Dynamically generated four-dimensional models in Lovelock cosmology

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We consider a class of D-dimensional Lovelock models provided with a positive cosmological constant whose induced metric is given by the product of the metrics of a three-dimensional (external) and a $(D-4)$-dimensional (internal) maximally symmetric space. When all the Lovelock coefficients are non-negative, we show that these models admit classical solutions with a constant internal scale factor, and that for these solutions the evolution of the external dimensions can be described by a four-dimensional Einstein theory with a positive effective Hilbert-Einstein coefficient and non-negative effective cosmological constant. In addition, we prove that the perturbative formalism for the treatment of the Lovelock model is always well defined in the region of the gravitational configuration space covered by the considered solutions with a constant internal scale factor. We also examine the dynamically generated four-dimensional theory that is obtained when the internal scale factor remains constant, and discuss the role played by the no-boundary condition in the corresponding process of reducing the degrees of freedom of the minisuperspace model.

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I. INTRODUCTION

Unification field theories assume the existence of more than four dimensions in our Universe [1,2]. The extra dimensions are necessary for the consistency of the physical theory [2]. However, these extra dimensions remain unobservable in our present Universe. A possible mechanism for explaining this fact can be the compactification of all of these dimensions to Planck length scales. Of course, any tentative compactification mechanism that could be physically acceptable needs to be studied from the point of view of multidimensional cosmology.

In multidimensional frameworks there exists a most natural generalization of Einstein gravity which, on the other hand, seems to be related to the low-energy limit of stringy gravity [3]. This generalization is provided by Lovelock theories of gravity [4]. Unlike in Einstein theory, the Lovelock Lagrangian generally includes quadratic and higher-order corrections in the Riemann tensor. Actually, the Lovelock Lagrangian is formed up by a linear combination of dimensionally continued Euler forms, thus generalizing the Hilbert-Einstein Lagrangian [5]. The associated Lovelock gravitational equations can be formulated by means of the more general symmetric and conserved tensor constructed from the metric and its first and second spacetime derivatives [4]. This generic Lovelock tensor contains the Einstein tensor as a particular case, and in fact reduces to the latter when the spacetime possesses four dimensions. In addition, linearized Lovelock gravity (around Minkowski space) is ghost-free [5,6], and the number of gravitational degrees of freedom in Lovelock and Einstein gravity turns out to be the same [7].

In this work we will concentrate on D-dimensional models in Lovelock gravity with the metric

$$ds^2 = -N^2(t)\, dt^2 + a^2(t) d\Omega_3^2 + b^2(t) d\Omega_{D-4}^2. \quad (1.1)$$

$N(t)$ is the lapse function and $d\Omega_3^2$ and $d\Omega_{D-4}^2$ denote, respectively, the metrics of a three-dimensional and a $(D-4)$-dimensional maximally symmetric space with associated scale factors $a(t)$ and $b(t)$. We will restrict ourselves to the cases in which the three-dimensional space is spherically symmetric or flat, with the space of dimension $D-4$ corresponding to a $(D-4)$-sphere or to a $(D-4)$-torus.

These models can provide classical solutions in which all the extra dimensions become spontaneously compactified. We will refer to the $D-4$ dimensions as the internal dimensions, while the rest of dimensions will be referred to as external. Of course, the existence of cosmological solutions with spontaneous compactification of the internal dimensions depends not only on the gravitational dynamics, but also on the specific matter content of the system. In this paper we will consider a simple Lovelock model with a positive cosmological term and no matter fields. Our study can be thought of as a preliminary step before analyzing more realistic cosmological models.

Some Lovelock models with metric of the form (1.1) have been already considered in the literature [8–10]. In most of the cases, only corrections in curvature up to quadratic or cubic order have been taken into account in the Lovelock Lagrangian, although certain cosmological models of the form (1.1) have been studied in generic Lovelock gravity, either with a constant $(D-4)$-torus as the internal space [11], with static spaces both in the external and the internal dimensions [12], or with a constant internal sphere and a external de Sitter space [12].

In spite of the interest devoted to this type of Lovelock models, there exist serious difficulties in the straightforward application of Lovelock gravity to multidimensional cosmology. In generic Lovelock gravity, the relations between the time derivatives of the induced metric and the gravitational momenta cannot be global and single-valuedly inverted in the whole gravitational configuration space [13]. Therefore, the Hamiltonian formalism of Lovelock gravity cannot be defined in the conventional way. The above problem can nevertheless be solved by means of a perturbative treatment of Lovelock gravity [14–16]. In fact, this perturbative formalism has been recently implemented in Ref. [16] in a class of minisuperspace models containing those of the form (1.1). We now briefly summarize some of the results of that reference which will be of use in the present work.

Let us first introduce the notation

$$x = \frac{\dot{a}}{aN}, \quad y = \frac{\dot{b}}{bN}, \quad \bar{g} = \frac{k_a}{a^2}, \quad \bar{h} = \frac{k_b}{b^2}, \tag{1.2}$$

where $\dot{a}$ and $\dot{b}$ denote the time derivatives of the scale factors, $k_a = 1$ if the external space is spherically symmetric, or $k_a = 0$ if it is flat, and $k_b = 1$ or 0 according to whether the internal space is a $(D-4)$-sphere or a $(D-4)$-torus, respectively.

In the region of low derivatives x and y and spatial curvatures $\bar{g}$ and $\bar{h}$, it is possible to single-valuedly invert the derivatives-momenta Lovelock relations in such a way that the obtained local inversion corresponds to the perturbed single-valued Einsteinian inversion in the presence of the higher-order Lovelock corrections. This local inversion can be analytically continued in the complex plane of the gravitational momenta and the spatial curvatures, getting a single valued inversion which straightforwardly provides us with a single-valued Hamiltonian. In order to preserve the single valuedness of the considered inversion in the process of analytic continuation, one must in general introduce cuts in the complex plane of its associated variables. These cuts restrict the range of the constructed inversion, which no longer runs over the whole configuration space.

In a parallel way, it is possible to invert the Hamiltonian constraint and the expression of one of the gravitational momenta in terms of the derivatives x and y. As the corresponding Einsteinian inversion turns out to be double valued (except when $D = 5$ and the inverted momentum is that of the internal scale factor), one first determines the two local inversion branches which correspond to the perturbed Einsteinian inversion in the region of low spatial curvatures $\bar{g}$ and $\bar{h}$ and derivatives x and y. These two local inversions can then be extended in the complex plane of their variables, i.e., of the matter energy density, the spatial curvatures and the inverted gravitational momentum. To keep the single-valuedness in the process of analytic continuation, it is again necessary to introduce cuts that restrict the ranges of the so-obtained inversions. From the constructed double-valued inversion and the expression of the gravitational momentum that has not been inverted, one can get a constraint which quadratically depends on that momentum. Such a constraint turns out to be at least semiclassically equivalent to the perturbative Hamiltonian constraint which results from the insertion of the single-valued derivatives-momenta inversion in the Hamiltonian constraint of the system, provided the cuts and definition domains of the different introduced inversions have been properly chosen. The quantization of this perturbative Hamiltonian constraint leads then to the generalized Wheeler-DeWitt equation for the Lovelock models under consideration. When $D = 5$, the Hamiltonian constraint and the internal scale factor momentum can be single-valuedly inverted in Einstein gravity in terms of x and y. Therefore, the corresponding perturbative Lovelock inversion must then be single valued, instead of double valued. The constraint coming from this perturbative inversion and from the expression of the external scale factor momentum is thus linear in the momentum which was not inverted. It can be seen that such a constraint can be obtained from the single-valued perturbative inversion of the derivatives-momenta relations, so that it suffices to construct this last inversion.

This paper has the following outline. In Sec. II we present the explicit expressions of the dynamical equations and gravitational momenta for the Lovelock model (1.1). We then investigate in Sec. III the possible existence of classical solutions with a constant internal scale factor, concentrating in a model with a positive cosmological constant. The cosmological behavior of these classical solutions is also discussed in Sec. III, showing that the dynamics of the external dimensions can be described by an effective Einsteinian theory when the internal scale factor remains constant. In Sec. IV we prove that the perturbative formalism for the treatment of the studied Lovelock model is always well defined in the whole region of the configuration space covered by the constant internal scale factor solutions considered in Sec. III. Section V deals with the analysis of the reduced minisuperspace model obtained by eliminating the degree of fredom corresponding to the internal scale factor, which is assumed to be constant. Solving then for both the reduced and unreduced minisuperspaces the Wheeler-DeWitt equation in semiclassical approximation, we examine the role of the no-boundary condition in the process of freezing some of the degrees of freedom of the model. Results are summarized in Sec. VI. Finally, Appendix A contains some useful calculations needed for the study of the constant internal scale factor solutions.

II. LOVELOCK MODEL AND DYNAMICAL EQUATIONS

In this section we will get the Lovelock dynamical equations and the gravitational momenta for models with metric of the form (1.1). These expressions can be obtained by applying the formulas presented in Ref. [16].

We first assume that the gravitational action for a generic D-dimensional manifold $\mathcal{M}$, with D-tetrad $\{e^a\}$ $(a = 1, ..., D)$ and curvature two-form R^{ab} [17], is given by the Lovelock action

$$S = \int_{\mathcal{M}} \sum_{m=1}^{M} \frac{L_m}{3!(D-4)!(D-2m)} \times R^{a_1 a_2} \cdots R^{a_{2m-1} a_{2m}} e^{a_{2m+1}} \cdots e^{a_D}, \tag{2.1}$$

where $\epsilon_{a_1 \cdots a_D}$ is the Levi-Civita tensor in D dimensions, the Lovelock coefficients L_m are real constants and $M = \mathrm{int}(\frac{D-1}{2})$.

We suppose that the Hilbert-Einstein coefficient L_1 is strictly positive, so that the action (2.1) provides an attractive theory of gravity in the Einsteinian limit. We will use dimensionless variables, with numerical values expressed in Planck units.

