Dynamical systems approach to tilted Bianchi cosmologies: Irrotational models of type V

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We write the Einstein field equations for the irrotational tilted Bianchi type-V cosmological models as an autonomous differential equation in terms of expansion-normalized dimensionless variables. The theory of dynamical systems is then used to give a complete qualitative description of the evolution of the models with nonextreme tilt.

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I. INTRODUCTION

The Bianchi cosmologies play an important role in theoretical cosmology and have been much studied since the 1960s. A Bianchi cosmology represents a spatially homogeneous universe, since by definition the spacetime admits a three-parameter group of isometries whose orbits are spacelike hypersurfaces. These models can be used to analyze aspects of the physical Universe which pertain to or which may be affected by anisotropy in the rate of expansion, for example, the cosmic microwave background radiation, nucleosynthesis in the early Universe, and the question of the isotropization of the universe itself [1].

A Bianchi cosmology is said to be *orthogonal* if the fluid velocity vector is orthogonal to the group orbits. Otherwise the model is said to be *tilted* [2]. A tilted model is spatially homogeneous relative to observers whose world lines are orthogonal relative to the group orbits, but is spatially inhomogeneous relative to observers comoving with the fluid. Recently the present authors used expansion-normalized variables to write the Einstein field equations (EFE's) for the orthogonal Bianchi cosmologies with a perfect fluid source as an autonomous differential equation (DE) [3]. This enables one to use the theory of dynamical systems to describe the evolution of the models qualitatively, in a way which is particularly simple from a physical and geometrical point of view. In the present paper we show that this choice of variables also provides an effective method for studying the evolution of tilted Bianchi cosmologies.

Since our objective is to illustrate the method in a simple context, we consider a class of models for which the resulting state space is three-dimensional, namely the tilted Bianchi cosmologies of group type V with an irrotational fluid. These models are anisotropic generalizations of the Friedmann-Roberston-Walker (FRW) models with negative spatial curvature. In a tilted Bianchi cosmology the tilt can become extreme in a finite time as measured along the fluid congruence, with the result that the group orbits become timelike. This means that the models are no longer spatially homogeneous [4]. While a complete analysis of all orbits in the three-dimensional state space could be given, we have decided to concentrate on the orbits which correspond to models in which the tilt does not become extreme.

In this paper, we assume that the cosmological fluid satisfies a linear equation of state

$$\hat{p} = (\gamma - 1)\hat{\mu}\,,$$

where γ is a constant. Of particular interest are the values $\gamma = 1$ (pressure-free matter) and $\gamma = \frac{4}{3}$ (a radiation fluid). In addition, the value $\gamma = 2$ (a stiff fluid) has been studied in connection with the early Universe. Furthermore, values of γ which satisfy $0 \le \gamma < \frac{2}{3}$, while physically unrealistic as regards a classical fluid, are of interest in connection with inflationary models of the Universe. In particular, the value $\gamma = 0$, for which the fluid can be interpreted as a positive cosmological constant, corresponds to exponential inflation, while the values $0 < \gamma < \frac{2}{3}$ correspond to power law inflation, in FRW models [5].

The locally rotationally symmetric (LRS) tilted Bianchi V models, which correspond to orbits in a two-dimensional subset of our state space, have been studied qualitatively by a number of authors. The most detailed work is that of Collins and Ellis [4], who write the EFE's for these models as a two-dimensional autonomous DE. The right-hand side of the DE is not analytic, however, and this results in the different FRW models, which are an important special case, being identified. This difficulty is avoided in our approach, as will be explained later.

The plan of the paper is as follows. In Sec. II we use the orthonormal frame equations of King and Ellis [2] to derive the basic autonomous DE in terms of expansion-normalized variables. In Sec. III we give the equilibrium points of the DE, and derive general results concerning the asymptotic behavior of the orbits. We analyze the two-dimensional invariant subset which corresponds to the locally rotationally symmetric (LRS) models, and use a monotone function to show that the dynamics in this subset determines, to a large extent, the dynamics in the three-dimensional state space. This enables us to give a complete analysis or the orbits for which the tilt is nonextreme. In Sec. IV we provide the cosmological interpretation of the phase portraits, and in Sec. V we comment on some mathematical aspects of the problem. The Appendixes contain some of the technical details.

As regards motivation and background, this paper depends to a large extent on Wainwright and Hsu [3] (WH). In Sec. II, it is also assumed that the reader is familiar with the orthonormal frame formalism of Ellis and MacCallum [6]. We use geometrized units with $c = 1$, $8\pi G = 1$, and sign conventions of MacCallum [7]. In addition, familiarity with some basic concepts and results in the theory of dynamical systems is assumed in Secs. III and V. We refer to WH for a brief summary and further references.

II. DERIVATION OF THE AUTONOMOUS DE

In the WH approach to the Bianchi cosmologies, the basic variables are the commutation functions associated with a group-invariant orthonormal frame. These variables are then rescaled using the rate of expansion scalar so as to make them dimensionless.

