# Entropy versus action in the (2+1)-dimensional Hartle-Hawking wave function

S. Carlip*

*Department of Physics, University of California, Davis, California 95616*

(Received 18 May 1992)

In most attempts to compute the Hartle-Hawking "wave function of the Universe" in Euclidean quantum gravity, two important approximations are made: the path integral is evaluated in a saddle-point approximation, and only the leading (least action) extremum is taken into account. In (2+1)-dimensional gravity with a negative cosmological constant, the second assumption is shown to lead to incorrect results: although the leading extremum gives the most important single contribution to the path integral, topologically inequivalent instantons with larger actions occur in great enough numbers to predominate. One can thus say that in 2+1 dimensions, and possibly in 3+1 dimensions as well, entropy dominates the action in the gravitational path integral.

## I. INTRODUCTION

Quantum cosmology is a difficult subject, not least because we do not yet have a consistent quantum theory of gravity. In the absence of such a theory, cosmologists must rely on plausible, but necessarily speculative, approaches to gravity in the very early Universe. One attractive approach is Hawking's Euclidean path integral [1], which describes the wave function of Universe in terms of a "Wick rotated" gravitational path integral over Riemannian (positive-definite) metrics $g$, with an action

$$I_E[g,\phi] = -\frac{1}{16\pi G}\int_M d^n x \sqrt{g}\,(R\,[g]-2\Lambda)$$

$$-\frac{1}{8\pi G}\int_{\partial M} d^{(n-1)}x\sqrt{h}\,K + I_{\text{matter}}[\phi]\,. \quad (1.1)$$

Here $R\,[g]$ is the scalar curvature for an $n$-dimensional manifold $M$ ($n=4$ for standard physics), $\Lambda$ is the cosmological constant, $h$ is the induced metric on $\partial M$, and $K$ is the trace of the intrinsic curvature of $\partial M$, while $\phi$ represents a generic set of matter fields.

A path integral ordinarily determines a transition amplitude between an initial and a final configuration, and to specify a unique wave function one must select appropriate initial conditions. The Hartle-Hawking "no boundary" proposal [2] is that there should be *no* initial geometry—the path integral should be evaluated for compact manifolds with only a single boundary component $\Sigma$. If we specify a metric $h$ and a set of matter fields $\phi|_\Sigma$ on $\Sigma$, the path integral

$$\psi[h,\phi|_\Sigma] = \int [dg][d\phi]\exp\{-I_E[g,\phi]\} \quad (1.2)$$

can be interpreted as a wave function, giving an amplitude for the Universe to have spatial geometry $h$ and matter configuration $\phi|_\Sigma$. There seems to be no natural way to select any one particular topology in such a path integral, so in the Hartle-Hawking approach one sums over all manifolds $M$, subject to the condition that $\Sigma$ be the sole boundary component.

To obtain interesting physics, a further restriction on $M$ must be imposed. The integration in (1.2) is over Riemannian metrics, and it is necessary to "analytically continue" to obtain the observed Lorentzian structure of spacetime. This will be possible if the Riemannian metrics in the path integral can be joined to Lorentzian metrics to the future of $\Sigma$ (see Fig. 1). Gibbons and Hartle [3] have shown that a finite action continuation across $\Sigma$ exists only if the extrinsic curvature of $\Sigma$ vanishes. One should therefore limit the sum to manifolds $M$ and metrics $g$ such that the boundary $\partial M = \Sigma$ is totally geodesic.

It is perhaps time to admit that no one knows how to evaluate a path integral of the form (1.2). Ordinary general relativity is nonrenormalizable, so the standard perturbative expansions make no sense. A standard procedure in quantum cosmology is to compute the path integral in a saddle-point approximation, in the hope that the resulting estimate may be useful even if higher-order corrections are not well defined. The result is a kind of instanton picture, describing the semiclassical tunneling from "nothing" to a Universe. An extra implicit assumption is usually, although not always, made: one considers
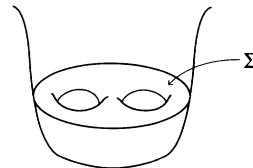


FIG. 1. A spacetime that will contribute to the Hartle-Hawking wave function. The metric is Riemannian to the past of $\Sigma$ and Lorentzian of the future.

---

*Electronic address: carlip@dirac.ucdavis.edu

only the extrema with the smallest actions,[1] which are assumed to dominate the path integral.

It would clearly be useful to have a well-defined model in which these assumptions could be tested more carefully. Gravity in $2+1$ dimensions provides such a model. It is by now fairly well known that pure $(2+1)$-dimensional gravity is renormalizable [6,7], and that a canonically quantized theory can be formulated with no approximations [6,8,9]. At the same time, many of the basic features of $(3+1)$-dimensional gravity—the form of the action, diffeomorphism invariance, the possibility of summing over topologies—remain unaltered.

In a series of very interesting papers, Fujiwara et al. [10–12] have begun to investigate quantum cosmology for pure $(2+1)$-dimensional gravity with a negative cosmological constant. They have found the leading saddle-point contribution to the path integral, and have investigated the possibilities of topology change and the spontaneous creation of particlelike defects in spacetime. The purpose of this paper is to extend this work by examining the contributions of higher action extrema. We shall see that the classical solutions discussed by Fujiwara et al. do not, in fact, determine the overall structure of the Hartle-Hawking wave function; instead, the wave function is peaked at certain highly symmetric two-geometries for which an infinite number of distinct topologies contribute to the path integral.

## II. THE PATH INTEGRAL IN 2+1 DIMENSIONS

Let us begin with a brief discussion of the path integral (1.2) in $2+1$ dimensions with a negative cosmological constant and no matter. The classical equations of motion coming from the action (1.1) are

$$R_{ik} = -2|\Lambda|g_{ik} \ . \tag{2.1}$$

In three dimensions, the Ricci tensor completely determines the curvature, and (2.1) implies that $g_{ik}$ is a constant negative curvature metric, i.e., that $(M,g)$ is a hyperbolic three-manifold. The Hartle-Hawking wave function depends on the boundary value $h$, which is itself a hyperbolic metric on the totally geodesic boundary $\Sigma$. This is in accord with the canonically quantized theory [6,8,13], where wave functions are quite naturally expressed as square-integrable functions on the moduli space $\mathcal{M}_\Sigma$ of hyperbolic metrics on $\Sigma$.

