solutions to the various equations. In general, only the expressions in $2+1$ dimensions will be presented; the straightforward generalizations to $3+1$ dimensions will be given when required for comparison purposes.

The spacetime produced by a single point mass with deficit angle π may be described by a half plane, as shown in Fig. 1, with the edge identification $x \doteq -x$ at $y=0$. This physical space may be extended to a covering space consisting of the entire x–y plane, and an arbitrary function on the physical space may be represented by a function on the covering space which satisfies the condition

$$f(x,y,t) = f(-x,-y,t)\,. \tag{2.1}$$

Such a function is uniquely determined by the function on the physical space. An arbitrary C^∞ function $\phi(x,y,t)$ on the covering space yields a C^∞ function on the physical space provided that $\phi(x,0,t) = \phi(-x,0,t)$. The eigenfunctions of the wave operator on the physical space are eigenfunctions of the wave operator on the covering space which satisfy this continuity condition. Given $\phi(x,y,t)$, an arbitrary eigenfunction of the wave operator on the covering space, an eigenfunction $f(x,y,t)$ on the

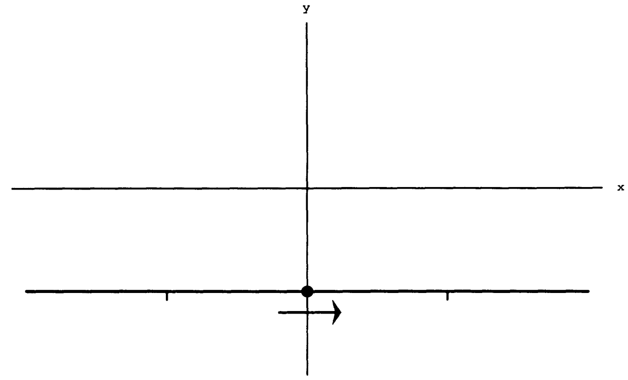

FIG. 1. The space for a single point mass of deficit angle π. The space consists of the region above the line with the tick marks denoting the identified lines.

physical space may be constructed as

$$f(x,y,t) = \phi(x,y,t) + \phi(-x,-y,t)\,. \tag{2.2}$$

In order to find the functions on Gott spacetime which is generated by two relatively moving point masses, this condition must be expressed in a frame in which the point mass is moving. The Lorentz transform to the moving frame is most conveniently done in null coordinates:

$$x^\pm \equiv x \pm t\,; \tag{2.3}$$

then, in boosted null coordinates,

$$x'^\pm = e^{\pm\alpha} x^\pm\,, \quad y' = y - Y_0\,. \tag{2.4}$$

The mass point that was at $x=0=y$ in the original coordinates is at $x' - t' \tanh\alpha = 0 = y' + Y_0$ in the new coordinates, corresponding to a mass point moving with velocity $\tanh\alpha$ in the new coordinates. Since the metric was Minkowskian in the original coordinates, it is still the Minkowski metric but with the boosted identification

$$x^\pm \doteq -e^{\pm 2\alpha} x^\mp\,, \quad \text{at } y = -Y_0\,, \tag{2.5}$$

where the primes on the coordinates have been dropped. The condition that a function on the covering space defines a function on the physical space is

$$f(x^+,x^-,y) = f(-e^{2\alpha}x^-, -e^{-2\alpha}x^+, -y-2Y_0)\,, \tag{2.6}$$

where f is now regarded as a function of the null coordinates $x^\pm$ rather than as a function of x and t.

If a second mass point moving in the opposite direction is added at $y = Y_0$, there is then the further identification

$$x^\pm \doteq -e^{\mp 2\alpha} x^\mp \qquad \text{at} \qquad y = Y_0\,. \tag{2.7}$$

The resultant space shown in Fig. 2 is restricted to the region $-Y_0 < y < Y_0$. For a total deficit angle less than 2π, the space is asymptotically a cone. In this case, the cone has a zero opening angle; i.e., it is asymptotically one end of a cylinder for point masses relatively at rest. Because of the relative motion of the point masses, the cylinder must be regarded as two cylinder halves joined with a relative boost.

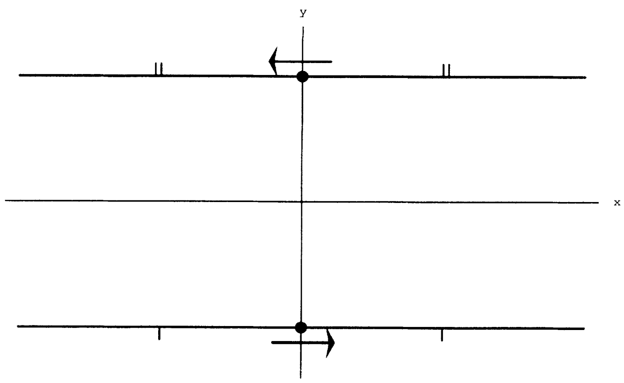

FIG. 2. The space for two point masses of deficit angle π. The space consists of the region between the lines with the single and double tick marks respectively denoting the identified lines.

The continuity conditions for functions on this space are those of the original string, Eq. (2.6), as well as

$$f(x^+,x^-,y) = f(-e^{-2\alpha}x^-,-e^{2\alpha}x^+,-y+2Y_0)\,; \tag{2.8}$$

these may be combined to yield the condition

$$f(x^+,x^-,y) = f(e^{-4\alpha}x^+,e^{4\alpha}x^-,y+4Y_0)\,, \tag{2.9}$$

for a function on the covering space to define a function on the physical space. This condition may be combined with either of the two previous conditions to form a sufficient set of conditions for a function on the covering space to define a function on the physical space. When the rapidity α vanishes, the masses are stationary, and the functions are periodic in y and even under inversion in the location of each of the masses. In the general case, the functions are even under boosted inversion in the location of each of the masses which implies that they are periodic under simultaneous translations in y by $4Y_0$ and in rapidity by -4α.

Just as with the single mass, an arbitrary function $\phi(x^+,x^-,y)$ on the covering space may be used to generate a function on the physical space which explicitly satisfies the continuity conditions [Eqs. (2.6), (2.8), and (2.9)]

$$\begin{aligned} f(x^+,x^-,y) = \sum_{n=-\infty}^{\infty} \big[&\phi(e^{-4n\alpha}x^+,e^{4n\alpha}x^-,y+4nY_0) \\ &+\phi(-e^{-(4n+2)\alpha}x^-,-e^{(4n+2)\alpha}x^+, \\ &\qquad -y+(4n+2)Y_0)\big], \end{aligned} \tag{2.10}$$

or

$$f(x^+,x^-,y) = \sum_{n=-\infty}^{\infty} \phi(x_n^+,x_n^-,y_n)\,, \tag{2.11}$$

where

$$\begin{aligned} x_n^{\pm} &= \begin{cases} e^{\mp 2n\alpha}x^{\pm}\,, & n \text{ even,} \\ -e^{\mp 2n\alpha}x^{\mp}\,, & n \text{ odd,} \end{cases} \\ y_n &= \begin{cases} y+2nY_0\,, & n \text{ even,} \\ -y+2nY_0\,, & n \text{ odd.} \end{cases} \end{aligned} \tag{2.12}$$

Two cases of particular interest are the point source and the Green's function which it produces. On the covering space a single source at the spacetime point x' is described by

$$\begin{aligned} \delta^{(3)}(x-x') &= \delta(t-t')\delta(x-x')\delta(y-y') \\ &= 2\delta(x^+-x'^+)\delta(x^--x'^-)\delta(y-y')\,. \end{aligned} \tag{2.13}$$

The point source on the physical space is thus given by

$$\delta_p^{(3)}(x-x') = \sum_{n=-\infty}^{\infty} 2\delta(x^+-x_n'^+)\delta(x^--x_n'^-)\delta(y-y_n')\,, \tag{2.14}$$

and the source in the physical spacetime is represented as a sum of image sources in the covering spacetime. Note that this expression is symmetric in x and x' so that either may be taken to be the independent variable, with the other being the location of the source point. As a result, this delta function defined on the covering space is a good function on the physical space when regarded as either a function of x or of x'. In $3+1$ dimensions, the delta function has an additional overall factor of $\delta(z-z')$.

The Green's function equation

$$(-\partial^2+m^2)G(x,x') = \delta(x-x')\,, \tag{2.15}$$

has the solution

$$G_0(x,x') = \begin{cases} \dfrac{i\exp[-ms(x,x')]}{4\pi s(x,x')} & 2+1 \text{ dimensions} \\ \dfrac{imK_1(ms(x,x'))}{2\pi^2 s(x,x')} & 3+1 \text{ dimensions} \end{cases} \tag{2.16}$$

in the covering space, where $s(x,x') = \sqrt{(x-x')^2}$, and the choice of analytic continuation into the region where $s<0$ determines which Green's function is obtained. The solution in the physical space then has the image form

$$G(x,x') = \sum_{n=-\infty}^{\infty} G_0(x,x_n')\,. \tag{2.17}$$

The choice of analytic continuation is critical; it will be discussed at length in Sec. V.

III. NONCAUSALITY

In the preceding section, a global set of coordinates for the spacetime with two moving point masses were found: The metric is the flat metric over the region $-Y_0 < y < Y_0$ with the identifications given by Eqs. (2.5) and (2.7). Since the causal properties of such spacetimes have been discussed extensively [6, 16, 17], a brief discussion will suffice here.

The conclusion is that this spacetime consists of three regions: A future region, a past region, and an acausal region. The past and future regions are causal in that no timelike or null curves intersect themselves in those regions, and they are respectively bounded in the future and the past by the Cauchy horizon which separates them from the acausal region.

The spacetime may be embedded in the covering spacetime, the causal properties of which are trivial. The source of a timelike or null curve has an infinite number of images in the covering space, Eq. (2.12), and a given spacetime point is within the future light cone of the source point in the physical spacetime if it is within the future light cone of any one of the images in the covering spacetime. A given point in the physical space may be identified with the $n=0$ image in the covering space. The point can be connected to itself by a nonspacelike curve in the physical space if and only if the image of the curve in the covering space is a nonspacelike curve. If the path in the physical space wraps n' times around the world line(s) of the point masses, its image in the covering space goes from the $n=n'$ image of the source point to the identified, $n=0$, image point.

Consider the past region P shown in Fig. 3. A source point in P has $x^+<0<x^-$, and its images at $(x_n^\pm, y_n)$ all lie on the same hyperbolic surface but displaced in y. The hyperbolic surface $x^+x^- = \text{const} < 0$, $x^->0$, and $-\infty<y<\infty$ is a spacelike surface so that all the images have spacelike separations from each other in the covering space. Thus, no future directed nonspacelike curve can connect the source to any of its images in the covering space, and, therefore, no future directed nonspacelike curve can connect the source point to itself in the physical space. A similar argument holds for the future region F defined by $x^+>0>x^-$. Hence both regions are causal; i.e., no nonspacelike curve can intersect itself in either region.

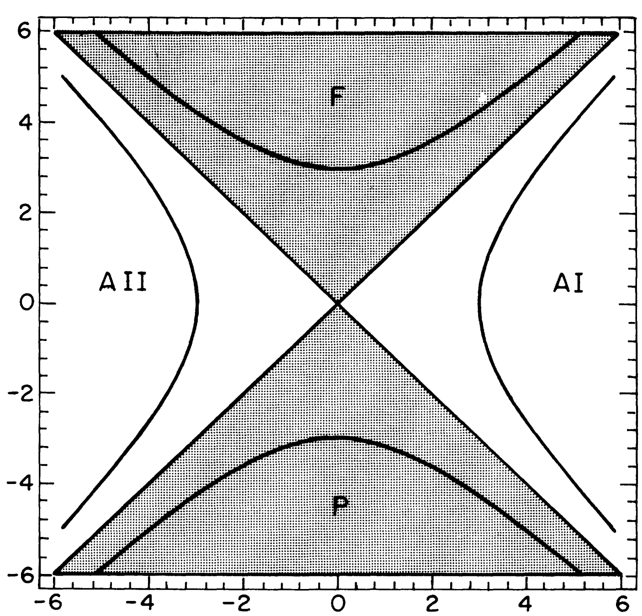

FIG. 3. The hyperbolae show the surfaces on which images of a point in the physical region lie. The past and future regions are shaded and the acausal region is unshaded. The y axis is perpendicular to the graph, and the physical space consists of the region $-Y_0<y<Y_0$.

