

Hereditary effects in gravitational radiation

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This paper derives the leading nonlinear hereditary effects in the generation of gravitational radiation, i.e., the terms in the wave form which depend in an irreducible manner on the entire past history of the source. At the quadratically nonlinear order there are two types of hereditary contributions. The first ones are due to the reradiation of gravitational waves by the stress-energy distribution of (linear) gravitational waves, and give rise to a net cumulative change in the wave form of bursts (“memory effect”). The second ones come from the backscattering of (linear) gravitational waves emitted in the past onto the constant curvature associated with the total mass of the source (“gravitational-wave tails”). An extension of a previously proposed multipole-moment wave generation formalism allows us to compute explicitly the wave form, including hereditary contributions, up to terms of fractional order $(v/c)^4$. Our results are derived for slow-moving systems of bodies, independently of the strength of their internal gravity. The tail contribution to the far wave-zone field is found to be fully consistent with a corresponding hereditary contribution to the gravitational radiation damping previously derived from a study of the near-zone field.

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I. INTRODUCTION

The present development of a worldwide network of gravitational-wave detectors makes it timely to deepen our theoretical understanding of the generation of gravitational radiation by material sources. Although some astrophysical sources, involving very strong gravitational fields, and undergoing very rapid time evolution, are so complex that they will have to be tackled by numerical simulations, a lot is still to be learned from analytical approximation methods. We have particularly in mind the emission of gravitational waves by in-spiraling compact binary systems where improvements in the computation of the wave form may be important for pulling the signal out of the noise.

The present paper is the continuation of a sequence of articles [1–5], that we shall refer to in the following as papers I–V, respectively, in which we expounded a new gravitational-wave generation formalism. This formalism decomposes the problem of relating the gravitational-wave form at infinity to the structure and motion of the source (“generation problem”) into three separate steps.

Step 1 consists in setting up an iterative algorithm which constructs the most general (past-stationary and past-asymptotically flat) solution of the vacuum Einstein equations in the form of a double, nonlinearity and multipolar, expansion. The arbitrary elements entering this construction are two sets of time-dependent symmetric and trace-free (STF) Cartesian tensors, referred to as the “algorithmic multipole moments:”

$$\mathcal{M} = \{M_L(t) ; \ell \geq 0\} \cup \{S_L(t) ; \ell \geq 1\}, \quad (1.1)$$

where $\{M_L ; \ell \geq 0\} = \{M, M_i, M_{i_1 i_2}, \dots\}$ denote the “mass” algorithmic moments, while $\{S_L ; \ell \geq 1\} = \{S_i, S_{i_1 i_2}, \dots\}$ denote the “spin” ones. In Eq. (1.1) the index L is shorthand notation for a multispatial index of order ℓ , $L \equiv i_1 i_2 \dots i_\ell$ (see Ref. [6] for our notation).

The (formal) solution of the vacuum Einstein equations constructed by this algorithm will be referred to as the “external metric,” because it is expected to represent the metric everywhere in a domain $D_e = \{(\mathbf{x}, t) ; r > r_0\}$ exterior to the source (which is located near the origin of our spatial coordinate system). Using as basic gravitational variables the densitized contravariant metric $\mathcal{G}^{\alpha\beta} \equiv \sqrt{g} g^{\alpha\beta}$, the result of the algorithm is to give

$$\mathcal{G}_{\text{ext}}^{\alpha\beta} = \mathcal{G}_{\text{ext}}^{\alpha\beta}[\mathcal{M}], \quad (1.2)$$

where the square brackets denote a general functional dependence. [Note that $\mathcal{G}_{\text{ext}}^{\alpha\beta}$ are functions of four variables, while \mathcal{M} is a set of functions of one variable.] Building up on previous work [7–9], we have shown in paper I how to define the right-hand side of Eq. (1.2) to all orders in a combined multipolar post-Minkowskian expansion. [In principle this construction can be implemented to any (finite) order by algebraic manipulation programs.]

Step 2 consists in extracting from the algorithmically constructed external metric (1.2) the “radiative multipole moments”

$$\mathcal{R} = \{J_L^{\text{rad}[\ell]}(t) ; \ell \geq 2\} \cup \{J_L^{\text{rad}[\ell]}(t) ; \ell \geq 2\}, \quad (1.3)$$

that describe the leading ($1/R$) behavior of the gravitational field in the far wave zone (“future null infinity”). More precisely, it was shown in paper II that the general multipolar post-Minkowskian metric (1.2) admitted, under the assumption of past stationarity, a regular conformal structure at future null infinity (“asymptotic simplicity” [10, 11]). This means in particular that there exist some “radiative” coordinates $X^\mu = (cT, X^i)$ with respect to which the metric coefficients, say $G_{\alpha\beta}^{\text{ext}}(X^\gamma)$ [where

$ds_{\text{ext}}^2 = g_{\alpha\beta}^{\text{ext}} dx^\alpha dx^\beta = G_{\alpha\beta}^{\text{ext}} dX^\alpha dX^\beta$], admit an asymptotic expansion in powers of R^{-1} , when $R = |\mathbf{X}| \rightarrow \infty$ with $U \equiv T - R/c$ and $\mathbf{N} = \mathbf{X}/R$ being fixed. Then the radiative multipole moments (1.3) are defined as the coefficients of the multipole decomposition of the $1/R$ part of the external metric in radiative coordinates [9]. The transverse-traceless (TT) part of the asymptotic spatial metric (gravitational-wave amplitude “ h_{ij}^{TT} ”) reads

$$h_{ij}^{\text{TT}} \equiv (G_{ij}^{\text{ext}} - \delta_{ij})^{\text{TT}} = + \frac{4G}{c^2 R} P_{ijhk}(\mathbf{N}) \sum_{\ell \geq 2} \frac{1}{\ell! c^\ell} \left\{ N_{L-2} I_{hkL-2}^{\text{rad}[\ell]}(U) - \frac{2\ell}{(\ell+1)c} N_{\alpha L-2} \varepsilon_{ab(h} J_{k)bL-2}^{\text{rad}[\ell]}(U) \right\} + O\left(\frac{1}{R^2}\right), \quad (1.4)$$

where $T_{(hk)} \equiv \frac{1}{2}(T_{hk} + T_{kh})$, and where $P_{ijhk}(\mathbf{N}) = P_{ih}P_{jk} - \frac{1}{2}P_{ij}P_{hk}$, with $P_{ih}(\mathbf{N}) \equiv \delta_{ih} - N_i N_h$, denotes the TT algebraic projection operator onto the plane orthogonal to \mathbf{N} .

The transformation between the original (“canonical”) coordinates in which the external metric is constructed by the algorithm (1.2) and the radiative coordinates can be algorithmically constructed (see paper II and below, where the construction is implemented to order G^2). Therefore, the radiative moments (1.3) can be functionally expressed in terms of the algorithmic moments (1.1), at least as a formal power series in the gravitational coupling constant:

$$\mathcal{R} = \mathcal{R}[\mathcal{M}]. \quad (1.5)$$

In the linearized approximation, the radiative multipole moments are the ℓ th-order time derivatives of the algorithmic moments:

$$I_L^{\text{rad}[\ell]} = d^\ell M_L / dU^\ell + O(G), \quad (1.6a)$$

$$J_L^{\text{rad}[\ell]} = d^\ell S_L / dU^\ell + O(G). \quad (1.6b)$$

For this reason, it has been suggested [9] to introduce also the ℓ th-order antiderivatives of the basic radiative moments, say I_L^{rad} and J_L^{rad} such that $I_L^{\text{rad}[\ell]} \equiv d^\ell I_L^{\text{rad}} / dU^\ell$, $J_L^{\text{rad}[\ell]} \equiv d^\ell J_L^{\text{rad}} / dU^\ell$. However, the definition of the objects I_L^{rad} and J_L^{rad} leads to ambiguities and difficulties when one considers sources that were not stationary in the remote past. In the present paper, we shall consider only the moments (1.3) which are directly related to the observable wave form h_{ij}^{TT} . The superscripts $[\ell]$ must be viewed as a mere notation reminding us that, in restricted physical situations, $I_L^{\text{rad}[\ell]}$ and $J_L^{\text{rad}[\ell]}$ may be equal to the ℓ th time derivatives of other objects.

The first two steps of our formalism give, in principle, a complete picture of the nonlinear structure of the gravitational field everywhere outside the source (at least, in the domain where the field is weak enough for the nonlinearity expansion to make sense). However, this knowledge is totally disconnected from the actual material source, and must be complemented by a different, source-rooted, approach. Indeed, the aim of the third

step of the formalism is to provide the link between the algorithmic moments (1.1) and the structure and motion of the source. Symbolically,

$$\mathcal{M} = \mathcal{M}[\text{source}]. \quad (1.7)$$

Step 3, Eq. (1.7), can be tackled by different means, depending upon which type of source one is considering. For instance, if one considers a mildly relativistic source, the link (1.7) will be derived by matching the (multipolar-post-Minkowskian-expanded) external metric (1.2) to the inner metric obtained by describing the near-zone gravitational field of the source by means of a combined weak-field-slow-motion (“post-Newtonian”) approximation scheme. This matching procedure has been studied in paper III, and we implemented it in papers IV and V at an accuracy which goes well beyond the usual “quadrupole formalism” [12–14]. More precisely, we showed how to obtain the link (1.7), for both the mass [4] and spin [5] moments, up to fractional corrections of order $(v/c)^4 \sim (GM/c^2 r)^2$:

$$M_L = I_L[\text{source}] + O(c^{-4}) \quad (\ell \geq 0), \quad (1.8a)$$

$$S_L = J_L[\text{source}] + O(c^{-4}) \quad (\ell \geq 1), \quad (1.8b)$$

where I_L and J_L denote some explicit, compact-support integral expressions involving only the stress-energy tensor of the matter and its time derivatives.

By eliminating the algorithmic moments between the results of Steps 2 and 3, i.e., Eqs. (1.5) and (1.7), one can finally relate the gravitational wave form to the structure and motion of the source. The accuracy with which this can be done is limited by the accuracy of each separate step. The main object of the present paper is to refine the accuracy of Step 2, i.e., to work out explicitly the leading nonlinear $O(G)$ contributions in Eqs. (1.6a) and (1.6b). These contributions exhibit a new feature which has been hitherto neglected in generation formalisms. Indeed, we shall find that these $O(G)$ terms are, following the terminology of paper III, “hereditary” in the sense that they depend on the full past history of the system, in other terms, they keep a “memory” of the past activity of the source. (In paper III we derived the leading hereditary contributions in the *near-zone* gravitational fields.

The link with our present *wave-zone* hereditary effects will be discussed below). The physical origin of these hereditary contributions to the wave form is twofold. The leading contribution, for slow-motion sources, comes from the scattering of linear gravitational waves off the background curvature generated by the total mass of the system (“wave tails”). A second contribution, which enters at the same order of nonlinearity, comes from the reradiation of gravitational waves by the stress-energy distribution of linear waves. The relative characteristics of these two types of hereditary effects will be discussed below. (Note that our results on nonlinear hereditary effects were already contained in an earlier paper [15].)

The inclusion of wave tails will boost the accuracy of our generation formalism by one power of the slow-motion parameter v/c . Moreover, we shall extend the domain of applicability of our results by showing how to generalize Eqs. (1.8) to systems containing strongly self-gravitating bodies, e.g., an in-spiraling binary neutron star.

Finally, let us note that it is conceivable that one could match directly the analytically known metric (1.2) to a numerically computed metric considered in a finite grid around some astrophysical source. This could be helpful in extracting from finite-grid results the true wave form at infinity. We shall comment below on the impact of the results we shall derive in this paper on this program.

The organization of this paper is as follows. In Sec. II we investigate the nonlinear hereditary functional dependence of the quadratic external metric and of the radiative multipole moments on the algorithmic moments. The hereditary effects that we obtain (“memory” and “tail” effects) are discussed and we study their sensitivity on the remote past of the source. In Sec. III we extend our wave generation formalism to the inclusion of “tail” effects (the only hereditary effects to be included at lowest order in the slow-motion approximation), and to systems of strongly self-gravitating bodies. The “tail” effects are finally shown to be consistent with correspond-

ing hereditary effects in the radiation reaction force acting within the source.