On general grounds, we expect the contribution of an extremum $(M,\bar{g})$ to the path integral to take the form

$$\Psi_M[\bar{g}|_\Sigma] = \Delta_M e^{-I_E[\bar{g}]} \ , \tag{2.2}$$

where $I_E[\bar{g}]$ is the classical action at the extremum and the prefactor $\Delta_M$ is a combination of determinants coming from small fluctuation around $\bar{g}$ and from gauge fixing. Using (2.1), we see that the exponent is

$$I_E[\bar{g}] = \frac{1}{4\pi G|\Lambda|^{1/2}} \mathrm{Vol}(M) \ , \tag{2.3}$$

where $\mathrm{Vol}(M)$ is the volume of the three-manifold $M$ with the metric rescaled to constant curvature $-1$. This volume is a well-known topological invariant in three-manifold theory: a given manifold will admit at most one hyperbolic metric, and it can be shown that at most a finite number of three-manifolds have any given hyperbolic volume [14].

To evaluate the prefactor $\Delta_M$, we can appeal to the relationship between the three-dimensional Einstein action and the Chern-Simons action for the gauge group PSL(2,$\mathbb{C}$). As Witten has observed [15], the first order form of the Einstein action in three dimensions with $\Lambda < 0$ can be rewritten as

$$I_{\mathrm{CS}} = \frac{1}{64\pi i G|\Lambda|^{1/2}} \int_M d^3x \ \epsilon^{ijk}[A_i^a(\partial_j A_{ka} - \partial_k A_{ja}) + \tfrac{2}{3}\epsilon_{abc} A_i^a A_j^b A_k^c] + \mathrm{c.c.} \ , \tag{2.4}$$

where the complex gauge field

$$A_i^a = \tfrac{1}{2}\epsilon^{abc}\omega_{ibc} + i|\Lambda|^{1/2}e_i^a \tag{2.5}$$

is a PSL(2,$\mathbb{C}$) connection, expressed in terms of a "dreibein" or frame field $e_i^a$ and a spin connection $\omega_{ib}^a$ on $M$. The extrema of (2.4) are flat PSL(2,$\mathbb{C}$) connections, and it is easy to check that the condition of flatness is equivalent to the field equations (2.1) for the metric $g_{ik} = e_i^a e_{ka}$. At the same time, (2.4) may also be recognized as the standard Chern-Simons action for PSL(2,$\mathbb{C}$), allowing us to use known results for Chern-Simons path

integrals to evaluate $\Delta_M$.

To understand this correspondence better, it is helpful to know a bit more about the geometric significance of the connection $A_i^a$. Recall first that PSL(2,$\mathbb{C}$) is the isometry group of hyperbolic three-space $\mathbb{H}^3$. Any three-manifold $M$ with a constant negative curvature metric is locally isometric to $\mathbb{H}^3$, and can be covered by coordinate charts $U_\alpha$ isometric to $\mathbb{H}^3$ with transition functions $\phi_{\alpha\beta}$ in PSL(2,$\mathbb{C}$). This "geometric structure" allows us to define a natural flat $sl(2,\mathbb{C})$ bundle $E$ over $M$ as follows [16]: we first construct the product bundle $sl(2,\mathbb{C}) \times U_\alpha$ on each chart, and then identify the fibers in the overlap $U_\alpha \cap U_\beta$ by means of the adjoint action of the transition function $\phi_{\alpha\beta}$. It can then be shown that the connection $A_i^a$ is precisely the flat connection on $E$.

Equivalently, at least if $M$ is geodesically complete, we can write

$$M = \mathbb{H}^3/\Gamma \tag{2.6}$$

---

[1]In wormhole sums [4,5], one includes a class of higher action metrics as well, essentially an instanton gas of the leading extrema, but most of the classical solutions are still neglected.

for some discrete subgroup $\Gamma \subset \mathrm{PSL}(2,\mathbb{C})$, unique up to conjugation. $\Gamma$ is called the holonomy group of $M$; it is isomorphic to the fundamental group $\pi_1(M)$, and can be viewed as a representation of $\pi_1(M)$ in $\mathrm{PSL}(2,\mathbb{C})$. The bundle $E$ is then

$$E = (sl(2,\mathbb{C}) \times \mathbb{H}^3)/\Gamma , \qquad (2.7)$$

where the quotient is by the simultaneous action of $\Gamma$ as a group of isometries of $\mathbb{H}^3$ and the adjoint action of $\Gamma$ on $sl(2,\mathbb{C})$. This description makes it clear that $E$ is completely determined by the group $\Gamma$.

Standard results from Chern-Simons theory now tell us that

$$\Delta_M = T^{1/2}(M,E) , \qquad (2.8)$$

where $T(M,E)$ is the Ray-Singer torsion, or equivalently[2] the Reidemeister-Franz torsion, associated with the flat bundle $E$. Strictly speaking, we must use a slight modification of the standard definition of Ray-Singer torsion, as discussed in Ref. [17], because $\mathrm{PSL}(2,\mathbb{C})$ is noncompact. For a complex gauge group, the effect of this change is essentially to replace the torsion computed in terms of $A$ alone with its absolute square; heuristically, the partition function receives separate contributions from the path integrals over $A$ and $\bar{A}$ in the action (2.4). To obtain (2.8) from Ref. [17], we have also used the fact that the connection $A_i^a$ is isolated (by rigidity theorems for hyperbolic structures [19]) and irreducible, and that the framing anomaly and the phase of the prefactor $\Delta_M$ both vanish for $\mathrm{PSL}(2,\mathbb{C})$, essentially because of a cancellation between left- and right-moving modes [17,20].

The Chern-Simons formulation can potentially give us information about higher-order corrections to the saddle-point approximation as well. In particular, although the higher-order terms have not been computed explicitly, we know they will be of order $G|\Lambda|^{1/2}$, and will thus be small if the cosmological constant is sufficiently small [15].

Combining (2.3) and (2.8), we obtain a contribution to the wave function of the form

$$\Psi_M[h] = T^{1/2}(M,E) \exp\left\{ -\frac{\mathrm{Vol}(M)}{4\pi G|\Lambda|^{1/2}} \right\} \qquad (2.9)$$

for each extremum $(M,\bar{g})$ with $\bar{g}|_\Sigma = h$. These contributions must be summed over extrema:

$$\Psi[h] = \sum_{M \in \mathcal{J}(\Sigma,h)} \Psi_M[h] , \qquad (2.10)$$

where the "space of instantons" $\mathcal{J}(M,h)$ comprises all hyperbolic manifolds with induced hyperbolic metric $h$ on a

single totally geodesic boundary $\Sigma$. Our next task is to categorize this space.

## III. COUNTING HYPERBOLIC MANIFOLDS

The classification of hyperbolic three-manifolds is one of the most active areas of research in modern topology, and for now we should not expect to completely understand the space of extrema of the (2+1)-dimensional Einstein action. We may still look for particular points in $\mathcal{J}(M,h)$, however, that can give us useful information about the Hartle-Hawking wave function (2.10).