The argument does not hold for the noncausal region A shown as two separate regions AI and AII in Fig. 3. (Note that although the two regions appear to be separated, they are not since they are connected at $y=\pm Y_0$.) Since these regions are defined by, respectively, $x^\pm>0$, and $x^\pm<0$, the images given by Eq. (2.12) appear alternately in the left and right branches on the hyperbolas shown. In addition, the images are displaced in y perpendicular to the diagram. The two surfaces in the covering space on which the images lie are each timelike surfaces which are asymptotically null at infinity, and each surface is everywhere spacelike relative to the other surface. As a result, no nonspacelike curve can connect a point to an image on the reflected hyperbola, but, since each surface is timelike, the images on the same hyperbola can in general be connected to each other by a nonspacelike curve. That is, nonspacelike curves which start out in A and wrap around both masses some number of times may intersect themselves. Because successive images are displaced in y, there is a spacelike component of the separation in the y direction. As a result, the projection of the light cones into the x-t plane is narrower than 45°, and some, but not all, images can be connected by a nonspacelike curve.

Points in the noncausal region can actually be connected to themselves by nonspacelike goedesics. To see this, note that the image of the geodesic in the physical space is the usual straight line in the covering space. The tangent vector of the curve is $\vec{\partial}/\partial\tau=(k^+,k^-,\pm1)$, where the affine parameter τ is normalized so that $dy/d\tau=\pm1$ in the covering space. The geodesic is a future directed nonspacelike curve if $k^+>0>k^-$, and $-k^+k^-\geq1$. Thus $k^\pm=\pm e^{\pm\eta}v$, where $v\geq1$. The curve starts at the image point given by (2.12), $(e^{-4n\alpha}x^+, e^{4n\alpha}x^-, y+4nY_0)$, and then, in the covering space, is given by

$$\begin{aligned} x^+(\tau) &= e^{-4n\alpha}x^+ + ve^{\eta}\tau\,,\\ x^-(\tau) &= e^{4n\alpha}x^- - ve^{-\eta}\tau\,,\\ y(\tau) &= y+4nY_0\pm\tau\,. \end{aligned} \tag{3.1}$$

The geodesic reaches $(x^\pm,y)$ provided there is a value of τ such that $x^+(\tau)=x^+$, $x^-(\tau)=x^-$, and $y(\tau)=y$. A solution exists only if the plus (minus) sign is chosen in (3.1), and n and $x^\pm$ are all negative (positive). In the case where the initial point is in the left segment AII, let $x^\pm=-Xe^{\pm\beta}$, $n=-N$, and the solution is

$$\begin{aligned} \tau &= 4NY_0\,,\\ v &= X\sinh 2N\alpha/(2NY_0)\,,\\ e^{\eta} &= e^{\beta+2N\alpha}\,. \end{aligned} \tag{3.2}$$

The number N is just the winding number of the path which connects the point $(x^\pm,y)$ to itself. Typical curves for winding numbers 1 and 2 are shown in the physical space in Fig. 4. A future-directed nonspacelike curve with winding number N connects the point x with itself

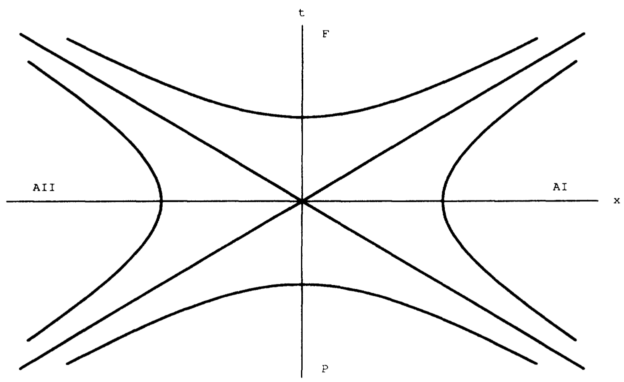

FIG. 4. The left and right figures respectively show closed future-directed timelike curves of winding number 1 and 2. The identified points are labeled in increasing temporal order from A to B (D) along the respective curves.

in the physical space provided that the curve is nonspacelike. This is true if $v = X \sinh 2N\alpha/2NY_0 \geq 1$. For large enough winding number every point with $X > 0$ can be connected to itself. Thus, the acausal region consists of all those points with $x^+x^- > 0$. The Cauchy horizons are, by definition, the boundary of the acausal region; these are the null surfaces $x^+ = 0$ and $x^- = 0$. The Cauchy horizons are not part of the acausal region because the images of a source point on the Cauchy horizon are all spacelike with respect to the source point, hence no nonspacelike curve can connect the image to the source point.

The surfaces for which $v = 1$ ($X = 2NY_0/\sinh 2N\alpha$) are the surfaces consisting of points at which null geodesics intersect themselves; Kim and Thorne [14] refer to them as polarized hypersurfaces. They will be discussed further in Sec. VI, where it will be shown that they are the loci of weak singularities of the stress-energy.

The surfaces $x^+x^- = \text{const} < 0$ are spacelike surfaces restricted to the causal region F (P) if $x^+ > 0$ ($x^+ < 0$); however, they are not Cauchy surfaces because there are future-directed nonspacelike curves which do not intersect the surfaces. (For example, a null curve coming in along the Cauchy horizon does not intersect a surface in the past region.) It is true that every timelike geodesic intersects every such surface in both the past and the future regions and, as will be discussed in Sec. IV, arbitrary solutions to the wave equation may be specified in terms of data on these surfaces.

IV. EIGENFUNCTIONS

For various purposes it is convenient to have an explicit set of complete orthonormal functions on the physical space; it is even more convenient if they are eigenfunctions of the wave operator. In the case of the Gott space considered here, it is easy to construct eigenfunctions on the covering space. To construct a complete set of orthonormal eigenfunctions of the wave operator on the physical space is not much more difficult. The main problem is showing that they are, in fact, a complete set.

On the covering space the eigenfunction ψ of the wave operator satisfies the equation

$$\partial^2\psi(x^+, x^-, y) = w\psi(x^+, x^-, y)\,, \tag{4.1}$$

where w is the eigenvalue of the wave operator. The general eigenfunction is given by

$$\psi(x^+, x^-, y) = \frac{1}{2}\int dk^+\, dk^-\, dk_y e^{i(k^+x^- + k^-x^+ + k_y y)} \times\delta(k^+k^- + k_y^2 + w) f(k^+, k^-, k_y)\,. \tag{4.2}$$

A special case is

$$\psi_{w,k_y}(x^+, x^-, y) = \frac{1}{2}\int dk^+\, dk^- e^{i(k^+x^- + k^-x^+)/2}\delta(k^+k^- + k_y^2 + w)e^{ik_y y} f_{w,k_y}(k^+/k^-)\,, \tag{4.3}$$

where f now depends only on the ratio k^+/k^-. This function must be invariant under the transformation, (2.9), $x^\pm \to e^{\mp 4\alpha}x^\pm$ and $y \to y + 4Y_0$, or

$$f_{w,k_y}(k^+/k^-) = e^{ik_y 4Y_0} f_{w,k_y}(e^{-8\alpha}k^+/k^-)\,, \tag{4.4}$$

which implies that a special solution with periodicity n and homogeneous in k^+/k^- with power $i\eta$ is

$$f_{w,k_y}(k^+/k^-) = e^{i\frac{y}{Y_0}(\frac{n\pi}{2} + \alpha\eta)}(k^+/k^-)^{i\eta/2}\,. \tag{4.5}$$

The general solution is then a superposition of the special solutions

$$\psi_{w',\eta,n}(x^+, x^-, y) = \frac{1}{4\pi}\int dk^+ dk^- e^{i(k^+x^- + k^-x^+)/2}\delta(k^+k^- + w')e^{i\frac{y}{Y_0}(\frac{n\pi}{2} + \alpha\eta)}(k^+/k^-)^{i\eta/2}, \tag{4.6}$$

where $w' = w + (n\pi/2 + \alpha\eta)^2/Y_0^2$ and $k_y = (n\pi/2 + \alpha\eta)/Y_0$. The function still does not satisfy the reflection conditions, (2.6) and (2.8), around the world lines of the two masses separately, and the ranges of the $k^\pm$ integrations are not yet determined. Under the reflections $x^\pm \to -e^{\mp 2\alpha}x^\mp$ and $y \to -y + 2Y_0$, the corresponding change of variables

$$k^\pm \to -e^{\mp 2\alpha}k^\mp\,, \tag{4.7}$$

which, along with $n \to -n$ and $\eta \to -\eta$, yields, modulo the question of the phases in the factor $(k^+/k^-)^{i\eta/2}$,

$$\psi_{w',\eta,n}(x^+,x^-,y) \to (-1)^n \psi_{w',-\eta,-n}(x^+,x^-,y)\,. \tag{4.8}$$

If $w' > 0$, the two-dimensional vector (k^+,k^-) is timelike, and the transformation (4.7) preserves the signs of $k^\pm$. The positive-frequency solution may then be taken to be

$$\psi^{(+)}_{w',\eta,n}(x^+,x^-,y) = \frac{1}{(2\pi)^{3/2}}\frac{1}{2}\int_0^\infty dk^+ \int_{-\infty}^\infty dk^- \delta(k^+k^- + w')e^{i(k^+x^- + k^-x^+)/2} \times \left[\frac{e^{i\frac{y}{Y_0}(\frac{n\pi}{2}+\alpha\eta)}}{\sqrt{2Y_0}i^n}\left(\frac{k^+}{-k^-}\right)^{i\eta/2} + \frac{e^{-i\frac{y}{Y_0}(\frac{n\pi}{2}+\alpha\eta)}}{\sqrt{2Y_0}i^{-n}}\left(\frac{k^+}{-k^-}\right)^{-i\eta/2}\right], \tag{4.9}$$

where, because of the explicit minus sign in the $k^+/(-k^-)$ factor, there are no phase ambiguities. The expression for negative η and $n \to -n$ is the same as Eq. (4.9); hence η may be taken to be positive. The overall normalization is shown to be correct in Appendix A, where the orthonormality and completeness of the functions are shown.

The complex conjugate of $\psi^{(+)}$ is a negative-frequency solution which is independent of the positive-frequency solution; it may be written in the form

$$\begin{aligned}\psi^{(-)}_{w',\eta,n}(x^+,x^-,y) &= \left(\psi^{(+)}_{w',\eta,n}(x^+,x^-,y)\right)^* \\ &= \frac{1}{(2\pi)^{3/2}}\frac{1}{2}\int_0^\infty dk^+ \int_{-\infty}^\infty dk^- \delta(k^+k^- + w')e^{-i(k^+x^- + k^-x^+)/2} \\ &\quad\times \left[\frac{e^{i\frac{y}{Y_0}(\frac{n\pi}{2}+\alpha\eta)}}{\sqrt{2Y_0}i^n}\left(\frac{k^+}{-k^-}\right)^{i\eta/2} + \frac{e^{-i\frac{y}{Y_0}(\frac{n\pi}{2}+\alpha\eta)}}{\sqrt{2Y_0}i^{-n}}\left(\frac{k^+}{-k^-}\right)^{-i\eta/2}\right].\end{aligned} \tag{4.10}$$

For $w' < 0$ the momentum (k^+,k^-) is spacelike, and the transformation, Eq. (4.7), changes the sign of k; instead of Eq. (4.7), the change of variables

$$k^\pm \to e^{\mp 2\alpha}k^\mp \tag{4.11}$$

is used, and the solution to the wave equation on the physical space, Eq. (4.1), is

$$\begin{aligned}\psi^{s}_{w',\eta,n}(x^+,x^-,y) = \frac{1}{(2\pi)^{3/2}}\frac{1}{2}\int_0^\infty dk^+ \int_{-\infty}^\infty dk^- \delta(k^+k^- + w') &\times \left[\frac{e^{i(k^+x^- + k^-x^+)/2}e^{i\frac{y}{Y_0}(\frac{n\pi}{2}+\alpha\eta)}}{\sqrt{2Y_0}i^n}\left(\frac{k^+}{k^-}\right)^{i\eta/2}\right. \\ &\left. + \frac{e^{-i(k^+x^- + k^-x^+)/2}e^{-i\frac{y}{Y_0}(\frac{n\pi}{2}+\alpha\eta)}}{\sqrt{2Y_0}i^{-n}}\left(\frac{k^+}{k^-}\right)^{-i\eta/2}\right]\end{aligned} \tag{4.12}$$

where $w' < 0$. This solution is real, and changing the sign of η does not yield an equivalent solution, hence $-\infty < \eta < \infty$.