II. NONLINEAR HEREDITARY STRUCTURE OF GRAVITATIONAL RADIATION

A. Nonlinear hereditary structure of radiative multipole moments

Equation (1.5) symbolizes the link between the radiative multipole moments \mathcal{R} , Eq. (1.3), which are directly measurable in the asymptotic wave zone [see Eq. (1.4)], and the algorithmic multipole moments which are convenient functional parameters encoding the structure of the metric outside the source, Eq. (1.2). As both the multipolar post-Minkowskian algorithm and the construction of radiative coordinates proceed by expansion in powers of the gravitational constant G the relation (1.5) will admit a formal nonlinearity expansion of the type

$$\mathcal{R} = \mathcal{D}\mathcal{M} + G\mathcal{X}_2[\mathcal{M}] + \cdots + G^{n-1}\mathcal{X}_n[\mathcal{M}] + \cdots, \quad (2.1)$$

where \mathcal{D} is a linear differential operator and \mathcal{X}_n denotes a multilinear functional (homogeneous of order n) of the algorithmic moments. More precisely, one can write (for $\ell \geq 2$)

$$I_L^{\text{rad}[\ell]} = M_L^{(\ell)}(u) + \sum_{n \geq 2} G^{n-1} X_{nL}(u), \quad (2.2a)$$

$$\varepsilon_{ai\ell i_{\ell-1}} J_{aL-2}^{\text{rad}[\ell-1]}(u) = \varepsilon_{ai\ell i_{\ell-1}} S_{aL-2}^{(\ell-1)}(u) + \sum_{n \geq 2} G^{n-1} Y_{nL}(u), \quad (2.2b)$$

where a superscript within parentheses denotes a multitime differentiation, $F^{(\ell)}(u) \equiv d^\ell F(u)/du^\ell$ and where the n -tuple nonlinear functionals X_n and Y_n have the general structure

$$X_{nL}(u) = \sum \int_{-\infty}^u \cdots \int_{-\infty}^u du_1 \cdots du_n \mathcal{K}_{LL_1 \cdots L_n}(u, u_1, \dots, u_n, P^{\text{alg}}, P^{\text{rad}}) \mathcal{M}_{L_1}^{(p_1)}(u_1) \cdots \mathcal{M}_{L_n}^{(p_n)}(u_n). \quad (2.2c)$$

In Eq. (2.2c), \mathcal{K} denotes a multi-time kernel whose index structure is made out only of Kronecker deltas and $\mathcal{M}_{\underline{L}}$ denotes either a mass moment M_L (in which case $\underline{L} \equiv \ell$) or a spin moment endowed with its natural Levi-Civita symbol, $\varepsilon_{ai\ell+1 i_\ell} S_{aL-1}$ (in which case $\underline{L} = \ell + 1$). Only quantities having the dimension of time enter the kernel \mathcal{K} : the time argument u of the left-hand side, n intermediate time arguments (all restricted to be anterior to u because of the multiretarded nature of the algorithm), the time scale P^{alg} entering the definition of the algorithm [i.e., $P^{\text{alg}} \equiv r_1/c$ where r_1 is the length scale used to adimensionalize the radius in the analytic-continuation factors $(r/r_1)^B$ present in the algorithm, see, e.g., paper III, Eq. (3.6b)], and the time scale P^{rad} entering the algorithm of construction of radiative coordinates, starting from the (harmonic) algorithmic ones. (P^{alg} and P^{rad} can be chosen at will. In particular, one could arrange

to have $P^{\text{rad}} \equiv P^{\text{alg}}$. However, it is clearer to keep these two time scales separate.)

It is useful to distinguish two types of terms in the multilinear functionals $\mathcal{X}_n[\mathcal{M}]$ appearing in Eq. (2.1) [and more explicitly in Eq. (2.2c)]. Namely

$$\mathcal{X}_n[\mathcal{M}] = \mathcal{S}_n[\mathcal{M}] + \mathcal{H}_n[\mathcal{M}], \quad (2.3)$$

where $\mathcal{S}_n[\mathcal{M}]$ denotes a “synchronous” (or “snapshot”) functional, i.e., a sum of terms that depend only on the values of the $\mathcal{M}_{\underline{L}}(u)$ ’s and their time derivatives at the same time argument where one evaluates $\mathcal{R}(u)$, while $\mathcal{H}_n[\mathcal{M}]$ denotes a “hereditary” functional, i.e., a sum of terms that depend, in an irreducible manner, upon the values of some $\mathcal{M}_{\underline{L}_i}(u_i)$ for all the time arguments $u_i < u$. The hereditary terms are physically interesting for two reasons. On the one hand, their study casts

a light on the way the nonlinear structure of Einstein's equations generates a functional dependence of the outgoing radiation on the entire past history of the source, and, on the other hand, these terms could be numerically important in some radiation processes because they build up gradually during all the time when the system radiates. In particular, these terms might be crucial to an accurate computation of the phase of the wave form emitted by in-spiraling binary systems.

In the following, we shall determine explicitly the leading hereditary terms in the radiative multipoles, i.e., those coming from the quadratic functionals $G\mathcal{H}_2[\mathcal{M}]$ in the nonlinearity expansion (2.1).

B. Hereditary structure of the quadratic external metric

The nonlinear expansion of the external ("gothic") metric (1.2) reads

$$\mathcal{G}_{\text{ext}}^{\alpha\beta}[\mathcal{M}] = f^{\alpha\beta} + Gh_1^{\alpha\beta}[\mathcal{M}] + G^2h_2^{\alpha\beta}[\mathcal{M}] + \dots, \quad (2.4)$$

where $f^{\alpha\beta}$ is the Minkowski metric with signature +2, $f^{\alpha\beta} = \text{diag}(-1, 1, 1, 1)$, and where $Gh_1^{\alpha\beta}$, $G^2h_2^{\alpha\beta}$, ... are the linearized, quadratic, ... approximations to the metric, that depend functionally on the set of algorithmic moments \mathcal{M} . The linearized external metric $h_1^{\alpha\beta}$ in the expansion (2.4) is given explicitly by the multipole expansion

$$h_1^{00}[\mathcal{M}] = -\frac{4}{c^2} \sum_{\ell \geq 0} \frac{(-)^\ell}{\ell!} \partial_L \left[r^{-1} M_L \left(t - \frac{r}{c} \right) \right], \quad (2.5a)$$

$$h_1^{0i}[\mathcal{M}] = \frac{4}{c^3} \sum_{\ell \geq 1} \frac{(-)^\ell}{\ell!} \partial_{L-1} \left[r^{-1} M_{iL-1}^{(1)} \left(t - \frac{r}{c} \right) \right] + \frac{4}{c^3} \sum_{\ell \geq 1} \frac{(-)^\ell \ell}{(\ell+1)!} \varepsilon_{iab} \partial_{aL-1} \times \left[r^{-1} S_{bL-1} \left(t - \frac{r}{c} \right) \right], \quad (2.5b)$$

$$h_1^{ij}[\mathcal{M}] = -\frac{4}{c^4} \sum_{\ell \geq 2} \frac{(-)^\ell}{\ell!} \partial_{L-2} \left[r^{-1} M_{ijL-2}^{(2)} \left(t - \frac{r}{c} \right) \right] - \frac{8}{c^4} \sum_{\ell \geq 2} \frac{(-)^\ell \ell}{(\ell+1)!} \partial_{aL-2} \times \left[r^{-1} \varepsilon_{ab(i} S_{j)bL-2}^{(1)} \left(t - \frac{r}{c} \right) \right] \quad (2.5c)$$

[see, e.g., Eqs. III (3.3), by which we denote Eqs. (3.3) of paper III]. The linearized external metric (2.5) is the "seed" of the entire algorithm. For instance, the quadratic piece $h_2^{\alpha\beta}$ is defined as the sum of two contributions:

$$h_2^{\alpha\beta}[\mathcal{M}] = p_2^{\alpha\beta} + q_2^{\alpha\beta}. \quad (2.6)$$

The first contribution $p_2^{\alpha\beta}$ is the retarded integral (computed using a procedure of analytic continuation) of the effective quadratic source $N_2^{\alpha\beta}(h_1)$ given by Eq. III (3.5):

$$p_2^{\alpha\beta} = FP\Box_R^{-1} N_2^{\alpha\beta}, \quad (2.7)$$

where $FP\Box_R^{-1}$ ("finite part of the retarded integral") is the operator defined by Eqs. I (3.13) and (3.14) (in which enters the length scale $r_1 \equiv cP^{\text{alg}}$). As for the second contribution in (2.6), $q_2^{\alpha\beta}$, it is a solution of the homogeneous wave equation of the retarded type:

$$q_2^{\alpha\beta} = \sum_{\ell \geq 0} \partial_L \left[r^{-1} K_L^{\alpha\beta} \left(t - \frac{r}{c} \right) \right], \quad (2.8)$$

which serves to ensure the condition of harmonicity of the coordinates. In Eq. (2.8), the functions $K_L^{\alpha\beta}(u)$ denote some quadratic functionals of the algorithmic moments $M_L(u)$ and $S_L(u)$ that can be explicitly computed, if necessary, by means of the algorithm of paper I [see Eqs. I (4.12) and (4.13)]. They will be partially computed below. The nature of the functional dependence of $h_2^{\alpha\beta}$ in terms of the algorithmic moments has been investigated in paper III. There it was found (extending the terminology used above for the functionals $\mathcal{M}^{\text{rad}}[\mathcal{M}]$), that, contrary to $h_1^{\alpha\beta}$ which is a snapshot functional of \mathcal{M} [in the sense that $h_1^{\alpha\beta}(t, \mathbf{x})$ depends only on the values of the $\mathcal{M}_L(u)$'s and their time-derivatives at the retarded time-argument $u = t - r/c$, where $r \equiv |\mathbf{x}|$], $h_2^{\alpha\beta}$ contains, in addition to many snapshot terms, an irreducibly hereditary dependence on the values of the $\mathcal{M}_L(v)$'s for $v < u$. From Eqs. III (4.31) we can separate explicitly the hereditary terms:

$$h_2^{\alpha\beta}[\mathcal{M}] = \Box_R^{-1}(r^{-2} Q^{\alpha\beta}) + \sum_{\ell=0,1} \partial_L \left[r^{-1} T_L^{\alpha\beta}(t - r/c) \right] + S_2^{\alpha\beta}[\mathcal{M}]. \quad (2.9)$$

In Eq. (2.9) $Q^{\alpha\beta}(u, \mathbf{n})$, where $u = t - r/c$, denotes the coefficient of $1/r^2$ in the effective nonlinear source $N_2^{\alpha\beta}(r, u, \mathbf{n})$, $T_L^{\alpha\beta}(u)$ (present only for the multipoles $\ell = 0$ and $\ell = 1$) are some antiderivatives of products of derivatives of multipole moments [that were contained in the $K_L^{\alpha\beta}(u)$ terms in Eq. (2.8)], while the remainder $S_2^{\alpha\beta}[\mathcal{M}]$ denotes some synchronous functional of \mathcal{M} (i.e., a sum of terms of the type $F(u) \hat{n}_L/r^k$ where $F(u)$ is a product of derivatives of multipole moments taken at the same retarded time u).