Let us first note that for most hyperbolic metrics $h$ on $\Sigma$, there are no saddle-point contributions—the field equations (2.1) usually have no solution with a specified boundary value for the metric. It is known, however, that solutions exist for a dense set of values of $h$ in the moduli space $\mathcal{M}_\Sigma$ [21]. If the full path integral behaves reasonably smoothly, the saddle-point approximation for $\Psi[h]$ on such a dense set should be adequate for physics.

Kojima and Miyamoto [22] have recently found the hyperbolic manifolds of least volume with a single totally geodesic boundary of any given genus. These are the extrema $M_R$ considered by Fujiwara et al. [10]. For a given spatial topology $\Sigma$, the corresponding manifold $M_R(\Sigma)$ will have some definite boundary metric $h_R$, and since the wave function (2.9) is exponentially suppressed for large volumes, we might expect $\Psi[h]$ to be peaked at the corresponding two-geometry.

On the other hand, a given extremum $M_R$ makes only a single contribution to $\Psi[h_R]$. We must also ask whether other spatial metrics $h$ are boundary values for large numbers of extrema of the action. If this is the case, the number of instantons, the "entropy," may overcome the exponential volume suppression.

To see that this is possible, we consider a family of hyperbolic three-manifolds discovered by Neumann and Reid [23,24]. (The manifolds most relevant to physics are actually not quite the ones described in these references, but rather a closely related family found by Reid; see the Appendix for details.) The family consists of an infinite number of manifolds $\tilde{M}_{(p,q)}$, where $p$ and $q$ are relatively prime integers, with the following characteristics: (1) each of the $\tilde{M}_{(p,q)}$ has a single totally geodesic boundary, with a fixed hyperbolic metric $h_\infty$ that is independent of $p$ and $q$; (2) the volumes of the $\tilde{M}_{(p,q)}$ are bounded above by a finite number $\mathrm{Vol}(\tilde{M}_\infty)$, and converge to $\mathrm{Vol}(\tilde{M}_\infty)$ as $p^2+q^2 \to \infty$; and (3) the Ray-Singer torsions $T(\tilde{M}_{(p,q)}, E_{(p,q)})$ considered in the previous section do not converge as $p^2+q^2 \to \infty$, but instead take on a dense set of values in the interval $(0, cT_\infty]$, where $cT_\infty$ is a positive constant.

These properties imply that the $\tilde{M}_{(p,q)}$ all give positive contributions to the Hartle-Hawking wave function at $h = h_\infty$. Indeed, conditions (2) and (3) guarantee that the sum over topologies diverges: the volumes converge to $\mathrm{Vol}(\tilde{M}_\infty)$, while infinitely many of the prefactors are bounded below by some $\epsilon > 0$. The Hartle-Hawking wave function is thus infinitely peaked at $h_\infty$.

The construction of the families $\tilde{M}_{(p,q)}$ is discussed in more detail in the Appendix, but the basic procedure is

---

[2]For noncompact groups, this equivalence is discussed in [17]. Schwarz and Tyupkin [18] have pointed out that an additional anomaly can occur because $M$ has a boundary. This problem will not arise when the boundary is totally geodesic, however, since one can then compute everything on the (closed) double of $M$.

fairly easy to describe. Neumann and Reid start with a finite volume hyperbolic orbifold $M_\infty$ (note no tilde here) with two essential characteristics: a totally geodesic boundary consisting of a sphere with three conical singularities, and a cusp $K$ that is separated from this boundary. The singularity at $K$ can be "filled in" by a standard procedure called hyperbolic Dehn surgery—essentially by replacing a neighborhood of the cusp with a solid torus—to obtain a set of new orbifolds $M_{(p,q)}$, where the integers $(p,q)$ describe the way the torus is twisted before it is glued in. This surgery procedure cannot change the hyperbolic structure on the boundary, however, since a sphere with three cone points admits only one hyperbolic metric.

Neumann and Reid then consider a covering space $\tilde{M}_\infty$ of $M_\infty$ in which the orbifold singularities, including the conical singularities on the boundary, are "unwrapped." The cusp $K$ of $M_\infty$ is lifted to a set of cusps $\tilde{K}$ on $\tilde{M}_\infty$, and Dehn surgeries on $M_\infty$ lift to surgeries on $\tilde{M}_\infty$. These lifted surgeries must again leave the boundary of $\tilde{M}_\infty$ fixed, since otherwise their projections to $M_\infty$ would change the boundary there. We thus obtain an infinite family of nonsingular three-manifolds all having an identical totally geodesic boundary.

It can be shown that Dehn surgery on a cusp of a hyperbolic manifold always decreases the volume [28]. This fact gives us condition (2) above: the volume of $\tilde{M}_{(p,q)}$ is bounded above by the volume of the cusped manifold $\tilde{M}_\infty$. For $p^2+q^2$ large, Neumann and Zagier [25] have found a fairly simple description of the rate of convergence of the volume $\text{Vol}(\tilde{M}_{(p,q)})$ to $\text{Vol}(\tilde{M}_\infty)$:

$$\#\{(p,q):\text{Vol}(M_{(p,q)})<\text{Vol}(M_\infty)-1/x\}$$
$$=6\pi x+O(x^{1/2}) , \quad (3.1)$$

where $\#A$ denotes the number of elements in the set $A$.

To obtain condition (3), we must understand the behavior of the Ray-Singer torsion under hyperbolic Dehn surgery. Our basic strategy parallels Witten's computation of Chern-Simons amplitudes on manifolds that have undergone surgery [15]: we separately compute the torsions of $\tilde{M}_{(p,q)}-V$ and $V$, where $V$ is the solid torus added by the surgery. We can then use the "gluing theorem" for torsion [26] to obtain

$$T(\tilde{M}_{(p,q)},E_{(p,q)})=T(\tilde{M}_{(p,q)}-V,E_{(p,q)}|_{\tilde{M}-V})T(V,E_{(p,q)}|_V) .$$
$$(3.2)$$

By construction, $\tilde{M}_{(p,q)}-V$ is diffeomorphic to $\tilde{M}_\infty$; its torsion differs from that of $\tilde{M}_\infty$ only because the hyperbolic structure—the flat bundle $E_{(p,q)}$—differs. But for $p^2+q^2$ large, the holonomy groups $\Gamma(\tilde{M}_{(p,q)}-V)$ converge to $\Gamma(\tilde{M}_\infty)$, which is sufficient to show that the torsions converge. For $V$, on the other hand, the Ray-Singer torsion can be calculated explicitly (see the Appendix for details). The result again depends on the flat bundle, and thus on $p$ and $q$; one finds that

$$T(V,E_{(p,q)}|_V)=\frac{c}{4}(\cosh 2l-\cos 2t)^2 , \quad (3.3)$$

where $l$ and $t$ are the length and torsion of the core geo-

desic of $V$. ("Torsion" here means not the Ray-Singer torsion, but the ordinary geometric torsion of the geodesic as a curve in three-space.) For $p$ and $q$ large, $l$ converges to zero, but it is known [27] that $t$ takes on a dense set of values in the interval $[0,2\pi]$, so $T(V,E_{(p,q)})$ varies rapidly in the range $(0,c]$. Our result then follows directly from inserting (3.3) into (3.2).