As is shown in Appendix A these functions form a complete orthonormal set,

$$\int dx\, \psi^{a'}_{w',\eta',n'}(x)^*\psi^{a}_{w,\eta,n}(x) = \delta_{a',a}\delta(w'-w)\delta(\eta'-\eta)\delta_{n',n}\,, \tag{4.13}$$

for integration over the physical space, and

$$\begin{aligned}2\delta(x'^+ - x^+)\delta(x'^- - x^-)\delta(y'-y) = &\int_0^\infty dw' \int_0^\infty d\eta \sum_{n=-\infty}^{\infty}\left(\psi^{(+)}_{w',\eta,n}(x')^*\psi^{(+)}_{w',\eta,n}(x) + \psi^{(-)}_{w',\eta,n}(x')^*\psi^{(-)}_{w',\eta,n}(x)\right) \\ &+ \int_{-\infty}^0 dw' \int_{-\infty}^\infty d\eta \sum_{n=-\infty}^{\infty} \psi^{s}_{w',\eta,n}(x')^*\psi^{s}_{w',\eta,n}(x)\,,\end{aligned} \tag{4.14}$$

for x' and x both in the physical space.

As a corollary to these results, an arbitrary function ϕ on the physical space may be written as

$$\phi(x) = \sum_{n=-\infty}^{\infty} \int_0^{\infty} dw' \int_0^{\infty} d\eta \left(\psi^{(+)}_{w',\eta,n}(x) f^{(+)}_n(w',\eta) + \psi^{(-)}_{w',\eta,n}(x) f^{(-)}_n(w',\eta) \right)$$
$$+ \sum_{n=-\infty}^{\infty} \int_{-\infty}^{0} dw' \int_{-\infty}^{\infty} d\eta \psi^{s}_{w',\eta,n}(x) f^{s}_n(w',\eta) \,. \tag{4.15}$$

The solutions to the wave equation, Eq. (4.1), for a given value of $w = m^2$ are, of course, not complete on the spacetime. However, they are complete on a given spacelike surface in Minkowski spacetime. That is, using positive- and negative-frequency solutions, a solution with an arbitrary initial value and an arbitrary initial time derivative on the surface can be constructed. Alternatively, using positive-frequency solutions, a positive-frequency solution with an arbitrary initial value (or an arbitrary initial time derivative) can be constructed. In the Gott space, the same is true with some qualifications. An arbitrary solution can be constructed for an arbitrary spacelike surface solely within either the future region or the past region. That is, an arbitrary value of either the function or its normal derivative may be specified. If both positive and negative solutions are used, both the value of the function and its normal derivative may be specified.

The positive-frequency solution for the wave equation is

$$\phi^{(+)}_{\eta,n}(x^+,x^-,y) \equiv \sqrt{2\pi}\psi^{(+)}_{w,\eta,n}(x^+,x^-,y)$$
$$= \frac{1}{(2\pi)}\frac{1}{2}\int_0^{\infty} dk^+ \int_{-\infty}^{\infty} dk^- \delta(k^+k^- + m^2 + (n\pi/2+\alpha\eta)^2/Y_0^2) e^{i(k^+x^-+k^-x^+)/2}$$
$$\times \left[\frac{e^{i\frac{y}{Y_0}(\frac{n\pi}{2}+\alpha\eta)}}{\sqrt{2Y_0}i^n} \left(\frac{k^+}{-k^-}\right)^{i\eta/2} + \frac{e^{-i\frac{y}{Y_0}(\frac{n\pi}{2}+\alpha\eta)}}{\sqrt{2Y_0}i^{-n}} \left(\frac{k^+}{-k^-}\right)^{-i\eta/2} \right], \tag{4.16}$$

where $w = m^2 + (n\pi/2 + \alpha\eta)^2/(4Y_0)^2$, and the conservation condition for solutions to the wave equation reads

$$\partial_\nu \left[\phi'(x) \overleftrightarrow{\partial^\nu} \phi(x) \right] = 0 \,. \tag{4.17}$$

Then the integral over a surface

$$(\phi', \phi) = \int d\sigma_\nu \phi'(x) \frac{1}{i} \overleftrightarrow{\partial^\nu} \phi(x) \,, \tag{4.18}$$

is constant, provided that the functions (ϕ', ϕ) drop off fast enough at infinity. In Appendix A, this integral, in the case of the basis functions (4.16) and spacelike surfaces restricted to either the future or the past, is shown to be just the orthonormality relation.

The integral over a surface is most easily done by changing into coordinates appropriate to the surface, $x \to (\tau, \xi)$, where τ is the coordinate labeling the surface, and ξ are the coordinates in the surface. The normal to the surface is $\mathbf{n} = \boldsymbol{\partial}/\partial\tau$, and the integral becomes

$$(\phi', \phi) = \int d\xi n_\nu \sqrt{-g} \phi'(x) \frac{1}{i} \overleftrightarrow{\partial^\nu} \phi(x) \,, \tag{4.19}$$

where $-g$ is the absolute value of the determinant of the metric. It is shown in Appendix A that, for an arbitrary $x^+x^- =$ const surface in either the past or the future region,

$$\int d\sigma_\nu \phi^{(+)}_{\eta,n}(x)^* \frac{1}{i} \overleftrightarrow{\partial^\nu} \phi'^{(+)}_{\eta',n'}(x) = \delta(\eta - \eta')\delta_{n,n'} \,. \tag{4.20}$$

Thus an arbitrary positive-frequency solution to the wave equation,

$$\phi(x) = \int_0^{\infty} d\eta \sum_{n=-\infty}^{\infty} \phi^{(+)}_{\eta,n}(x) \phi(\eta, n) \,, \tag{4.21}$$

has the same (positive definite) norm in both the past and in the future regions:

$$(\phi', \phi) = \int_0^{\infty} d\eta \sum_{n=-\infty}^{\infty} \phi'(\eta, n)^* \phi(\eta, n) \,. \tag{4.22}$$

That is, every particle that starts out in the past region eventually ends up in the future region, and every particle which ends up in the future region started out in the past region.

V. GREEN'S FUNCTION

The formulation of a quantum field is well understood in Minkowski space. There are several alternative formulations: one may use the Wightman functions, the time-ordered product, the Green's function, or a functional integral to calculate the matrix elements of the field, its spectrum, and any scattering which may occur.

In summary, the basic quantity is the Wightman function

$$\Delta^{(+)}(x, x') = \langle 0+ | \phi(x)\phi(x') | 0- \rangle$$
$$= \int \frac{d^3p}{2p^0} \frac{e^{ip(x-x')}}{(2\pi)^3} \,, \tag{5.1}$$

where the initial and final states are respectively the initial and final vacua. In the case of the free field in Minkowski space, these vacua are the same. The frequency $p^0 = \sqrt{\mathbf{p}^2 + m^2}$. Since the vacuum is the lowest-energy state, the matrix element has positive (negative) frequency with respect to t (t'). The time-ordered product

$$\begin{aligned}\Delta_F(x,x') &= i\,\langle 0+|(\phi(x)\phi(x'))_+|\,0-\rangle \\ &= i\theta(t-t')\,\langle 0+|\phi(x)\phi(x')|\,0-\rangle \\ &\quad + i\theta(t'-t)\,\langle 0+|\phi(x')\phi(x)|\,0-\rangle \end{aligned} \tag{5.2}$$

both yields the Wightman function and can be constructed from it. It satisfies the Green's function equation

$$(-\partial^2+m^2)\,\Delta_F(x,x') = \delta(x-x')\,, \tag{5.3}$$

with the boundary condition that it be of positive (negative) frequency for $t > t'$ ($t < t'$). This equation and boundary condition uniquely determine the Green's function to be

$$\Delta_F(x,x') = \int \frac{d^4k}{(2\pi)^4}\frac{e^{ikx}}{k^2+m^2-i\epsilon}\,, \tag{5.4}$$

where the $-i\epsilon$ enforces the positive-frequency boundary conditions. This expression in turn may be represented as the functional integral over all field configurations on M^4:

$$\Delta_F(x,x') = i\int [d\phi]\, e^{iW[\phi]}\phi(x)\phi(x')\,, \tag{5.5}$$

where

$$W[\phi] = -\frac{1}{2}\int d^4x\,[(\partial\phi)^2+(m^2-i\epsilon)\phi^2]. \tag{5.6}$$

The $-i\epsilon$ in the integrand is needed to allow an analytic continuation to complex times which enforces convergence of the functional integral and also yields the correct boundary conditions on Δ_F. The validity of this formulation may be established either (1) by explicitly performing the functional integral or (2) by summing over all possible field configurations at each time (Cauchy surface) using the Hamiltonian to propagate the fields from one time to the next.

In Gott space the above line of argument does not work. The Wightman function (5.1) must be a solution to the homogeneous equation. If the state $\langle 0+|$ ($|0-\rangle$) is assumed to be the lowest energy state in the future (past), then the Wightman function must be a positive-frequency function; hence it must be expressible as a superposition of the functions $\phi^{(+)}_{\eta,n}(x)$ $[\phi^{(-)}_{\eta,n}(x')]$. Because the mass points are moving, the space is not static, and there is no conserved energy; nonetheless, for a free field there is a conserved particle number, and the zero particle state has zero energy which is less than that of any other state. Since there is particle conservation, one can require that the Wightman function have positive (negative) frequency in t (t') on a spacelike surface in the future region F (P) in Fig. 3. The Green's function derived below does satisfy this criterion and yields the following explicit expression for the Wightman function in Gott space:

$$\begin{aligned}\Delta^{(+)}(x,x') &= \langle 0+|\phi(x)\phi(x')|\,0-\rangle \\ &= \int_0^\infty d\eta \sum_{n=-\infty}^{\infty}\phi^{(+)}_{\eta,n}(x)\phi^{(-)}_{\eta,n}(x')\,, \end{aligned} \tag{5.7}$$

where ϕ is the on-mass-shell eigenfunction defined in Eq. (4.16).

Because there is no time ordering of the points in the noncausal region A, one cannot construct the time-ordered product, even given the Wightman function, unless at least one of the points is outside this region. That is, one cannot construct an expression like Eq. (5.2) for Gott space. One can, however, look for solutions to the Green's function equation (5.3); the solution to the equation is

$$\begin{aligned}G(x,x') &= \int_0^\infty d\lambda\int_0^\infty d\eta \sum_{n=-\infty}^{\infty}\frac{\psi^{(+)}_{\lambda,\eta,n}(x')^*\psi^{(+)}_{\lambda,\eta,n}(x)+\psi^{(-)}_{\lambda,\eta,n}(x')^*\psi^{(-)}_{\lambda,\eta,n}(x)}{-\lambda+m^2-i\epsilon+(n\pi/2+\alpha\eta)^2/Y_0^2} \\ &\quad + \int_{-\infty}^0 d\lambda\int_{-\infty}^\infty d\eta\sum_{n=-\infty}^{\infty}\frac{\psi^{s}_{\lambda,\eta,n}(x')^*\psi^{s}_{\lambda,\eta,n}(x)}{-\lambda+m^2+(n\pi/2+\alpha\eta)^2/Y_0^2}\,. \end{aligned} \tag{5.8}$$

This expression is well defined by virtue of the $-i\epsilon$ in the denominator which tells how to go around the pole as in the Minkowski case. Its inclusion assures that the answer satisfies the conditions given above as long as x or x' is in the causal region. The expression (5.8) solves the Green's function equation (5.3) and satisfies the positive-frequency condition in the regions where it can be applied. Since Green's functions can only differ by a solution to the homogeneous equation, and, by the argument of Sec. IV, a solution to the homogeneous equation is uniquely determined by its value in the past and future regions, this must be the unique Green's function which satisfies the positive-frequency conditions in the causal regions.

Alternatively, the functional integral definition (5.5) of the matrix element of the product of two field operators may be used; however, it can no longer be derived from an operator formulation of the theory. No foliation of the spacetime using spacelike Cauchy surfaces exists; hence there is no global time parameter which can be used to describe the time evolution of the system, and, by the same token, there is no Hamiltonian which can be used to generate the time evolution. Thus, the matrix element cannot be written as a sum over a complete set of states (field configurations) at each time. However, one can imagine defining the quantum field theory by the functional integral. The nontrivial causal properties mean that the classical field theory also has no Hamiltonian and no global time parameter. This does not prevent one from defining the field theory and deriving its equations of motion by varying an action defined over the space. That procedure yields all the usual structures in the case of a causal space and provides a generalization in the case of an acausal space. The corresponding generalization of the quantum theory is to define the matrix elements as the functional integral over field configurations with the complex measure given in Eq. (5.5).