The quantity $Q^{\alpha\beta}$ consists of two separate terms:

$$Q^{\alpha\beta}(u, \mathbf{n}) = \frac{k^\alpha k^\beta}{c^2} \Pi + \frac{4M}{c^4} \frac{d^2 z^{\alpha\beta}}{du^2}, \quad (2.10)$$

where $k^\alpha \equiv (1, \mathbf{n})$ denotes the (Minkowskian) outgoing radial null direction, and where

$$\Pi(u, \mathbf{n}) = \frac{1}{2} \frac{dz^{\mu\nu}}{du} \frac{dz_{\mu\nu}}{du} - \frac{1}{4} \frac{dz_\mu^\mu}{du} \frac{dz_\nu^\nu}{du}. \quad (2.11a)$$

The quantity $\Pi(u, \mathbf{n})$ is proportional to the gravitational-wave luminosity [computed with the linearized metric (2.5)]. Namely

$$\Pi(u, \mathbf{n}) = \frac{16\pi}{Gc^3} \frac{dL^{\text{grav}}}{d\Omega} \Big|_{h_1} = \frac{16\pi}{Gc^3} \frac{dE^{\text{grav}}}{du d\Omega} \Big|_{h_1}. \quad (2.11b)$$

In Eqs. (2.10), (2.11a) $z^{\alpha\beta}(u, \mathbf{n})$ denotes the nonstatic

part of the coefficient of $1/r$ in the (linearized) wavezone expansion of the linearized metric $h_1^{\alpha\beta}$, Eqs. (2.5). Namely

$$z^{00}(u, \mathbf{n}) = -4 \sum_{\ell \geq 2} \frac{n_L}{c^{\ell+2}\ell!} M_L^{(\ell)}(u), \quad (2.12a)$$

$$z^{0i}(u, \mathbf{n}) = -4 \sum_{\ell \geq 2} \frac{n_{L-1}}{c^{\ell+2}\ell!} M_{iL-1}^{(\ell)}(u) + 4 \sum_{\ell \geq 2} \frac{\ell}{c^{\ell+3}(\ell+1)!} \varepsilon_{iab} n_{aL-1} S_{bL-1}^{(\ell)}(u), \quad (2.12b)$$

$$z^{ij}(u, \mathbf{n}) = -4 \sum_{\ell \geq 2} \frac{n_{L-2}}{c^{\ell+2}\ell!} M_{ijL-2}^{(\ell)}(u) + 8 \sum_{\ell \geq 2} \frac{\ell}{c^{\ell+3}(\ell+1)!} n_{aL-2} \varepsilon_{ab(i} S_{j)L-2}^{(\ell)}(u). \quad (2.12c)$$

[Equations (2.12) follow from Eqs. (2.5) by letting the spatial derivatives $\partial_L \equiv \partial_{i_1} \cdots \partial_{i_\ell}$ act on the retarded times $t-r/c$ present in the various functions $F(t-r/c)/r$, and by deleting the time-independent mass term M/r .] Moreover, we see that the term $k^\alpha k^\beta \Pi/c^2$ in Eq. (2.10) can be thought of as the stress-energy tensor of the outgoing gravitational radiation (“bundle of gravitons”).

The quadratically nonlinear metric can thus be naturally decomposed into four contributions:

$$h_2^{\alpha\beta}[\mathcal{M}] = u^{\alpha\beta} + v^{\alpha\beta} + w^{\alpha\beta} + S_2^{\alpha\beta}[\mathcal{M}]. \quad (2.13)$$

In Eq. (2.13), $S_2^{\alpha\beta}[\mathcal{M}]$ is the synchronous remainder of Eq. (2.9), while

$$u^{\alpha\beta} \equiv \square_R^{-1} \left(\frac{k^\alpha k^\beta}{c^2 r^2} \Pi \right) \quad (2.14)$$

is the hereditary term coming from the fact that the gravitational energy radiated in the past by the source acts itself as a nonlinear source for the gravitational field (this term was studied in [15] under the name of the “bundle” term),

$$v^{\alpha\beta} \equiv \square_R^{-1} \left(\frac{4M}{c^4 r^2} \frac{d^2 z^{\alpha\beta}}{du^2} \right) \quad (2.15)$$

is the hereditary term due to the fact that the curved light cones differ by $O(M \ln r)$ from the flat ones (it can also be thought of as a scattering of the gravitational waves on the background curvature associated with the ADM mass M), and

$$w^{\alpha\beta} \equiv \sum_{\ell=0,1} \partial_L \left(\frac{1}{r} T_L^{\alpha\beta}(t-r/c) \right) \quad (2.16)$$

is the “semihereditary” term associated with the secular variations of mass, linear momentum and angular momentum (as will appear from its expression below).

Thanks to the various technical tools introduced in our previous papers (plus those discussed in Appendix A below), one can give the explicit expressions of the various hereditary components of $h_2^{\alpha\beta}$. To evaluate the term (2.14), one first introduces the quantity

$$\lambda^\alpha \equiv \square_R^{-1} \left(\frac{k^\alpha}{2cr^2} \Pi^{(-1)}(u, \mathbf{n}) \right), \quad (2.17)$$

where $\Pi^{(-1)}(u)$ is the past-zero antiderivative of $\Pi(u)$,

$$\Pi^{(-1)}(u, \mathbf{n}) \equiv \int_{-\infty}^u dv \Pi(v, \mathbf{n}) = \frac{16\pi}{Gc^3} \frac{dE^{\text{grav}}(u, \mathbf{n})}{d\Omega} \Big|_{h_1}, \quad (2.18)$$

where $dE^{\text{grav}}/d\Omega$ is the angular distribution of the total energy radiated in the form of gravitational waves between the infinite past and the retarded time u . Then one notices that the combination $u^{\alpha\beta} + \partial^\alpha \lambda^\beta + \partial^\beta \lambda^\alpha - f^{\alpha\beta} \partial_\gamma \lambda^\gamma$ is the retarded integral of a term of the form $F(u, \mathbf{n})/r^3$ which can be explicitly evaluated by using formulas III (4.24) and III (4.26). Finally, one gets the term (2.14) in the form

$$u^{\alpha\beta} = \frac{1}{cr} \int_{-\infty}^{t-r/c} dv U^{\alpha\beta}(v, \mathbf{n}) - \partial^\alpha \lambda^\beta - \partial^\beta \lambda^\alpha + f^{\alpha\beta} \partial_\gamma \lambda^\gamma, \quad (2.19)$$

where

$$U^{00} = -\frac{1}{2} \Pi_0, \quad (2.20a)$$

$$U^{0i} = \sum_{\ell \geq 0} \frac{1}{2(\ell+1)} n_{iL} \Pi_L + \sum_{\ell \geq 1} \frac{-1}{2(\ell+1)} n_{L-1} \Pi_{iL-1}, \quad (2.20b)$$

$$U^{ij} = \sum_{\ell \geq 0} \frac{1}{\ell+2} n_{ijL} \Pi_L + \sum_{\ell \geq 1} \frac{-1}{(\ell+1)(\ell+2)} \delta_{ij} n_L \Pi_L + \sum_{\ell \geq 1} \frac{-(\ell-2)}{(\ell+1)(\ell+2)} n_{L-1(i} \Pi_{j)L-1} + \sum_{\ell \geq 2} \frac{-2}{(\ell+1)(\ell+2)} n_{L-2} \Pi_{ijL-2}. \quad (2.20c)$$

In Eqs. (2.20) the quantities $\Pi_L(u)$ represent the coefficients of the expansion of the gravitational luminosity $\Pi(u, \mathbf{n})$ in STF spherical harmonics:

$$\Pi(u, \mathbf{n}) \equiv \sum_{\ell \geq 0} \hat{n}_L \Pi_L(u) \quad (2.21)$$

[$\Pi_L(u)$ being a symmetric and trace-free Cartesian tensor; $\Pi_0(u)$ in Eq. (2.20a) denotes the $\ell = 0$ piece of $\Pi(u, \mathbf{n})$, i.e., its spherical average].

Turning our attention to the term (2.15), one notices that it is a sum of retarded integrals of terms of the form $\hat{n}_L G(u)/r^2$. The latter can be evaluated by using the formula III (4.21), namely,

$$\square_R^{-1} \left[\frac{\hat{n}_L}{r^2} G \left(t - \frac{r}{c} \right) \right] = \frac{(-)^\ell}{2\ell!} \int_r^{+\infty} dz G \left(t - \frac{z}{c} \right) \hat{\delta}_L \left\{ \frac{(z-r)^\ell \ln(z-r) - (z+r)^\ell \ln(z+r)}{r} \right\}. \quad (2.22)$$

We are interested in studying the asymptotic behavior of the right-hand side of Eq. (2.22) when $r \rightarrow \infty$, with $t - r/c$ fixed. For this purpose, it is convenient to use an alternate form of the retarded integral (2.22), involving the Legendre function of the second kind $Q_\ell(x)$. Let us recall that the latter function is defined, when $x > 1$, by

$$Q_\ell(x) = \frac{1}{2} P_\ell(x) \ln \left(\frac{x+1}{x-1} \right) - \sum_{i=1}^{\ell} \frac{1}{i} P_{\ell-i}(x) P_{i-1}(x), \quad (2.23)$$

where $P_\ell(x)$ is the usual Legendre polynomial, with $P_\ell(1) = 1$ (see, e.g., Ref. [16], page 333). Then it is shown in Appendix A that the form (2.22) of the retarded integral can be rewritten as

$$\square_R^{-1} \left[\frac{\hat{n}_L}{r^2} G \left(t - \frac{r}{c} \right) \right] = -\frac{\hat{n}_L}{r} \int_r^{+\infty} dz G \left(t - \frac{z}{c} \right) Q_\ell \left(\frac{z}{r} \right). \quad (2.24)$$

Now, the desired expansion at infinity, $r \rightarrow \infty$ with $t - r/c$ fixed, is easily obtained from the fact that, from Eq. (2.23), the Legendre function $Q_\ell(x)$ behaves, when $x \rightarrow 1^+$, as

$$Q_\ell(x) = -\frac{1}{2} \ln \left(\frac{x-1}{2} \right) - \sum_{i=1}^{\ell} \frac{1}{i} + O[(x-1) \ln(x-1)]. \quad (2.25)$$

Inserting the latter expansion into (2.24) yields immediately, when $r \rightarrow \infty$ with $t - r/c$ fixed,

$$\square_R^{-1} \left[\frac{\hat{n}_L}{r^2} G \left(t - \frac{r}{c} \right) \right] = \frac{c\hat{n}_L}{2r} \int_0^{+\infty} dy G \left(t - \frac{r}{c} - y \right) \left[\ln \left(\frac{cy}{2r} \right) + 2 \sum_{i=1}^{\ell} \frac{1}{i} \right] + O \left(\frac{\ln r}{r^2} \right). \quad (2.26)$$

By summing over the multipolarity ℓ , one gets the following formula for a source $G(u, \mathbf{n}) = \sum_{\ell \geq 0} n_L G_L(u)$, where the $G_L(u)$ are STF:

$$\square_R^{-1} \left[\frac{1}{r^2} G \left(t - \frac{r}{c}, \mathbf{n} \right) \right] = \frac{c}{2r} \int_0^{+\infty} dy \left[G \left(t - \frac{r}{c} - y, \mathbf{n} \right) \ln \left(\frac{cy}{2r} \right) + \sum_{\ell \geq 0} \left(\sum_{i=1}^{\ell} \frac{2}{i} \right) n_L G_L \left(t - \frac{r}{c} - y \right) \right] + O \left(\frac{\ln r}{r^2} \right). \quad (2.27)$$

These results can be directly applied to the evaluation of Eq. (2.15). Since the source of $v^{\alpha\beta}$, namely $4Mr^{-2}c^{-4}d^2z^{\alpha\beta}/du^2$, is a time derivative, the second term in (2.27) yields only synchronous terms that we can ignore. Hence, the leading asymptotic behavior of the hereditary part of $v^{\alpha\beta}$ can be written as

$$v^{\alpha\beta} = \frac{2M}{rc^3} \int_0^\infty dy \frac{d^2z^{\alpha\beta}}{du^2} \left(t - \frac{r}{c} - y, \mathbf{n} \right) \ln \left(\frac{cy}{2r} \right) + \frac{\mathcal{S}_v^{\alpha\beta}(t-r/c, \mathbf{n})}{r} + O \left(\frac{\ln r}{r^2} \right). \quad (2.28)$$

In Eq. (2.28) $\mathcal{S}_v^{\alpha\beta}(u)$ denotes some synchronous functional of the algorithmic moments $\mathcal{M}(u)$.