It would be interesting to find a more detailed description of the behavior of the parameter $t$ for $p^2+q^2$ large, ideally leading to a result for the Ray-Singer torsion analogous to (3.1). Such a description would allow us to approximate the sum over $p$ and $q$ by an integral, perhaps permitting a more quantitative description of the divergence of the wave function.

## IV. IMPLICATIONS

We have seen that the Hartle-Hawking wave function (2.10) diverges for at least one value of the spatial metric $h$. A natural question is how often this behavior occurs. If $\Psi[h]$ diverges for most values of $h$, our result is essentially negative—we will have merely shown that the "leading instanton" approximation is not valid. If such divergences are relatively rare, on the other hand, we may have learned a good deal about $\Psi[h]$.

Let us first consider the exponent in (2.9). The volume $\text{Vol}(M)$ may be viewed as a real-valued function on the space of instantons $\mathcal{I}(M,h)$. In our example,we produced a bounded sequence of volumes by hyperbolic Dehn surgery on a cusped manifold. For closed hyperbolic three-manifolds, it can be shown that this is the only way to produce such a sequence; in particular, the only accumulation points of the volume in $\mathbb{R}$ correspond to manifolds with at least one cusp [14,28].

By a simple doubling argument, the same is true for manifolds with a totally geodesic boundary. Hence the kind of divergence we saw above will only occur for values of $h$ that can be realized as boundary values of hyperbolic metrics on cusped manifolds. Most hyperbolic manifolds have no cusps, of course, so this is likely to be a significant restriction, although as far as I know this issue has not been investigated by topologists.

The Neumann-Reid construction suggests, although it does not prove, a much stronger restriction. In general, one expects hyperbolic Dehn surgery to change the boundary metric of a three-manifold, smearing out any divergence in the sum over topologies. This did not occur in our example for a very specific reason: the boundary we have been considering can be realized as a covering space of a rigid surface, the two-sphere with three conical singularities. ("Rigid" means that the surface admits only one constant negative curvature metric, i.e., that its moduli space is a single point.) Only a few, highly symmetric surfaces occur as covering spaces of rigid surfaces, and it is plausible that the sum over topologies will diverge only for such surfaces. If this is the case, it may be possible to give a complete description of the normalized Hartle-Hawking wave function as a sum of delta functions at isolated points in moduli space.

The key question, of courses, is whether such results can be extended to 3+1 dimensions. For $n=4$, the ex-

trema of the action (1.1) need not have constant negative curvature, and a detailed analysis becomes much more difficult. But we can at least ask how the constant negative curvature manifolds contribute to the wave function.

Four-dimensional hyperbolic manifolds have a discrete set of volumes, with only finitely many manifolds having the same volume. In contrast with the three-dimensional case, there are no longer any accumulation points. It is still true, however, that the number of manifolds of a given volume can grow rapidly as the volume increases. According to Gromov [14], the number of hyperbolic four-manifolds with volume less than $x$ may grow as fast as

$$x \exp\{\exp[\exp(4+x)]\} \ . \qquad (4.1)$$

If the increase is nearly this rapid, we may again expect entropy to dominate action in the Hartle-Hawking wave function of the Universe.

Moreover, it is plausible that the sum over topologies will again be dominated by highly symmetric spacetimes. Topologically distinct hyperbolic manifolds with the same volume typically arise when the boundaries of a fundamental polyhedron can be glued together to form a manifold in more than one way. As in the (2+1)-dimensional case, this may be viewed as an indication of underlying symmetry. This connection is admittedly speculative, however; a more quantitative description would clearly be of interest.

## ACKNOWLEDGMENTS

## APPENDIX: MATHEMATICAL DETAILS

The purpose of this appendix is to fill in some of the mathematical details omitted from the text. We will discuss hyperbolic Dehn surgery, the Neumann-Reid construction, and the computation of Ray-Singer torsion.

### 1. Hyperbolic Dehn surgery

In this section, we briefly summarize hyperbolic Dehn surgery [28,16]. A cusp of a hyperbolic three-manifold $M$ can be viewed as an embedded circle that is "infinitely far away" from the rest of the manifold in the hyperbolic metric. Topologically, a neighborhood of a cusp is diffeomorphic to $T^2 \times [t_0, \infty)$, where $T^2$ is a two-dimensional torus. (The circle itself is not part of $M$,

which is complete but not compact.) Metrically, we can take the upper half space model for $\mathbb{H}^3$, with the standard constant negative curvature metric

$$ds^2 = t^{-2}(dx^2+dy^2+dt^2) \ ; \qquad (A1)$$

a neighborhood of a cusp then looks like a region

$$N = \{(x,y,t):t > t_0, z \sim z+1, z \sim z+\tau\} \qquad (A2)$$

(see Fig. 2). Note that for fixed $t$, the metric (3.2) is Euclidean, so the constant $t$ cross sections of (A2) are ordinary flat tori with modulus $\tau$.

To perform (topological) Dehn surgery, we remove a small neighborhood $K$ of a cusp, or more generally of any embedded circle in $M$, and replace it with a solid torus $V = S^1 \times D^2$. $M - K$ has a toroidal boundary where $K$ has been cut out, and associated with this boundary are two commuting generators of $\pi_1(M-K)$, say $m$ and $l$. To glue in the solid torus $V$, we first choose a closed curve $\gamma$ on the boundary along which to attach a cross section $\{p\} \times D^2$ of $V$. Once this disk is glued in, there is no remaining topological freedom, since $V - \{p\} \times D^2$ is topologically simply a ball, which can be glued in uniquely.

The curve $\gamma$ can be written in terms of the generators $l$ and $m$ as

$$\gamma = m^p l^q \ . \qquad (A3)$$

This expression will represent a simple closed curve if $p$ and $q$ are relatively prime integers. The effect of surgery on the fundamental group is to add one relation $m^p l^q = 1$ to $\pi_1(M-K)$, that is, to kill one generator.