Action (2.1) must be corrected with appropriate surface terms when the manifold $\mathcal{M}$ presents a boundary [18]. Assuming that this boundary is composed of a final $(D-1)$-dimensional surface of constant time $t=t_f$ and of an initial surface with $t=t_0$, the boundary corrections that must be added to action (2.1) for models of the form (1.1) are [16]

$$-V_3 V_{D-4} a^3 b^{D-4} \sum_{m=1}^{M} \frac{L_m 2m}{D-2m} \sum_{n=p_1}^{q_1} \sum_{r=0}^{s_0} \sum_{w=0}^{m-1-s_0} A^1_{Dmnrw} \frac{\bar{g}^r \bar{h}^w x^{n-2r} y^{2m-1-n-2w}}{2m-1-2r-2w} \Bigg|_{t_0}^{t_f}, \tag{2.2}$$

where V_3 and V_{D-4} are, respectively, the volumes of the maximally symmetric spaces of dimension 3 and $D-4$, and

$$p_l = \max(0, 2m+4-D-l), \quad q_l = \min(2m-l, 3), \quad s_l = \mathrm{int}\left(\frac{n+l}{2}\right), \tag{2.3}$$

$$A^l_{Dmnrw} = \binom{D-2m-1+l}{3-n} \binom{m-l}{w} \binom{m-w-l}{r} \binom{2m-2r-2w-l}{n-2r}. \tag{2.4}$$

Including the boundary corrections (2.2), the Lovelock action (2.1) can be written for our minisuperspace model as [16]

$$S = -\int dt V_3 V_{D-4} N a^3 b^{D-4} \sum_{m=1}^{M} L_m \sum_{n=p_0}^{q_0} \sum_{r=0}^{s_0} \sum_{w=0}^{m-s_1} A^0_{Dmnrw} \frac{\bar{g}^r \bar{h}^w x^{n-2r} y^{2m-n-2w}}{2m-1-2r-2w}. \tag{2.5}$$

Defining then the matter Lagrangian $S_{\mathrm{matt}} = \int dt V_3 V_{D-4} a^3 b^{D-4} \mathcal{L}_{\mathrm{matt}}$ and, similarly, the gravitational Lagrangian $\mathcal{L}$ from (2.5), one can introduce the gravitational momenta

$$p_x = \frac{\partial \mathcal{L}}{\partial(\dot{a}/a)} = -\sum_{m=1}^{M} L_m \sum_{n=p_0}^{q_0} \sum_{r=0}^{s_0} \sum_{w=0}^{m-s_1} \frac{n-2r}{2m-2r-2w-1} A^0_{Dmnrw} \bar{g}^r \bar{h}^w x^{n-2r-1} y^{2m-n-2w}, \tag{2.6}$$

$$p_y = \frac{\partial \mathcal{L}}{\partial(\dot{b}/b)} = -\sum_{m=1}^{M} L_m \sum_{n=p_0}^{q_0} \sum_{r=0}^{s_0} \sum_{w=0}^{m-s_1} \frac{2m-n-2w}{2m-2r-2w-1} A^0_{Dmnrw} \bar{g}^r \bar{h}^w x^{n-2r} y^{2m-n-2w-1}, \tag{2.7}$$

by assuming that the matter Lagrangian does not depend on $\dot{a}$ and $\dot{b}$. From p_x and p_y we have the canonical momenta conjugate to the scale factors:

$$p_a = V_3 V_{D-4} a^2 b^{D-4} p_x, \quad p_b = V_3 V_{D-4} a^3 b^{D-5} p_y. \tag{2.8}$$

In the variables

$$g = \frac{k_a}{a^2} + \frac{\dot{a}^2}{a^2 N^2} = \bar{g} + x^2, \quad h = \frac{k_b}{b^2} + \frac{\dot{b}^2}{b^2 N^2} = \bar{h} + y^2, \quad f = \frac{2\dot{a}\dot{b}}{abN^2} = 2xy, \tag{2.9}$$

the Hamiltonian constraint $\partial(\mathcal{L} + \mathcal{L}_{\mathrm{matt}})/\partial N = 0$ can be written [16]

$$-\mathcal{H} \equiv \sum_{m=1}^{M} L_m \sum_{n=p_0}^{q_0} \mathcal{G}_{Dmn}(g,h,f) = \rho, \tag{2.10}$$

where $\rho = -\partial \mathcal{L}_{\mathrm{matt}}/\partial N$ is the energy density of the matter content, $\mathcal{H} = -\partial \mathcal{L}/\partial N$ and

$$\mathcal{G}_{Dmn}(g,h,f) = \binom{D-2m-1}{3-n} \sum_{r=w_0}^{s_0} \frac{g^r h^{m+r-n} f^{n-2r} m!}{(n-2r)! r! (m+r-n)!}, \tag{2.11}$$

$$w_l = \max(0, n-m+l). \tag{2.12}$$

The Lovelock action (2.5) can now be rewritten

$$S = \int dt\, V_3 V_{D-4} a^3 b^{D-4} \left[\frac{\dot{a}}{a} p_x + \frac{\dot{b}}{b} p_y - N\mathcal{H} \right]. \tag{2.13}$$

As to the Euler-Lagrange equations of the scale factors a and b, they can be expressed as [16]

$$\begin{pmatrix} \partial_x p_x & \partial_y p_x \\ \partial_x p_y & \partial_y p_y \end{pmatrix} \begin{pmatrix} \ddot{a}/(aN^2) \\ \ddot{b}/(bN^2) \end{pmatrix} = \begin{pmatrix} 3(\sigma_a + \rho) - \hat{B} \\ (D-4)(\sigma_b + \rho) - \hat{C} \end{pmatrix}, \tag{2.14}$$

with

$$\sigma_a = \frac{1}{3a^2 N} \frac{\partial (a^3 \mathcal{L}_{\rm matt})}{\partial a},$$
$$\sigma_b = \frac{1}{(D-4) b^{D-5} N} \frac{\partial (b^{D-4} \mathcal{L}_{\rm matt})}{\partial b}, \tag{2.15}$$

$$\hat{B} = \sum_{m=1}^{M} L_m \sum_{n=p_0}^{q_0} n \mathcal{G}_{Dmn}(g,h,f), \tag{2.16}$$

$$\hat{C} = \sum_{m=1}^{M} L_m \sum_{n=p_0}^{q_0} (2m-n) \mathcal{G}_{Dmn}(g,h,f). \tag{2.17}$$

The matrix elements in the left-hand side of Eq. (2.14) are given by [16]

$$\partial_x p_x = \sum_{m=1}^{M} L_m \sum_{n=p_2}^{q_0-2} \binom{D-2m-1}{1-n} \mathcal{F}_{mn}(g,h,f), \tag{2.18}$$

$$\partial_y p_y = \sum_{m=1}^{M} L_m \sum_{n=p_0}^{q_2} \binom{D-2m-1}{3-n} \mathcal{F}_{mn}(g,h,f), \tag{2.19}$$

$$\partial_x p_y = \partial_y p_x = \sum_{m=1}^{M} L_m \sum_{n=p_1}^{q_1-1} \binom{D-2m-1}{2-n} \mathcal{F}_{mn}(g,h,f), \tag{2.20}$$

where

$$\mathcal{F}_{mn}(g,h,f) = -2 \sum_{r=w_1}^{s_0} \frac{g^r h^{m+r-n-1} f^{n-2r} m!}{(n-2r)! r! (m+r-n-1)!} \tag{2.21}$$

and one can check that, for the values of index n allowed in (2.18), the dependence on g of $\partial_x p_x$ actually drops off.

The Euler-Lagrange equations (2.14) and the Hamiltonian constraint (2.10) contain all the relevant information of the Lovelock equations of the model. On the other hand, σ_a and σ_b play the role of pressures in the respective external and internal maximally symmetric spaces [9,10,16].

III. SOLUTIONS WITH CONSTANT INTERNAL SCALE FACTOR

We will now discuss the possible existence of classical solutions of the form (1.1) with constant internal scale factor, $b(t) = b_0$. We will mainly analyze the simple case of models in Lovelock gravity provided with a cosmological term.

For solutions with constant $b(t) = b_0$, we will have, from (1.2) and (2.9)

$$y = 0, \quad f = 0, \quad h = \bar{h} = \frac{k_b}{b_0^2} = h_0. \tag{3.1}$$

We will only consider compact internal spaces, with $k_b = 1$ if this space corresponds to a $(D-4)$-sphere or $k_b = 0$ if it is a $(D-4)$-torus. For physical solutions, b_0 will be real, and hence $h_0 \geq 0$ in all the cases.

For $b(t)$ to be constant in a classical solution, $\ddot{b}$ must identically vanish during the evolution. Therefore, we obtain from Eqs. (2.14) the compatibility condition

$$\partial_x p_x(h_0, f=0)[(D-4)(\sigma_b + \rho) - \hat{C}(g, h_0, f=0)]$$
$$= \partial_x p_y(g, h_0, f=0)[3(\sigma_a + \rho) - \hat{B}(g, h_0, f=0)]. \tag{3.2}$$

In (3.2), $\partial_x p_y(g, h_0, f=0)$ denotes the evaluation of $\partial_x p_y$ at g, $h = h_0$ and $f = 0$. We have used similar notation for $\hat{B}$ and $\hat{C}$, and also for $\partial_x p_x$, which does not depend on g.