Finally, the semihereditary term (2.16) is obtained by a straightforward application of the algorithm [Eqs. I (4.12) and (4.13)]. This yields

$$w^{00} = -\frac{4}{r} m \left(t - \frac{r}{c} \right) + 4\partial_i \left[\frac{1}{r} m_i \left(t - \frac{r}{c} \right) \right], \quad (2.29a)$$

$$w^{0i} = -\frac{4}{r} m_i^{(1)} \left(t - \frac{r}{c} \right) - 2\varepsilon_{iab} \partial_a \left[\frac{1}{r} s_b \left(t - \frac{r}{c} \right) \right], \quad (2.29b)$$

$$w^{ij} = 0, \quad (2.29c)$$

where the functions $m(u)$, $m_i(u)$, and $s_i(u)$ are some semihereditary functions linked to the scalar part Π_0 of Π [first term in (2.21)], to the vector part Π_i of Π [second term in (2.21)] and to some other synchronous vectorial functions F_i and G_i (whose explicit expressions will not be needed) by

$$m(u) = -\frac{1}{4c} \Pi_0^{(-1)}(u), \quad (2.30a)$$

$$m_i(u) = -\frac{1}{12} \Pi_i^{(-2)}(u) + F_i^{(-1)}(u), \quad (2.30b)$$

$$s_i(u) = G_i^{(-1)}(u). \quad (2.30c)$$

By expanding the spatial derivatives in Eqs. (2.29) we get the leading asymptotic behavior of $w^{\alpha\beta}$ in the form

$$w^{\alpha\beta} = \frac{1}{cr} \int_{-\infty}^{t-r/c} dv W^{\alpha\beta}(v, \mathbf{n}) + \frac{\mathcal{S}_w^{\alpha\beta}(t-r/c, \mathbf{n})}{r} + O \left(\frac{1}{r^2} \right), \quad (2.31)$$

where $\mathcal{S}_w^{\alpha\beta}$ denotes another synchronous functional of \mathcal{M} , and where

$$W^{00} = \Pi_0 + \frac{1}{3} n_i \Pi_i, \quad (2.32a)$$

$$W^{0i} = \frac{1}{3} \Pi_i, \quad (2.32b)$$

$$W^{ij} = 0. \quad (2.32c)$$

C. Hereditary terms in the radiative multipole moments

The sum of the right-hand sides of Eqs. (2.19), (2.28), and (2.31) gives the hereditary piece of the leading term in the far wave-zone expansion of the quadratic external metric. In order to extract from this leading term the radiative multipole moments, we still need to transform the (harmonic) coordinates x^α used in the algorithm to some radiative coordinates X^α adapted to a smooth description of the metric structure at null infinity. The proof that such a transformation $X^\alpha = X^\alpha(x^\beta)$ exists was given in paper II. From inspection of Eqs. (2.19) and (2.28), one sees that this transformation at order G^2 can be taken as

$$X^\alpha = x^\alpha + G\xi^\alpha + G^2\lambda^\alpha + O(G^3), \quad (2.33)$$

where

$$\xi^\alpha \equiv -\frac{2M}{c^2} \delta_0^\alpha \ln(r/cP^{\text{rad}}), \quad (2.34)$$

and where λ^α was defined in Eq. (2.17). The quantity P^{rad} in Eq. (2.34) denotes a new arbitrary time scale (*a priori* independent of the time scale $P^{\text{alg}} \equiv r_1/c$ entering the algorithm). Roughly speaking, the ξ^α term serves to correct for the difference between flat and curved cones, while λ^α takes care of the influence of the radiated gravitational energy on harmonic coordinate systems. (As has been well known since the work of Fock [14] both types of terms generate logarithms in the harmonic-coordinate components of the wave-zone metric.) Under the transformation (2.33) the gothic metric components change

from $\sqrt{g_{\text{ext}}}g_{\text{ext}}^{\alpha\beta}(x^\gamma) = f^{\alpha\beta} + Gh_1^{\alpha\beta} + G^2h_2^{\alpha\beta} + O(G^3)$ to

$$\sqrt{G_{\text{ext}}}G_{\text{ext}}^{\alpha\beta}(X^\gamma) = f^{\alpha\beta} + Gh_1^{\prime\alpha\beta}(X^\gamma) + G^2h_2^{\prime\alpha\beta}(X^\gamma) + O(G^3), \quad (2.35)$$

where

$$h_1^{\prime\alpha\beta}(X^\gamma) = \left[h_1^{\alpha\beta} + \partial^\alpha \xi^\beta + \partial^\beta \xi^\alpha - f^{\alpha\beta} \partial_\gamma \xi^\gamma \right]_{x^\alpha = X^\alpha}, \quad (2.36)$$

$$h_2^{\prime\alpha\beta}(X^\gamma) = \left[h_2^{\alpha\beta} + \partial^\alpha \lambda^\beta + \partial^\beta \lambda^\alpha - f^{\alpha\beta} \partial_\gamma \lambda^\gamma - \xi^\mu \partial_\mu h_1^{\alpha\beta} \right]_{x^\alpha = X^\alpha}. \quad (2.37)$$

Using Eqs. (2.13), (2.19), (2.28), and (2.31), one finds indeed that the leading-logarithmic terms disappear from Eq. (2.37) to leave a $1/R$ falloff in the limit $R \rightarrow \infty$ with $T - R/c$ fixed. (The results given above would allow still for the presence of subdominant $\ln R/R^2$ terms, which a more complete treatment [2] proves to be altogether absent in radiative coordinates.) More precisely, one finds for the leading $1/R$ terms in the radiative-coordinates linearized metric

$$h_1^{\prime\alpha\beta}(X^\gamma) = \frac{1}{R} \left[z^{\alpha\beta}(U, \mathbf{N}) - 2M(\delta_0^\alpha K^\beta + \delta_0^\beta K^\alpha) \right] + O\left(\frac{1}{R^2}\right), \quad (2.38)$$

where $U \equiv T - R/c$, $\mathbf{N} \equiv \mathbf{X}/R$, where $z^{\alpha\beta}(U, \mathbf{N})$ denotes the quantities (2.12) with the replacement everywhere of $u \equiv t - r/c$ by $U \equiv T - R/c$ and of $\mathbf{n} = \mathbf{x}/r$ by $\mathbf{N} \equiv \mathbf{X}/R$, and where $K^\alpha \equiv (1, \mathbf{N})$ is the radiative-coordinates Minkowskian outgoing null vector. As for the leading $1/R$ term in the radiative quadratic metric (2.37) it has the form

$$h_2^{\prime\alpha\beta}(X^\gamma) = \frac{1}{R} \left\{ \frac{1}{c} \int_{-\infty}^U dV K^{\alpha\beta}(V, \mathbf{N}) + \frac{2M}{c^3} \int_0^{+\infty} dY \ln \left(\frac{Y}{2P^{\text{rad}}} \right) \frac{d^2}{dU^2} z^{\alpha\beta}(U - Y, \mathbf{N}) + \mathcal{S}_2^{\alpha\beta}(U, \mathbf{N}) \right\} + O\left(\frac{1}{R^2}\right), \quad (2.39)$$

where $\mathcal{S}_2^{\alpha\beta}(U, \mathbf{N})$ is some synchronous quadratic functional of $\mathcal{M}(U)$, and where $K^{\alpha\beta} \equiv [U^{\alpha\beta} + W^{\alpha\beta}]_{x=X}$ is given by

$$K^{00}(U, \mathbf{N}) = \frac{1}{2} \Pi_0 + \frac{1}{3} N_i \Pi_i, \quad (2.40a)$$

$$K^{0i}(U, \mathbf{N}) = \frac{1}{3} \Pi_i + \sum_{\ell \geq 0} \frac{1}{2(\ell+1)} N_{iL} \Pi_L + \sum_{\ell \geq 1} \frac{1}{2(\ell+1)} N_{L-1} \Pi_{L-1}, \quad (2.40b)$$

$$K^{ij}(U, \mathbf{N}) = \sum_{\ell \geq 0} \frac{1}{\ell+2} N_{ijL} \Pi_L + \sum_{\ell \geq 1} \frac{-1}{(\ell+1)(\ell+2)} \delta_{ij} N_L \Pi_L + \sum_{\ell \geq 1} \frac{-(\ell-2)}{(\ell+1)(\ell+2)} N_{L-1(i} \Pi_{j)L-1} + \sum_{\ell \geq 2} \frac{-2}{(\ell+1)(\ell+2)} N_{L-2} \Pi_{ijL-2}. \quad (2.40c)$$

In Eqs. (2.40) $\Pi_L \equiv \Pi_L(U)$ denote as in Eq. (2.21) above the STF spherical harmonics expansion coefficients of the gravitational luminosity $\Pi(U, \mathbf{N})$ [obtained by replacing $(u, \mathbf{n}) \rightarrow (U, \mathbf{N})$ in the definition (2.21)]. Let us note the following algebraic identity satisfied by $K^{\alpha\beta}$:

$$K^{\alpha\beta} K_\beta \equiv 0, \quad (2.41)$$

where $K_\beta = f_{\beta\alpha} K^\alpha = (-1, \mathbf{N})$.

$$I_L^{\text{rad}[\ell]}(U) = M_L^{(\ell)}(U) + \frac{Gc^{\ell+1}\ell!}{2(\ell+1)(\ell+2)} \int_{-\infty}^U dV \Pi_L(V) + \frac{2GM}{c^3} \int_0^{+\infty} dY \ln\left(\frac{Y}{2P^{\text{rad}}}\right) M_L^{(\ell+2)}(U-Y) + GS'_{2L}(U) + O(G^2), \quad (2.42a)$$

$$J_L^{\text{rad}[\ell]}(U) = S_L^{(\ell)}(U) + \frac{2GM}{c^3} \int_0^{+\infty} dY \ln\left(\frac{Y}{2P^{\text{rad}}}\right) S_L^{(\ell+2)}(U-Y) + GS''_{2L}(U) + O(G^2), \quad (2.42b)$$

where S'_{2L} and S''_{2L} are some synchronous quadratic functionals of \mathcal{M} [while the remainders $O(G^2)$ contain at least cubically nonlinear hereditary functionals].

Equations (2.42) constitute one of the central results of the present paper. Before completing it in the next section, it can be useful to comment upon the nature of the various hereditary effects entering Eqs. (2.42).

D. “Memory” versus “tail” effects

A striking feature of the results (2.42) is that they display the presence of two, and only two, different types of hereditary terms entering, at the quadratic approximation, the link between algorithmic moments and radiative ones, $\mathcal{R} = \mathcal{R}[\mathcal{M}]$: namely Π terms and M terms [the latter containing an integral over $\ln(Y/2P^{\text{rad}})$]. Moreover, although both types of contributions are hereditary, they contain quite different weightings of the past activity of the system.

From the results (2.38) and (2.39) it is a simple matter to extract the radiative multipole moments defined by Eq. (1.4) above. (One must keep in mind the minus sign appearing in the relation between the covariant and the gothic metric deviations.) Upon application of the TT projection operator $P_{ijk}(\mathbf{N})$ many terms in Eqs. (2.38), (2.39) drop out, and one gets, for the radiative multipole moments,

Let us first consider the Π terms, i.e., the nonlinear hereditary influence of the emission of gravitational radiation by the system in the past. The kernel entering these contributions [as written in Eq. (2.42a)],

$$\int_{-\infty}^U dV \mathcal{K}_\Pi(U, V) \mathcal{M}_{L_1}^{(p_1)}(V) \mathcal{M}_{L_2}^{(p_2)}(V),$$

is a flat function of $U - V$, more precisely a step function: $\mathcal{K}_\Pi(U, V) \propto \theta(U - V)$. (The convergence of the integral is ensured by the fact that the time derivatives of the moments tend to zero in the remote past.) As a consequence, even after the system has ceased to emit radiation (in the sense that $M_L^{(\ell+1)}$ and $S_L^{(\ell+1)}$ tend to zero at late times), the cumulative effect of the emission of gravitational radiation will produce a constant (DC) contribution to the gravitational-wave amplitude (1.4) (“memory effect”). More explicitly, one reads off Eq. (2.42a):

$$\left[I_L^{\text{rad}[\ell]}(+\infty) - I_L^{\text{rad}[\ell]}(-\infty) \right] = \left[M_L^{(\ell)}(+\infty) - M_L^{(\ell)}(-\infty) \right] + \frac{Gc^{\ell+1}\ell!}{2(\ell+1)(\ell+2)} \int_{-\infty}^{+\infty} dV \Pi_L(V) + O(G^2), \quad (2.43a)$$

$$\left[J_L^{\text{rad}[\ell]}(+\infty) - J_L^{\text{rad}[\ell]}(-\infty) \right] = \left[S_L^{(\ell)}(+\infty) - S_L^{(\ell)}(-\infty) \right] + O(G^2). \quad (2.43b)$$