King and Ellis [2] (see page 225) have shown that for tilted Bianchi models of group type V with an irrotational fluid, an invariant orthonormal frame can be chosen so that the commutators have the form

$$[\mathbf{e}_0,\,\mathbf{e}_\alpha] = \theta_\alpha^{\ \beta}(t)\,\mathbf{e}_\beta\,,$$

$$[\mathbf{e}_\alpha,\,\mathbf{e}_\beta] = [a_\alpha(t)\delta_\beta^{\ \nu} - a_\beta(t)\delta_\alpha^{\ \nu}]\mathbf{e}_\nu\,,$$

where $\theta_\alpha^{\ \beta}(t)$ is diagonal, and $a_\alpha(t) = \delta_\alpha^{\ 1} a_1(t)$. Greek indices take on values 1 to 3, and are raised by means of $\delta^{\alpha\beta}$. In addition, $\mathbf{e}_0$ is the unit normal to the group orbits $t =$ const, and the fluid velocity $\mathbf{u}$ lies in the 2-space spanned by $\mathbf{e}_0$ and $\mathbf{e}_1$. Following King and Ellis [2], we write

$$\mathbf{u} = \cosh\beta\,\mathbf{e}_0 + \sinh\beta\,\mathbf{e}_1\,, \tag{2.1}$$

where β is called the hyperbolic angle of tilt.

The $\theta_{\alpha\beta}(t)$ are the frame components of the rate of expansion tensor of the normal congruence to the group orbits, and determine the rate of shear tensor according to

$$\sigma_{\alpha\beta} = \theta_{\alpha\beta} - \tfrac{1}{3}\theta\,\delta_{\alpha\beta}\,,$$

where $\theta = \theta_\alpha^{\ \alpha}$ is the rate of expansion scalar. The quantity $a_1(t)$ determines the curvature of the group orbits.

It is convenient to introduce new shear variables according to

$$\sigma_+ = \tfrac{3}{2}(\sigma_{22} + \sigma_{33}), \qquad \sigma_- = \tfrac{\sqrt{3}}{2}(\sigma_{22} - \sigma_{33})\,, \tag{2.2}$$

and a tilt variable v according to

$$v = \tanh\beta\,. \tag{2.3}$$

The basic variables $(\theta,\,\sigma_+,\,\sigma_-,\,a_1,\,v)$ describe the physical state of the cosmological model at time t. The Einstein field equations lead to an autonomous DE in $\mathbb{R}^5$ for these variables, together with one constraint. When we rescale the basic variables, the evolution equation for θ decouples, leaving an autonomous DE in $\mathbb{R}^4$ subject to one constraint.

The stress-energy tensor for a perfect fluid is

$$T_{ab} = \hat{\mu} u_a u_b + \hat{p}(u_a u_b + g_{ab})\,, \tag{2.4}$$

and as discussed in the Introduction, we assume that

$$\hat{p} = (\gamma - 1)\hat{\mu}\,. \tag{2.5}$$

In order to write the EFE's in terms of the basic variables, it is necessary to decompose the stress-energy tensor with respect to the unit normal to the group orbits $\mathbf{n} = \mathbf{e}_0$. Following King and Ellis [2] we write

$$T_{ab} = \mu\, n_a n_b + p(n_a n_b + g_{ab}) + 2q_{(a}n_{b)} + \pi_{ab}\,,$$

where $q_a n^a = 0$, $\pi_{ab} n^b = 0$. It follows from Eqs. (2.1), (2.4), and (2.5) that

$$\begin{aligned} \mu &= (1 + \gamma\,\sinh^2\beta)\hat{\mu}, \\ p &= (\gamma - 1 + \tfrac{1}{3}\gamma\,\sinh^2\beta)\hat{\mu}\,, \end{aligned} \tag{2.6}$$

and that the nonzero components of q_a and π_{ab} are

$$q_1 = \gamma\hat{\mu}\,\sinh\beta\,\cosh\beta\,, \qquad \pi_{11} = -\tfrac{2}{3}\gamma\hat{\mu}\,\sinh^2\beta\,.$$

The autonomous DE for the basic variables $(\theta,\,\sigma_+,\,\sigma_-,\,a_1,\,v)$ may now be obtained from the field equations as given by King and Ellis [2] [see Eqs. (2.16) and (1.34)] or MacCallum [7] [see Eqs. (3.14) and (3.15)]. In either case, it is necessary to do some manipulations in order to obtain an evolution equation for v.