On the other hand, from Eqs. (2.16)–(2.19) it follows that

$$\hat{B}(g, h_0, f=0) = -g \partial_x p_x(h_0, f=0),$$
$$\hat{C}(g, h_0, f=0) = -h_0 \partial_y p_y(g, h_0, f=0). \tag{3.3}$$

Let us restrict our discussion to Lovelock models of the type (1.1) with a cosmological term

$$S_{\rm matt} = -\int dt\, V_3 V_{D-4} N a^3 b^{D-4} \Lambda. \tag{3.4}$$

Then

$$\rho = \Lambda, \quad \sigma_a = -\Lambda, \quad \sigma_b = -\Lambda. \tag{3.5}$$

Inserting Eqs. (3.3) and (3.5) in (3.2), we get the compatibility condition

$$\partial_x p_x(h_0, f=0)[h_0 \partial_y p_y(g, h_0, f=0) - g\partial_x p_y(g, h_0, f=0)] = 0. \quad (3.6)$$

In addition to this relation, the solutions with a constant internal scale factor must verify the Hamiltonian constraint (2.10), which, for $f = 0$ and $h = h_0$, can be rewritten as

$$g\,\tilde{L}(h_0) + \tilde{\Lambda}(h_0) = \Lambda, \quad (3.7)$$

where we have introduced the definitions

$$\tilde{L}(h_0) = \sum_{m=1}^{M_1} L_m (D - 2m - 1) m h_0^{m-1},$$
$$\tilde{\Lambda}(h_0) = \sum_{m=1}^{M_3} L_m \left(\frac{D - 2m - 1}{3} \right) h_0^m, \quad (3.8)$$

M_1 and M_3 being given by the generic expression

$$M_l = \mathrm{int}\left(\frac{D - 1 - l}{2} \right). \quad (3.9)$$

Constraint (3.7) is formally the same as a Hamiltonian constraint in Einstein gravity for an isotropic and homogeneous model in four dimensions [9]. $\tilde{L}(h_0)$ plays then the role of an effective Hilbert-Einstein constant; i.e., $\tilde{L}(h_0)$ is proportional to the inverse of the effective Newton constant corresponding to the dynamically generated four-dimensional model [9]. Likewise, $\Lambda - \tilde{\Lambda}(h_0)$ can be regarded as the effective cosmological constant of such a four-dimensional model.

In order to get a well-defined effective dynamics for the external dimensions, we must require that $\tilde{L}(h_0)$ does not vanish in the considered solution of constant internal scale factor. Since, from (2.18) and the first definition in (3.8),

$$\tilde{L}(h_0) = -\frac{1}{2}\partial_x p_x(h_0, f=0), \quad (3.10)$$

we will demand that, in the studied classical solution,

$$\partial_x p_x(h_0, f=0) \neq 0. \quad (3.11)$$

Using (3.11) and (3.6), we finally attain the condition for the existence of an admissible solution with constant internal scale factor:

$$g\,\partial_x p_y(g, h_0, f=0) = h_0 \partial_y p_y(g, h_0, f=0), \quad (3.12)$$

with h_0 and g satisfying relation (3.7) and inequality (3.11).

On the other hand, it can be checked that, when (3.11) is verified, equations of motion (2.14) are equivalent to the time derivative of the Hamiltonian constraint (3.7) and condition (3.12). Thus, when Eqs. (3.11) and (3.12) are satisfied, all the dynamics of the external dimensions are contained in the Hamiltonian constraint (3.7), as it would be for a four-dimensional Robertson-Walker model in Einstein theory with a cosmological constant.

Employing expressions (2.19) and (2.20), condition (3.12) can be rewritten

$$\mathcal{F} \equiv g^2\, c(h_0) + g\, d(h_0) - e(h_0) = 0, \quad (3.13)$$

where the coefficients $c(h_0)$, $d(h_0)$ and $e(h_0)$ are polynomials of the internal scalar curvature h_0:

$$c(h_0) = \sum_{m=2}^{M} L_m m(m-1) h_0^{m-2}, \quad (3.14)$$

$$d(h_0) = \frac{1}{2}\sum_{m=1}^{M_1} L_m m(D - 2m - 1)(D - 4m) h_0^{m-1}, \quad (3.15)$$

$$e(h_0) = \sum_{m=1}^{M_3} L_m m \left(\frac{D - 2m - 1}{3} \right) h_0^m, \quad (3.16)$$

with M_1 and M_3 given by (3.9). Expressions as (3.7) and (3.13), restricted to the case $k_a = 0$ and $\Lambda = 0$, were found by Deruelle and Fariña-Busto [12].

We notice that relation (3.13) is in general quadratic in g, except for Einsteinian gravity, where $c(h_0)$ identically vanishes. Taking into account that, from (3.7), $g = [\Lambda - \tilde{\Lambda}(h_0)]/\tilde{L}(h_0)$ [with $\tilde{L}(h_0) \neq 0$], Eq. (3.13) can be considered as a constraint, for the admissible classical solutions of the system, between the D-dimensional cosmological constant and the constant internal scalar curvature.

We are especially interested in Lovelock theories such that the corresponding effective Hilbert-Einstein constant $\tilde{L}(h_0)$ turns out to be strictly positive whichever the allowed value of the constant internal scalar curvature, $h_0 \geq 0$, can be in the classical evolution. Moreover, we want the value of g in the considered solutions with a constant internal scale factor always to be positive; otherwise the effective dynamics of the external dimensions would correspond to a four-dimensional model with a negative cosmological constant, leading then to all the problems that this type of model poses both in classical and quantum Einstein cosmology [19,20].

As to the former requirement, it is easy to check from the first expression in (3.8) that, at least in Lovelock theories with non-negative coefficients,

$$L_m \geq 0 \quad \forall m > 1, \quad L_1 > 0, \quad (3.17)$$

the effective constant $\tilde{L}(h_0)$ is always positive for any $h_0 \geq 0$. It follows then from (3.10) that in these cases

$$\partial_x p_x(h_0, f=0) < 0 \quad \forall h_0 \geq 0, \ \forall D \geq 5. \quad (3.18)$$

We notice that, when (3.17) is satisfied, the associated Lovelock polynomial

$$P(z) = \left(\frac{D-1}{3} \right) \sum_{m=1}^{M} L_m z^m \quad (3.19)$$

is strictly increasing in the whole positive real axis $z \geq 0$. It was shown [14] that Lovelock theories which verify this condition on their associated polynomials lead to appealing physical behavior. This is still another reason for emphasizing the interest of theories satisfying (3.17). From now on, we assume condition (3.17) holds.

Using expressions (3.14)–(3.16), we then check that the solution to (3.13)

$$g_+(h_0) = \frac{2e(h_0)}{d(h_0) + \sqrt{d^2(h_0) + 4c(h_0)e(h_0)}}, \qquad (3.20)$$

verifies that, for $D \geq 6$,

$$g_+(h_0 = 0) = 0 \quad \text{and} \quad g_+(h_0) > 0 \quad \forall h_0 > 0, \qquad (3.21)$$

so that $g_+(h_0) \geq 0 \ \forall h_0 \geq 0$, as we wanted. The positiveness of $g_+(h_0)$ is marked by the subscript plus. It is worth noting that, for fixed h_0, $g_+(h_0)$ is constant. Later on, we will comment on the case $D = 5$.

Inserting solution (3.20) in (3.7) we obtain the relation that must satisfy the internal scalar curvature h_0 and the D-dimensional cosmological constant Λ of the model:

$$\begin{aligned}\Lambda = \Lambda_+(h_0) &\equiv \frac{2e(h_0)\tilde{L}(h_0)}{d(h_0) + \sqrt{d^2(h_0) + 4c(h_0)e(h_0)}} + \tilde{\Lambda}(h_0) \\ &= \frac{2[e(h_0)\tilde{L}^2(h_0) + d(h_0)\tilde{L}(h_0)\tilde{\Lambda}(h_0) - c(h_0)\tilde{\Lambda}^2(h_0)]}{d(h_0)\tilde{L}(h_0) - 2c(h_0)\tilde{\Lambda}(h_0) + \tilde{L}(h_0)\sqrt{d^2(h_0) + 4c(h_0)e(h_0)}}. \end{aligned} \qquad (3.22)$$

From Eqs. (3.14)–(3.17) and (3.8), it follows that, for $D \geq 6$,

$$\Lambda_+(h_0 = 0) = 0 \quad \text{and} \quad \Lambda_+(h_0) > 0, \quad \forall h_0 > 0. \qquad (3.23)$$

Hence, the D-dimensional cosmological constant must be non-negative if we want any of the considered solutions with constant internal scale factor to exist.

Finally, using (2.19), (3.17) and (3.21) it is easily shown that, in the studied solutions with $b(t) = b_0$,

$$\partial_y p_y(g_+(h_0), h_0, f = 0) < 0 \quad \forall h_0 \geq 0, \ \forall D \geq 6. \qquad (3.24)$$

The case $D = 5$ compulsorily requires $k_b = 0$. So, $h_0 = 0$ for any constant value b_0 of $b(t)$. The D-dimensional cosmological constant must vanish if there exist solutions with a constant internal scale factor, since $\Lambda_+(h_0 = 0) = 0$, and the effective cosmological constant of the dynamically generated four-dimensional model turns out to be also equal to zero, $g_+(h_0 = 0) = 0$.

On the other hand, in the limit $L_m \to 0 \ \forall m > 1$, $c(h_0)$ [given by (3.14)] vanishes, and $d(h_0)$, $e(h_0)$, $\tilde{L}(h_0)$, and $\tilde{\Lambda}(h_0)$ [as given by (3.15), (3.16), and (3.8)] tend to the corresponding Einsteinian values

$$d(h_0) = \frac{1}{2}L_1(D-3)(D-4), \quad \tilde{L}(h_0) = L_1(D-3), \qquad (3.25)$$

$$e(h_0) = \tilde{\Lambda}(h_0) = L_1(D-3)(D-4)(D-5)\frac{h_0}{6}. \qquad (3.26)$$

From Eqs. (3.20) and (3.22), $g_+(h_0)$ and $\Lambda_+(h_0)$ tend then to

$$\begin{aligned} g_+(h_0) &= \frac{D-5}{3}h_0, \\ \Lambda_+(h_0) &= L_1(D-2)(D-3)(D-5)\frac{h_0}{6}. \end{aligned} \qquad (3.27)$$

Actually, (3.27) coincides with the only existing solution to relations (3.7) and (3.13) in Einstein gravity. Therefore, the solutions with a constant internal scale factor and $g = g_+(h_0)$ which satisfy (3.22) can be interpreted as perturbed Einsteinian solutions of constant internal scale factor in the presence of the higher-order Lovelock corrections. Nevertheless, if the D-dimensional cosmological constant is assumed to remain fixed, the constant internal scale factor of the analyzed Lovelock solutions does not exactly coincide with that of the corresponding Einsteinian solutions, for the value of $\Lambda_+(h_0)$ [given by (3.22)] differs in Lovelock gravity from its Einsteinian counterpart.