In general, one can say little about the geometry of a manifold resulting from surgery. In fact, any three-manifold can be obtained from Dehn surgery on a link in the three-sphere [29]. However, Thurston [28] has shown that if one performs Dehn surgery on a cusp of a hyperbolic three-manifold, the resulting manifold will itself admit a hyperbolic metric for all but a finite number of choices of $p$ and $q$. The process of creating this new hyperbolic manifold is called hyperbolic Dehn surgery.

Thurston's result can be restated as follows. A cusped manifold admits a unique *complete* hyperbolic metric. But it also admits an infinite number of incomplete hyperbolic metrics, parametrized by relatively prime integers $p$ and $q$, that can be completed to give nonsingular,
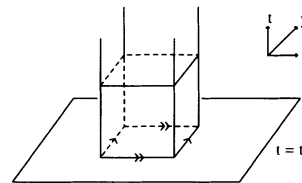


FIG. 2. The neighborhood of a cusp of a hyperbolic manifold, represented in the upper half space model of $\mathbb{H}^3$. Note that the area of a toroidal cross section is proportional to $t^{-2}$, while the distance from $t_0$ to $t$ is $d = \ln(t/t_0)$, so the area decreases exponentially with proper distance.

complete hyperbolic metrics by adding an appropriate solid torus. This formulation makes it easy to describe one sense in which the manifolds $M_{(p,q)}$ converge to $M_\infty$ as $p^2+q^2 \to \infty$: it can be shown that the corresponding distance functions converge [28,32]. Pictorially, a cusp may be visualized as an infinitely long, exponentially shrinking tube with a toroidal cross section (Fig. 2); as $p$ and $q$ become large, the surgery affects the manifold farther and farther out on this tube, leaving an increasingly large piece of $M_\infty$ essentially unchanged.

This convergence is also reflected algebraically in the holonomy groups $\Gamma_{(p,q)}$. An element of $\Gamma_{(p,q)}$ represents a closed geodesic in $M_{(p,q)}$, and as $p^2+q^2 \to \infty$, the geodesics converge to the geodesics of $M_\infty$. Consequently, the holonomy groups also converge; in particular, if $g_{(p,q)} \in \Gamma_{(p,q)}$ represents a curve $\gamma$ in $M_{(p,q)}-K$, then the $g_{(p,q)}$ will converge to the element $g \in \Gamma_\infty$ representing $\gamma$ in the holonomy of $M_\infty$. This result will be important below in the analysis of Ray-Singer torsion.

## 2. The Neumann-Reid construction

The fundamental construction used in this paper is based on the papers [23] and [24] of Neumann and Reid. The particular variation used here is unpublished, and was explained to me by Alan Reid.

An orbifold is a space locally modeled on $\mathbb{R}^n/\Gamma$, where $\Gamma$ is a finite group that acts properly discontinuously but not necessarily freely. The set of points of $\mathbb{R}^n$ at which the action is not free projects down to a set called the singular locus of the orbifold. A typical two-dimensional orbifold is a cone of order $n$, $\mathcal{O}=\mathbb{R}^2/C_n$, where $C_n$ is the group generated by rotations by $2\pi/n$ around some point $p$. $C_n$ acts freely except at $p$, and the singular locus is thus the apex of the cone. $C_n$ also acts on $\mathbb{R}^3$ by rotation around an axis; the singular locus is then a line corresponding to this axis. By a (three-dimensional) hyperbolic orbifold, we mean an orbifold locally modeled on $\mathbb{H}^3/\Gamma$, where $\Gamma$ is now a group of isometries of hyperbolic three-space. Orbifold singularities then come from torsion elements of $\Gamma$.

In [23], Neumann and Reid construct a set of hyperbolic three-orbifolds $M_\infty(m,n)$, each having an underlying space $S^2\times[0,\infty)$ and a boundary consisting of a totally geodesic two-sphere $\Sigma(m,n)$ with three conical points. The singular locus of one of these orbifolds is shown in Fig. 3. A line in this diagram labeled by an integer $n$ corresponds to a cone angle of $2\pi/n$; a vertex at which lines labeled 2, 2, and $n$ meet is locally $\mathbb{R}^3/D_n$,
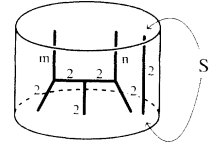


FIG. 3. The orbifold that forms the starting point of the Neumann-Reid construction. A line labeled by an integer $n$ represents a singularity with cone angle $2\pi/n$. The underlying space of this orbifold is $S^2\times[0,\infty)$; horizontal cross sections of the diagram should be interpreted as two-spheres.

where $D_n$ is the dihedral group of order $2n$.

The top boundary of Fig. 3 is thus a sphere with cone points of cone angles $2\pi/2$, $2\pi/m$, and $2\pi/n$. The bottom boundary, on the other hand, is a "pillow cusp," with a neighborhood of the form $K\approx F\times[t_0,\infty)$, where $F$ is a sphere with four cone points of cone angle $\pi$. $F$ has Euler characteristic zero, and can be expressed as the quotient of a torus by the cyclic group of order 2; that is, a pillow cusp has an ordinary cusp as a double cover.

We previously defined hyperbolic Dehn surgery for an ordinary cusp, but it can be shown that a similar procedure is possible for a pillow cusp. As in the case of an ordinary cusp, surgery on the cusp of $M_\infty(m,n)$ produces a family of orbifolds $M_{(p,q)}(m,n)$, all but a finite number of which admit hyperbolic structures. Moreover, this surgery cannot affect the boundary $\Sigma(m,n)$, since a sphere with three conical points is rigid, that is, it admits a unique hyperbolic metric.

Now suppose that for some $m$ and $n$, we can find a new manifold $\tilde{M}_\infty(m,n)$ that is a finite covering space of $M_\infty(m,n)$, such that the boundary $\Sigma(m,n)$ lifts to a single connected surface $\tilde{\Sigma}(m,n)$. In such a covering, the cusp $K$ will lift to a set of cusps $\tilde{K}$, and $(p,q)$ surgery on $K$ will lift to some $(p',q')$ surgery on $\tilde{K}$. The boundary $\tilde{\Sigma}(m,n)$ must be left invariant by such $(p',q')$ surgery, since it projects back to the fixed boundary $\Sigma(m,n)$ of $M_\infty(m,n)$. The set of such $(p',q')$ surgeries on $\tilde{K}$, that is, surgeries that are equivariant with respect to the cover, is an infinite subset of the set of all surgeries, and we will have thus obtained an infinite family of manifolds with identical totally geodesic boundary $\tilde{\Sigma}(m,n)$.