If this procedure is adopted, the resultant propagator from Eq. (5.5),

$$G(x,x') = i\int [d\phi]\, e^{iW[\phi]}\phi(x)\phi(x')\,, \tag{5.9}$$

where, using the expansion, Eq. (4.15),

$$\begin{aligned} W[\phi] = \sum_{\lambda,\eta,n} & \left[m^2 - i\epsilon - \lambda + (n\pi/2 + \alpha\eta)^2/Y_0^2\right] \\ & \times[\phi_n^{(+)}(\lambda,\eta)^*\phi_n^{(+)}(\lambda,\eta) \\ & +\phi_n^{(-)}(\lambda,\eta)^*\phi_n^{(-)}(\lambda,\eta) \\ & +\phi_n^{s}(\lambda,\eta)^*\phi_n^{s}(\lambda,\eta)]/2\,, \end{aligned} \tag{5.10}$$

with the η sum running over positive values for the $\phi^{(\pm)}$ terms and over all values for the ϕ^s terms. The λ sum runs over positive (negative) values for $\phi^{(\pm)}$ (ϕ^s). The result of the integration is precisely that of Eq. (5.8). It is shown in Appendix B that Eq. (5.8) is precisely equal to the sum of images given in Eq. (2.17) with the additional information that the intervals $s(x,x'_n)$ are to be defined by the analytic continuation

$$s(x,x') = \sqrt{(y-y')^2 + (x-x')^2 - (|t-t'| - i\epsilon)^2}\,, \tag{5.11}$$

with the requirement that the real part of the square root be positive. As a result of this, the Green's function cannot be regarded as a function of x with t given a fixed imaginary part; the different terms in the sum have different imaginary parts for t because t may be greater than t'_n for some terms and less than t'_n for other terms. However, $s(x,x')$ is an analytic function of t: Suppose that x' is in the noncausal region and that x is in the future region. Then all the terms in the sum may be defined by giving t a negative imaginary part. The function is analytic in t with branch points at $t'_n \pm \sqrt{(y-y'_n)^2 + (x-x'_n)^2}$. As t is continued past each of the branch points there is a perfectly well-defined continuation: t goes below every branch point with a positive square root and above every branch point with a negative square root. The result is that those terms that are associated with images to the past (future) of x have positive (negative) frequency in t. Those terms that are associated with images that are spacelike with respect to x are independent of the choice of frequency.

The particles created by the operators propagate forward in time with positive frequencies. That is, if a field operator acts at x' creating a particle, it can be annihilated at any point within the future light cone of x', and it carries a positive frequency if that happens. Of course, as in Minkowski space, operators at spacelike separations can create and annihilate a particle. This is because a positive-frequency excitation cannot be localized, but is spread out over the Compton wavelength of the particle with an amplitude that decays exponentially with distance. Those terms in the Green's function that correspond to a spacelike separation between x and the image of x' decay exponentially with increasing separation.

Although there is a perfectly good interpretation of the Green's function in terms of the particles produced, there is no such interpretation in terms of the ordering of the field operators. As long as at least one point is outside the acausal region, there is a well-defined time ordering of the points, and the field operators may be interpreted as having the corresponding order. However, when both points are in the acausal region, the fields cannot be ordered in accord with the points; the ordering is that annihilation occurs to the future of creation. Thus the "ordering" is an ordering of propagation, not an ordering of the field operators. It is precisely this ordering that occurs when an interacting field theory is considered. This will be discussed further in the following section.

VI. OPERATOR PRODUCTS

The Green's function derived in Sec. V gives an explicit representation for the vacuum matrix element of free-field operator products. It may be written as

$$\langle(\phi(x)\phi(x'))_+\rangle = -i\sum_{n=-\infty}^{\infty} G_0(x,x'_n)\,, \tag{6.1}$$

where x'_n is the position of the nth image of the point x' in the covering space. If either x or x' are outside the acausal region, the ordering is a time ordering. Otherwise it is not an ordering of the field operators, but rather an ordering of the propagation of the field excitations.

As $x' \to x$, the $n = 0$ term in the sum diverges; this is just the standard divergence of field theory in Minkowski space. It is associated with various renormalizations which are not of concern here and will be dropped.

The remaining terms are in general finite as the points approach each other; in particular

$$s(x,x'_n) \to s_n(x) = \begin{cases} \sqrt{(-x^+x^-)\sinh^2 n\alpha + (2nY_0)^2}\,, & n \text{ even,} \\ \sqrt{(x^+e^{n\alpha} + x^-e^{-n\alpha})^2 + (2y + 2nY_0)^2}\,, & n \text{ odd,} \end{cases} \tag{6.2}$$

As a cursory examination of Fig. 3 will show, the terms with odd n are always spacelike, and no choice of n can make

s_n go to zero except along the world lines of the point masses, where $x^+ = -e^{\mp 2\alpha} x^-$, $y = \mp Y_0$. The terms with even n are spacelike for x outside the acausal region.

The expectation value of ϕ^2 can now be written as

$$\langle \phi^2(x) \rangle = -i \sum_{n=1}^{\infty} [G_0(x, x_n) + G_0(x, x_{-n})] \equiv -i\tilde{G}(x, x) . \tag{6.3}$$

Each term is finite for x in the causal regions. The first question is the convergence of the sum. For nonzero mass, the sum is always convergent because $-iG_0 \sim e^{-ms}/s^q$, where q is 1 for $2+1$ dimensions and 3/2 for $3+1$ dimensions. In either case, the exponential assures convergence of the sum. In the zero mass case $-iG_0 \sim 1/s^q$, where q is 1 for $2+1$ dimensions, and 2 for $3+1$ dimensions. Again, the sum converges because the asymptotic behavior of s_n^{-q} is $e^{-qn\alpha}$.

As the Cauchy horizon is approached, the convergence is more delicate. The odd n terms still have the same asymptotic behavior in n as before (except at the intersection of the Cauchy horizons $x^+ = 0 = x^-$). The even n terms still converge for a nonzero mass

$$\left\langle \left(\phi^2(x)\right)_+ \right\rangle = \begin{cases} \displaystyle\sum_{n=1}^{\infty} \frac{2\exp(-4nY_0 m)}{4\pi 4nY_0} = \frac{1}{8\pi Y_0} \ln\left(\frac{1}{1-e^{-4Y_0 m}}\right) , & 2+1 \text{ dim,} \\ \displaystyle\sum_{n=1}^{\infty} \frac{2mK_1(4nY_0 m)}{2\pi^2 4nY_0} , & 3+1 \text{ dim.} \end{cases} \tag{6.4}$$

In the massless case, the sum diverges as $x^+x^- \to 0$ in $2+1$ dimensions but is finite in $3+1$ dimensions. This divergence is logarithmic and therefore integrable; it appears for low dimension and reflects the limited phase space available for the wave to spread in.

In the acausal region the situation is more complicated. For large n, s_n is imaginary with an infinitesimal positive real part which assures convergence just as in the causal region. There are series of surfaces defined by $x^+x^- = (4nY_0/\sinh 4n\alpha)^2$ on which the interval $s_{2n}(x)$ vanishes. These are precisely the polarized hypersurfaces [14] discussed in Section III, where the point x can be connected to itself by a lightlike geodesic with winding number n. The expectation value of $\phi^2(x)$ is singular on these surfaces since the point x is on its own light cone. Furthermore, the surfaces are dense as the Cauchy horizon is approached; that is, there are an infinite number of such surfaces between every point in the acausal region and the Cauchy horizon. The singularity is the standard light-cone singularity s^{-q}, where q is 1 (2) in $2+1$ $(3+1)$ dimensions. An imaginary $s_n(x)$ indicates that x is within its own future light cone (for paths with winding number n), and, therefore, a particle can be created at x to propagate forward until it is annihilated at the same point. The propagator is then complex with a positive phase (because of the positive-frequency condition). This will be important in determining the properties of an interacting field in Sec. VII.

The stress energy of the field is given by

$$T_{\mu\nu} = \phi(x)_{,\mu}\phi(x)_{,\nu} - \eta_{\mu\nu}\tfrac{1}{2}\left[\phi(x)^{,\lambda}\phi(x)_{,\lambda} + m^2\phi^2(x)\right] . \tag{6.5}$$

To compute its matrix elements, note that derivatives of the function

$$s_{2n}(x, x') = \sqrt{(x^+ - e^{4n\alpha}x'^+)(x^- - e^{-4n\alpha}x'^-) + (y - y' - 4nY_0)^2} \tag{6.6}$$

are given by

$$\begin{aligned} \frac{\partial s_{2n}(x,x')}{\partial x^\pm} &= \frac{x^\mp - e^{\mp 4n\alpha}x'^\mp}{2s_{2n}(x,x')} , & \frac{\partial s_{2n}(x,x')}{\partial x'^\pm} &= \frac{x'^\mp - e^{\pm 4n\alpha}x^\mp}{2s_{2n}(x,x')} , \\ \frac{\partial s_{2n}(x,x')}{\partial y} &= \frac{y - y' - 4nY_0}{s_{2n}(x,x')} , & \frac{\partial s_{2n}(x,x')}{\partial y'} &= \frac{-(y - y' - 4nY_0)}{s_{2n}(x,x')} , \end{aligned} \tag{6.7}$$

which imply that

$$
\begin{aligned}
\left.\frac{\partial^2 f(s_{2n}(x,x'))}{\partial x^{\pm}\partial x'^{\pm}}\right|_{x'=x} &= -\left[\frac{(x^{\mp})^2\sinh^2 2n\alpha}{s_{2n}}\right]\left[\frac{f'(s_{2n})}{s_{2n}}\right]',\\
\left.\frac{\partial^2 f(s_{2n}(x,x'))}{\partial x^{\pm}\partial y'}\right|_{x'=x} &= \left[\frac{\pm(4nY_0)e^{\mp 2n\alpha}x^{\mp}\sinh 2n\alpha}{s_{2n}}\right]\left[\frac{f'(s_{2n})}{s_{2n}}\right]',\\
\left.\frac{\partial^2 f(s_{2n}(x,x'))}{\partial y\partial x'^{\pm}}\right|_{x'=x} &= \left[\frac{\pm(4nY_0)e^{\pm 2n\alpha}x^{\mp}\sinh 2n\alpha}{s_{2n}}\right]\left[\frac{f'(s_{2n})}{s_{2n}}\right]',\\
\left.\frac{\partial^2 f(s_{2n}(x,x'))}{\partial x^{\pm}\partial x'^{\mp}}\right|_{x'=x} &= \left[\frac{x^{\mp}x^{\pm}e^{\mp 4n\alpha}\sinh^2 2n\alpha}{s_{2n}}\right]\left[\frac{f'(s_{2n})}{s_{2n}}\right]' - \left[\frac{e^{\mp 4n\alpha}}{2}\right]\left[\frac{f'(s_{2n}(x,x'))}{s_{2n}(x,x')}\right],\\
\left.\frac{\partial^2 f(s_{2n}(x,x'))}{\partial y\partial y'}\right|_{x'=x} &= -\left[\frac{(4nY_0)^2}{s_{2n}}\right]\left[\frac{f'(s_{2n})}{s_{2n}}\right]' - \left[\frac{f'(s_{2n}(x,x'))}{s_{2n}(x,x')}\right],
\end{aligned}
\tag{6.8}
$$

which in turn yield

$$
\partial^\lambda\partial'_\lambda f(s_{2n}(x,x')) + m^2 f(s_{2n})\big|_{x'=x} = (\cosh 4n\alpha - 1)\left[\frac{4x^+x^-\sinh^2 2n\alpha}{s_{2n}^2}\left(m^2 f(s_{2n}) - \frac{3f'(s_{2n})}{s_{2n}}\right) - \frac{2}{s_{2n}}f'(s_{2n})\right].
\tag{6.9}
$$

Moreover, for $f(s_{2n}(x,x')) = G(x,x'_{2n})$, it obeys the wave equation

$$
0 = (-\partial^2 + m^2)f(s) = -3\left(\frac{f'(s)}{s}\right) - s\left(\frac{f'(s)}{s}\right)' + m^2 f(s)\,,
\tag{6.10}
$$

where $f(s_{2n}) = -iG_0(x,x_n)$.