In this sense, we notice that relation (3.22) determines the value of the D-dimensional cosmological constant for which a solution with a given constant internal scalar curvature h_0 exists. In practice, we would prefer a relation that could provide the value of h_0 once the D-dimensional cosmological constant of the model is known. The case $D = 5$ is trivial, since Λ must identically vanish and any constant value of the internal scale factor is then admissible. Moreover, assuming that condition (3.17) holds, we show in Appendix A that relation (3.22) can be inverted in the whole positive axis $h_0 \geq 0$, at least if the total dimension of the spacetime is not too high ($6 \leq D \leq 14$). Although we have not been able to prove that relation (3.22) is invertible $\forall h_0 \geq 0$ and $\forall D \geq 6$, we think that such an inversion can always be carried out. Restricting to the cases considered in Appendix A, $6 \leq D \leq 14$, the discussed inversion of (3.22) $\forall h_0 \geq 0$ leads then to a single-valued relation $h_0 = h_0(\Lambda)$ for $\Lambda \geq 0$, which essentially determines the admissible solutions with constant internal scale factor for the models with a fixed positive D-dimensional cosmological constant.

For solutions in which the internal dimensions can accept a physical compactification in a $(D-4)$-sphere ($k_b = 1$), we expect the constant internal scalar curvature h_0 to be of the order unity in Planck units. $g_+(h_0)$ will be then also of this order, as can be checked from expressions (3.20) and (3.14)–(3.16), except for special fine-tunings of the Lovelock coefficients. One of these possible fine-tunings is that for which the highest order Lovelock coefficient L_M is considerably large, and the remaining Lovelock coefficients are close to unity [12]. Although the cosmological behavior of the analyzed solutions with a constant internal scale factor seems to be unsatisfactory, it can be regarded as a standpoint to discuss more realistic physical models in which a dynamical compactification with the desired characteristics can be acheived.

If the internal space is a $(D-4)$-torus, we will have $h_0 = 0$ for any constant internal scale factor. These solutions are possible only if the D-dimensional cosmological constant of the model vanishes. Since $g_+(h_0 = 0) = 0$, the external dimensions evolve as in a four-dimensional model with a zero cosmological constant. An admissible classical solution would be the four-dimensional Minkowski space. Of course, one should still explain the splitting of D dimensions in two distinct flat spaces; one physically observable and the other not [21,22].

IV. PERTURBATIVE FORMALISM OF LOVELOCK GRAVITY FOR SOLUTIONS WITH CONSTANT INTERNAL SCALE FACTOR

In Refs. [14,16] we introduced a perturbative formalism of Lovelock gravity which solved the problems associated with the almost nondegenerate character of this kind of gravity theory. The implementation of such a formalism in models with metric of the form (1.1) is contained in Ref. [16]. In this section we will prove that this perturbative formalism is well defined in the whole region of the gravitational configuration space covered by the constant internal scale factor solutions analyzed in Sec. III.

The perturbative single-valued inversion of the derivatives-momenta relations is always well defined (in an analytic way) in any simply connected region around the origin of the configuration space $(x, y, \bar{g}, \bar{h})$ in which the associated Jacobian

$$|J| = \partial_x p_x \partial_y p_y - (\partial_x p_y)^2 \tag{4.1}$$

is everywhere different from zero, and such that the image of this region under $(p_x, p_y, \bar{g}, \bar{h})$ turns out to be simply connected [with p_x and p_y given by (2.6) and (2.7) and $\bar{g}$ and $\bar{h}$ the identity transformations) [16]. Also, the perturbative inversion of the Hamiltonian constraint and one of the gravitational momenta can always be well defined in such a way that its range contains any simply connected region around the origin of the space $(x, y, \bar{g}, \bar{h})$ in which the Jacobian (4.1) never vanishes and which satisfies the following in addition.

(a) If the inverted momentum is p_x, i) the image of that region under $(p_x, \rho, \bar{g}, \bar{h})$ [considering (2.10) as the definition of a function $\rho(x, y, \bar{g}, \bar{h})$] turns out to be simply connected, and (ii) within the studied region, all the points with $x = 0$ can be connected with the origin by a path along which $x = 0$ and $\partial_y p_y \neq 0$.

(b) If the inverted momentum is p_y, (i) the image of the mentioned region under $(p_y, \rho, \bar{g}, \bar{h})$ is simply connected, and (ii) within the considered region, all the points with $y = 0$ are connectable with the origin by a path along which $y = 0$ and $\partial_x p_x \neq 0$.

The analyzed inversion of the Hamiltonian constraint and one of the gravitational momenta can then be analytically defined except at those points for which $x = 0$, if we are inverting the momentum p_x, or for which $y = 0$, if the inverted momentum corresponds to p_y. At such points, however, the inversion is always well defined as an algebraic function [16].

The case $D = 5$ is somewhat special, for the perturbative inversion of the Hamiltonian constraint and momentum p_x is then single-valued. Nevertheless, the constraint associated with that single-valued inversion can indeed be obtained from the perturbative inversion of the derivatives-momenta relations [16]. Therefore, it suffices in this case to show that the latter inversion is well defined.

Let us consider then the region of the gravitational configuration space covered by the analyzed solutions with constant internal scale factor, which is given by

$$\Omega \equiv \{(x, y, \bar{g}, \bar{h})/x^2 + \bar{g} = g_+(\bar{h}), y = 0, \bar{g} \geq 0, \bar{h} \in \Delta\}, \tag{4.2}$$

$$\Delta \equiv \begin{cases} [0, \infty) & \text{if } D \geq 6, \\ \{0\} & \text{if } D = 5, \end{cases} \tag{4.3}$$

in which $g_+(\bar{h})$ is defined by means of (3.20). This region is simply connected and contains the origin $x = y = \bar{g} = \bar{h} = 0$. Moreover, assuming that (3.17) holds, we show in Appendix A that

$$|J(g_+(h_0), h_0, f = 0)| < 0 \quad \forall h_0 \geq 0, \tag{4.4}$$

and therefore the Jacobian (4.1) is strictly negative in the whole region Ω.

On the other hand, the images of Ω under $(p_x, p_y, \bar{g}, \bar{h})$, $(p_x, \rho, \bar{g}, \bar{h})$ and $(p_y, \rho, \bar{g}, \bar{h})$ are simply connected. Ω being simply connected, we only have to prove that two points in the image of Ω never happen to coincide. Suppose the opposite, i.e., that the images corresponding to two different points in Ω, $z_1 \equiv (x_1, y = 0, \bar{g}_1, \bar{h}_1)$ and $z_2 \equiv (x_2, y = 0, \bar{g}_2, \bar{h}_2)$ are identical. We will then have that $\bar{g}_1 = \bar{g}_2$ and $\bar{h}_1 = \bar{h}_2$, and from the definition of Ω it follows that $x_1^2 = g_+(\bar{h}_1) - \bar{g}_1 = g_+(\bar{h}_2) - \bar{g}_2 = x_2^2$. Thus, either $x_1 = x_2$, and the two considered points are indeed the same, or $x_1 = -x_2 \neq 0$. In the latter case, however, it can be checked from expressions (2.6), (2.7) and condition (3.17) that $p_x(x_1, y = 0, \bar{g}_1, \bar{h}_1) = -p_x(x_2, y = 0, \bar{g}_2, \bar{h}_2) \neq 0$ and $p_y(x_1, y = 0, \bar{g}_1, \bar{h}_1) = -p_y(x_2, y = 0, \bar{g}_2, \bar{h}_2) \neq 0$. This clearly implies that none of the images of the two points z_1 and z_2, either under $(p_x, p_y, \bar{g}, \bar{h})$, $(p_x, \rho, \bar{g}, \bar{h})$ or $(p_y, \rho, \bar{g}, \bar{h})$, can ever coincide. Therefore, all the considered images of Ω are simply connected.

As a consequence, it can be now asserted that the perturbative inversion of the derivatives-momenta relations (2.6) and (2.7) can be analytically defined in the image of Ω under $(p_x, p_y, \bar{g}, \bar{h})$, its range in that region being equal to Ω. When $D = 5$, in addition, the constraint associated with the single-valued perturbative inversion of the Hamiltonian constraint and momentum p_x will therefore be well defined in the image of Ω under $(p_x, \rho, \bar{g}, \bar{h})$.

In order to prove that the perturbative double-valued inversions of the Hamiltonian constraint and one of the momenta can be defined in such a way that their ranges contain the region Ω, we finally have to demonstrate that all the points in Ω with $x = 0$ (for $D \geq 6$) or with $y = 0$ (for $D \geq 5$) can, respectively, be connected with the origin by a path contained in Ω along which $x = 0$ and $\partial_y p_y \neq 0$, or by a path along which $y = 0$ and $\partial_x p_x \neq 0$.

For, let us assume that (for $D \geq 6$) z_1 is a certain point in Ω for which $x = 0$, i.e., $z_1 \equiv (x = 0, y = 0, \bar{g} = g_+(\bar{h}_1), \bar{h} = \bar{h}_1)$ with $\bar{h}_1 \geq 0$. We can always construct in Ω the path $\{(x = 0, y = 0, \bar{g} = g_+(\bar{h}), \bar{h})/\bar{h} \in [0, \bar{h}_1]\}$, which connects z_1 with the origin $x = y = \bar{g} = \bar{h} = 0$. Along such a path, $x = 0$ and $\partial_y p_y$ is different from zero, because $\partial_y p_y(g_+(h), h, f = 0) < 0 \ \forall h \geq 0$ and $D \geq 6$, according to (3.24). In a similar way, all the points with $y = 0$ in Ω (for any $D \geq 5$) can be connected with the origin by a path contained in Ω along which $y = 0$ and $\partial_x p_x$ does not vanish, for Ω is simply connected, $y = 0$ in Ω and, from (3.18), $\partial_x p_x$ is always negative in this region. We thus conclude that the ranges of all the introduced perturbative inversions can always contain the region Ω of the gravitational configuration space.