We must thus show that such a covering can exist. To do so, let us choose $m=3$ and $n>6$ a prime number such that $n=1(\bmod 4)$. We shall need a presentation of the orbifold fundamental group of $M_\infty(3,n)$:

$$\Gamma(3,n)=\langle a,b,c,d\,|\,a^2=b^2=c^3=d^n=(ab)^2=(bc)^2=(cd)^2=(da)^2=1\rangle\,, \tag{A4}$$

where the subgroup representing the fundamental group of the boundary $\Sigma(3,n)$ is the triangle group

$$\Delta(2,3,n)=\langle c,d\,|\,c^3=d^n=(cd)^2=1\rangle\,. \tag{A5}$$

This presentation can be obtained by applying standard

techniques [30] to the construction of Ref. [23], in which $M_\infty(m,n)$ is described explicitly in terms of reflections in the faces of a polyhedron. Alternatively, it can be read off from Fig. 3; a loop around a line with cone angle $2\pi/r$ represents an element in $\Gamma(m,n)$ of order $r$, while rela-

tions come from the requirement that the product of loops completely encircling a vertex be contractible.

We will have completed our proof if the following conditions are satisfied: (1) there exists an epimorphism $\phi$ from $\Gamma(3,n)$ to a finite group $G_n$ such that $\Delta(2,3,n)$ surjects onto $G$; and (2) the kernel $\kappa$ of $\phi$ is torsion free. To see that these conditions are sufficient, observe that the cover corresponding to the subgroup $\kappa \subset \Gamma$ is a finite cover of $M_\infty(3,n)$ with fundamental group $\kappa$; by condition (2), this covering space is a manifold, with no orbifold singularities. Then by standard covering space theory, condition (1) implies that the preimage of $\Sigma(3,n)$ is a connected surface (see, for instance, section 5.11 of Ref. [31]).

To complete the proof, we must therefore construct the map $\phi$ to a finite group. Let $G_n$ be the finite simple group $PSL(2,\mathbf{F}_n)$, where $\mathbf{F}_n$ is the field of $n$ elements. For $n$ prime such that $n = 1 \pmod 4$, some elementary number theory shows that $-1$ is a square in $\mathbf{F}_n$. Let $x \in \mathbf{F}_n$ be such that $x^2 = -1$, and let $z = -1 + 2x, t = -4x(1+2x)^{-1}$. We then define the epimorphism $\phi$ from $\Gamma(3,n)$ to $PSL(2,\mathbf{F}_n)$ by

$$a \mapsto \begin{bmatrix} x & t \\ 0 & -x \end{bmatrix}, \quad b \mapsto \begin{bmatrix} 2 & z \\ 2+z & -2 \end{bmatrix},$$
$$c \mapsto \begin{bmatrix} 0 & 1 \\ -1 & 1 \end{bmatrix}, \quad d \mapsto \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}. \tag{A6}$$

Note that these each have determinant 1 in $PSL(2,\mathbf{F}_n)$. We then have

$$ab \mapsto \begin{bmatrix} -2x & xz-2t \\ 2-x & 2x \end{bmatrix}, \quad bc \mapsto \begin{bmatrix} -z & 2+z \\ 2 & z \end{bmatrix},$$
$$cd \mapsto \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \quad da \mapsto \begin{bmatrix} x & t-x \\ 0 & -x \end{bmatrix}, \tag{A7}$$

and it is easily checked that the relations in (A4) are all satisfied. Moreover, $PSL(2,\mathbf{F}_n)$ is generated by $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ and $\begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix}$, so $\Delta(2,3,n)$ surjects onto $G_n$. Finally, as every element of finite order is conjugate in $\Gamma(3,n)$ to one of the elements (A6), (A7), the kernel is torsion-free [30]. Our two requirements are thus met, and the proof is completed.

For these examples, we can compute the genus of the boundary $\tilde{\Sigma}(3,n)$ explicitly. The Euler characteristic of $\Sigma(3,n)$ is $\chi(\Sigma(3,n)) = (6-n)/6n$ [28], while by construction, the degree of the covering is equal to the order of $PSL(2,\mathbf{F}_n)$. Thus

$$2 - 2g = \chi(\tilde{\Sigma}(3,n)) = \frac{6-n}{6n} |PSL(2,\mathbf{F}_n)|$$
$$= \frac{(6-n)(n^2-1)}{12}. \tag{A8}$$

It may checked that the resulting genus is always an integer, although typically a rather large one, when $n = 1 \pmod 4$.

### 3. Computing torsions

In this section, we compute Ray-Singer torsions for surgered manifolds. (See [18,33] for more detailed

definitions of these torsions.) Our starting point is a "gluing formula." Let $E$ be a flat bundle over a manifold $M$, and suppose that $M$ can be written as a union of two pieces $M_1$ and $M_2$ joined along a common boundary $\Sigma$. Vishik [26] has then shown that

$$T(M,E) = T(M_1, E|_{M_1}) T(M_2, E|_{M_2}) T(\Sigma, E|_\Sigma), \tag{A9}$$

where the determinants on $M_1$ and $M_2$ are defined with relative (Dirichlet) boundary conditions. In particular, for hyperbolic $(p,q)$ Dehn surgery on a cusp of $M_\infty$, we can take $M_1$ to be the solid torus $V$ added by surgery, $M_2$ to be $M_\infty - K$, and $E$ to be the flat $sl(2,\mathbb{C})$ bundle (see Sec. 2) corresponding to the new hyperbolic structure on the manifold $M_{(p,q)}$ resulting from the surgery. The common boundary $\Sigma$ is then a two-dimensional torus, and the last term in (A9) can be omitted, since the torsion of a closed even-dimensional manifold is always trivial.