Evolution equations

$$\dot{\theta} = -\frac{1}{3}\theta^2 - \frac{2}{3}(\sigma_+^{\ 2} + \sigma_-^{\ 2}) - \frac{1}{2}\,\frac{(3\gamma - 2) + (2 - \gamma)v^2}{1 + (\gamma - 1)v^2}\,\mu,$$

$$\dot{\sigma}_+ = -(\theta - 2va_1)\sigma_+,$$

$$\dot{\sigma}_- = -\theta\,\sigma_-,$$

$$\dot{a}_1 = \tfrac{1}{3}(2\sigma_+ - \theta)a_1,$$

$$\dot{v} = \frac{v(1 - v^2)}{3[1 - v^2(\gamma - 1)]}[2\sigma_+ + (3\gamma - 4)\theta - 6(\gamma - 1)a_1 v].$$

Constraint equation

$$\gamma v\mu + 2[1 + (\gamma - 1)v^2]a_1\sigma_+ = 0.$$

Defining equation for μ

$$\mu = \tfrac{1}{3}(\theta^2 - \sigma_+^{\ 2} - \sigma_-^{\ 2} - 9a_1^{\ 2}).$$

Auxiliary equation

$$\dot{\hat{\mu}} = \frac{\mu}{1 + v^2(\gamma - 1)}[-\gamma\theta + 2\gamma a_1 v + \tfrac{1}{3}\gamma(2\sigma_+ - \theta)v^2].$$

In these equations, an overdot denotes the derivative $\frac{d}{dt}$ along the $\mathbf{e}_0$ congruence.

Motivated by WH, we now introduce expansion-normalized dimensionless variables Σ_+, Σ_-, and A according to

$$\Sigma_+ = \frac{\sigma_+}{\theta}, \qquad \Sigma_- = \frac{\sigma_-}{\theta}, \qquad A = \frac{3a_1}{\theta}. \tag{2.7}$$

The density μ is replaced by the density parameter Ω, which is defined by

$$\Omega = \frac{3\mu}{\theta^2}. \tag{2.8}$$

We introduce a dimensionless time variable τ by

$$\frac{dt}{d\tau} = \frac{3}{\theta}. \tag{2.9}$$

Finally, the deceleration parameter q of the normal congruence is defined by

$$\frac{d\theta}{d\tau} = -(1+q)\theta \tag{2.10}$$

[see WH, Eq. (2.22)].

The evolution equations can now be written in dimensionless form, with a prime denoting $\frac{d}{d\tau}$.

Evolution equations

$$\begin{aligned} \Sigma_+' &= -(2-q-2Av)\Sigma_+, \\ \Sigma_-' &= -(2-q)\Sigma_-, \\ A' &= (q+2\Sigma_+)A, \\ v' &= \frac{v(1-v^2)}{1-(\gamma-1)v^2}[2\Sigma_+ + (3\gamma-4) - 2(\gamma-1)Av]. \end{aligned} \tag{2.11}$$

Constraint equation

$$\gamma v\Omega + 2[1+(\gamma-1)v^2]A\Sigma_+ = 0. \tag{2.12}$$

Defining equations for Ω and q

$$\Omega = 1 - A^2 - \Sigma_+^2 - \Sigma_-^2, \tag{2.13}$$

$$q = 2 - 2A^2 - \frac{1}{2}\frac{[3(2-\gamma)+(5\gamma-6)v^2]\Omega}{1+(\gamma-1)v^2} \tag{2.14}$$

Auxiliary equation

$$\Omega' = \frac{\Omega}{1+(\gamma-1)v^2}[\,2q-(3\gamma-2)+2\gamma Av + \{2q(\gamma-1)+2\gamma\Sigma_+ - (2-\gamma)\}v^2]. \tag{2.15}$$

III. QUALITATIVE ANALYSIS

We have shown that the evolution of the tilted irrotational Bianchi V cosmologies is governed by a DE in $\mathbb{R}^4$:

$$\frac{d\mathbf{X}}{d\tau} = \mathbf{F}(\mathbf{X}), \qquad \mathbf{X} = (\Sigma_+, \Sigma_-, A, v), \tag{3.1}$$

given by Eq. (2.11), subject to a constraint

$$G(\mathbf{X}) = 0, \tag{3.2}$$

given by Eq. (2.12), and subject to the inequalities

$$\Omega \geq 0, \qquad v^2 \leq 1. \tag{3.3}$$

Since the DE (2.11) is invariant under the transformations

$$(\Sigma_+, \Sigma_-, A, v) \to (\Sigma_+, \Sigma_-, -A, -v)$$

and

$$(\Sigma_+, \Sigma_-, A, v) \to (\Sigma_+, -\Sigma_-, A, v),$$

we can restrict our considerations without loss of generality to the invariant set defined by

$$A \geq 0, \quad \text{and} \quad \Sigma_- \geq 0. \tag{3.4}$$

The state space for the DE is the compact subset D of $\mathbb{R}^4$ defined by the restrictions (3.2), (3.3) and (3.4). The subsets of D defined by $\Sigma_+ > 0$ and $\Sigma_+ < 0$ are invariant, as follows from the evolution equation for Σ_+. We shall denote these invariant sets by D_+ and D_-, respectively. Since the state space D is compact, the solutions of the DE are defined for all real values of τ. This implies that the solutions of the DE define a dynamical system on D (see WH, section 3.1), and we can thus apply the theory of dynamical systems.