We have seen that the derivatives-momenta relations (2.6) and (2.7) can always be analytically and single-valuedly inverted in the whole region of the configuration space covered by the constant scale factor solutions considered in Sec. III. Substituting this perturbative inversion of the derivatives-momenta relations in Eq. (2.10), we can, in particular, obtain a perturbative Hamiltonian constraint which is a function of p_x, p_y ρ, $\bar{g}$, and $\bar{h}$. The quantization of this constraint leads then to a perturbative Wheeler-DeWitt equation for the minisuperspace model with gravitational degrees of freedom the scale factors a and b. The remaining perturbative constraints of the model must be at least semiclassically equivalent to the perturbative Hamiltonian constraint, once the cuts and definition domains of the different perturbative inversions have been properly chosen [16].

V. DYNAMICALLY GENERATED FOUR-DIMENSIONAL MODEL AND NO-BOUNDARY CONDITION

In this section we will discuss the dynamically generated four-dimensional theory obtained when the internal scale factor remains constant in the evolution. We will still assume that condition (3.17) is satisfied.

In the model with positive D-dimensional cosmological constant considered in Sec. III, one can freeze the internal scale factor by simply imposing that $b(t) = b_0$ holds identically, provided that b_0 and the D-dimensional cosmological constant Λ are related by Eq. (3.22), i.e., $\Lambda = \Lambda_+(h_0)$, with $h_0 = k_b b_0^{-2}$. This relation guarantees the consistency of freezing the degree of freedom b, for it allows us to impose the restrictions $b = b_0$ and $\dot{b} = 0$. We get in this way a reduced minisuperspace model whose only gravitational degree of freedom is the external scale factor a.

The Hamiltonian constraint and the gravitational momentum p_x for this reduced minisuperspace are given, respectively, by Eqs. (3.7) and (2.6), the latter evaluated at $y = 0$ and $h = k_b b_0^{-2}$:

$$g\,\tilde{L}(h_0) + \tilde{\Lambda}(h_0) = \Lambda, \tag{5.1}$$

$$p_x = -2x\,\tilde{L}(h_0). \tag{5.2}$$

$\tilde{L}(h_0)$ and $\tilde{\Lambda}(h_0)$ play the role of effective constants, depending on the internal scalar curvature, with $\tilde{L}(h_0) > 0$ and $\tilde{\Lambda}(h_0) \geq 0 \ \forall h_0 > 0$. The momentum p_x is related to the canonical momentum conjugate to a by $p_a = V_3 V_{D-4} a^2 b_0^{D-4} p_x$.

We will restrict our analysis to cosmological models with compact spherically symmetric spaces, i.e., $k_a = k_b = 1$. In this case h_0, $\tilde{\Lambda}(h_0)$ and $\bar{g}$ are always strictly positive.

From (2.9), (5.1), and (5.2), it follows that the wave functions of the reduced minisuperspace corresponding to the effective four-dimensional theory, $\Psi(a)$, must satisfy the Wheeler-DeWitt equation

$$\left[\frac{-1}{4\tilde{L}(h_0)}\frac{\partial^2}{\partial a^2} + (V_3 V_{D-4} a^2 b_0^{D-4})^2 \left(\tilde{\Lambda}(h_0) - \Lambda + \frac{1}{a^2}\tilde{L}(h_0)\right)\right]\Psi(a) = 0, \tag{5.3}$$

In semiclassical approximation $\Psi \simeq e^{-I}$, we will have

$$I(a) = \mp 2 V_3 V_{D-4} b_0^{D-4} \tilde{L}(h_0) \int_{a_0}^{a} d\tilde{a}\,\tilde{a}^2 \left[\frac{1}{\tilde{a}^2} - \frac{\Lambda - \tilde{\Lambda}(h_0)}{\tilde{L}(h_0)}\right]^{1/2}, \tag{5.4}$$

a_0 being a certain constant. Supposing that the time coordinate varies in the interval $[t_0, t_f]$, a_0 can be interpreted as the fixed initial value of the scale factor $a(t)$, $a_0 = a(t_0)$, and a can be thought of as the value of $a(t)$ in the final surface of constant time $t = t_f$.

In addition, it is obvious from (3.20) and (3.22) that

$$\frac{\Lambda - \tilde{\Lambda}(h_0)}{\tilde{L}(h_0)} = g_+(h_0). \tag{5.5}$$

In what follows we will see that the semiclassical action (5.4) coincides with the Lovelock action of the classical solution with $b(t)=b_0$, $\dot{b}=0$ (i.e, $y=0$) and $g=g_+(h_0)$ integrated over $t\in[t_0,t_f]$. Adding the cosmological term (3.4) to the expression (2.13) of the gravitational Lovelock action, we obtain the following action for the unreduced minisuperspace with degrees of freedom a and b:

$$S=V_3V_{D-4}\int_{t_0}^{t_f}dt\,Na^3b^{D-4}\left[p_x x+p_y y-\mathcal{H}-\Lambda\right], \tag{5.6}$$

with $\mathcal{H}$ given by (2.10). In the considered classical solution, $\mathcal{H}=-\Lambda$ and $y=0$, so that action (5.6) can be rewritten

$$S_{\rm cl}=V_3V_{D-4}b_0^{D-4}\int_{a_0}^{a}d\tilde{a}\,\tilde{a}^2p_x, \tag{5.7}$$

where we have used $a(t_0)=a_0$ and $a(t_f)=a$. Moreover, in the solution with a constant internal scale factor, p_x adopts the expression (5.2), with $x^2=g_+(h_0)-a^{-2}$. We thus get

$$S_{\rm cl}=\mp 2V_3V_{D-4}b_0^{D-4}\tilde{L}(h_0)\int_{a_0}^{a}d\tilde{a}\,\tilde{a}^2\left[g_+(h_0)-\frac{1}{\tilde{a}^2}\right]^{1/2}. \tag{5.8}$$

Inserting now relation (5.5), and defining the Euclidean action of the classical solution as $I_{\rm cl}=-iS_{\rm cl}$, we finally conclude that $I_{\rm cl}$ and the semiclassical action (5.4) coincide.

Since action (5.4) implicitly depends on the internal scale factor b_0, we can investigate whether the associated semiclassical approximation $\Psi\simeq e^{-I}$ can be somehow considered as a solution of the perturbative Wheeler-DeWitt equation obtained by quantizing the Hamiltonian constraint (2.10) in the unreduced minisuperspace with gravitational degrees of freedom a and b. This perturbative Wheeler-DeWitt equation will be well defined at least in the whole region of the gravitational configuration space covered by the discussed solutions with constant internal scale factor.

As (5.4) corresponds to the action of a classical solution [with $b(t)=b_0$] of the unreduced minisuperspace model, we could expect that the wave function $\Psi=e^{-I(a,b_0)}$, with $I(a,b_0)$ given by (5.4), should coincide with the value, evaluated at $b=b_0$, of a semiclassical approximation to the mentioned perturbative Wheeler-DeWitt equation.

Note now that the wave functions of the minisuperspace with degrees of freedom a and b adopt the generic dependence

$$\hat{\Psi}=\hat{\Psi}(a,b|a_0,\hat{b}_0), \tag{5.9}$$

with (a,b) and $(a_0,\hat{b}_0)$, respectively, the final and initial values of the scale factors $(a(t),b(t))$ on the two assumed existing boundaries of the manifold. Equation (5.9) reflects the fact that the wave function solely depends on the values of the induced metric on the boundaries of the manifold. Moreover, owing to the invariance of Lovelock theories of gravity under D-dimensional diffeomorphisms, the wave function (5.9) should satisfy the perturbative Wheeler-DeWitt equation associated with the Hamiltonian constraint (2.10) both on the final and initial boundaries of the manifold [23].

If $\Psi=e^{-I(a,b_0)}$ must coincide with the value of a semiclassical approximation to any wave function $\hat{\Psi}$ of the form (5.9), for b identically equal to b_0, we would obviously have

$$\Psi=e^{-I(a,b_0)}\simeq\hat{\Psi}(a,b|a_0,\hat{b}_0)\Big|_{b=\hat{b}_0=b_0}, \tag{5.10}$$

and the dependence of Ψ on b_0 would then simultaneously come from the dependence of $\hat{\Psi}$ on b and $\hat{b}_0$, thus preventing us from studying in detail whether the expected coincidence of the values of the different semiclassical approximations at $b=b_0$ actually holds.

We can nevertheless eliminate the dependence on b_0 coming from the initial boundary of the D-dimensional manifold by adopting a no-boundary condition for the wave function (5.9) [20,24]. We will demand the vanishing of both the initial value of $a(t)$ and then of the corresponding initial boundary. This no-boundary condition can always be implemented in the analyzed D-dimensional minisuperspace model, as any spatial section of our manifold is the product of a three-dimensional and a $(D-4)$-dimensional spheres and hence is cobordant to zero [21,25].

The Lovelock action (5.6) had been obtained by adding surface terms corrections which are due to the existence of boundaries. Our no-boundary condition implies that the surface terms corresponding to the initial boundary should be substracted from action (5.6). For the classical solution with $b(\tau)=b_0$ and $a_0=a(\tau_0)=0$ (in Euclidean time), $\dot{a}(\tau_0)$ must be equal to ± 1, and the contributions that must be reinserted in the Euclidean Lovelock action in the absence of the initial boundary are, from (2.2),

$$\mp\frac{4}{3}V_3V_{D-4}b_0^{D-4}\sum_{m=2}^{M}L_m\frac{m(m-1)}{D-2m}h_0^{m-2}, \tag{5.11}$$

where we have used $k_a=1$, $\bar{h}=h_0$ and that the Euclidean action corresponds to the Wick-rotated Lorentzian action $iS(-iN)=-I$ [20].

If we preserve the above terms in the action of the reduced minisuperspace, we obtain from Eq. (5.4) the modified semiclassical action

$$I_{\rm (NB)}(a,b_0)=\mp 2V_3V_{D-4}b_0^{D-4}\left(\tilde{L}(h_0)\int_0^a d\tilde{a}\,\tilde{a}^2\left[\frac{1}{\tilde{a}^2}-\frac{\Lambda-\tilde{\Lambda}(h_0)}{\tilde{L}(h_0)}\right]^{1/2}+\frac{2}{3}\sum_{m=2}^{M}L_m\frac{m(m-1)}{D-2m}h_0^{m-2}\right), \tag{5.12}$$

where we have also employed that $a_0 = 0$.