Equation (2.43a) can be directly expressed [using Eq. I(A.29a)] in terms of the angular distribution $dE^{\text{grav}}(\mathbf{N})/d\Omega \equiv \int_{-\infty}^{+\infty} dU dE^{\text{grav}}/dU d\Omega$ of the total energy radiated by the system:

$$\begin{aligned} I_L^{\text{rad}[\ell]}(+\infty) - I_L^{\text{rad}[\ell]}(-\infty) &= M_L^{(\ell)}(+\infty) - M_L^{(\ell)}(-\infty) \\ &+ \frac{2c^{\ell-2}(2\ell+1)!!}{(\ell+1)(\ell+2)} \int \hat{N}_L \frac{dE^{\text{grav}}}{d\Omega}(\mathbf{N}) d\Omega + O(G^2). \end{aligned} \quad (2.43c)$$

The DC memory effects in Eqs. (2.43) associated with

the differences $M_L^{(\ell)}(+\infty) - M_L^{(\ell)}(-\infty)$ and $S_L^{(\ell)}(+\infty) - S_L^{(\ell)}(-\infty)$ have been discussed by [17] and [18]. These differences generically do not vanish if the source contains free moving masses in its initial or final state (scattering situation). (In setting up our formalism we restricted our attention to systems that become stationary in the remote past; we shall admit here that we can extend the applicability of the results of our formalism to cover scattering situations.) The DC memory associated with the Π terms was first noticed, in a particular context, by Payne [19] (see also [20]). It was included in the results of [15], where it was noted to be (formally) of higher order in the slow-motion parameter v^{source}/c than the other hereditary contributions in (2.42) (see below). Recently,

this effect has been rediscovered in a general, and rigorous, guise [21] (see also [22]). In Refs. [23, 24] the connection of the result of Ref. [21] with the effect of the past emission of gravitational radiation was made explicit, and Ref. [24] derived (independently of the work [15] that had gone unnoticed) formulas equivalent to Eqs. (2.43). Note however that the practical interest of the pure DC effects (2.43) is rather small because gravitational-wave detectors have a limited frequency bandwidth, $[f_{\min}, f_{\max}]$, especially on the low-frequency side, i.e., $f_{\min} > 0$. Therefore, in order to assess the possible physical relevance of (2.43), one needs to complete it by including all hereditary effects having a heredity time scale comparable to f_{\min}^{-1} . The answer to the latter question is given, at the quadratic approximation, by Eqs. (2.42).

In addition to the “memory” hereditary effects coming from the Π terms (which comprise both DC and AC con-

tributions) Eqs. (2.42) contain also the contributions of the backscattering of the gravitational waves emitted in the past onto the constant curvature associated with the total mass M . These contributions, i.e., the M terms in Eqs. (2.42), are often referred to as “gravitational-wave tails,” and have been studied in various contexts (notably in Ref. [25] which is closest in spirit to our approach). However, we are not aware of results as explicit as ours, notably Eqs. (2.42) and its completions given below. In the form in which they are written in Eqs. (2.42), these M terms seem to contain the kernel $\mathcal{K}_M^{\log}(U-V) \propto \ln(U-V)\theta(U-V)$, exhibiting a logarithmic blowup for large time intervals. However, this logarithmic behavior, though mathematically correct (within the assumptions of our framework) is, from the physical point of view, slightly misleading. Indeed, let us integrate twice by parts the M terms:

$$\int_{-\infty}^U dV \ln\left(\frac{U-V}{2P^{\text{rad}}}\right) M_L^{(\ell+2)}(V) = \frac{1}{2P^{\text{rad}}} M_L^{(\ell)}(U - 2P^{\text{rad}}) + \int_{U-2P^{\text{rad}}}^U dV \ln\left(\frac{U-V}{2P^{\text{rad}}}\right) M_L^{(\ell+2)}(V) - \int_{-\infty}^{U-2P^{\text{rad}}} \frac{dV}{(U-V)^2} M_L^{(\ell)}(V). \quad (2.44)$$

The new form (2.44) shows that the influence of the remote-past activity of the source enters radiative moments via a quadratically decreasing kernel $\mathcal{K}_M^{\text{quad}}(U-V) \propto (U-V)^{-2}$. Therefore, in a scattering situation, where $M_L^{(\ell)}(V)$ is expected to have a finite, nonzero, limit as $V \rightarrow -\infty$ [see the expressions below relating $M_L(V)$ to the matter distribution] the remote past history of the system gives a “tail” contribution to the radiative moments which falls off only as the inverse of the time span between now and the considered period in the past. Strictly speaking, the M terms give no DC contributions to Eqs. (2.43). However, for what concerns realistic, band-limited detectors of gravitational waves the rather slow falloff of the kernel $\mathcal{K}_M^{\text{quad}}$ indicates that the M terms might contribute, for some sources, numerically important hereditary effects.

III. GENERATION OF GRAVITATIONAL WAVES, INCLUDING TAIL EFFECTS

A. Relation between the radiative and algorithmic quadrupole moments for slow-motion systems

In the previous section, we investigated the relation between the radiative moments \mathcal{R} and the algorithmic ones \mathcal{M} in the form of a nonlinearity expansion, Eq. (2.1). Our final result (2.42) solved the problem of getting the first nonlinear terms in the relation $\mathcal{R} = \mathcal{R}[\mathcal{M}]$ that depended upon the past history of \mathcal{M} . These hereditary terms $\mathcal{H}_2[\mathcal{M}]$ appear at the quadratic order, and at the same order there are other “synchronous” terms, \mathcal{S}'_{2L} and \mathcal{S}''_{2L} in Eqs. (2.42), that were left undetermined by the investigation of the previous section.

In the present section we shall consider slow-motion radiating systems, for which it is meaningful to order the terms in the relation $\mathcal{R} = \mathcal{R}[\mathcal{M}]$ according to the powers of $1/c$ (slow-motion expansion), instead of those of G (nonlinearity expansion). (Physically the small parameter of the slow-motion expansion is $v/c \sim r_0/\lambda$ where v is a characteristic internal velocity of the source, and r_0/λ the ratio between the size of the source and the characteristic wavelength of the emitted radiation.) First, let us recall from Eq. (1.4) that the contributions of the successive radiative multipoles to the directly observable gravitational wave form h_{ij}^{TT} are of decreasing slow-motion order:

$$h_{ij}^{\text{TT}} \sim \sum_{\ell \geq 2} \left\{ \frac{1}{c^{\ell+2}} I_L^{\text{rad}[\ell]} + \frac{1}{c^{\ell+3}} J_L^{\text{rad}[\ell]} \right\}. \quad (3.1)$$

In view of Eq. (3.1), we shall consider with particular attention the leading contribution to the wave form, i.e., the electriclike radiative quadrupole $I_{ij}^{\text{rad}[2]}$. The slow-motion expansion of the relation $\mathcal{R} = \mathcal{R}[\mathcal{M}]$ is obtained by putting back the needed powers of $1/c$ in Eqs. (2.2). From the fact that the kernel $\mathcal{K}_{L\mathcal{L}_1 \dots \mathcal{L}_n}$ contains only quantities having the dimension of time, dimensional analysis (see paper IV) shows that the n th-order nonlinearities contribute to $I_L^{\text{rad}[\ell]}$ and $\varepsilon_{ai\ell i\ell-1} J_{aL-2}^{\text{rad}[\ell-1]}$ terms of order

$$\frac{G^{n-1}}{c^{3(n-1)+\Sigma \ell_i - \ell}} \int_{-\infty}^u \dots \int_{-\infty}^u du_1 \dots du_n \times \mathcal{K}_{L\mathcal{L}_1 \dots \mathcal{L}_n}(u, u_1, \dots, u_n, P^{\text{alg}}, P^{\text{rad}}) \times \mathcal{M}_{\mathcal{L}_1}^{(p_1)}(u_1) \dots \mathcal{M}_{\mathcal{L}_n}^{(p_n)}(u_n). \quad (3.2)$$

Moreover, the knowledge that the index structure of

$\mathcal{K}_{L_1 \dots L_n}$ is made only of Kronecker δ 's is easily seen to imply the equality

$$\ell = \sum_{i=1}^n \ell_i - 2k, \quad (3.3)$$

where the natural integer k denotes the number of contractions among the indices born by the \mathcal{M}_{L_i} 's. Therefore, the n th-order nonlinearities contribute terms of order $O(1/c^{3(n-1)+2k})$. In particular, quadratic nonlinearities contribute terms of order $O(1/c^{3+2k})$ while cubic and higher nonlinearities are at least $O(1/c^6)$. Now, as all the

$$I_{ij}^{\text{rad}[2]}(U) = M_{ij}^{(2)}(U) + \frac{2GM}{c^3} \int_0^{+\infty} dY M_{ij}^{(4)}(U-Y) \left[\ln \left(\frac{Y}{2P_{\text{rad}}} \right) + K' \right] + O \left(\frac{1}{c^5} \right), \quad (3.4a)$$

$$J_{ij}^{\text{rad}[2]}(U) = S_{ij}^{(2)}(U) + \frac{2GM}{c^3} \int_0^{+\infty} dY S_{ij}^{(4)}(U-Y) \left[\ln \left(\frac{Y}{2P_{\text{rad}}} \right) + K'' \right] + O \left(\frac{1}{c^5} \right), \quad (3.4b)$$

where K' and K'' are two numerical constants. The latter constants denote the coefficients appearing in the most general $O(1/c^3)$ quadratically nonlinear synchronous quadrupolar terms S'_{2ij} and S''_{2ij} in Eqs. (2.42). (Dimensional analysis shows that the only other possible synchronous term would involve at least one derivative of the spin moment S_i , which vanishes as a consequence of the field equations, or of the mass dipole M_i , which we can put to zero by working in the center-of-mass frame of the ingoing system.)

To compute the constants K' and K'' we need to implement in detail the algorithmic construction of the part of the external metric which is generated by the nonlinear interplay between the monopole (M) and the quadrupole moments M_{ij} and S_{ij} . This implementation is explicitly done in Appendix B for the mass quadrupole M_{ij} . The final result for the constant K' appearing in Eq. (3.4a) is

$$K' = \frac{11}{12}. \quad (3.5)$$

B. Relation between the gravitational-wave form and the matter distribution for slow-motion, weakly self-gravitating systems

Equations (3.4) solve, within the indicated accuracy, the second step of our formalism, i.e., that symbolized by Eq. (1.5) of the Introduction. As we emphasized there, this result must be completed by a source-rooted approach providing the missing link between the algorithmic moments and the structure and motion of the source,

moments entering the angular distribution coefficients, Π_L , of the gravitational energy flux are of order $\ell_i \geq 2$, we see that the Π terms in Eq. (2.42a) can contribute at order $1/c^3$ only for radiative multipoles of order $\ell \geq 4$. Conversely, we see that the Π terms contribute only at order $O(1/c^5)$ to the quadrupole and octupole radiative moments.

In the case of the electriclike and magneticlike quadrupole moments $I_{ij}^{\text{rad}[2]}$ and $J_{ij}^{\text{rad}[2]}$, one concludes from the arguments above that their slow-motion expansion up to order $O(1/c^5)$ takes the form

as symbolized in Eq. (1.7). In the case of slow-motion systems having an everywhere weak self-gravitational field we have already provided this link with the fractional accuracy $O(1/c^4)$. Namely, we have shown in paper IV that

$$M_{ij}(t) = I_{ij}(t) + O(1/c^4), \quad (3.6)$$

where $I_{ij}(t)$ is the post-Newtonian “source” mass quadrupole

$$I_{ij}(t) = \int d^3\mathbf{x} \hat{x}_{ij} \sigma(\mathbf{x}, t) + \frac{1}{14c^2} \frac{d^2}{dt^2} \int d^3\mathbf{x} \hat{x}_{ij} \mathbf{x}^2 \sigma(\mathbf{x}, t) - \frac{20}{21c^2} \frac{d}{dt} \int d^3\mathbf{x} \hat{x}_{ijk} \sigma_k(\mathbf{x}, t), \quad (3.7)$$

involving an effective active gravitational mass density

$$\sigma = c^{-2} (T^{00} + T^{ss}), \quad (3.8)$$

and an effective active current density

$$\sigma_i = c^{-1} T^{0i}. \quad (3.9)$$

In these equations \hat{x}_{ij} and \hat{x}_{ijk} denote the STF parts of $x^i x^j$ and $x^i x^j x^k$, respectively (e.g., $\hat{x}_{ij} \equiv x^i x^j - \mathbf{x}^2 \delta^{ij}/3$), and $T^{\mu\nu}$ denote the contravariant components of the stress-energy tensor of the source in a harmonic coordinate system (see paper IV).