The structure of the DE and of the constraint also implies the existence of various lower dimensional invariant sets.

(1) The two-dimensional (2D) subset defined by $v = 0$ is invariant, and describes the orthogonal (i.e., untilted) models. It follows from the constraint equation that either $A = 0$, giving the untilted Bianchi I models, or $\Sigma_+ = 0$, giving the untilted Bianchi V models. The phase portraits for these invariant sets are given in Figs. 1 and 2.

(2) The 2D subset defined by $\Sigma_- = 0$ is invariant, and describes the tilted LRS Bianchi V models. This invariant set is analyzed in detail later in this section (see Figs. 5–11).

(3) The one-dimensional subset defined by $\Sigma_+ = \Sigma_- =$

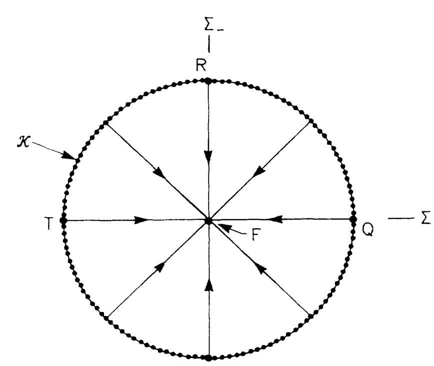

FIG. 1. This figure shows the phase portrait for the untilted Bianchi I models ($v = 0$, $A = 0$), for $0 < \gamma < 2$.

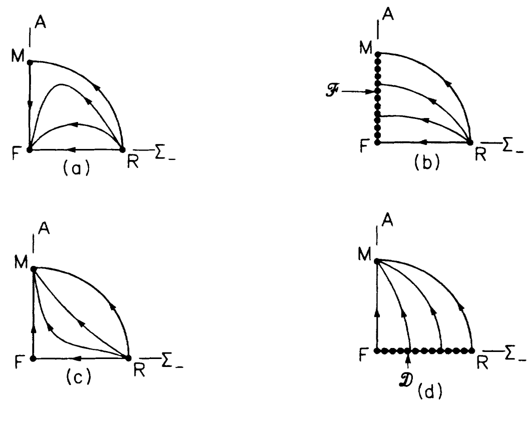

FIG. 2. This figure shows the phase portraits for the untilted Bianchi V models ($v = 0 = \Sigma_+$). The four cases are (a) $0 < \gamma < \frac{2}{3}$, (b) $\gamma = \frac{2}{3}$, (c) $\frac{2}{3} < \gamma < 2$, (d) $\gamma = 2$.

0 is invariant, and describes the FRW models with zero or negative spatial curvature (see the orbit $F \to M$ in Figs. 2 and 5–11 to follow).

These invariant subsets play an important role in describing the qualitative properties of the orbits of the DE.

We now begin the qualitative analysis by investigating the local stability of the equilibrium points of the DE. The equilibrium points are found by solving the system of equations $\mathbf{F}(\mathbf{X}) = \mathbf{0}$, $G(\mathbf{X}) = 0$ [refer to Eqs. (3.1), (3.2), (2.11), and (2.12)]. The procedure is routine, and we state the results below.

Equilibrium points with zero tilt ($v = 0$)

$$F : \Sigma_+ = \Sigma_- = A = 0 \ \text{ with } 0 < \gamma \leq 2 ,$$

$$M : \Sigma_+ = \Sigma_- = 0, \ A = 1 \ \text{ with } 0 < \gamma \leq 2 ,$$

$$\mathcal{F} : \Sigma_+ = \Sigma_- = 0, \ 0 < A < 1 \ \text{ with } \gamma = \frac{2}{3} ,$$

$$\mathcal{K} : \Sigma_+^2 + \Sigma_-^2 = 1, \ A = 0 \ \text{ with } 0 < \gamma \leq 2 ,$$

$$\mathcal{D} : 0 < \Sigma_+^2 + \Sigma_-^2 < 1, \ A = 0 \ \text{ with } \gamma = 2 .$$

The isolated equilibrium points F and M represent, respectively, the flat FRW model with equation of state parameter γ and the (vacuum) Milne model. $\mathcal{K}$ is a circle of equilibrium points which correspond to the Kasner vacuum solutions (see WH, Table 3). $\mathcal{F}$ is a line of equilibrium points which represent special FRW models which exist only if $\gamma = \frac{2}{3}$. $\mathcal{D}$ is a disc of equilibrium points which correspond to the Jacobs stiff fluid solutions (see WH, Table 3).