We must first compute the torsion $T(V,E_{(p,q)})$ for $V = S^1 \times D^2$. Since $D^2$ is simply connected, we can use the product formula[3]

$$T(S^1 \times D^2, E) = T(S^1, E|_{S^1})^{\chi(D^2)}, \tag{A10}$$

where $\chi$ is the Euler number, $\chi(D^2) = 1$. We thus need merely evaluate the Ray-Singer torsion for a circle. As Bar-Natan and Witten have noted [17], there are some subtleties involved in the definition of the Laplacian because $sl(2,\mathbb{C})$ is noncompact, but it is not hard to check that the appropriate Ray-Singer torsion is

$$T(S^1, E) = |\det' \Delta_0|, \tag{A11}$$

where the Laplacian acts on $sl(2,\mathbb{C})$-valued functions $\phi = \phi^a t_a$ twisted by the holonomy $H$:

$$\phi(\theta + 2\pi) = H^{-1} \phi(\theta) H. \tag{A12}$$

Up to an overall conjugation that does not affect the determinants, the holonomy around $S^1$ takes the form

$$H = \begin{bmatrix} e^{l+it} & 0 \\ 0 & e^{-(l+it)} \end{bmatrix}, \tag{A13}$$

where $l$ and $t$ have a geometrical interpretation as the length and torsion of the "core geodesic" $S^1$ of $V$. The evaluation of the determinant (A11) is then reasonably straightforward: eigenfunctions take the form

$$\phi(\theta) = \begin{bmatrix} a(\theta) & b(\theta) \\ c(\theta) & -a(\theta) \end{bmatrix} \tag{A14}$$

with

$$a(\theta + 2\pi) = a(\theta),$$
$$b(\theta + 2\pi) = e^{-2(l+it)} b(\theta),$$
$$c(\theta + 2\pi) = e^{2(l+it)} c(\theta), \tag{A15}$$

---

[3]This relationship is proven by Ray and Singer for closed manifolds [33], but the extension to manifolds with boundary with relative boundary conditions is straightforward.

and $\zeta$-function regularization gives

$$T(S^1,E)=16\pi^2(\cosh2l-\cos2t)^2 \ . \tag{A16}$$

An important caveat is necessary here: Eq. (A16) is not quite independent of the metric in $S^1$. The Laplacian (A11) has zero modes, constant matrices $\phi$ that commute with the holonomy, and the Ray-Singer torsion should really be viewed as a section of the line bundle $(\det H^0)^{-1}(\det H^1)$. A canonical section exists only if one has a metric with which to normalize harmonic forms; the expression (A16) was computed with respect to the metric $d\theta^2$ on the circle. This metric dependence must drop out of the final expression for the torsion of the manifold $M_{(p,q)}$ resulting from the surgery, since the Laplacians there have no zero modes (see below), but it would be interesting to understand how this happens in more detail. It was because of this ambiguity that the constant $c$ was left unspecified in Eq. (3.3).

It is instructive to recompute (A16) in terms of Reidemeister-Franz torsion (see [34] or [35] for a clear explanation of this invariant; note, however, that the quantity $\tau$ in Ref. [34] is equal to $-\ln T$ in our conventions). One begins with the cellular version of the flat bundle (2.7):

$$\mathcal{C}(\tilde{V},\partial\tilde{V};E)=sl(2,\mathbb{C})\otimes_\Gamma \mathcal{C}(\tilde{V},\partial\tilde{V}) \ , \tag{A17}$$

where $\mathcal{C}(\tilde{V},\partial\tilde{V})$ is a (relative) chain complex for the universal covering space $\tilde{V}$ of $V$, and the holonomy group $\Gamma=\langle H\rangle\approx\pi_1(V)$ acts by deck transformations on $\tilde{V}$ and by the adjoint action on $sl(2,\mathbb{C})$. We can take the Pauli matrices $\sigma^a$ and $i\sigma^a$ as generators for $sl(2,\mathbb{C})$, and it is evident from Fig. 4 that a basis for $\mathcal{C}(\tilde{V},\partial\tilde{V};E)$ consists of the three-cells $\{\sigma^a\otimes S,(i\sigma^a)\otimes S\}$ and the two-cells $\{\sigma^a\otimes F,(i\sigma^a)\otimes F\}$, with a boundary operator

$$\partial(\sigma^a\otimes S)=\begin{cases}(H^{-1}\sigma^a H-\sigma^a)\otimes F & \text{if } a=1,2, \\ 0 & \text{if } a=3\end{cases} \tag{A18}$$

(with a similar expression for the cells involving $i\sigma^a$). The relative homology group $H_3$ is clearly generated by $\{\sigma^3\otimes S,(i\sigma^3)\otimes S\}$, while $H_2$ is generated by $\{\sigma^3\otimes F,(i\sigma^3)\otimes F\}$.

To compute Reidemeister-Franz torsion, we need volume forms for $C_q$ and $H_q$. The volume forms for the chain groups are determined by the preferred basis fixed by the cell decomposition of $V$:

$$\omega_2=\wedge(\sigma^a\otimes F)\wedge((i\sigma^a)\otimes F) \ ,$$
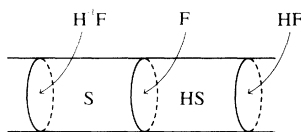$$\omega_3=\wedge(\sigma^a\otimes S)\wedge((i\sigma^a)\otimes S) \ , \tag{A19}$$



FIG. 4. A cell decomposition for the universal covering space $\tilde{V}$ of the solid torus $V$. Relative to the boundary $\partial\tilde{V}$, the only cells are the three-cell $S$, the two-cell $F$, and their translates by deck transformations.

while volume forms for the homologies take the form

$$\mu_2=h(\sigma^3\otimes F)\wedge((i\sigma^3)\otimes F) \ ,$$
$$\mu_3=k(\sigma^3\otimes S)\wedge((i\sigma^3)\otimes S) \tag{A20}$$

for some constants $h$ and $k$. We now choose an arbitrary volume form for $B_2=\partial C_3$, say

$$\rho=\partial(\sigma^1\otimes S)\wedge\partial(\sigma^2\otimes S)\wedge\partial((i\sigma^1)\otimes S)\wedge\partial((i\sigma^2)\otimes S) \ , \tag{A21}$$

and write

$$\omega_2=m_{\text{even}}\,\rho\wedge\mu_2 \ ,$$
$$\omega_3=m_{\text{odd}}\,(\partial^{-1}\rho)\wedge\mu_3 \ . \tag{A22}$$

The Reidemeister-Franz torsion is then defined to be

$$T(\mathcal{C}(\tilde{V},\partial\tilde{V};E))=m_{\text{odd}}/m_{\text{even}} \ . \tag{A23}$$

It is easy to check that this expression agrees with (A16), up to a constant factor depending on $h$ and $k$. This ambiguity again reflects the existence nontrivial homology; the two expressions will agree completely if we define the volume forms $\mu_2$ and $\mu_3$ on the homology to be dual to normalized volume forms on the cohomology, where the normalization once again depends on the choice of metric. It is worth noting that the dependence of the torsion on the holonomy group $\langle H\rangle$ comes entirely through the boundary operator (A18), and can be expressed in terms of the determinant of the "combinatorial Laplacian" $\partial^\dagger\partial+\partial\partial^\dagger$.