Combining Eqs. (3.4)–(3.7) we get the following explicit link between the radiative electriclike quadrupole and the source:

$$I_{ij}^{\text{rad}[2]}(U) = I_{ij}^{(2)}(U) + \frac{2GM}{c^3} \int_0^{+\infty} dY I_{ij}^{(4)}(U-Y) \left[\ln \left(\frac{Y}{2P_{\text{rad}}} \right) + \frac{11}{12} \right] + O \left(\frac{1}{c^4} \right). \quad (3.10)$$

Note that, if we consider sources that were so “quiet” in the infinite past that the second derivative of $I_{ij}(t)$ was tending to zero strictly faster than t^{-1} [say like $(-t)^{-1-\epsilon}$ when $t \rightarrow -\infty$], we can integrate twice the right-hand side of Eq. (3.10), and thereby give a meaning to a quan-

tity I_{ij}^{rad} (having the dimension of a mass quadrupole) such that $I_{ij}^{\text{rad}[2]} = I_{ij}^{\text{rad}(2)}$. However, it is physically better to work only with the (time differentiated) radiative moment $I_{ij}^{\text{rad}[2]}$, because, on the one hand, it is directly

linked with the observable wave form h_{ij}^{TT} , and on the other hand, there are many interesting physical situations where $I_{ij}(t)$ is not expected to tend to zero in the infinite past (in particular, scattering situations). In such cases, the impossibility to define an undifferentiated I_{ij}^{rad} is probably linked to a loss of differentiability of the conformal structure at future null infinity. More precisely, a related calculation of Damour [26] indicates that in a scattering situation the curvature component Ψ_0 (in the notation of Newman and Penrose) peels only as $1/r^4$ instead of the usually assumed $1/r^5$ behavior connected

$$J_{ij}^{\text{rad}[2]}(U) = J_{ij}^{(2)}(U) + \frac{2GM}{c^3} \int_0^\infty dY J_{ij}^{(4)}(U - Y) \left[\ln \left(\frac{Y}{2P^{\text{rad}}} \right) + K'' \right] + O \left(\frac{1}{c^4} \right), \quad (3.11)$$

where K'' is the (yet uncalculated) constant appearing in Eq. (3.4b), and where J_{ij} is a post-Newtonian “source” spin quadrupole which has been explicitly given in paper V as a compact-support integral involving only the stress-energy tensor of the matter:

$$J_{ij}(t) = J_{ij}[T^{\mu\nu}] \quad (3.12)$$

(see Sec. VB of paper V).

Actually the truncation of the result (3.11) at order $1/c^3$ is sufficient, when combined with the c^{-4} -accurate result (3.10) and the c^{-3} -accurate results for the higher multipoles,

$$I_L^{\text{rad}[\ell]} = M_L^{(\ell)} + O(1/c^3) = I_L^{(\ell)} + O(1/c^3), \quad (3.13a)$$

$$J_L^{\text{rad}[\ell]} = S_L^{(\ell)} + O(1/c^3) = J_L^{(\ell)} + O(1/c^3) \quad (3.13b)$$

(where the post-Newtonian-accurate source multipoles $I_L[T^{\mu\nu}]$ and $J_L[T^{\mu\nu}]$ were obtained in papers IV and V, respectively), to obtain the link between the gravitational-wave form and the structure and motion of the source within the fractional accuracy $O(1/c^4)$:

$$h_{ij}^{\text{TT}}(T, \mathbf{X}) = \frac{G}{c^4 R} \left[\mathcal{H}_{ij}[T^{\mu\nu}] + O \left(\frac{1}{c^4} \right) \right] + O \left(\frac{1}{R^2} \right). \quad (3.14)$$

The hereditary functional $\mathcal{H}_{ij}[T^{\mu\nu}]$ is obtained by first inserting the results of papers IV and V in Eqs. (3.10)–(3.13), and then inserting the latter equations in the radiative multipole expansion (1.4). Up to the fractional accuracy $O(1/c^4)$ there is only one hereditary contribution in the outgoing wave form, namely the “tail” appearing in Eq. (3.10), caused by the scattering of a linearized quadrupolar wave off the curvature of spacetime associated with the ADM mass M . Note that at the next order $1/c^4$ the wave form will contain two independent hereditary contributions: the tail term in the radiative spin quadrupole, Eq. (3.11), as well as a tail term in the radiative mass octupole, as is clear from Eq. (2.42a).

The explicit appearance of the (arbitrary) time scale P^{rad} in Eq. (3.10), as well as in the final result (3.14) is to be noted. This time scale was introduced in Eq. (2.34) as part of the definition of the transformation between the algorithmic coordinate system x^μ (linked to the source

with a C^3 differentiability of the conformal structure at future null infinity. Such calculations exhibiting an explicit loss of peeling behavior may shed a light on the physical meaning of the nonpeeling estimates rigorously derived by Christodoulou and Klainerman [27].

Continuing our study of tail effects, we find by combining the result (3.4b) above with the results of paper V about the analogue of Eq. (3.6) for the spin moments, that we can derive the following link between the radiative magneticlike quadrupole and the source:

by the results of papers IV and V) and the radiative coordinate system X^μ . We see from Eq. (2.34) that a multiplicative change $P^{\text{rad}} \rightarrow \lambda P^{\text{rad}}$ induces an additive shift in $X^0 \equiv cT$ (for some fixed x^μ), and thereby an additive shift in $U \equiv T - R/c$, namely

$$U_{\lambda P^{\text{rad}}}(x^\mu) = U_{P^{\text{rad}}}(x^\mu) + 2(\ln \lambda) GM/c^3. \quad (3.15)$$

It is easy to see that the effect of the shift (3.15) in the first term on the right-hand side of Eq. (3.10) is exactly compensated by the λ -dependent contribution due to the integral on the right-hand side:

$$\frac{2GM}{c^3} \int_0^\infty dY I_{ij}^{(4)}(U - Y) \ln(1/\lambda), \quad (3.16)$$

thereby ensuring that $I_{ij}^{\text{rad}[2]}[U(x^\mu)]$ and $h_{ij}^{\text{TT}}(x^\mu)$, where we recall that x^μ are source-rooted coordinates, are independent of the choice of P^{rad} (within the accuracies with which they have been derived). Thereby we see that the numerical value 11/12 of the coefficient K' computed in Eq. (3.5) has an intrinsic meaning, independent of conventional choices.

One can note that the computation of Eq. (3.14), with the indicated accuracy, necessitates the knowledge of the evolution of the material source with post-Newtonian accuracy [i.e., neglecting only terms $= O(1/c^4)$]. We see therefore, that one can now boost up the precision of the evolution-and-generation scheme of Ref. [28] by including in the computation of the outgoing wave form the fractional $O(1/c^3)$ terms displayed in Eq. (3.10).

C. Relation between the gravitational-wave form and the source for systems of well-separated strongly self-gravitating bodies

The results of the previous subsection have been derived for material sources that contain everywhere weak gravitational fields. *A priori*, this excludes the application of our final results to the very interesting sources which consist of systems of strongly self-gravitating bodies, say an in-spiraling binary neutron star. However, because of the “modular” structure of our formalism (which consists of three separate steps), we can extend the application of our results to such a case. Indeed, the essence

of the third step of our formalism is to identify (modulo a coordinate transformation) the near-zone expansion of the (algorithmically constructed) external metric (1.2) with the expression taken in the external near-zone by the metric generated by the matter, as given by some source-rooted, post-Newtonian-type, approximation method. See, in particular, Eq. (4.17) of paper V, whose left-hand side is the near-zone expansion of the algorithmic metric, while its right-hand side comes from solving, in the near-zone, the inhomogeneous field equations

$$\square h_{in}^{\alpha\beta} = \frac{16\pi G}{c^4} \bar{T}^{\alpha\beta} + N_2^{\alpha\beta}(h) + O(6, 7, 6). \quad (3.17)$$

In Eq. (3.17) $\bar{T}^{\alpha\beta}$ denotes the contravariant tensor density of weight +2 representing the stress-energy distribution of the matter, $\bar{T}^{\alpha\beta} \equiv gT^{\alpha\beta}$ where $g \equiv -\det(g_{\alpha\beta})$, $N^{\alpha\beta}$ denotes as above the quadratically nonlinear terms in the harmonically reduced Einstein equations (written in terms of $h^{\alpha\beta} \equiv g^{\alpha\beta} - f^{\alpha\beta}$), and the symbol $O(6, 7, 6)$ means that the allowed error terms are $O(c^{-6})$ in h^{00} , $O(c^{-7})$ in h^{0i} and $O(c^{-6})$ again in h^{ij} .

At this point, we can make use of the fact that Damour [29] has shown that the metric generated by a system of well-separated strongly self-gravitating bodies could be obtained, everywhere outside the bodies, by iteratively solving inhomogeneous equations of the form (3.17), with an effective $\bar{T}^{\alpha\beta}$ of the form

$$\begin{aligned} \bar{T}_\varepsilon^{\alpha\beta} = \sum_A m_A \int ds_A Z_\varepsilon[x^\mu - z_A^\mu(s_A)] \\ \times u_A^\alpha u_A^\beta [\mathcal{G}(z_A)]^{1/4} [\mathcal{G}_{\mu\nu}(z_A) u_A^\mu u_A^\nu]^{-1/2}. \end{aligned} \quad (3.18)$$

In Eq. (3.18) $A = 1, \dots, N$ labels the N compact bodies, m_A denotes the (constant) Schwarzschild mass of the A th compact body (defined when considering an isolated body), $z_A^\mu(s)$ denotes some ‘‘center of field’’ world line associated with the A th body (with $u_A^\mu \equiv dz_A^\mu/ds_A$ and $f_{\mu\nu} u_A^\mu u_A^\nu = -1$), \mathcal{G} denotes minus the determinant of $\mathcal{G}^{\alpha\beta} = g^{1/2} g^{\alpha\beta}$, $\mathcal{G}_{\alpha\beta}$ denotes the matrix inverse of $\mathcal{G}^{\alpha\beta}$ and $Z_\varepsilon(x)$ is the function

$$Z_\varepsilon(x) \equiv H_4^{-1}(\varepsilon) (-f_{\alpha\beta} x^\alpha x^\beta)^{(\varepsilon-4)/2}, \quad (3.19a)$$

where

$$H_4(\varepsilon) = \pi 2^{\varepsilon-1} \Gamma\left(\frac{\varepsilon}{2}\right) \Gamma\left(\frac{\varepsilon-2}{2}\right). \quad (3.19b)$$