The introduced correction (5.11) is actually a constant in the reduced minisuperspace, and must be included in (5.4) if we want the value of the semiclassical action for the reduced minisuperspace to coincide with that of the no-boundary D-dimensional Lovelock action for the classical solution with $b(t) = b_0$.

The no-boundary terms (5.11) identically vanish in Einstein gravity, so, once we have fixed $a_0 = 0$, there is no difference between actions (5.4) and (5.12).

Performing the integration in Eq. (5.12), we arrive at

$$I_{(\mathrm{NB})}(a,b_0) = \mp 2V_3V_{D-4}b_0^{D-4}\Bigg(-\frac{\tilde{L}(h_0)}{3}\frac{\tilde{L}(h_0)}{\Lambda-\tilde{\Lambda}(h_0)}\left[1-\frac{\Lambda-\tilde{\Lambda}(h_0)}{\tilde{L}(h_0)}a^2\right]^{3/2} + \frac{\tilde{L}(h_0)}{3}\frac{\tilde{L}(h_0)}{\Lambda-\tilde{\Lambda}(h_0)} + \frac{2}{3}\sum_{m=2}^{M} L_m \frac{m(m-1)}{D-2m}h_0^{m-2}\Bigg). \tag{5.13}$$

Beyond what we expected, the wave function associated with the semiclassical action (5.13), $\Psi = e^{-I_{(\mathrm{NB})}(a,b_0)}$, has such an implicit dependence on b_0 that it corresponds to a solution of the perturbative Wheeler-DeWitt equation, evaluated at $b = b_0$, of the unreduced minisuperspace. We shall show now that, in order to prove this assertion, it is enough to verify that the following equalities hold

$$\begin{aligned} i\frac{\partial I_{(\mathrm{NB})}(a,b_0)}{\partial a} &= V_3V_{D-4}a^2b_0^{D-4}p_x(x=\pm\sqrt{g_+(h_0)-a^{-2}}, y=0, \bar{g}=a^{-2}, \bar{h}=h_0) \\ &= \mp 2V_3V_{D-4}a^2b_0^{D-4}\tilde{L}(h_0)\sqrt{g_+(h_0)-a^{-2}}, \end{aligned} \tag{5.14}$$

$$\begin{aligned} i\frac{\partial I_{(\mathrm{NB})}(a,b_0)}{\partial b_0} &= V_3V_{D-4}a^3b_0^{D-5}p_y(x=\pm\sqrt{g_+(h_0)-a^{-2}}, y=0, \bar{g}=a^{-2}, \bar{h}=h_0) \\ &= \mp 2V_3V_{D-4}a^3b_0^{D-5}\sqrt{g_+(h_0)-a^{-2}}\Bigg[(g_+(h_0)+2a^{-2})\frac{1}{3}\sum_{m=2}^{M}L_m m(m-1)h_0^{m-2} \\ &\qquad + \sum_{m=1}^{M_2}L_m\left(\frac{D-2m-1}{2}\right)mh_0^{m-1}\Bigg], \end{aligned} \tag{5.15}$$

where we have used the explicit expressions of p_x, p_y, and $\tilde{L}(h_0)$, given by (2.6) and (2.7) and (3.8), and introduced the symbolic notation $\partial I_{(\mathrm{NB})}(a,b_0)/\partial b_0 = (\partial I_{(\mathrm{NB})}(a,b)/\partial b)|_{b=b_0}$, with $I_{(\mathrm{NB})}(a,b)$ given by (5.13) and evaluated at b, instead of b_0. We adopt from now on this type of compact notation.

We proved in Sec. IV that the perturbative inversion of the derivatives-momenta relations (2.6) and (2.7) is well defined all over the region

$$\{(p_x(x,y,\bar{g},\bar{h}), p_y(x,y,\bar{g},\bar{h}), \bar{g}, \bar{h})/x=\pm\sqrt{g_+(h_0)-a^{-2}}, y=0, \bar{g}=a^{-2}>0, \bar{h}=h_0>0\}. \tag{5.16}$$

At any point of the region (5.16), the perturbative inversion of (2.6) and (2.7) provides us precisely with the corresponding values $x = \pm\sqrt{g_+(h_0)-a^{-2}}$ and $y = 0$. Since the Hamiltonian constraint (2.10), with $\rho = \Lambda$, is satisfied at $\bar{g} = a^{-2}$, $\bar{h} = h_0$, $x = \pm\sqrt{g_+(h_0)-a^{-2}}$, and $y = 0$ [assumed, as we do, that condition (5.5) is satisfied], it follows that, when relations (5.14) and (5.15) are verified, the wave function $\hat{\Psi} = e^{-I_{(\mathrm{NB})}(a,b)}$, associated with (5.13), is a semiclassical solution to the quantum version of constraint (2.10) at $b = b_0$.

Let us then prove (5.14) and (5.15). Relation (5.14) can be straightforwardly obtained from Eq. (5.12) and condition (5.5). The demonstration of (5.15) is not so immediate. Taking into account that $h_0 = b_0^{-2}$, we check from (3.8) that

$$\left[\frac{d}{db_0}\left(\frac{\Lambda-\tilde{\Lambda}(h_0)}{\tilde{L}(h_0)}\right)\right]\tilde{L}(h_0)b_0 = 2\frac{\Lambda-\tilde{\Lambda}(h_0)}{\tilde{L}(h_0)}\sum_{m=2}^{M_1}L_m m(m-1)(D-2m-1)h_0^{m-1} + 2\sum_{m=1}^{M_3}L_m\left(\frac{D-2m-1}{3}\right)h_0^{m}. \tag{5.17}$$

Employing now condition (5.5) and Eqs. (3.13)–(3.16) [with $g = g_+(h_0)$], we have that

$$\left[\frac{d}{db_0}\left(\frac{\Lambda-\tilde{\Lambda}(h_0)}{\tilde{L}(h_0)}\right)\right]\tilde{L}(h_0)b_0 = 2\left(\frac{\Lambda-\tilde{\Lambda}(h_0)}{\tilde{L}(h_0)}\right)^2\sum_{m=2}^{M}L_m m(m-1)h_0^{m-2} + 2\frac{\Lambda-\tilde{\Lambda}(h_0)}{\tilde{L}(h_0)}\sum_{m=1}^{M_2}L_m m\left(\frac{D-2m-1}{2}\right)h_0^{m-1} \tag{5.18}$$

at $b = b_0$. By using Eq. (5.18), condition (5.5) and the relations

$$\frac{d}{db_0}(b_0^{D-4}\tilde{L}(h_0)) = 2b_0^{D-5}\sum_{m=1}^{M_2} L_m m \left(\frac{D-2m-1}{2}\right) h_0^{m-1}, \tag{5.19}$$

$$\frac{d}{db_0}\left(b_0^{D-4}\sum_{m-2}^{M} L_m \frac{m(m-1)}{D-2m} h_0^{m-2}\right) = b_0^{D-5}\sum_{m=2}^{M} L_m m(m-1) h_0^{m-2}, \tag{5.20}$$

one can show that (5.15) is satisifed at $b = b_0$.

In conclusion, $\Psi = e^{-I_{(\mathrm{NB})}(a,b)}$, given by (5.13), is a semiclassical solution, at $b = b_0$, of the perturbative Wheeler-DeWitt equation associated with the Hamiltonian constraint (2.10), provided b_0^{-2} and the D-dimensional cosmological constant Λ are related by (5.5).

We will have then that, around $b = b_0$, the action

$$\hat{I}_{(\mathrm{NB})}(a,b) = I_{(\mathrm{NB})}(a,b_0) + \frac{\partial I_{(\mathrm{NB})}(a,b_0)}{\partial b_0}(b - b_0), \tag{5.21}$$

obtained from (5.13), is a valid approximation to the no-boundary semiclassical action of the minisuperspace with degrees of freedom a and b up to terms of order $(b-b_0)^2$. This is equivalent to asserting that, if the D-dimensional model with a positive cosmological constant Λ admits a classical solution with a constant internal scale factor $b = b_0$ [such that $\Lambda = \Lambda_+(h_0)$], the semiclassical action of the reduced minisuperspace modified with the no-boundary terms (5.11) contains implicit information about the values at $b = b_0$ of both the semiclassical action of the unreduced minisuperspace and its first derivative with respect to the internal scale factor.

It seems, therefore, that the no-boundary condition might play an important role in the proccess of dimensional reduction of the theory.

VI. CONCLUSIONS

In this paper we have considered a class of D-dimensional Lovelock models with an induced metric given by the product of the metrics of a three-dimensional external and a $(D-4)$-dimensional internal maximally symmetric space, both of them spherically symmetric or flat. We have first presented the Lovelock action and dynamical equations of motion for these models, discussing then the possible existence of classical solutions with a constant internal scale factor.

We have concentrated on Lovelock theories with non-negative coefficients provided with a positive cosmological constant. By tuning the D-dimensional cosmological constant, these theories admit classical solutions with any constant and real value of the internal scale factor. At least when the total dimension of the spacetime is not too high, we have seen that the D-dimensional cosmological constant of the model determines the constant value of the internal scalar curvature, which gives the internal scale factor in the mentioned classical solutions. For such solutions, the dynamical evolution of the external dimensions turns out to be described by an effective four-dimensional Einstein theory. The associated effective Hilbert-Einstein coefficient is strictly positive for all the considered solutions with constant internal scale factor, and the effective four-dimensional cosmological constant is always non-negative. These two effective constants of the dynamically generated four-dimensional model depend only on the constant value taken by the internal scalar curvature, and this dependence has been shown to be polynomial.

On the other hand, we have proved that the perturbative formalism of Lovelock gravity [14,16] can be implemented in the studied models at least in the whole region of the gravitational configuration space which is covered by the considered solutions with constant internal scale factor.