In $2+1$ dimensions the matrix elements of the stress tensor are therefore given by

$$
\begin{aligned}
\langle T^{\pm\pm}(x)\rangle &= \sum_{n=1}^{\infty}\left[\frac{-8x^{\pm}x^{\pm}\sinh^2 2n\alpha}{s_{2n}^2}\right]\left[m^2 f(s_{2n}) - \frac{3f'(s_{2n})}{s_{2n}}\right],\\
\langle T^{\pm y}(x)\rangle &= 0\,,\\
\langle T^{\pm\mp}(x)\rangle &= \sum_{n=1}^{\infty}2\left[\frac{4x^+x^-\sinh^2 2n\alpha}{s_{2n}^2}\right]\left[m^2 f(s_{2n}) - \frac{3f'(s_{2n})}{s_{2n}}\right] - 4\frac{f'(s_{2n})}{s_{2n}},\\
\langle T^{yy}(x)\rangle &= \sum_{n=1}^{\infty}\left[\frac{-2(4nY_0)^2}{s_{2n}^2}\right]\left[m^2 f(s_{2n}) - \frac{3f'(s_{2n})}{s_{2n}}\right] - 2\frac{f'(s_{2n})}{s_{2n}}\\
&\quad -(\cosh 4n\alpha - 1)\left\{\frac{4x^+x^-\sinh^2 2n\alpha}{s_{2n}^2}\left[m^2 f(s_{2n}) - \frac{3f'(s_{2n})}{s_{2n}}\right] - 2\frac{f'(s_{2n})}{s_{2n}}\right\}.
\end{aligned}
\tag{6.11}
$$

For $m \neq 0$ and $x^+x^- \neq 0$, these sums are convergent, yielding a finite stress energy tensor at all points in the causal regions. As $m \to 0$, the convergence provided by the exponential e^{-ms} disappears, but the sums still converge because $f'/s \sim 1/s^3$, and the asymptotic behavior in n is $e^{-2n\alpha}$ leading to convergence. As $x^+x^- \to 0$ the sums are more delicate. For $m \neq 0$ and $x^+x^- = 0$, $s_{2n} = 4nY_0$, and the asymptotic behavior of the terms in the sums is $\sim e^{4n(\alpha - mY_0)}$; the sum converges provided $\alpha < mY_0$. Thus, for sufficiently small relative rapidity of the mass points, the stress tensor is regular on the Cauchy horizons. For $m = 0$ the sums diverge on the Cauchy horizons. The sums, which are somewhat tricky to evaluate for large rapidity α, are easily estimated for small α. In that case they are of the form

$$
S = \sum_n \frac{e^{4n\alpha}n^q}{[(-x^+x^-)e^{4n\alpha} + n^2(4Y_0)^2]^p}\,,
\tag{6.12}
$$

which may be approximated as an integral over n, and then evaluated by using a steepest-descent approximation. The result is

$$
S \sim \frac{\bar{n}^q}{(-x^+x^-)[(4Y_0)^2\bar{n}^2]^{(p-1)}}\,,
\tag{6.13}
$$

where $\bar{n} \sim \ln[(4Y_0)^2/(-x^+x^-)]$. The matrix elements of the stress-energy tensor are then, as the Cauchy horizons are approached,

$$
\begin{aligned}
\langle T^{\pm\pm}(x)\rangle &\sim \frac{-2x^{\pm}x^{\pm}}{(-x^+x^-)\left[(4Y_0)^2\ln^2[(4Y_0)^2/(-x^+x^-)]\right]^{3/2}}\,,\\
\langle T^{\pm y}(x)\rangle &\sim 0\,,\\
\langle T^{\pm\mp}(x)\rangle &\sim \text{finite}\,,\\
\langle T^{yy}(x)\rangle &\sim \frac{1}{2(-x^+x^-)\left[(4Y_0)^2\ln^2[(4Y_0)^2/(-x^+x^-)]\right]^{1/2}}\,.
\end{aligned}
\tag{6.14}
$$

VII. INTERACTING FIELDS

For an interacting scalar field, the action (5.6) of the free field is replaced by

$$W[\phi] = -\frac{1}{2}\int d^4x\left[(\partial\phi)^2 + (m^2 - i\epsilon)\phi^2 + \frac{\lambda}{4!}\phi^4\right]. \tag{7.1}$$

When this is used in the functional integral expression for the propagator

$$\Delta_F(x,x') = i\int [d\phi]\, e^{iW[\phi]}\phi(x)\phi(x'), \tag{7.2}$$

the resultant expression for Δ_F to first order in the coupling constant λ is, after renormalization,

$$\Delta_F(x,x') = G(x,x') + i\frac{\lambda}{2}\int dy G(x,y)\tilde{G}(y,y)G(y,x'), \tag{7.3}$$

where $\tilde{G}$, as defined in Eq. (6.3), denotes the Green's function with the Minkowski space, zero winding number term, removed. This is nonsingular as the points approach each other, and the dropped term simply renormalizes the mass of the scalar particle.

If x' is in one of the causal regions and in the past of x, then Δ_F may be integrated over a spacelike hyperboloid with $\phi^{(+)}_{\eta,n}(x')$, and the reduction formula, (A38), together with the orthonormality relation, (A26), yields the result

$$\langle 0|\phi(x)|\eta,n;\sigma(x')\rangle = \phi^{(+)}_{\eta,n}(x) + i\frac{\lambda}{2}\int_{y>\sigma(x')} dy\, G(x,y)\tilde{G}(y,y)\phi^{(+)}_{\eta,n}(y), \tag{7.4}$$

where $\sigma(x')$ denotes the spacelike surface over which the integral was done, and $y > \sigma(x')$ means that the integral is restricted to points in the future of $\sigma(x')$.

Applying the reduction formula (A38) again with $\phi^{(+)^*}$, the matrix element becomes

$$\langle \eta_1,n_1;\sigma(x)|\eta,n;\sigma(x')\rangle = \delta_{n_1,n}\delta(\eta_1-\eta) - i\frac{\lambda}{2}\int_{\sigma(x)>y>\sigma(x')} dy\phi^{(+)}_{\eta_1,n_1}(y)^* V(y)\phi^{(+)}_{\eta,n}(y), \tag{7.5}$$

where $V(y) \equiv -i\tilde{G}(y,y)$. As was discussed in Sec. VI, $V(y)$ is real for y in the causal regions. As long as the region of integration does not include any of the acausal region, the additional contribution to the matrix element is purely imaginary and only contributes terms of order λ^2 to the unitarity relation. But $V(y)$ is complex in the acausal region, and the matrix element contains a real part coming from the integration over that region. As a result, the eigenvalues of the scattering matrix (7.5) are not associated with real phases. Furthermore, the imaginary parts of the phases have no definite sign so that some probabilities will be greater than 1 (but of first order in λ), and the failure of unitarity cannot be associated with a failure to include all possible final states. Unitarity, expressed in terms of the particles that start in the past causal region and end in the future causal region, fails because real (rather than virtual) particles are created in the acausal region, are propagated around a closed timelike path, and are annihilated at the spacetime point at which they were created. If one could treat the terms with different winding numbers as physically distinct events so that there was no interference between them, there would be no problem with unitarity: The specification of the state would include a specification of whether an on-shell particle was created and, if so, what its winding number was. Then, the sum over states would include the sum over winding numbers and the phase would be a kind of final-state phase which would cancel in the sum over final states. The interference term between no production and production with some winding number would not enter and, in this approximation, there would be no lack of unitarity. Since the on-shell particles appear in neither the future causal region nor the past causal region, there is no way to specify, in terms of data on the initial or final surfaces, what on-shell particles were created with what winding numbers never to emerge from the acausal region.

Deutsch [18] has discussed this problem by describing the behavior of the system in the acausal region by means of a density matrix. Although the density matrix does provide partial information about what happens in the acausal region and is precisely the kind of information that is required in order to address the unitarity problem, it is additional information imposed from the outside (subject to some consistency conditions) and does not arise naturally from the theory.

The unitarity problem is directly associated with the absence of a global Cauchy surface. If there were such a surface one could specify data or the state of the system on that surface, and the propagation would uniquely determine the state for the entire spacetime. Here, the data on spacelike surfaces in the causal regions completely determines the noninteracting field but not the interacting quantum field. (The spacelike surface is not a global Cauchy surface.)

To summarize, the potential V in Eq. (7.5) may be written as a sum of terms arising from paths with different winding numbers. The contribution of each term to the amplitude contains information about a real particle which was created, then annihilated at the same spacetime point after winding around the singularities n times; but the specification of the state does not include the information about what particles were produced. It is hard to see how the specification of the state could be enlarged to include such information: The real potential that arises from the virtual creation and annihilation of particles cannot and should not be separated into contributions with different winding numbers, and one expects that the analytic continuation from virtual to real production be valid here as well as in causal spaces. If one could extend the surface on which the data is specified into the acausal region, one could then expect to have a unitarity relation in which the probabilities added to unity. However, there is no surface in the acausal region on which unrestricted data may be specified, and no way to separate the contributions from different winding numbers so that they do not interfere with the zero winding number contribution. The inclusion of higher order corrections in λ cannot save the situation because the failure is first order in λ. Furthermore, the work of Friedman *et al.* [19] shows similar results in order λ^2; the theory is not unitary by any usual standard.

VIII. BACK REACTION

The matrix element of the stress tensor is of the form Eq. (6.14) for the divergent parts. These terms are invariant under boosts in the (x,t) plane and reflections in y, and they are independent of y. The most general metric in the $(2+1)$-dimensional space that satisfies these symmetries is

$$ds^2 = e^{2\psi(\zeta)}dx^+dx^- + e^{2\phi(\zeta)}dy^2\,, \tag{8.1}$$

where $\zeta = -x^+x^-$.

The Einstein tensor for this metric is

$$\begin{aligned} G^{\pm\pm} &= -4x^\pm x^\pm\left(\phi' e^{(\phi-2\psi)}\right)' e^{-\phi}\,,\\ G^{\pm y} &= 0\,,\\ G^{\pm\mp} &= 4\left(-\zeta\phi' e^{\phi}\right)' e^{-(\phi+2\psi)}\,,\\ G^{yy} &= 4\left(-\zeta\psi'\right)' e^{-2\psi}\,. \end{aligned} \tag{8.2}$$

The expression for the stress energy must be equal to this to lowest order; hence,

$$\begin{aligned} (-\zeta\psi')' &\sim \frac{G}{2\zeta[(4Y_0)^2\ln^2[(4Y_0)^2/\zeta]]^{1/2}},\\ \phi'' &\sim \frac{2G}{\zeta[(4Y_0)^2\ln^2[(4Y_0)^2/\zeta]]^{3/2}}. \end{aligned} \tag{8.3}$$

The $\pm\mp$ equation is nonsingular at $\zeta \sim 0$, and it does not provide any additional information. The coefficients of the right-hand sides of Eqs. (8.3) are nontrivial functions of α which cannot be determined by the methods used here; however, Newton's constant G is included so that the units come out correctly. In $3+1$ dimensions, the powers of $1/2$ and $3/2$ become 1 and 2 respectively because the propagator is, for $m=0$, $1/s^2$ rather than $1/s$. The solutions to the equations are then

$$\begin{aligned} \psi(\zeta) &\sim \begin{cases} -C(G/Y_0)\ln[(4Y_0)^2/\zeta]\{1+\ln\ln[(4Y_0)^2/\zeta]\}\,, & 2+1 \text{ dim},\\ -C(G/Y_0^2)\ln\ln[(4Y_0)^2/\zeta]\,, & 3+1 \text{ dim}, \end{cases}\\ \phi(\zeta) &\sim \text{finite}, \end{aligned} \tag{8.4}$$

where C is a positive constant depending upon the dimension and upon α.

Since $\psi \sim -\infty$ as $\zeta \sim 0$, the resultant metric is singular at the Cauchy horizons (the coefficient of dx^+dx^-, $e^{2\psi}$, vanishes there). The calculation of the stress tensor as modified by the change in the metric is beyond the scope of this paper.