Equilibrium points with intermediate tilt ($0 < v^2 < 1$)

$$\tilde{M} : \Sigma_+ = \Sigma_- = 0, \ A = 1, \ v = \frac{3\gamma - 4}{2(\gamma - 1)}$$
$$\text{with } \tfrac{6}{5} < \gamma < 2, \quad \text{and} \quad \gamma \neq \tfrac{4}{3} .$$

$$\mathcal{C}_\pm^\pm : \Sigma_+ = \tfrac{1}{2}(4 - 3\gamma),$$
$$\Sigma_- = \pm\tfrac{1}{2}\sqrt{3(2-\gamma)(3\gamma-2)}, \quad A = 0 ,$$
$$0 < v < 1 \quad \text{or} \quad -1 < v < 0 \ \text{ with } \ \tfrac{2}{3} < \gamma < 2 .$$

The isolated equilibrium point $\tilde{M}$ represents flat spacetime. It depends on γ, and if $\gamma = \frac{4}{3}$ it coincides with the Milne equilibrium point M. The four lines of equilibrium points $\mathcal{C}_\pm^\pm$ represent particular Kasner vacuum solutions.

Equilibrium points with extreme tilt ($v^2 = 1$)

$$M^+ : \Sigma_+ = \Sigma_- = 0, \ A = 1, \ v = 1 \ \text{ with } 0 < \gamma \leq 2 ,$$

$$M^- : \Sigma_+ = \Sigma_- = 0, \ A = 1, \ v = -1 \ \text{ with } 0 < \gamma < 2 ,$$

$$\mathcal{K}^\pm : \Sigma_+^2 + \Sigma_-^2 = 1, \ A = 0, \ v = \pm 1 \ \text{ with } 0 < \gamma < 2 ,$$

$$\mathcal{H} : \Sigma_- = 0, \ A = 1 + \Sigma_+, \ v = 1, \ -1 < \Sigma_+ < 0$$
$$\text{with } 0 < \gamma \leq 2 .$$

Since the tilt is extreme, these equilibrium points do not correspond to exact Bianchi solutions.

Local stability of the equilibrium points

We are interested in equilibrium points which are sources or sinks. The analysis of the local stability of the equilibrium points is complicated by the presence of the constraint equation $G(\mathbf{X}) = 0$. One has to use the constraint to eliminate one of the variables, thereby reducing the DE to three dimensions. Unfortunately the constraint is such that one cannot globally eliminate one variable: which variable can be eliminated depends on the equilibrium point. At the FRW equilibrium point F, the gradient of the constraint function G is

$$\nabla G = (0, 0, 0, \gamma) , \tag{3.5}$$

and hence in a neighborhood of F one can uniquely solve for v and reduce the DE to a DE in Σ_+, Σ_- and A. At the Milne equilibrium point M, we have

$$\nabla G = (2, 0, 0, 0) , \tag{3.6}$$

and hence in a neighborhood of M one can uniquely solve for Σ_+, and reduce the DE to a DE in Σ_-, A and v. On the Kasner circle $\mathcal{K}$, we have

$$\nabla G = (0, 0, 2\Sigma_+, 0) . \tag{3.7}$$

It follows that there are two exceptional points, namely, $\Sigma_+ = 0, \Sigma_- = \pm 1$, at which the constraint surface is singular (it is easy to show that these are the only points in state space at which $\nabla G = \mathbf{0}$). Except at these two exceptional points on $\mathcal{K}$, we can eliminate the variable A on $\mathcal{K}$ and reduce the DE to one in Σ_+, Σ_- , and v.

In this way, we can calculate the eigenvalues of the linearization of the DE at the equilibrium points, which are given in Appendix B. We can use the eigenvalues to identify all sinks and sources as follows: (1) If $0 < \gamma < \frac{2}{3}$, the point F is a sink; (2) if $\frac{2}{3} < \gamma < \frac{4}{3}$, the point M is a sink; (3) if $\frac{4}{3} < \gamma < 2$, the point $\tilde{M}$ is a sink; (4) if $\frac{2}{3} < \gamma < 2$, the arc of the Kasner circle $\mathcal{K}$ defined by $\Sigma_+ > \frac{1}{2}(4-3\gamma)$ is a source (see Fig. 3); (5) if $0 < \gamma < 2$ the arcs of the Kasner circles $\mathcal{K}^\pm$ defined by $\Sigma_+ < \frac{1}{2}(4-3\gamma)$ are sources (see Fig. 3).

A Monotone function

Since $\Omega > 0$ implies $q < 2$ by Eq. (2.14), the evolution equation for Σ_- implies that Σ_- is a monotone decreasing function along all orbits with $\Sigma_- > 0$ and $\Omega > 0$. This fact significantly restricts the evolution at late times, since as proved in Proposition C.1, it implies that

$$\lim_{\tau\to+\infty} \Sigma_- = 0 \tag{3.8}$$

for all orbits with $\Omega > 0$. Thus the asymptotic behavior as $\tau \to +\infty$ of all orbits with $\Omega > 0$ is determined by the asymptotic behavior of the orbits in the 2D invariant set $\Sigma_- = 0$. which we now study in detail.