We are now left with the second term in (A9), the torsion of the manifold $M_2=M_\infty-K$ with flat bundle $E_{(p,q)}$. This quantity is rather difficult to compute in general, but some conclusions about its behavior can be reached. Note first that the topology of $M_2$ is independent of the choice of the surgery coefficients $p$ and $q$; in fact, $M_2$ is diffeomorphic to the cusped manifold $M_\infty$. Let us choose a cell decomposition for this manifold once and for all, with a corresponding set of deck transformations $\gamma_i\in\pi_1(M_\infty)$ that determine the gluing pattern of the cells. The Reidemeister-Franz torsion will again be computed from a finite set of determinants, whose entries are fixed by the cell decomposition and by the representation $\Gamma_{(p,q)}$ of $\pi_1(M_2)$ in $PSL(2,\mathbb{C})$. In particular, the only dependence of the torsion on the surgery coefficients will come through the dependence of the combinatorial Laplacian on a set of elements $g_{(p,q)}(\gamma_i)\in\Gamma_{(p,q)}$ that represent the gluing maps for $M_{(p,q)}-K$.

But we saw above that the representations $\Gamma_{(p,q)}$ converge to $\Gamma_\infty$ as $p^2+q^2\to\infty$. In particular, the elements $g_{(p,q)}(\gamma_i)$ converge. Hence the determinants of the combinatorial Laplacians must also converge, and the torsions $T(M_\infty-K,E_{(p,q)})$ will converge to some number $T_\infty$, which can be interpreted as the Reidemeister-Franz torsion for the original cusped manifold $M_\infty$ (with relative boundary conditions at the cusp). This guarantees that $T_\infty$ is nonvanishing, since Reidemeister-Franz torsion is always nonzero.

Finally, let us return to the question of zero modes and the possible metric dependence of the total torsion $T(M_{(p,q)}, E_{(p,q)})$. In principle, this torsion is again defined as a section of a line bundle:

$$(\det H^0)^{-1}(\det H^1)(\det H^2)^{-1}(\det H^3)$$

$$\approx (\det H^0)^{-2}(\det H^1)^2 \ . \qquad \text{(A24)}$$

In the case of interest to us, however, this bundle does not arise. Elements of $H^0(M_{(p,q)}; E_{(p,q)})$ represent global Killing vectors of $M_{(p,q)}$ [16], which are certainly generically absent. Elements of $H^1(M_{(p,q)}; E_{(p,q)})$ represent infinitesimal variations of the connection $A_i^a$ in the space of flat connections, that is, infinitesimal deformation of the hyperbolic structure. But the boundary of $M_{(p,q)}$ is totally geodesic, so $M$ can be replaced by its (closed) double for the purpose of analyzing these deformations. The Mostow rigidity theorem [19] then guarantees that no such deformations exist. Thus $H^0$ and $H^1$ are trivial, and any metric dependence must ultimately drop out of the torsion (A9).

---

[1] S. W. Hawking, in *General Relativity: An Einstein Centenary Survey*, edited by S. W. Hawking and W. Israel (Cambridge University Press, Cambridge, England, 1979).

[2] J. B. Hartle and S. W. Hawking, Phys. Rev. D **28**, 2960 (1983).

[3] G. W. Gibbons and J. B. Hartle, Phys. Rev. D **42**, 2458 (1990).

[4] S. Coleman, Nucl. Phys. **B310**, 643 (1988).

[5] S. B. Giddings and A. Strominger, Nucl. Phys. **B307**, 854 (1988).

[6] E. Witten, Nucl. Phys. **B311**, 46 (1988).

[7] S. Deser, J. McCarthy, and Z. Yang, Phys. Lett. B **222**, 61 (1989).

[8] V. Moncrief, J. Math. Phys. **30**, 2907 (1989).

[9] S. Carlip, Phys. Rev. D **42**, 2647 (1990).

[10] Y. Fujiwara *et al.*, Phys. Rev. D **44**, 1756 (1991).

[11] Y. Fujiwara *et al.*, Phys. Rev. D **44**, 1763 (1991).

[12] Y. Fujiwara *et al.*, Class. Quantum Grav. **9**, 867 (1992).

[13] A. Hosoya and K. Nakao, Class. Quantum Grav. **7**, 163 (1990).

[14] M. Gromov, in *Séminaire Bourbaki 1979/80, Exposés 543-560*, edited by A. Dold and B. Eckmann, Lecture Notes in Mathematics 842 (Springer, Berlin, 1981).

[15] E. Witten, Commun. Math. Phys. **121**, 351 (1989).

[16] C. Hodgson, Princeton University dissertation, 1986.

[17] D. Bar-Natan and E. Witten, Commun. Math. Phys. **141**, 423 (1991).

[18] A. S. Schwarz and Yu. S. Tyupkin, Nucl. Phys. **B242**, 436 (1984).

[19] G. D. Mostow, *Strong Rigidity of Locally Symmetric Spaces*, Annals of Mathematical Studies No. 78 (Princeton University Press, Princeton, NJ, 1973).

[20] E. Witten, Commun. Math. Phys. **137**, 29 (1991).

[21] W.P. Thurston, cited in M. Fujii, Osaka J. Math. **27**, 539 (1990).

[22] S. Kojima and Y. Miyamoto, J. Differential Geometry **34**, 175 (1991).

[23] W. D. Neumann and A. W. Reid, Math. Proc. Camb. Philos. Soc. **109**, 509 (1991).

[24] W. D. Neumann and A. W. Reid, "Rigidity of Cusps in Deformations of Hyperbolic 3-Orbifolds," Ohio State University report, 1991 (unpublished).

[25] W. D. Neumann and D. Zagier, Topology **24**, 307 (1985).

[26] S. M. Vishik, Sov. Math. Dokl. **36**, 174 (1988).

[27] R. Meyerhoff, in *Low Dimensional Topology and Kleinian Groups*, edited by D. B. A. Epstein, London Math. Society Lecture Notes Series Vol. 112 (Cambridge University Press, Cambridge, England, 1986).

[28] W. P. Thurston, Princeton lecture notes, 1979 (unpublished).

[29] W. B. R. Lickorish, Ann. Math. **76**, 531 (1962).

[30] A. M. Bruuner *et al.*, Topology Applications **20**, 289 (1985).

[31] W. S. Massey, *Algebraic Topology: An Introduction* (Springer, Berlin, 1977).

[32] J. Milnor, Bull. Am. Math. Soc. **6**, 9 (1982).

[33] D. B. Ray and I. M. Singer, Adv. Math. **7**, 145 (1971).

[34] J. Lott, in *Recent Developments in Geometry*, edited by S.-Y. Cheng, H. Choi, and R. E. Greene (American Mathematical Society, Providence, RI, 1989).

[35] J. Milnor, Bull. Am. Math. Soc. **173**, 358 (1966).