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APPENDIX A: ORTHONORMALITY

The functions defined in Eq. (4.9),

$$\begin{aligned} \psi^{(+)}_{w',\eta,n}(x^+,x^-,y) = \frac{1}{(2\pi)^{3/2}}\frac{1}{2}\int_0^\infty dk^+\int_{-\infty}^{\infty} dk^-\delta(k^+k^-+w')e^{i(k^+x^-+k^-x^+)/2}\\ \times\left[\frac{e^{i\frac{y}{Y_0}(\frac{n\pi}{2}+\alpha\eta)}}{\sqrt{2Y_0}i^n}\left(\frac{k^+}{-k^-}\right)^{i\eta/2} + \frac{e^{-i\frac{y}{Y_0}(\frac{n\pi}{2}+\alpha\eta)}}{\sqrt{2Y_0}i^{-n}}\left(\frac{k^+}{-k^-}\right)^{-i\eta/2}\right], \end{aligned} \tag{A1}$$

Eq. (4.10),

$$\psi^{(-)}_{w',\eta,n}(x^+,x^-,y) = \left(\psi^{(+)}_{w',\eta,n}(x^+,x^-,y)\right)^* = \frac{1}{(2\pi)^{3/2}}\frac{1}{2}\int_0^\infty dk^+ \int_{-\infty}^\infty dk^- \delta(k^+k^- + w')e^{-i(k^+x^-+k^-x^+)/2} \times \left[\frac{e^{i\frac{y}{Y_0}(\frac{n\pi}{2}+\alpha\eta)}}{\sqrt{2Y_0}i^n}\left(\frac{k^+}{-k^-}\right)^{i\eta/2} + \frac{e^{-i\frac{y}{Y_0}(\frac{n\pi}{2}+\alpha\eta)}}{\sqrt{2Y_0}i^{-n}}\left(\frac{k^+}{-k^-}\right)^{-i\eta/2}\right], \tag{A2}$$

and Eq. (4.12),

$$\psi^s_{-v',\eta,n}(x^+,x^-,y) = \frac{1}{(2\pi)^{3/2}}\frac{1}{2}\int_0^\infty dk^+ \int_{-\infty}^\infty dk^- \delta(k^+k^- - v') \times \left[\frac{e^{i(k^+x^-+k^-x^+)/2}e^{i\frac{y}{Y_0}(\frac{n\pi}{2}+\alpha\eta)}}{\sqrt{2Y_0}i^n}\left(\frac{k^+}{k^-}\right)^{i\eta/2} + \frac{e^{-i(k^+x^-+k^-x^+)/2}e^{-i\frac{y}{Y_0}(\frac{n\pi}{2}+\alpha\eta)}}{\sqrt{2Y_0}i^{-n}}\left(\frac{k^+}{k^-}\right)^{-i\eta/2}\right], \tag{A3}$$

form a complete orthonormal set, where w and v are both positive. To see this, calculate the inner product over the physical space

$$I(w',\eta',n',a';w,\eta,n,a) = \frac{1}{2}\int_{-\infty}^\infty dx^+ \int_{-\infty}^\infty dx^- \int_{-Y_0}^{Y_0} dy\, \psi^{a'}_{w',\eta',n'}(x)\psi^a_{w,\eta,n}(x)\,. \tag{A4}$$

For each combination of functions, the integrations over (x^+,x^-) yield a δ function of the momenta (k^+,k^-), which in turn forces the momenta in the definitions of the ψ's to be equal. The integral thus vanishes unless $a'=a$ (otherwise the ranges of the k integrations do not overlap). Since the two momenta are equal, a factor of $\delta(w'-w)$ appears. Hence

$$I(w',\eta',n',a';w,\eta,n,a) = \delta_{a',a}\delta(w'-w)I^a(w;\eta',n';\eta,n), \tag{A5}$$

where

$$I^a(w;\eta',n';\eta,n) = \int_{-Y_0}^{Y_0} dy \int_0^\infty dk^+ \int_{-\infty}^\infty dk^- \delta(k^+k^- + w)E^a_w(y;\eta',n',\eta,n), \tag{A6}$$

and

$$E^a_w(y;\eta',n',\eta,n) = \begin{cases} \left[\left(e^{i(y/Y_0)[(n-n')\pi/2+\alpha(\eta-\eta')]}/8\pi Y_0 i^{n-n'}\right)(k^+/-k^-)^{i(\eta-\eta')/2} \right. \\ \left. +(n',\eta')\to(-n',-\eta')\right] + \left[(n,\eta)\to(-n,-\eta)\right], \quad a=\pm, \\ \left[\left(e^{i(y/Y_0)[(n-n')\pi/2+\alpha(\eta-\eta')]}/8\pi Y_0 i^{n-n'}\right)(k^+/k^-)^{i(\eta-\eta')/2}\right] \\ +\left[(n',\eta',n,\eta)\to(-n',-\eta',-n,-\eta)\right], \quad a=s\,. \end{cases} \tag{A7}$$

The integrations over $k^\pm$ now yield $2\pi\delta(\eta\mp\eta')$, with the terms with the plus sign appearing only for $a=\pm$, in which case they vanish because $\eta>0$. The integral becomes

$$I(w',\eta',n',a';w,\eta,n,a) = \delta_{a',a}\delta(w'-w)\delta(\eta'-\eta)\int_{-Y_0}^{Y_0} dy\, E^a_{w,\eta}(y,n',n), \tag{A8}$$

where

$$E^a_{w,\eta}(y,n',n) = \left(e^{i(y/Y_0)[(n-n')\pi/2]}/4Y_0 i^{n-n'}\right) + (n,n')\to(-n,-n')\,. \tag{A9}$$

The integration over y then yields $\delta_{n',n}$ and

$$I(w',\eta',n',a';w,\eta,n,a) = \delta_{a',a}\delta(w'-w)\delta(\eta'-\eta)\delta_{n',n}. \tag{A10}$$

The functions thus form an orthonormal set.

Completeness is established by calculating

$$I(x'^+,x'^-,y';x^+,x^-,y) = \int_0^\infty dw \int_0^\infty d\eta \sum_{n=-\infty}^{\infty} \left[\psi^{(+)}_{w,\eta,n}(x')^*\psi^{(+)}_{w,\eta,n}(x) + \psi^{(-)}_{w,\eta,n}(x')^*\psi^{(-)}_{w,\eta,n}(x)\right] + \int_{-\infty}^0 dw \int_{-\infty}^\infty d\eta \sum_{n=-\infty}^{\infty} \psi^s_{w,\eta,n}(x')^*\psi^s_{w,\eta,n}(x)\,. \tag{A11}$$

The sums over n are of the form

$$\sum_{n=-\infty}^{\infty} \frac{e^{in\pi(y-y')/2}}{4Y_0} = \delta(y-y') \tag{A12}$$

or

$$\sum_{n=-\infty}^{\infty} \frac{e^{in\pi(y+y'+2Y_0)/2}}{4Y_0} = \delta(y+y'+2Y_0)\,; \tag{A13}$$

the second form vanishes for $-Y_0 < (y,y') < Y_0$. Hence the sum over n yields

$$\int_0^\infty d\eta \sum_{n=-\infty}^{\infty} \left[\psi^{(+)}_{w,\eta,n}(x')^*\psi^{(+)}_{w,\eta,n}(x) + \psi^{(-)}_{w,\eta,n}(x')^*\psi^{(-)}_{w,\eta,n}(x)\right] = \delta(y'-y) I^\pm(w;x'^+,x'^-;x^+,x^-)\,, \tag{A14}$$

where

$$\begin{aligned} I^\pm(w,x'^+,x'^-,x^+,x^-) = & \int_0^\infty d\eta \frac{1}{(2\pi)^3}\frac{1}{2}\int_0^\infty dk^+ \int_{-\infty}^\infty dk^- \frac{1}{2}\int_0^\infty dk'^+ \int_{-\infty}^\infty dk'^- \delta(k^+k^- + w) \\ & \times\delta(k'^+k'^- + w)\left[e^{i(k^+x^- + k^-x^+ - k'^+x'^- - k'^-x'^+)/2} + e^{-i(k^+x^- + k^-x^+ - k'^+x'^- - k'^-x'^+)/2}\right] \\ & \times\left[(k^+k'^-/k^-k'^+)^{i\eta/2} + (k^+k'^-/k^-k'^+)^{-i\eta/2}\right] \\ = & \frac{1}{(2\pi)^2}\frac{1}{2}\int_0^\infty dk^+ \int_{-\infty}^\infty dk^- \frac{1}{2}\int_0^\infty dk'^+ \int_{-\infty}^\infty dk'^- \\ & \times\delta(k^+k^- + w)\delta(k'^+k'^- - k^+k^-)\delta\big(\ln(k^+k'^-/k^-k'^+)^{1/2}\big) \\ & \times\left[e^{i(k^+x^- + k^-x^+ - k'^+x'^- - k'^-x'^+)/2} + e^{-i(k^+x^- + k^-x^+ - k'^+x'^- - k'^-x'^+)/2}\right) \\ = & \frac{1}{(2\pi)^2}\frac{1}{4}\int_{-\infty}^\infty dk^+ \int_{-\infty}^\infty dk^- \delta(k^+k^- + w)\left[e^{i[k^+(x^- - x'^-) + k^-(x^+ - x'^+)]/2}\right]. \end{aligned} \tag{A15}$$

A similar argument leads to the result

$$\begin{aligned} \int_{-\infty}^\infty d\eta \sum_{n=-\infty}^{\infty} \psi^s_{w,\eta,n}(x')^*\psi^s_{w,\eta,n}(x) &= \delta(y'-y) I^s(w;\eta,x'^+,x'^-;x^+,x^-) \\ &= \delta(y'-y)\frac{1}{(2\pi)^2}\frac{1}{4}\int_{-\infty}^\infty dk^+ \int_{-\infty}^\infty dk^- \delta(k^+k^- + w)\left[e^{i(k^+(x^- - x'^-) + k^-(x^+ - x'^+))/2}\right]. \end{aligned} \tag{A16}$$

If Eqs. (A11), (A14), (A15), and (A16) are combined the result

$$\begin{aligned} I(x'^+,x'^-,y';x^+,x^-,y) &= \int_0^\infty dw \int_0^\infty d\eta \sum_{n=-\infty}^{\infty} \left[\psi^{(+)}_{w,\eta,n}(x')^*\psi^{(+)}_{w,\eta,n}(x) + \psi^{(-)}_{w,\eta,n}(x')^*\psi^{(-)}_{w,\eta,n}(x)\right] \\ &\quad + \int_{-\infty}^0 dw \int_{-\infty}^\infty d\eta \sum_{n=-\infty}^{\infty} \psi^s_{w,\eta,n}(x')^*\psi^s_{w,\eta,n}(x) \\ &= 2\delta(y'-y)\delta(x'^+ - x^+)\delta(x'^- - x^-) \end{aligned} \tag{A17}$$

is obtained, and completeness is established.

In order to calculate the functional integral in Sec. V, the integral

$$I'(w',\eta',n',a';w,\eta,n,a) = \frac{1}{2}\int_{-\infty}^\infty dx^+ \int_{-\infty}^\infty dx^- \int_{-Y_0}^{Y_0} dy\, \partial^\mu \psi^{a'}_{w',\eta',n'}(x)\partial_\mu \psi^a_{w,\eta,n}(x) \tag{A18}$$

is required. The terms with derivatives with respect to $x^\pm$ yield integrals that are the same as (A4) with extra factors of $k^\pm$ inside the integrals which, after the $x^\pm$ integrations are done, simply become a factor of $-w$ multiplying (A4). The term with derivatives with respect to y yields an integral which is again the same but with an extra factor of $(n\pi/2+\alpha\eta)(n'\pi/2+\alpha\eta')/Y_0^2$; this factor does not affect the previous arguments, and the result is an overall factor of $(n\pi/2+\alpha\eta)^2/Y_0^2$ in (A4). Thus,

$$I'(w',\eta',n',a';w,\eta,n,a) = \left[-w + (n\pi/2+\alpha\eta)^2/Y_0^2\right]\delta_{a',a}\delta_{n',n}\delta(\eta'-\eta)\delta(w'-w)\,. \tag{A19}$$

The eigenfunctions ψ may be expressed in terms of Bessel functions of imaginary order: The $k^\pm$ integrations may be done explicitly in (4.9); for $w>0$, first calculate

$$\begin{aligned} I(x,\eta) &= \int_0^\infty dk^+ \int_{-\infty}^{\infty} dk^- \delta(k^+k^- + w)\left(\frac{k^+}{-k^-}\right)^{i\eta/2} e^{i(k^+x^- + k^-x^+)/2} \\ &= \int_0^\infty \left(\frac{dk^+}{k^+}\right)\left(\frac{k^+}{\sqrt{w}}\right)^{i\eta} e^{i(k^+x^- - wx^+/k^+)/2}\,. \end{aligned} \tag{A20}$$

For $x^\pm > 0$, the change of variables $k^+ \to \sqrt{wx^+/x^-}e^\sigma$ followed by a translation of σ by $i\pi/2$ leads to the expression

$$\begin{aligned} I(x,\eta) &= \int_{-\infty}^{\infty} d\sigma \left(\frac{x^+}{x^-}\right)^{i\eta/2} e^{i\eta\sigma} e^{i\sqrt{wx^+x^-}\sinh\sigma} \\ &= \int_{-\infty}^{\infty} d\sigma \left(\frac{x^+}{x^-}\right)^{i\eta/2} e^{-\pi\eta/2} e^{i\eta\sigma} e^{-\sqrt{wx^+x^-}\cosh\sigma} \\ &= 2\left(\frac{x^+}{x^-}\right)^{i\eta/2} e^{-\pi\eta/2} K_{i\eta}\left(\sqrt{wx^+x^-}\right). \end{aligned} \tag{A21}$$