In those equations ε denotes a complex number, which is taken as different from zero to be able to solve Eq. (3.17) by iteration [including nonlinear terms and the values of $\mathcal{G}^{\alpha\beta}$ on the world lines, defined as formal power series in $h^{\alpha\beta}(z)$]. After the computation of the iteration, one must analytically continue ε down to zero (Ref. [29] showed that this process was mathematically well defined (no poles at $\varepsilon = 0$), and gave the physically unique metric outside N compact bodies). Thanks to this result, we can apply the method of papers IV and V to systems of compact bodies. (For simplicity, we have considered

nonspinning bodies in Eq. (3.18); the method is easily extended to slowly spinning bodies, in which case it is sufficient to add further spin contributions to Eq. (3.18), as indicated in Ref. [30]). We conclude that the algorithmic mass and spin moments of the metric outside a system of N compact bodies are given by

$$M_L = AC_{\varepsilon=0} \left\{ I_L[\bar{T}_\varepsilon^{\alpha\beta}] \right\} + O\left(\frac{1}{c^4}\right), \quad (3.20a)$$

$$S_L = AC_{\varepsilon=0} \left\{ J_L[\bar{T}_\varepsilon^{\alpha\beta}] \right\} + O\left(\frac{1}{c^4}\right), \quad (3.20b)$$

where $AC_{\varepsilon=0}$ stands for ‘‘analytic continuation at $\varepsilon = 0$,’’ and where the functionals $I_L[\bar{T}^{\mu\nu}]$, $J_L[\bar{T}^{\mu\nu}]$ are those defined by Eqs. V (5.11) [with Eqs. V(5.8), in which $(1 + 4U^{in}/c^2)T^{\mu\nu}$ must be everywhere replaced by $\bar{T}^{\mu\nu}$]. At the order at which we are working [defined by the error terms in Eq. (3.17)], it is a simple matter to work with the analytically extended effective source terms (3.18). Essentially, one finds that the analytic continuation in ε is a way to bypass all the ill-defined quantities that would arise if one was working with a formal ‘‘point-particle’’ stress-energy tensor, containing δ functions. More precisely, the use of the Riesz function Z_ε of Eqs. (3.19) is equivalent to replacing the usual δ distributions by the functions

$$\delta_\varepsilon(\mathbf{x}) = -\frac{1}{4\pi} \Delta(r^{\varepsilon-1}) = \frac{1}{4\pi} \varepsilon(1-\varepsilon)r^{\varepsilon-3}, \quad (3.21)$$

where $r \equiv |\mathbf{x}|$. With this notation, one can insert in Eqs. (3.20) the expressions of paper V:

$$\frac{1}{c^2} \bar{T}_\varepsilon^{00} = \sum_A m_A \left(1 + \frac{3}{c^2} U_A^\varepsilon + \frac{1}{2c^2} \mathbf{v}_A^2 \right) \delta_\varepsilon(\mathbf{x} - \mathbf{z}_A), \quad (3.22a)$$

$$\frac{1}{c} \bar{T}_\varepsilon^{0i} = \sum_A m_A v_A^i \left(1 + \frac{3}{c^2} U_A^\varepsilon + \frac{1}{2c^2} \mathbf{v}_A^2 \right) \delta_\varepsilon(\mathbf{x} - \mathbf{z}_A), \quad (3.22b)$$

$$\bar{T}_\varepsilon^{ij} = \sum_A m_A v_A^i v_A^j \left(1 + \frac{3}{c^2} U_A^\varepsilon + \frac{1}{2c^2} \mathbf{v}_A^2 \right) \delta_\varepsilon(\mathbf{x} - \mathbf{z}_A), \quad (3.22c)$$

where $v_A^i \equiv dz_A^i/dt$ and

$$U_A^\varepsilon = G \sum_{B \neq A} m_B |\mathbf{z}_A - \mathbf{z}_B|^{\varepsilon-1}. \quad (3.22d)$$

When ε is continued down to zero, $\delta_\varepsilon(\mathbf{x})$ tends (in the sense of distribution theory) towards $\delta(\mathbf{x})$, but this limit process works also for the nonlinear contributions in I_L and J_L , in which the direct use of δ functions would lead to undefined expressions. Finally, one finds that the mass multipole moments are explicitly given by

$$M_L(t) = I_L(t) + O(1/c^4), \quad (3.23)$$

where

$$\begin{aligned}
I_L(t) = & \sum_A m_A \left(1 - \frac{1}{c^2} U_A + \frac{3}{2c^2} \mathbf{v}_A^2 \right) \hat{z}_A^L \\
& + \frac{1}{2(2\ell+3)c^2} \frac{d^2}{dt^2} \sum_A m_A \mathbf{z}_A^2 \hat{z}_A^L \\
& - \frac{4(2\ell+1)}{(\ell+1)(2\ell+3)c^2} \frac{d}{dt} \sum_A m_A v_A^i \hat{z}_A^{iL}, \quad (3.24)
\end{aligned}$$

with

$$U_A = G \sum_{B \neq A} \frac{m_B}{|\mathbf{z}_A - \mathbf{z}_B|}. \quad (3.25)$$

In Eq. (3.24) \hat{z}_A^L denotes as usual the STF projection of $z_A^{i_1} \cdots z_A^{i_\ell}$. When $\ell = 2$ the result (3.24) agrees with the expression derived in the Appendix of [31].

Similarly, but with more work, one can write down explicit expressions for the spin moments:

$$S_L(t) = J_L(t) + O(1/c^4), \quad (3.26)$$

where J_L is a rather complicated expression which can be straightforwardly obtained from the results of paper V. In particular, Appendix C of paper V gives the fully explicit expression of the post-Newtonian spin quadrupole J_{ij} [Eq. V (C.5)].

Having obtained the explicit link between the algorithmic moments and the source we can straightforwardly use, as in the previous subsection, the other relations (3.4) and (3.13) to work out the analogue of Eq. (3.14) in the case of the generation of gravitational waves by systems of compact bodies.

D. Energy balance between the near-zone tail effects and the wave-zone ones

In paper III, we computed the leading terms in the *near-zone* metric which depended, in an irreducible manner, on the full past history of the system. We found that this dominant near-zone hereditary contribution had the physical effect of modifying the expression of the gravitational radiation damping force. More precisely, the lowest-order radiation reaction potential [32, 33], $V_R = x^a x^b Q_{ab}^{(5)}(t)/5c^5$, where $Q_{ab}(t)$ is any Newtonian-order quadrupole moment [e.g., $Q_{ab} = \int d^3x \sigma \hat{x}_{ab} + O(c^{-2}) = I_{ab} + O(c^{-2})$] was found to be modified by the addition

of the hereditary contribution

$$\delta Q_{ab}^{\text{hereditary}} V_R = \frac{1}{5c^5} x^a x^b \frac{d^3}{dt^3} \delta Q_{ab}^{[2]}(t), \quad (3.27)$$

where

$$\delta Q_{ab}^{[2]}(t) = \frac{4GM}{c^3} \int_0^{+\infty} dv \ln \left(\frac{v}{2P^{\text{rad}}} \right) Q_{ab}^{(4)}(t-v). \quad (3.28)$$

Note that, in paper III, we had used the time scale P^{alg} instead of P^{rad} to adimensionalize the integration time variable v in Eq. (3.28). This change of time scale introduces only synchronous terms in Eq. (3.28) and therefore leaves invariant the result (3.27). Clearly the term (3.28) must be related to the hereditary contributions we derived in the far-wave-zone metric, see Eq. (3.10). There is a useful way of looking at the relation between the two results, which consists of studying the energy balance between the energy extracted from the system by the hereditary-modified gravitational damping force derived from (3.27), and the energy lost at infinity in the form of hereditary-modified gravitational waves. More precisely, we already remarked in paper III [Eqs. III (7.20) and III (7.21)] that the irreversible energy losses in the near-zone caused by the modification (3.27) had the form

$$-\frac{d}{dt} E^{\text{source}} \Big|_{\text{leading hered}} = \frac{1}{5c^5} \left(\frac{d}{dt} \left[Q_{ab}^{(2)} + \frac{1}{2} \delta Q_{ab}^{[2]} \right] \right)^2 \quad (3.29)$$

[here and in the following, our calculations correctly include only the leading synchronous and hereditary contributions but neglect any other term, e.g., a $O(1/c^2)$ synchronous modification of the Newtonian quadrupole Q_{ab}].

On the other hand, we can write down the wave-zone losses computed from the Bondi formula applied to the tail-modified wave form (3.10). The multipolar expansion of

$$-\frac{dE^{\text{Bondi}}}{dU} = \frac{c^3}{32\pi G} \int \left(\frac{\partial h_{ij}^{\text{TT}}}{\partial U} \right)^2 R^2 d\Omega(\mathbf{N}), \quad (3.30a)$$

reads

$$-\frac{dE^{\text{Bondi}}}{dU} = \sum_{\ell \geq 2} \frac{(\ell+1)(\ell+2)}{(\ell-1)\ell!(2\ell+1)!!} \frac{G}{c^{2\ell+1}} \left(\frac{d}{dU} I_L^{\text{rad}[\ell]} \right)^2 + \sum_{\ell \geq 2} \frac{4\ell(\ell+2)}{(\ell-1)(\ell+1)!(2\ell+1)!!} \frac{G}{c^{2\ell+3}} \left(\frac{d}{dU} J_L^{\text{rad}[\ell]} \right)^2. \quad (3.30b)$$

From the results of the present paper we deduce that the leading contributions to the wave-zone losses (3.30) containing an influence of the past history of the source are simply

$$-\frac{dE^{\text{Bondi}}}{dU} \Big|_{\text{leading hered}} = \frac{1}{5c^5} \left(\frac{d}{dU} I_{ij}^{\text{rad}[2]} \right)^2 = \frac{1}{5c^5} \left(\frac{d}{dU} \left\{ I_{ij}^{(2)}(U) + \frac{2GM}{c^3} \int_0^\infty dY I_{ij}^{(4)}(U-Y) \left[\ln \left(\frac{Y}{2P^{\text{rad}}} \right) + \frac{11}{12} \right] \right\} \right)^2. \quad (3.31)$$

Comparing Eqs. (3.29) and (3.31), we see that the two results are nicely consistent (within the accuracy with which they are derived) with the expectation that there should be an energy balance between the energy extracted from the source and the radiation losses at infinity. In particular, one should note the role of the factor $\frac{1}{2}$ in Eq. (3.29) which ensures consistency between the result (3.28) (containing a prefactor $4GM/c^3$) and our wave-tail expression (3.10) (which contained a factor $2GM/c^3$).

APPENDIX A: LEGENDRE FORM OF RETARDED INTEGRALS

The aim of this appendix is to prove that the following expression of the retarded integral of a source behaving like r^{-2} ,

$$\square_R^{-1} \left[\frac{\hat{n}_L}{r^2} G \left(t - \frac{r}{c} \right) \right] = \frac{(-)^\ell}{2} \int_r^{+\infty} dz G \left(t - \frac{z}{c} \right) \times \hat{\partial}_L \varphi_\ell(r, z), \quad (\text{A1})$$

where we denote

$$\varphi_\ell(r, z) = \frac{(z-r)^\ell \ln(z-r) - (z+r)^\ell \ln(z+r)}{\ell! r}, \quad (\text{A2})$$

can be equivalently rewritten in the form

$$\square_R^{-1} \left(\frac{\hat{n}_L}{r^2} G \left(t - \frac{r}{c} \right) \right) = -\frac{\hat{n}_L}{r} \int_r^{+\infty} dz G \left(t - \frac{z}{c} \right) \times Q_\ell \left(\frac{z}{r} \right), \quad (\text{A3})$$

where $Q_\ell(x)$ denotes the Legendre function of the second kind. We shall use the following integral representation of the Legendre function of the second kind:

$$Q_\ell(x) = 2^\ell \int_x^{+\infty} \frac{(t-x)^\ell}{(t^2-1)^{\ell+1}} dt, \quad (\text{A4})$$

valid when $x > 1$ (see Ref. [16] page 318). [Another form of Q_ℓ is given by Eq. (2.23).]

We thus want to prove

$$\hat{\partial}_L \varphi_\ell(r, z) = \frac{2(-)^{\ell+1}}{r} \hat{n}_L Q_\ell \left(\frac{z}{r} \right), \quad (\text{A5})$$

or, using the form I(A30) of the operator $\hat{\partial}_L$,

$$\left(\frac{1}{r} \frac{\partial}{\partial r} \right)^\ell \varphi_\ell(r, z) = \frac{2}{(-r)^{\ell+1}} Q_\ell \left(\frac{z}{r} \right). \quad (\text{A6})$$

Let us first consider the partial derivatives with respect to z of the function $\varphi_\ell(r, z)$. From (A2) we find, after $\ell+1$ partial differentiations,

$$\left(\frac{\partial}{\partial z} \right)^{\ell+1} \varphi_\ell(r, z) = \frac{2}{z^2 - r^2}. \quad (\text{A7})$$

Applying to both sides of (A7) the operator $(r^{-1} \partial / \partial r)^\ell$ yields

$$\left(\frac{\partial}{\partial z} \right)^{\ell+1} \left(\frac{1}{r} \frac{\partial}{\partial r} \right)^\ell \varphi_\ell(r, z) = \frac{2^{\ell+1} \ell!}{(z^2 - r^2)^{\ell+1}}. \quad (\text{A8})$$

Finally, by integrating both sides of (A8) $\ell+1$ times with respect to z we get

$$\left(\frac{1}{r} \frac{\partial}{\partial r} \right)^\ell \varphi_\ell(r, z) = \left(\frac{-2}{r} \right)^{\ell+1} \int_{z/r}^{+\infty} \frac{(t - z/r)^\ell}{(t^2 - 1)^{\ell+1}} dt. \quad (\text{A9})$$

Note that in writing (A9) one has used the fact that the LHS of (A9) behaves like $z^{-\ell-1}$ when $z \rightarrow \infty$.