For the case of spherically symmetric spaces, we have analyzed the dynamically generated four-dimensional theory which is obtained by eliminating the degree of freedom of the internal scale factor, assumed to remain constant in the evolution. The minisuperspace corresponding to this four-dimensional theory has the external scale factor as the only gravitational degree of freedom. We have solved the Wheeler-DeWitt equation of this reduced minisuperspace in the semiclassical approximation, and checked that the value of the obtained semiclassical action coincides with that of the D-dimensional Lovelock action of the associated classical solution with a constant internal scale factor. Moreover, the semiclassical action of this reduced minisuperspace, modified with corrections coming from a D-dimensional no-boundary condition, turns out to contain implicit information about the no-boundary semiclassical action of the unreduced minisuperspace and its first derivative with respect to the internal scale factor, both evaluated at the constant value of the internal scale factor allowed by the classical evolution.

It is therefore possible that the no-boundary condition could play a relevant role in the process of freezing some of the degrees of freedom of the theory. In the models with a positive cosmological constant examined, we have seen that the surface corrections coming from the D-dimensional no-boundary condition do not vanish only for Lovelock theories other than Einstein gravity. Lovelock gravity can thus provide a specially fruitful framework for discussing a tentative connection between the mechanism of dimensional reduction and the no-boundary condition.

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APPENDIX A: SOLUTIONS WITH CONSTANT SCALE FACTOR. FURTHER REMARKS

In this Appendix we will analyze the constant internal scale factor solutions considerd in Sec. III for Lovelock models provided with a cosmological constant. We assume that condition (3.17) is satisfied. We will first prove that the Jacobian (4.1) is strictly negative in all the mentioned solutions with a constant internal scale factor, and show then that relation (3.22) between the D-dimensional cosmological constant Λ and the constant internal scalar curvature h_0 can be inverted in the whole semiaxis $h_0 \geq 0$, at least when the dimension of the spacetime is not too high ($6 \leq D \leq 14$). For the special case $D = 5$, the internal space has dimension one and must therefore be flat ($k_b = 0$). So, h_0 vanishes for any constant value b_0 of the internal scale factor. Moreover, from (3.22), the cosmological constant must be equal to zero if the model admits a solution with constant $b(t) = b_0$.

For the constant scale factor solutions investigated in Sec. III, we have $y = f = 0$, $h = h_0$ and $g = g_+(h_0)$, with $g_+(h_0) > 0 \; \forall h_0 > 0$ and $g_+(h_0 = 0) = 0$. In addition, h_0 and $g_+(h_0)$ verify relation (3.12):

$$g_+(h_0)\partial_x p_y(g_+(h_0), h_0, f = 0) = h_0 \partial_y p_y(g_+(h_0), h_0, f = 0). \tag{A1}$$

We want then to demonstrate that the Jacobian (4.1) is strictly negative at all points $(g_+(h_0), h_0, f = 0) \; \forall h_0 \geq 0$.

From expressions (2.18)–(2.20), one can check that $|J(g_+(h_0 = 0), h_0 = 0, f = 0)|$ is negative. For $D = 5$, h_0 is identically zero, and in this case we can already assure that the Jacobian (4.1) is negative for all the analyzed solutions with a constant internal scale factor. For $D \geq 6$, we still have to prove that $|J(g_+(h_0), h_0, f = 0)| < 0$ $\forall h_0 > 0$. Multiplying (4.1) by $g_+(h_0)$, which is strictly positive if $h_0 > 0$, and using (A1), we conclude that, at $(g_+(h_0), h_0, f = 0)$,

$$g_+(h_0)|J| = \partial_y p_y [g_+(h_0)\partial_x p_x - h_0 \partial_x p_y]. \tag{A2}$$

Taking into account inequality (3.24), our task reduces to show that, for $D \geq 6$ and $h_0 > 0$,

$$G \equiv h_0 \partial_x p_y(g_+(h_0), h_0, f = 0) - g_+(h_0)\partial_x p_x(g_+(h_0), h_0, f = 0) < 0. \tag{A3}$$

For $D = 6$,

$$G = \frac{3}{2} L_1 (g_+(h_0) - h_0); \tag{A4}$$

if $h_0 \geq 0$, it follows from (3.20) and (3.14)–(3.16) that

$$g_+(h_0) = \frac{2h_0 L_1}{3L_1 - 2L_2 h_0 + \sqrt{(3L_1 - 2L_2 h_0)^2 + 8L_1 L_2 h_0}} \leq \frac{h_0}{2}. \tag{A5}$$

Therefore, $G < 0$ if $h_0 > 0$.

For $D > 6$, let us suppose

$$g_+(h_0) \leq \frac{D-5}{3} h_0 \quad \forall h_0 \geq 0. \tag{A6}$$

Then, one can prove that

$$G \leq \sum_{m=1}^{M} L_m m h_0^m A_m, \tag{A7}$$

$$A_m = \frac{1}{6}[(D - 2m - 1)(-D + 6m - 4) - 2(D - 5)(m - 1)]. \tag{A8}$$

It is not difficult to see that the coefficients A_m have a maximum at $m = \frac{D+2}{4}$ (considering m as a real number), and that A_m is negative at that maximum if $D > 6$. Thus, $A_m < 0 \; \forall m \in \{1, ..., M\}$, and since $h_0 > 0$, we conclude that $G < 0$, as we wanted to prove.

We show now that inequality (A6) holds for $D > 6$, except when D =7, 9, or 10. Evaluating expressions (2.19) and (2.20) at $f = 0$,

$$-\frac{1}{2}\partial_y p_y(g_+(h_0),h_0,f=0)=\frac{D-5}{3}\left(-\frac{1}{2}\partial_x p_y(g_+(h_0),h_0,f=0)\right)-\frac{1}{3}X(h_0)+\frac{2}{3}g_+(h_0)Y(h_0), \tag{A9}$$

with the definitions

$$X(h_0)=4\sum_{m=2}^{M_2}L_m\binom{m}{2}\left(\frac{D-2m-1}{2}\right)h_0^{m-1}, \tag{A10}$$

$$Y(h_0)=2\sum_{m=2}^{M}L_m\binom{m}{2}(D-3m+1)h_0^{m-2}. \tag{A11}$$

From (2.19) and (A10), recalling that $g_+(h_0)\geq 0\ \forall h_0\geq 0$, we have that, for $D>6$,

$$-\frac{1}{2}\partial_x p_y(g_+(h_0),h_0,f=0)-\frac{1}{D-5}X(h_0)>0\quad\forall h_0\geq 0. \tag{A12}$$

Equations (A1) and (A9) lead to the equality

$$g_+(h_0)=h_0\frac{D-5}{3}\left[-\frac{1}{2}\partial_x p_y(g_+(h_0),h_0,f=0)-\frac{1}{D-5}X(h_0)\right]\left[-\frac{1}{2}\partial_x p_y(g_+(h_0),h_0,f=0)-\frac{2}{3}h_0Y(h_0)\right]^{-1}. \tag{A13}$$

Taking into account (A12), Eq. (A13) implies the inequality (A6) if and only if

$$-\frac{1}{D-5}X(h_0)\leq -\frac{2}{3}h_0Y(h_0). \tag{A14}$$

Relation (A14) can be equivalently written as

$$0\leq\sum_{m=2}^{M}L_m\binom{m}{2}h_0^{m-1}B_m, \tag{A15}$$

$$B_m=3\left(\frac{D-2m-1}{2}\right)-(D-5)(D-3m+1). \tag{A16}$$

Being $h_0\geq 0$, what remains to be seen is that the coefficients B_m are positive. These coefficients have a minimum at $\frac{D+2}{4}$ (taken as real functions of m) and, at that point, $B_m>0$ if $D\geq 12$. Moreover, for $D=8$ or 11, we have that the coefficients B_m are also positive $\forall m\in\{1,...,M\}$.

Let us consider now the remaining $D=7$, 9, or 10. We will study first the cases $D=9$ or 10. Suppose then that, for some $h_0>0$,

$$g_+(h_0)>\frac{D-5}{3}h_0. \tag{A17}$$

If (A17) is not satisfied for any $h_0>0$, inequality (A6) must hold [recall that $g_+(h_0=0)=0$], and therefore $G<0\ \forall h_0>0$. On the other hand, if (A17) is verified for some $h_0>0$, we get from expression (2.20) that, at $(g_+(h_0),h_0,f=0)$,

$$-\frac{1}{2}\partial_x p_y(g_+(h_0),h_0,f=0)>\sum_{m=1}^{M}L_m m h_0^{m-1}\left[\left(\frac{D-2m-1}{2}\right)+\frac{D-5}{3}(m-1)\right]. \tag{A18}$$

We show now that, for $D=9$ or 10 and $\forall h_0\geq 0$,

$$g_+(h_0)\leq\frac{D-5}{2}h_0. \tag{A19}$$

For all those $h_0\geq 0$ satisfying $g_+(h_0)\leq\frac{D-5}{3}h_0$, (A19) is trivial. From (A13), inequality (A19) will also be satisfied at all those points $h_0>0$ satisfying (A17) provided that, at these points,

$$-\frac{2}{D-5}X(h_0)\leq -\frac{1}{2}\partial_x p_y(g_+(h_0),h_0,f=0)-2h_0Y(h_0). \tag{A20}$$

Since (A18) is valid whenever $g_+(h_0)>\frac{D-5}{3}h_0$, (A20) is verified if, for $h_0>0$ and $D=9$ or 10,

$$\sum_{m=1}^{M} L_m m h_0^{m-1} C_m \geq 0, \tag{A21}$$

$$C_m = \left(\frac{D-2m-1}{2}\right)(3D+12m-27)-(D-5)(m-1)(5D-18m+11). \tag{A22}$$

For D =9 or 10 we have $M = 4$, and we can check that in these cases the coefficients C_m are positive for any $m \in \{1,...,4\}$. Then, since $h_0 > 0$, (A21) is satisfied, and hence inequality (A19) is valid $\forall h_0 \geq 0$.