The original expression (A20) is analytic for $\mp \mathrm{Im} x^\pm > 0$; hence the value for all x may be obtained by analytic continuation with the result

$$I(x,\eta) = \begin{cases} 2\left(\frac{x^+}{x^-}\right)^{i\eta/2} e^{-\pi\eta/2} K_{i\eta}\left(\sqrt{wx^+x^-}\right), & x^\pm > 0\,, \\ 2\left(\frac{-x^+}{-x^-}\right)^{i\eta/2} e^{+\pi\eta/2} K_{i\eta}\left(\sqrt{wx^+x^-}\right), & x^\pm < 0\,, \\ 2\left(\frac{x^+}{-x^-}\right)^{i\eta/2} K_{i\eta}\left(i\sqrt{wx^+(-x^-)}\right), & x^+ > 0 > x^-, \\ 2\left(\frac{-x^+}{x^-}\right)^{i\eta/2} K_{i\eta}\left(-i\sqrt{w(-x^+)x^-}\right). & x^- > 0 > x^+. \end{cases} \tag{A22}$$

Using these results in Eq. (4.9), the closed form expression for $\psi^{(+)}$ may then be immediately written as

$$\begin{aligned} &\phi^{(+)}_{\eta,n}(x^+,x^-,y,\eta) \\ &= \sqrt{2\pi}\psi^{(+)}_{w,\eta,n}(x^+,x^-,y,\eta) \\ &= \begin{cases} \left\{\left[\frac{e^{i\frac{y}{Y_0}\left(\frac{n\pi}{2}+\alpha\eta\right)}}{i^n(2\pi)\sqrt{2Y_0}}\right]\left(\frac{x^+}{x^-}\right)^{i\eta/2} e^{-\pi\eta/2} + \left[\frac{e^{-i\frac{y}{Y_0}\left(\frac{n\pi}{2}+\alpha\eta\right)}}{i^{-n}(2\pi)\sqrt{2Y_0}}\right]\left(\frac{x^+}{x^-}\right)^{-i\eta/2} e^{\pi\eta/2}\right\} K_{i\eta}\left(\sqrt{wx^+x^-}\right), & x^\pm > 0\,, \\ \left\{\left[\frac{e^{i\frac{y}{Y_0}\left(\frac{n\pi}{2}+\alpha\eta\right)}}{i^n(2\pi)\sqrt{2Y_0}}\right]\left(\frac{-x^+}{-x^-}\right)^{i\eta/2} e^{+\pi\eta/2} + \left[\frac{e^{-i\frac{y}{Y_0}\left(\frac{n\pi}{2}+\alpha\eta\right)}}{i^{-n}(2\pi)\sqrt{2Y_0}}\right]\left(\frac{-x^+}{-x^-}\right)^{-i\eta/2} e^{-\pi\eta/2}\right\} K_{i\eta}\left(\sqrt{wx^+x^-}\right), & x^\pm < 0\,, \\ \left\{\left[\frac{e^{i\frac{y}{Y_0}\left(\frac{n\pi}{2}+\alpha\eta\right]}}{i^n(2\pi)\sqrt{2Y_0}}\right]\left(\frac{x^+}{-x^-}\right)^{i\eta/2} + \left[\frac{e^{-i\frac{y}{Y_0}\left(\frac{n\pi}{2}+\alpha\eta\right)}}{i^{-n}(2\pi)\sqrt{2Y_0}}\right]\left(\frac{x^+}{-x^-}\right)^{-i\eta/2}\right\} K_{i\eta}\left(i\sqrt{wx^+(-x^-)}\right), & x^+ > 0 > x^-, \\ \left\{\left[\frac{e^{i\frac{y}{Y_0}\left(\frac{n\pi}{2}+\alpha\eta\right)}}{i^n(2\pi)\sqrt{2Y_0}}\right]\left(\frac{-x^+}{x^-}\right)^{i\eta/2} + \left[\frac{e^{-i\frac{y}{Y_0}\left(\frac{n\pi}{2}+\alpha\eta\right]}}{i^{-n}(2\pi)\sqrt{2Y_0}}\right]\left(\frac{-x^+}{x^-}\right)^{-i\eta/2}\right\} K_{i\eta}\left(-i\sqrt{w(-x^+)x^-}\right), & x^- > 0 > x^+, \end{cases} \end{aligned} \tag{A23}$$

where the fact that $K_{i\eta}$ is even in η has been used, and

$$w = w(n,\eta) = m^2 + (n\pi/2+\alpha\eta)^2/Y_0^2\,. \tag{A24}$$

Note that $K_{i\eta}$ is real for real values of its argument. The wave function has a branch point at $x^\pm = 0$, reflecting the causal anomalies which appear at the Cauchy horizons. However, wave packets constructed with ϕ^+ are well behaved along the horizons.

The wave functions $\phi_{\eta,n}$ defined in Eq. (4.16) form a complete set of functions on the spacelike hyperboloids in the past and in the future regions. They are orthonormal under integration over the surface in Eq. (4.18). In particular, for a spacelike hyperboloid in the past or future region, let $x^\pm = \pm\tau e^{\pm\xi}$, where τ is positive (negative) in the future (past) region; the metric becomes

$$d^2s = -d\tau^2 + \tau^2 d\xi^2 + dy^2 \,, \tag{A25}$$

and the integral over the spacelike hyperboloid τ = const is

$$\begin{aligned}
&\int_{-\infty}^{\infty} d\xi \int_{-Y_0}^{Y_0} dy\tau i \left[\phi^{(+)}_{\eta_1,n_1}(\tau e^{\xi}, -\tau e^{-\xi}, y)^* i \frac{\overleftrightarrow{\partial}}{\partial\tau} \phi^{(+)}_{\eta,n}(\tau e^{\xi}, -\tau e^{-\xi}, y)\right] \\
&= \tau \left[K_{i\eta_1}\left(i\tau\sqrt{w'}\right)^* i \frac{\overleftrightarrow{\partial}}{\partial\tau} K_{i\eta}\left(i\tau\sqrt{w}\right)\right] \\
&\quad \times \int_{-\infty}^{\infty} d\xi \int_{-Y_0}^{Y_0} dy \left\{\left[\frac{e^{i\frac{y}{Y_0}(\frac{n_1\pi}{2}+\alpha\eta_1)}}{i^{n_1}(2\pi)\sqrt{2Y_0}}\right] e^{i\xi\eta_1} + \left[\frac{e^{-i\frac{y}{Y_0}(\frac{n_1\pi}{2}+\alpha\eta_1)}}{i^{-n_1}(2\pi)\sqrt{2Y_0}}\right] e^{-i\xi\eta_1}\right\} \\
&\qquad \times \left\{\left[\frac{e^{i\frac{y}{Y_0}(\frac{n\pi}{2}+\alpha\eta)}}{i^{n}(2\pi)\sqrt{2Y_0}}\right] e^{i\xi\eta} + \left[\frac{e^{-i\frac{y}{Y_0}(\frac{n\pi}{2}+\alpha\eta)}}{i^{-n}(2\pi)\sqrt{2Y_0}}\right] e^{-i\xi\eta}\right\} \\
&= \delta(\eta_1 - \eta)\delta_{n_1,n} \,,
\end{aligned} \tag{A26}$$

where the Wronskian $z(K_\nu(-iz)i \overleftrightarrow{\partial} /\partial z K_\nu(iz)) = \pi$ is used to evaluate the overall factor. Note that the integral with $\phi^{(+)*}$ replaced by $\phi^{(-)*}$ vanishes because the Wronskian then vanishes.

In the noncausal region, the specification of a surface is more complicated. Because of the boost, the surface must have its intersection with the boundaries $y = \pm Y_0$ be continuous; that is, the identified point is also in the surface. This can be achieved in a variety of ways, but in no case is the resultant surface everywhere spacelike. One particularly simple choice is to let the surface be defined by

$$x^\pm = \begin{cases} \xi e^{\pm\tau} e^{\mp y\alpha/Y_0} \,, & x^\pm > 0 \,, \\ -\xi e^{\mp\tau} e^{\mp y\alpha/Y_0} \,, & x^\pm < 0 \,, \end{cases} \tag{A27}$$

where τ is constant. With this change of variables the metric for the space becomes

$$ds^2 = d\xi^2 + dy^2 - \xi^2(d\tau \mp \alpha dy/Y_0)^2 \,, \tag{A28}$$

where the $\mp$ sign is negative for the $x^\pm > 0$ region, and positive for the $x^\pm < 0$ region. The normal one-form is $d\tau$, and the normal derivative becomes

$$\mathbf{n}_\pm = \left(\frac{1}{\xi^2}\right) \frac{\overrightarrow{\partial}}{\partial\tau} \mp \left(\frac{\alpha}{Y_0}\right) \left[\frac{\overrightarrow{\partial}}{\partial y} \pm \left(\frac{\alpha}{Y_0}\right) \frac{\overrightarrow{\partial}}{\partial\tau}\right]. \tag{A29}$$

The integral over the surface τ = const is

$$\begin{aligned}
I_{n_1,\eta_1;n,\eta} = \int_0^\infty d\xi \int_{-Y_0}^{Y_0} dy\xi[\, &\phi^{(+)}_{\eta_1,n_1}(\xi e^{\tau-\alpha y/Y_0}, \xi e^{-\tau+\alpha}, y)^* i \overleftrightarrow{n}_+ \phi^{(+)}_{\eta,n}(\xi e^{\tau-\alpha y/Y_0}, \xi e^{-\tau+\alpha y/Y_0}, y) \\
&+ \phi^{(+)}_{\eta_1,n_1}(-\xi e^{-\tau-\alpha y/Y_0}, -\xi e^{\tau+\alpha}, y)^* i \overleftrightarrow{n}_- \phi^{(+)}_{\eta,n}(-\xi e^{-\tau-\alpha y/Y_0}, -\xi e^{\tau+\alpha y/Y_0}, y)],
\end{aligned} \tag{A30}$$

where

$$\phi^{(+)}_{\eta,n}(\xi e^{\tau-\alpha y/Y_0}, \xi e^{-\tau+\alpha y/Y_0}, y) = \left(\frac{1}{2\pi}\right) K_{i\eta}\left(\xi\sqrt{w(n,\eta)}\right) \left[e^{-\pi\eta} E_n(y,\tau;\eta) + e^{\pi\eta} E_{-n}(y,\tau;-\eta)\right] \,, \tag{A31}$$

and

$$\phi^{(+)}_{\eta,n}(-\xi e^{-\tau-\alpha y/Y_0}, -\xi e^{\tau+\alpha y/Y_0}, y) = \left(\frac{1}{2\pi}\right) K_{i\eta}\left(\xi\sqrt{w(n,\eta)}\right) \left[e^{\pi\eta} E_n(y,\tau;-\eta) + e^{-\pi\eta} E_{-n}(y,\tau;\eta)\right] \,, \tag{A32}$$

with

$$E_n(y,\tau;\eta) \equiv \frac{e^{in\pi y/2Y_0} e^{i\eta\tau}}{i^n\sqrt{2Y_0}}. \tag{A33}$$

The integral $I_{n_1,\eta_1;n,\eta}$ requires the y integral

$$\int_{-Y_0}^{Y_0} dy \left\{ \left[E_{n_1}(y,\tau;\eta_1)^* i \overleftrightarrow{n}_+ E_n(y,\tau;\eta) \right] + [(n,n_1) \to (-n,-n_1)] \right\}$$
$$= \delta_{n_1,n} e^{i\tau(\eta-\eta_1)} \left[-(\eta^2-\eta_1^2)/\xi^2 + w(n,\eta) - w(n,\eta_1) \right] / (\eta - \eta_1) \quad \text{(A34)}$$

and

$$\left[-(\eta^2-\eta_1^2)/\xi^2 + w(n,\eta) - w(n,\eta_1) \right] K_{i\eta_1}\left(\xi\sqrt{w(n,\eta_1)}\right) K_{i\eta}\left(\xi\sqrt{w(n,\eta)}\right)$$
$$= \frac{1}{\xi}\frac{\partial}{\partial\xi}\xi \left[K_{i\eta_1}\left(\xi\sqrt{w(n,\eta_1)}\right) \frac{\overleftrightarrow{\partial}}{\partial\xi} K_{i\eta}\left(\xi\sqrt{w(n,\eta)}\right) \right]. \quad \text{(A35)}$$

The integral over ξ of this result would vanish if it were not for the singular behavior as $\xi \to 0$ and the pole at $\eta = \eta_1$. The term that gives the singularity is

$$\begin{aligned} \int_0 \left[\frac{d\xi}{\xi}\right] K_{i\eta_1}(\xi a') K_{i\eta}(\xi a) &\simeq \int_0 d\xi \left(\frac{\eta+\eta_1}{4\xi}\right) \left[\Gamma(i\eta_1)(\xi a'/2)^{i\eta_1} + \Gamma(-i\eta_1)(\xi a'/2)^{-i\eta_1}\right] \\ &\quad \times \left[\Gamma(i\eta)(\xi a/2)^{i\eta} + \Gamma(-i\eta)(\xi a/2)^{-i\eta}\right] \\ &\simeq \left\{ \left[\left(\frac{-i\Gamma(i\eta)\Gamma(-i\eta_1)}{4(\eta-\eta_1-i\epsilon)} \right) + (\eta_1 \to -\eta_1) \right] + [(\eta \to -\eta)] \right\} \\ &= \left[\frac{\pi\delta(\eta-\eta_1)\Gamma(i\eta)\Gamma(-i\eta)}{2} \right] + (\eta_1 \to -\eta_1) \\ &= \left[\left(\frac{\pi^2\delta(\eta-\eta_1)}{\eta(e^{\pi\eta}-e^{-\pi\eta})} \right) + (\eta_1 \to -\eta_1) \right]. \end{aligned} \quad \text{(A36)}$$

These results may be combined to show that

$$I_{n_1,\eta_1;n,\eta} = \delta_{n_1,n}\delta(\eta_1 - \eta)\,. \quad \text{(A37)}$$

These results imply that the reduction formula can be written in the form

$$\int d\sigma_\mu \langle 0| \left[\phi^{(+)}_{\eta,n}(x)^* i \overleftrightarrow{\partial^\mu} \phi(x) \right] = \langle \eta, n; \sigma|\,, \quad \text{(A38)}$$

where the σ in the specification of the state denotes the surface over which the integral is done. In the case of the free field, the result is independent of the surface, but in the case of interactions the result depends upon the surface. These results also imply particle conservation for the free theory. The number of particles that reach the final surface is the same regardless of whether or not they traverse the acausal region between the initial and final surfaces.