The latter formula (A9) is, thanks to the integral representation (A4) of the Legendre function, precisely the formula (A6) we needed to prove.

APPENDIX B: MONOPOLE-QUADRUPOLE QUADRATIC METRIC

The case of the interaction between the mass monopole M and the mass quadrupole $M_{ij}(t)$ is the simplest case of nonstationary quadratically nonlinear metric. We present here the computation of this case, using the algorithm of paper I, from which we shall deduce the relation linking the radiative quadrupole to the algorithmic quadrupole.

The linearized metric composed with M and M_{ij} is, from Eqs. (2.5),

$$h_1^{00} = -\frac{4M}{c^2 r} - \frac{2}{c^2} \partial_{ab} \left[r^{-1} M_{ab} \left(t - \frac{r}{c} \right) \right], \quad (\text{B1a})$$

$$h_1^{0i} = \frac{2}{c^3} \partial_a \left[r^{-1} M_{ai}^{(1)} \left(t - \frac{r}{c} \right) \right], \quad (\text{B1b})$$

$$h_1^{ij} = -\frac{2}{c^4 r} M_{ij}^{(2)} \left(t - \frac{r}{c} \right). \quad (\text{B1c})$$

We insert this metric into the effective quadratic source $N_2^{\alpha\beta}$ [Eq. III (3.5)] and discard all terms which are not of the type " $M \times M_{ij}$ ". We find

$$N_2^{00} = \frac{\hat{n}_{ab}}{c^4 r^6} M \left\{ -126 M_{ab} - 126 \left(\frac{r}{c} \right) M_{ab}^{(1)} - 112 \left(\frac{r}{c} \right)^2 M_{ab}^{(2)} - 46 \left(\frac{r}{c} \right)^3 M_{ab}^{(3)} - 8 \left(\frac{r}{c} \right)^4 M_{ab}^{(4)} \right\} \left(t - \frac{r}{c} \right), \quad (\text{B2a})$$

$$N_2^{0i} = \frac{\hat{n}_{iab}}{c^5 r^5} M \left\{ 6 M_{ab}^{(1)} + 6 \left(\frac{r}{c} \right) M_{ab}^{(2)} + 2 \left(\frac{r}{c} \right)^2 M_{ab}^{(3)} \right\} \left(t - \frac{r}{c} \right) + \frac{n_a}{c^5 r^5} M \left\{ -\frac{108}{5} M_{ai}^{(1)} - \frac{108}{5} \left(\frac{r}{c} \right) M_{ai}^{(2)} - \frac{116}{5} \left(\frac{r}{c} \right)^2 M_{ai}^{(3)} - 8 \left(\frac{r}{c} \right)^3 M_{ai}^{(4)} \right\} \left(t - \frac{r}{c} \right), \quad (\text{B2b})$$

$$\begin{aligned}
N_2^{ij} = & \frac{\hat{n}_{ijab}}{c^4 r^6} M \left\{ 60 M_{ab} + 60 \left(\frac{r}{c}\right) M_{ab}^{(1)} + 24 \left(\frac{r}{c}\right)^2 M_{ab}^{(2)} + 4 \left(\frac{r}{c}\right)^3 M_{ab}^{(3)} \right\} \left(t - \frac{r}{c}\right) \\
& + \frac{\hat{n}_{a(i}}{c^4 r^6} M \left\{ \frac{72}{7} M_{j)a} + \frac{72}{7} \left(\frac{r}{c}\right) M_{j)a}^{(1)} - \frac{72}{7} \left(\frac{r}{c}\right)^2 M_{j)a}^{(2)} - \frac{96}{7} \left(\frac{r}{c}\right)^3 M_{j)a}^{(3)} \right\} \left(t - \frac{r}{c}\right) \\
& + \frac{\delta_{ij} n_{ab}}{c^4 r^6} M \left\{ -\frac{66}{7} M_{ab} - \frac{66}{7} \left(\frac{r}{c}\right) M_{ab}^{(1)} + \frac{24}{7} \left(\frac{r}{c}\right)^2 M_{ab}^{(2)} + \frac{46}{7} \left(\frac{r}{c}\right)^3 M_{ab}^{(3)} \right\} \left(t - \frac{r}{c}\right) \\
& + \frac{M}{c^6 c^4} \left\{ -\frac{24}{5} M_{ij}^{(2)} - \frac{24}{5} \left(\frac{r}{c}\right) M_{ij}^{(3)} - 8 \left(\frac{r}{c}\right)^2 M_{ij}^{(4)} \right\} \left(t - \frac{r}{c}\right). \tag{B2c}
\end{aligned}$$

Following the analytic continuation procedure of the algorithm, we now multiply $N_2^{\alpha\beta}$ by the factor r^B and compute $\square_R^{-1} \left(r^B N_2^{\alpha\beta} \right)$ by applying the formula III (4.23) to each of the terms in $r^B N_2^{\alpha\beta}$. Then we take the finite part (coefficient of B^0) of the Laurent expansion near $B = 0$ of the resulting expression [being careful to handle correctly the coefficients III (4.23bc)] and we get the first contribution $p_2^{\alpha\beta} = FP \square_R^{-1} N_2^{\alpha\beta}$ in the metric. As a check of the intermediate steps of the computation, one must find that the pole part (coefficient of B^{-1}) of $\square_R^{-1} \left(r^B N_2^{\alpha\beta} \right)$ in fact cancels out (see paper III). We find that $p_2^{\alpha\beta}$ so obtained is, in this monopole-quadrupole case, divergence-free: $\partial_\beta p_2^{\alpha\beta} = 0$. Hence, by application of the algorithm, we do not need to add any supplementary contribution, i.e., $q_2^{\alpha\beta} = 0$ and $h_2^{\alpha\beta} = p_2^{\alpha\beta}$. The result is

$$\begin{aligned}
h_2^{00} = & \frac{n_{ab}}{c^4 r^4} M \left\{ -21 M_{ab} - 21 \left(\frac{r}{c}\right) M_{ab}^{(1)} + 7 \left(\frac{r}{c}\right)^2 M_{ab}^{(2)} + 10 \left(\frac{r}{c}\right)^3 M_{ab}^{(3)} \right\} \left(t - \frac{r}{c}\right) \\
& - \frac{2n_{ab}}{c^8} M \int_1^\infty dx M_{ab}^{(4)} \left(t - \frac{rx}{c}\right) \left[(3x^2 - 1) \ln \left(\frac{x-1}{x+1}\right) + 6x \right], \tag{B3a}
\end{aligned}$$

$$\begin{aligned}
h_2^{0i} = & \frac{n_{iab}}{c^5 r^3} M \left\{ -M_{ab}^{(1)} - \left(\frac{r}{c}\right) M_{ab}^{(2)} - \frac{1}{3} \left(\frac{r}{c}\right)^2 M_{ab}^{(3)} \right\} \left(t - \frac{r}{c}\right) \\
& + \frac{n_a}{c^5 r^3} M \left\{ -5 M_{ai}^{(1)} - 5 \left(\frac{r}{c}\right) M_{ai}^{(2)} + \frac{19}{3} \left(\frac{r}{c}\right)^2 M_{ai}^{(3)} \right\} \left(t - \frac{r}{c}\right) \\
& - \frac{4n_a}{c^8} M \int_1^\infty dx M_{ai}^{(4)} \left(t - \frac{rx}{c}\right) \left[x \ln \left(\frac{x-1}{x+1}\right) + 2 \right], \tag{B3b}
\end{aligned}$$

$$\begin{aligned}
h_2^{ij} = & \frac{n_{ijab}}{c^4 r^4} M \left\{ -\frac{15}{2} M_{ab} - \frac{15}{2} \left(\frac{r}{c}\right) M_{ab}^{(1)} - 3 \left(\frac{r}{c}\right)^2 M_{ab}^{(2)} - \frac{1}{2} \left(\frac{r}{c}\right)^3 M_{ab}^{(3)} \right\} \left(t - \frac{r}{c}\right) \\
& + \frac{\delta_{ij} n_{ab}}{c^4 r^4} M \left\{ -\frac{1}{2} M_{ab} - \frac{1}{2} \left(\frac{r}{c}\right) M_{ab}^{(1)} - 2 \left(\frac{r}{c}\right)^2 M_{ab}^{(2)} - \frac{11}{6} \left(\frac{r}{c}\right)^3 M_{ab}^{(3)} \right\} \left(t - \frac{r}{c}\right) \\
& + \frac{n_{a(i}}{c^4 r^4} M \left\{ 6 M_{j)a} + 6 \left(\frac{r}{c}\right) M_{j)a}^{(1)} + 6 \left(\frac{r}{c}\right)^2 M_{j)a}^{(2)} + 4 \left(\frac{r}{c}\right)^3 M_{j)a}^{(3)} \right\} \left(t - \frac{r}{c}\right) \\
& + \frac{M}{c^4 r^4} \left\{ -M_{ij} - \left(\frac{r}{c}\right) M_{ij}^{(1)} - 4 \left(\frac{r}{c}\right)^2 M_{ij}^{(2)} - \frac{11}{3} \left(\frac{r}{c}\right)^3 M_{ij}^{(3)} \right\} \left(t - \frac{r}{c}\right) \\
& - \frac{4M}{c^8} \int_1^\infty dx M_{ij}^{(4)} \left(t - \frac{rx}{c}\right) \ln \left(\frac{x-1}{x+1}\right). \tag{B3c}
\end{aligned}$$

In these expressions we have used the Legendre representation (2.23) of the ‘‘hereditary’’ retarded integrals. The metric (B3) is essentially the harmonic coordinates version of the ‘‘2-2’’ metric of Refs. [8, 25].

Let us now write down the leading behavior at infinity $r \rightarrow \infty$, $u = t - \frac{r}{c} = \text{const}$, of this metric. Using Eq. (2.26) we find

$$\begin{aligned}
h_2^{00} = & -\frac{4n_{ab}}{c^7 r} M \int_0^\infty dy M_{ab}^{(4)}(u-y) \left[\ln \left(\frac{y}{2r}\right) + \frac{1}{2} \right] \\
& + O \left(\frac{\ln r}{r^2} \right), \tag{B4a}
\end{aligned}$$

$$\begin{aligned}
h_2^{0i} = & -\frac{4n_a}{c^7 r} M \int_0^\infty dy M_{ai}^{(4)}(u-y) \left[\ln \left(\frac{y}{2r}\right) + \frac{5}{12} \right] \\
& - \frac{n_{iab}}{3c^7 r} M M_{ab}^{(3)}(u) + O \left(\frac{\ln r}{r^2} \right), \tag{B4b}
\end{aligned}$$

$$\begin{aligned}
h_2^{ij} = & -\frac{4M}{c^7 r} \int_0^\infty dy M_{ij}^{(4)}(u-y) \left[\ln \left(\frac{y}{2r}\right) + \frac{11}{12} \right] \\
& - \frac{n_{ijab}}{2c^7 r} M M_{ab}^{(3)}(u) + \frac{4n_{a(i}}{c^7 r} M M_{j)a}^{(3)}(u) \\
& - \frac{11}{6} \frac{\delta_{ij} n_{ab}}{c^7 r} M M_{ab}^{(3)}(u) + O \left(\frac{\ln r}{r^2} \right). \tag{B4c}
\end{aligned}$$

Now we can follow the reasonings in Sec. II to put the latter metric into radiative form and then find the radiative quadrupole moment by TT projection onto the plane orthogonal to \mathbf{N} . The result can be immediately read off the first term of the right-hand side of Eq. (B4c). Finally we get

$$I_{ij}^{\text{rad}[2]}(t) = M_{ij}^{(2)}(t) + \frac{2GM}{c^3} \int_0^{+\infty} dy M_{ij}^{(4)}(t-y) \times \left[\ln\left(\frac{y}{2P_{\text{rad}}}\right) + \frac{11}{12} \right] \quad (\text{B5})$$

which is the formula used in the text.

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