The invariant set $\Sigma_- = 0$

The invariant set $\Sigma_- = 0$ is a surface in $\mathbb{R}^3$, defined by the constraint Eq. (3.2) with $\Sigma_- = 0$:

$$G(\Sigma_+, 0, A, v) = 0,$$

with the boundary of the surface defined by the inequalities

$$\Omega \equiv 1 - \Sigma_+^2 - A^2 \geq 0, \quad v^2 \leq 1, \qquad A \geq 0.$$

In Fig. 4 we have drawn the intersection of the constraint surface with the cylinder $\Omega = 0$ and the planes $v = 0$ and $v = \pm 1$, and have also labeled the equilibrium points for the case $\frac{2}{3} < \gamma < \frac{6}{5}$, all of which lie on the boundary. Note that only two points of the Kasner circle $\mathcal{K}$, labeled Q and T in Fig. 3, lie on this surface.

In order to sketch the orbits one would like to project this surface into one of the coordinate planes. This is not possible since some of the orbits in the boundary are parallel to the coordinate axes (see Fig. 4). We thus "open out" the surface in a continuous manner, by bending the lines TT^+ and MM^+ downwards, and bending the lines QQ^- and MM^- upwards, thereby obtaining a plane image of the surface, as in Figs. 5–11.

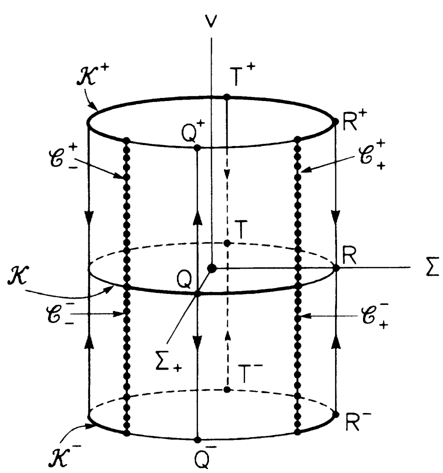

FIG. 3. This figure shows the stability properties of the Kasner circles $\mathcal{K}, \mathcal{K}^\pm$, which are given by $\Sigma_+^2 + \Sigma_-^2 = 1$. The lines of equilibrium points $\mathcal{C}_\pm^\pm$, which lie in the plane $\Sigma_+ = \frac{1}{2}(4-3\gamma)$, $\frac{2}{3} < \gamma < 2$, subdivide each Kasner circle into two arcs, the heavy arc being a source.

There are seven qualitatively distinct types of behavior, as γ varies from 0 to 2 (see Figs. 5–11). For $0 < \gamma < \frac{6}{5}$ the equilibrium point $\tilde{M}$ is not in the physical region of state space. When $\gamma = \frac{6}{5}$, the equilibrium point $\tilde{M}$ enters the physical region, coinciding with M^-, then moves along the line M^-M (Fig. 8), and merges with M when $\gamma = \frac{4}{3}$ (Fig. 9). Finally, M moves along the line MM^+ (Fig. 10) and merges with M^+ when $\gamma = 2$. The details of the portraits of the orbits can be deduced using the local stability of the equilibrium points and the main theorem on the asymptotic behavior of planar DE's [8]. The theorem can be applied separately to the invariant sets $\Sigma_+ \geq 0$ and $\Sigma_+ \leq 0$. Note that there are no interior equilibrium points, and hence no limit cycles. In addition, there are no heteroclinic cycles (a closed path consisting of sequentially oriented orbits), as follows by consideration of the orbits on the boundary. The presence of nonisolated equilibrium points complicates the analysis to some extent. However, if $0 < \gamma < 2$ the line of equilibrium points $\mathcal{H}$ is a source in the invariant set $\Sigma_+ \geq 0$. In the cases $\gamma = \frac{2}{3}$ and $\gamma = 2$ there are additional nonisolated equilibrium points, and it is necessary to use the DE itself in the analysis.

Behavior at early times

As mentioned earlier, if $\frac{2}{3} < \gamma < 2$, the arc of the Kasner circle defined by $\Sigma_+ > \frac{1}{2}(4-3\gamma)$ is a source (see Fig. 3), and so for these values of γ, there exists a set of orbits of positive measure in state space for which $\lim_{\tau\to-\infty} v = 0$. We observe that if $\gamma < \frac{4}{3}$, this arc is confined to the invariant set D_+ while if $\gamma > \frac{4}{3}$, the arc intersects both D_+ and D_-.

Behavior at late times

On account of Eq. (3.8), all orbits with $\Omega > 0$ are attracted to the invariant set $\Sigma_- = 0$ as $\tau \to +\infty$ (see Proposition C.1). We can now use Propositions C.2 and C.3, together with Figs. 5–11, to draw the following conclusions concerning the orbits with $\Omega > 0$, $v^2 < 1$ and $A > 0$ in the three-dimensional state space D.