APPENDIX B: IMAGES

In order to evaluate the sums over modes that appear at various places, the following expression must be evaluated:

$$I(y,a,\eta) = \sum_{n=-\infty}^{\infty} \frac{e^{i(n\pi/2+\alpha\eta)(y/Y_0)}}{a + (n\pi/2+\alpha\eta)^2/Y_0^2}\,, \quad \text{(B1)}$$

where $0 < y < 4Y_0$. It may be rewritten as

$$I(y,a,\eta) = \oint_C dz \frac{e^{i(z\pi/2+\alpha\eta)(y/Y_0)}}{e^{2\pi i z}-1} \frac{1}{a + (z\pi/2+\alpha\eta)^2/Y_0^2}\,, \quad \text{(B2)}$$

where the contour C encloses all the integers along the real axis, but does not enclose the zeros of the second denominator. Because of the bounds $0 < y < 4Y_0$, the integrand goes to zero exponentially as Im $z \to \pm\infty$, and the contour can be opened out to infinity, picking up the poles at the zeros of the second denominator. This evaluation yields

$$I(y,a,\eta)=\left(\frac{2Y_0}{\sqrt{a}}\right)\left[\frac{e^{-y\sqrt{a}}}{1-e^{-4Y_0\sqrt{a}}e^{-i4\alpha\eta}}+\frac{e^{y\sqrt{a}}}{e^{4Y_0\sqrt{a}}e^{-i4\alpha\eta}-1}\right]$$
$$=\sum_{n_1=0}^{\infty}\left(\frac{2Y_0}{\sqrt{a}}\right)\left[e^{-(y+4n_1Y_0)\sqrt{a}}e^{-i4n_1\alpha\eta}+e^{(y-4(n_1+1)Y_0)\sqrt{a}}e^{+i4(n_1+1)\alpha\eta}\right]$$
$$=\sum_{n_1=-\infty}^{\infty}\left(\frac{2Y_0}{\sqrt{a}}\right)e^{-|y+4n_1Y_0|\sqrt{a}}e^{-i4n_1\alpha\eta}\,, \tag{B3}$$

where $\mathrm{Re}\,\sqrt{a}>0$. If $-4Y_0<y<0$, the same result is obtained by noting that $I(y,a,\eta)=I(-y,a,-\eta)$ and replacing the summation index n_1 by $-n_1$; thus the result holds for $-4Y_0<y<4Y_0$.

Similarly, the sum

$$I^-(y,a,\eta)=\sum_{n=-\infty}^{\infty}\frac{e^{i(y/Y_0)(n\pi/2+\alpha\eta)}(-1)^n}{a+(n\pi/2+\alpha\eta)^2/Y_0^2}\,, \tag{B4}$$

where $0<y<4Y_0$, may be rewritten as

$$I^-(y,a,\eta)=\oint_C dz\frac{e^{i[(y+2Y_0)/Y_0](z\pi/2+\alpha\eta)}e^{-i2\alpha\eta}}{e^{2\pi iz}-1}\frac{1}{a+(z\pi/2+\alpha\eta)^2/Y_0^2}\,. \tag{B5}$$

Then,

$$I^-(y,a,\eta)=\left(\frac{2Y_0}{\sqrt{a}}\right)\sum_{n_1=-\infty}^{\infty}e^{-|y+(4n_1+2)Y_0|\sqrt{a}}e^{-i(4n_1+2)\alpha\eta}\,,$$

where $\mathrm{Re}\,\sqrt{a}>0$, and $-4Y_0<y+2Y_0<4Y_0$.

The sum over the $\psi^{(\pm)}$ modes in the expression for the Green's function may now be expressed as

$$\int_0^{\infty}d\eta\sum_{n=-\infty}^{\infty}\left[\psi^{(+)}_{\lambda,\eta,n}(x')^*\psi^{(+)}_{\lambda,\eta,n}(x)+\psi^{(-)}_{\lambda,\eta,n}(x')^*\psi^{(-)}_{\lambda,\eta,n}(x)\right]$$
$$=\frac{1}{4(2\pi)^3}\int_0^{\infty}d\eta\int_{-\infty}^{\infty}dk^+dk^-dk'^+dk'^-\delta(\lambda+k^+k^-)\delta(k^+k^--k'^+k'^-)$$
$$\times\sum_{n=-\infty}^{\infty}\frac{e^{i(kx-k'x')}}{k^+k^-+m^2-i\epsilon+(n\pi/2+\alpha\eta)^2/Y_0^2}$$
$$\times\left\{\frac{e^{i[(y-y')/Y_0](n\pi/2+\alpha\eta)}}{2Y_0}\left[\frac{k^+(-k'^-)}{(-k^-)k'^+}\right]^{i\eta/2}+\text{c.c.}\right.$$
$$\left.+\frac{e^{i[(y+y')/Y_0](n\pi/2+\alpha\eta)}(-1)^n}{2Y_0}\left[\frac{k^+k'^+}{(-k^-)(-k'^-)}\right]^{i\eta/2}+\text{c.c.}\right\}$$
$$=\frac{1}{4(2\pi)^3}\int_{-\infty}^{\infty}d\eta\int dk^+dk^-dk'^+dk'^-\delta(\lambda+k^+k^-)\delta(k^+k^--k'^+k'^-)$$
$$\times\sum_{n_1=-\infty}^{\infty}\frac{e^{i(kx-k'x')}}{\sqrt{a}}\left\{e^{-|y-y'+4n_1Y_0|\sqrt{a}}\left[\frac{(e^{-4n_1\alpha}k^+)(-k'^-)}{(-e^{4n_1\alpha}k^-)k'^+)}\right]^{i\eta/2}\right.$$
$$\left.+e^{-|y+y'+(4n_1+2)Y_0|\sqrt{a}}\left[\frac{(e^{-(4n_1+2)\alpha}k^+)k'^+}{(-e^{(4n_1+2)\alpha}k^-)(-k'^-)}\right]^{i\eta/2}\right\}\,, \tag{B6}$$

where $a=k^+k^-+m^2-i\epsilon$, $\mathrm{Re}\sqrt{a}>0$, and $kx=(k^+x^-+k^-x^+)/2$. The $k^\pm$ integration variables can then be scaled by $e^{\mp 4\alpha}$; the latter factors then appear multiplying $x^\pm$, and, using the image variables defined in Eq. (2.12) and letting $k^\pm\to -k^\mp$ in the second set of terms, the expression may be rewritten as

$$\int_0^\infty d\eta \sum_{n=-\infty}^{\infty} \left[\psi^{(+)}_{\lambda,\eta,n}(x')^* \psi^{(+)}_{\lambda,\eta,n}(x) + \psi^{(-)}_{\lambda,\eta,n}(x')^* \psi^{(-)}_{\lambda,\eta,n}(x)\right]$$

$$= \frac{1}{4(2\pi)^3} \int_{-\infty}^{\infty} d\eta\, dk^+ dk^- dk'^+ dk'^- \delta(\lambda + k^+k^-)\delta(k^+k^- - k'^+k'^-) \times \sum_{n_1=-\infty}^{\infty} \frac{e^{i(kx_{n_1} - k'x')}}{\sqrt{a}} e^{-|y_{n_1} - y'|\sqrt{a}} \left[\frac{k^+(-k'^-)}{(-k^-)k'^+}\right]^{i\eta/2}. \tag{B7}$$

The η and k' integrals can be done directly yielding

$$\int_0^\infty d\eta \sum_{n=-\infty}^{\infty} \left[\psi^{(+)}_{\lambda,\eta,n}(x')^* \psi^{(+)}_{\lambda,\eta,n}(x) + \psi^{(-)}_{\lambda,\eta,n}(x')^* \psi^{(-)}_{\lambda,\eta,n}(x)\right] = \frac{1}{4(2\pi)^2} \sum_{n_1=-\infty}^{\infty} \int_{-\infty}^{\infty} dk^+ dk^- \delta(\lambda + k^+k^-) \frac{e^{ik(x_{n_1} - x')}}{\sqrt{a}} e^{-|y_{n_1} - y'|\sqrt{a}}, \tag{B8}$$

where $\lambda > 0$, and a similar argument yields

$$\int_{-\infty}^{\infty} d\eta \sum_{n=-\infty}^{\infty} \psi^{s}_{\lambda,\eta,n}(x')^* \psi^{s}_{\lambda,\eta,n}(x) = \frac{1}{4(2\pi)^2} \sum_{n_1=-\infty}^{\infty} \int_{-\infty}^{\infty} dk^+ dk^- \delta(\lambda + k^+k^-) \frac{e^{ik(x_{n_1} - x')}}{\sqrt{a}} e^{-|y_{n_1} - y'|\sqrt{a}}, \tag{B9}$$

where $\lambda < 0$.

The familiar result for the Green's function in Minkowski spacetime is given by

$$\begin{aligned} G_M(x,x') &= \int \frac{d^3k}{(2\pi)^3} \frac{e^{ik\cdot(x-x')}}{k^2 + m^2 - i\epsilon} \\ &= \frac{1}{4(2\pi)^2} \int_{-\infty}^{\infty} \frac{dk^+ dk^-}{\sqrt{m^2 + k^+k^-}} e^{ik(x-x')} e^{-|y-y'|\sqrt{m^2+k^+k^-}}, \end{aligned} \tag{B10}$$

where $k \cdot x = kx + k_y y$. When these results are included in Eq. (5.8), the full Green's function becomes

$$G(x,x') = \sum_{n=-\infty}^{\infty} G_0(x_n, x') = \sum_{n=-\infty}^{\infty} G_0(x, x'_n), \tag{B11}$$

where the x'_n are the images given by Eq. (2.12).

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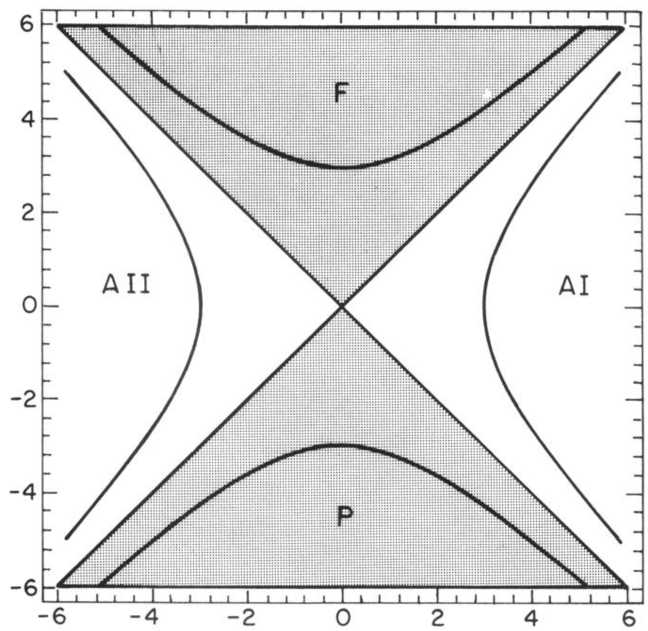

FIG. 3. The hyperbolae show the surfaces on which images of a point in the physical region lie. The past and future regions are shaded and the acausal region is unshaded. The y axis is perpendicular to the graph, and the physical space consists of the region $-Y_0 < y < Y_0$.