Employing now (A19), we conclude that G satisfies

$$G \leq \sum_{m=1}^{M} L_m m h_0^m E_m, \tag{A23}$$

$$E_m = \frac{1}{4}[(D-2m-1)(2m-3)-(D-5)(m-1)]. \tag{A24}$$

The coefficients E_m turn out to be nonpositive for $D =$ 9 or 10 and $\forall m \in \{1,...,4\}$, and in particular $E_1 < 0$. Then, $G < 0$ $\forall h_0 > 0$, and thus $|J(g_+(h_0),h_0,f=0)| < 0$ $\forall h_0 \geq 0$ for $D = 9$ or 10.

Let us finally discuss the case $D = 7$. Assume that $g_+(h_0) > h_0$ for some $h_0 > 0$. It follows from (2.20) and (A10) and (A11) that, if $D = 7$,

$$0 \leq -\frac{1}{2}\partial_x p_y(g_+(h_0),h_0,f=0) + X(h_0) - 2Y(h_0) \tag{A25}$$

at those h_0 for which $g_+(h_0) > h_0$. However, using (A13) and (A25), we conclude

$$g_+(h_0) \leq h_0, \tag{A26}$$

which contradicts our initial assumption. Therefore, (A26) must be satisfied $\forall h_0 \geq 0$ if $D = 7$. From this inequality, one can check that $G < 0$ $\forall h_0 > 0$ when $D = 7$.

We have thus demonstrated that, for Lovelock theories verifying (3.17),

$$|J(g_+(h_0),h_0,f=0)| < 0 \quad \forall h_0 \geq 0, \ \forall D \geq 5. \tag{A27}$$

This implies that the Jacobian (4.1) is strictly negative in all the constant internal scale factor solutions considered in Sec. III.

We now proceed to prove that relation (3.22) between the constant internal scalar curvature, $h_0 \geq 0$, and the D-dimensional cosmological constant, $\Lambda \geq 0$, can be inverted at least in spacetimes with not too high D ($D \neq 5$). Relation (3.22) can be rewritten

$$\Lambda = \Lambda_+(h_0) = g_+(h_0)\tilde{L}(h_0) + \tilde{\Lambda}(h_0), \tag{A28}$$

with $g_+(h_0)$ satisfying (A1), which can be reexpressed as [see (3.13)]

$$\mathcal{F}(g_+(h_0),h_0) = g_+^2(h_0)c(h_0) + g_+(h_0)d(h_0) - e(h_0) = 0. \tag{A29}$$

Using the implicit function theorem we get then

$$\frac{dg_+}{dh_0}(h_0) = -\frac{\partial_{h_0}\mathcal{F}}{\partial_g\mathcal{F}}(g_+(h_0),h_0), \tag{A30}$$

$$\partial_g\mathcal{F}(g_+(h_0),h_0) = 2g_+(h_0)c(h_0) + d(h_0). \tag{A31}$$

At $h_0 = 0$, $g_+(h_0 = 0) = 0$ and, from Eq. (3.15), $d(h_0 = 0) > 0$, so that $\partial_g\mathcal{F}(g_+(h_0=0),h_0=0) > 0$. On the other hand, if $h_0 > 0$ and $L_m = 0$ $\forall m > 1$ (Einstein gravity), we have, from (3.14) and (3.15), $d(h_0) > 0$ and $c(h_0) = 0$. So, again $\partial_g\mathcal{F}(g_+(h_0),h_0) > 0$. In any other case ($L_m \geq 0$ $\forall m > 1$, not all of them vanishing, $L_1 > 0$ and $h_0 > 0$), it is easy to see from (3.14) and (3.16) that $c(h_0) > 0$ and $e(h_0) > 0$ for $D \geq 6$. Rewritting then (3.20) as

$$g_+(h_0) = \frac{-d(h_0) + \sqrt{d^2(h_0) + 4c(h_0)e(h_0)}}{2c(h_0)}, \tag{A32}$$

it follows that, in these cases, $g_+(h_0) > -d(h_0)/[2c(h_0)]$, so that $\partial_g\mathcal{F}(g_+(h_0),h_0) > 0$. Therefore, we conclude that, for $D \geq 6$,

$$\partial_g\mathcal{F}(g_+(h_0),h_0) > 0 \quad \forall h_0 \geq 0. \tag{A33}$$

Differentiating (A28) with respect to h_0 and using (A30), we obtain

$$\frac{d\Lambda_+}{dh_0}(h_0) = g_+(h_0)\frac{d\tilde{L}}{dh_0}(h_0) + \frac{d\tilde{\Lambda}}{dh_0}(h_0) - \tilde{L}(h_0)\frac{\partial_{h_0}\mathcal{F}}{\partial_g\mathcal{F}}(g_+(h_0),h_0). \tag{A34}$$

In addition, from (3.8), (2.18), and (2.19),

$$g_+(h_0)\frac{d\tilde{L}}{dh_0}(h_0) + \frac{d\tilde{\Lambda}}{dh_0}(h_0) = -\frac{1}{2}\partial_y p_y(g_+(h_0),h_0,f=0) > 0 \quad \forall h_0 \geq 0, \tag{A35}$$

$$\tilde{L}(h_0) = -\frac{1}{2}\partial_x p_x(h_0,f=0) > 0 \quad \forall h_0 \geq 0, \tag{A36}$$

where we have also employed inequalities (3.18) and (3.24), valid for $D \geq 6$. Moreover, making use of (A29) and (3.16), at $h_0 = 0$,

$$\partial_{h_0}\mathcal{F}(g_+(h_0=0)=0,h_0=0)=-L_1\begin{pmatrix}D-3\\3\end{pmatrix}<0 \quad \text{for } D\geq 6. \quad \text{(A37)}$$

We conclude finally

$$\frac{d\Lambda_+}{dh_0}(h_0=0)>0 \quad \text{for } D\geq 6. \quad \text{(A38)}$$

We can now assert that relation (3.22) can be inverted in the whole semiaxis $h_0 \geq 0$, for $D \geq 6$, if $(d\Lambda_+/dh_0)(h_0) > 0\ \forall h_0 \geq 0$. (A38) states that this is at least verified at $h_0 = 0$. For $D \geq 6$ and $h_0 > 0$, proving that

$$W(h_0)\equiv -\frac{1}{2}\partial_y p_y(g_+(h_0),h_0,f=0)\partial_g\mathcal{F}(g_+(h_0),h_0) +\frac{1}{2}\partial_x p_x(h_0,f=0)\partial_{h_0}\mathcal{F}(g_+(h_0),h_0)>0 \quad \text{(A39)}$$

guarantees, taking into account (A33)–(A36), that $(d\Lambda_+/dh_0)(h_0) > 0$. Using again (A33), (A35), and (A36), it follows that a sufficienct though not necessary condition for $W(h_0)$ to be strictly positive in the whole semiaxis $h_0 > 0$ is

$$\partial_{h_0}\mathcal{F}(g_+(h_0),h_0)<0 \quad \forall h_0>0. \quad \text{(A40)}$$

Multiplying (A39) by h_0^2 and using relation (A1) and the already proved inequalities (A3), (A33), and (3.18), we obtain another sufficient condition for $W(h_0)>0\ \forall h_0>0$:

$$Z(h_0)\equiv h_0^2\partial_{h_0}\mathcal{F}(g_+(h_0),h_0) -g_+^2(h_0)\partial_g\mathcal{F}(g_+(h_0),h_0)<0 \quad \forall h_0>0. \quad \text{(A41)}$$

In conclusion, (A39) holds if (A40) or (A41) is satisfied.

We have checked by explicit calculation that (A39) is verified $\forall h_0 > 0$ at least when $6 \leq D \leq 14$.

For $D = 6$, we have from (A5) that $g_+(h_0) \leq \frac{1}{2}h_0$. It is then straightforward to see that (A40) is satisfied. For D =7 or 8, we have that $h_0\partial_{h_0}\mathcal{F} - \mathcal{F}$ is strictly negative at $(g_+(h_0), h_0)\ \forall h_0 > 0$, and hence (A40) holds, because $\mathcal{F}$ vanishes at $(g_+(h_0), h_0)$. Taking into account that, from (A19), $g_+(h_0) \leq 2h_0$ for $D = 9$, $W(h_0) - 3L_3h_0\mathcal{F}(g_+(h_0),h_0)$ turns out then to be strictly positive $\forall h_0 > 0$. Since $\mathcal{F}(g_+(h_0),h_0) = 0$, we conclude then that (A39) holds if $D = 9$.

For $D = 10$ or 12, it can be shown that $Z(h_0) - h_0\mathcal{F}(g_+(h_0),h_0)$ is smaller than zero $\forall h_0 > 0$, provided that, from (A19) and (A6), we respectively have $g_+(h_0) \leq \frac{5}{2}h_0$ if $D = 10$ and $g_+(h_0) \leq \frac{7}{3}h_0$ if $D = 12$. (A41) is then satisfied if $D = 10$ or 12.

Finally, for $D = 11$, 13 or 14 we will use the feature that $g_+(h_0) \leq \frac{D-5}{3}h_0\ \forall h_0 > 0$ is valid for these dimensions. For $D = 11$, $W(h_0) - 12L_4h_0^2\mathcal{F}(g_+(h_0),h_0)$ is greater than zero for positive h_0, and from (A29) it follows that (A39) is then satisfied. In a similar way, for $D = 13$ and $D = 14$, respectively, $W(h_0) - 25L_5h_0^3\mathcal{F}(g_+(h_0),h_0)$ and $W(h_0) - [24L_6h_0^4 + \frac{80}{3}L_5h_0^3]\mathcal{F}(g_+(h_0),h_0)$ turn out to be strictly positive. Therefore, inequality (A39) holds also $\forall h_0 > 0$ in these cases.

Owing to complication of the expression in (A39) for $W(h_0)$, we have not been able to find a general demonstration which allows us to assert that (A39) holds $\forall h_0 > 0$ for any Lovelock theory satisfying (3.17) whichever $D \geq 6$. Nevertheless, as (A39) is verified $\forall h_0 > 0$ for $6 \leq D \leq 14$, we think that it does so probably for any $D \geq 6$.

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