(1) If $0 < \gamma < \frac{2}{3}$, all orbits are asymptotic to the equilibrium point F as $\tau \to +\infty$.

(2) If $\frac{2}{3} < \gamma \leq \frac{6}{5}$, all orbits are asymptotic to the equilibrium point M as $\tau \to +\infty$.

(3) If $\frac{6}{5} < \gamma < \frac{4}{3}$, all orbits in D_- and a set of orbits of nonzero measure in D_+, are asymptotic to M as $\tau \to \infty$.

(4) If $\gamma = \frac{4}{3}$, all orbits in D_-, and none in D_+, are asymptotic to M as $\tau \to +\infty$.

(5) If $\frac{4}{3} < \gamma < 2$, all orbits in D_-, and none in D_+, are asymptotic to $\tilde{M}$ as $\tau \to +\infty$.

Apart from a few exceptional orbits, the remaining orbits are asymptotic to equilibrium points with extreme tilt as $\tau \to +\infty$.

Orbits with nonextreme tilt

We can now draw the following conclusions concerning the existence of orbits with nonextreme tilt.

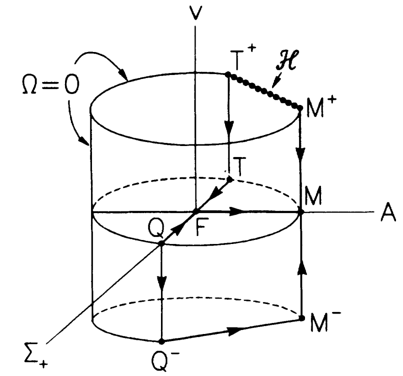

FIG. 4. This figure shows the invariant subset $\Sigma_- = 0$ in the case $\frac{2}{3} < \gamma < \frac{6}{5}$. The bold lines and curves are the intersection of the constraint surface $G = 0$ with the vacuum boundary $\Omega = 0$ and with the planes $v = 0$ and $v = \pm 1$.

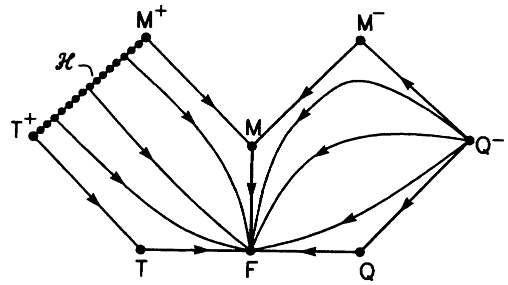

FIG. 5. This figure shows the orbits in the invariant set $\Sigma_- = 0$ in the case $0 < \gamma < \frac{2}{3}$.

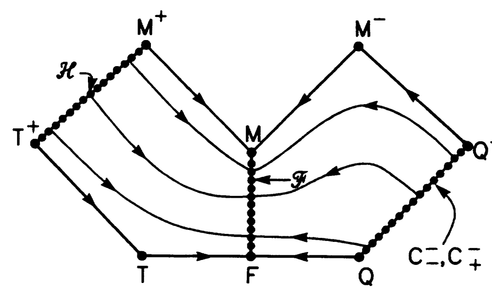

FIG. 6. This figure shows the orbits in the invariant set $\Sigma_- = 0$ in the case $\gamma = \frac{2}{3}$.

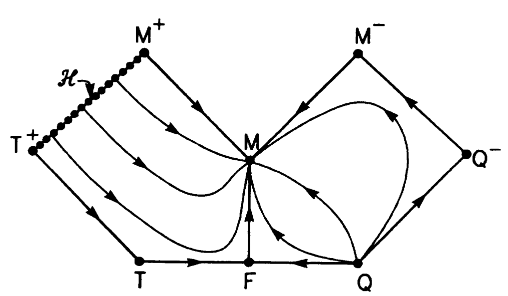

FIG. 7. This figure shows the orbits in the invariant set $\Sigma_- = 0$ in the case $\frac{2}{3} < \gamma \leq \frac{6}{5}$.

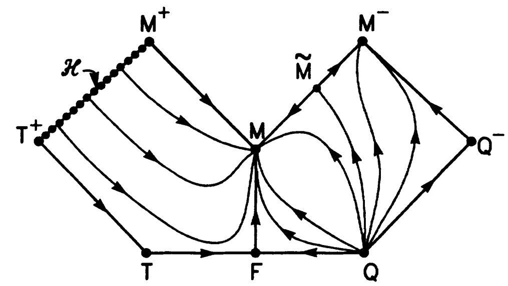

FIG. 8. This figure shows the orbits in the invariant set $\Sigma_- = 0$ in the case $\frac{6}{5} < \gamma < \frac{4}{3}$.

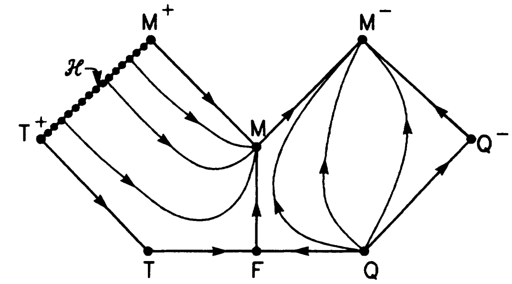

FIG. 9. This figure shows the orbits in the invariant set $\Sigma_- = 0$ in the case $\gamma = \frac{4}{3}$.

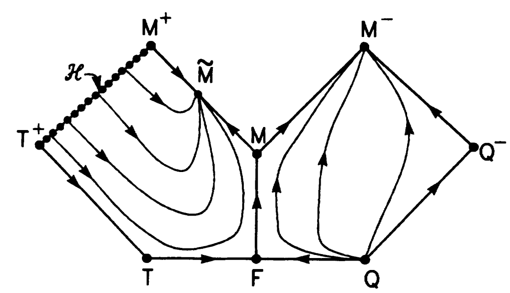

FIG. 10. This figure shows the orbits in the invariant set $\Sigma_- = 0$ in the case $\frac{4}{3} < \gamma < 2$.

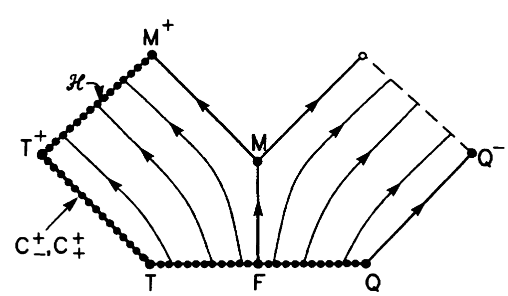

FIG. 11. This figure shows the orbits in the invariant set $\Sigma_- = 0$ in the case $\gamma = 2$.

(1) If $\frac{2}{3} < \gamma < \frac{4}{3}$, there exists a set of orbits in D_+ which are asymptotic to the Kasner circle $\mathcal{K}$ as $\tau \to -\infty$, and which are asymptotic to the Milne equilibrium point M as $\tau \to +\infty$. This set of orbits has nonzero measure in D. We shall refer to these orbits as *class I orbits*, and describe them symbolically by writing $K_1 \to M$. Here K_1 denotes an equilibrium point on the Kasner circle $\mathcal{K}$ with $\Sigma_+ > \frac{1}{2}(4-3\gamma) > 0$.

(2) If $\frac{4}{3} < \gamma < 2$, there exists an open set of orbits in D_- which are asymptotic to the Kasner circle as $\tau \to -\infty$ and which are asymptotic to the tilted equilibrium point $\tilde{M}$ as $\tau \to +\infty$. This set of orbits has nonzero measure in D. We shall refer to these orbits as *class II orbits*, and describe them symbolically by writing $K_2 \to \tilde{M}$. Here K_2 denotes an equilibrium point on the Kasner circle $\mathcal{K}$ with $0 > \Sigma_+ > \frac{1}{2}(4-3\gamma)$.

IV. MODELS WITH NONEXTREME TILT

We regard the variables $\mathbf{X} = (\Sigma_+, \Sigma_-, A, v)$ in the DE (3.1) as determining the (dimensionless) *dynamical state* of a cosmological model at an instant of time. In order to specify the *physical state* $(\theta, \sigma_+, \sigma_-, a_1, v)$ one has to give in addition the rate of expansion $\theta(\tau)$. This is determined according to

$$\theta(\tau) = \theta(\tau_0)\exp\left(\int_{\tau_0}^{\tau}[1+q(u)]du\right), \quad -\infty < \tau < \infty, \quad (4.1)$$

as follows from Eq. (2.10). Here q is expressed in terms of $(\Sigma_+, \Sigma_-, A, v)$ through Eqs. (2.13) and (2.14). Finally the clock time t along the normal congruence is determined by $\theta(\tau)$ through Eq. (2.9).

Equation (4.1) shows that to each orbit there corresponds a one-parameter family of cosmological models (unless the orbit is an equilibrium point; see WH, p. 1419). We choose the parameter $\theta(\tau_0)$ to be positive, so that the normal congruence is expanding. It then follows from Proposition A.2 that the fluid congruence is expanding ($\hat{\theta} > 0$).

In a tilted Bianchi cosmology, an initial big-bang singularity does not necessarily occur (see, for example, Ref. [4]). However, if an orbit approaches the Kasner circle $\mathcal{K}$ as $\tau \to -\infty$ the corresponding cosmological model does have a big-bang singularity at a finite time in the past. We justify this as follows. Since $v \to 0$ and $q \to 2$ as $\tau \to -\infty$, the evolution equations for Ω and θ imply that $\Omega \approx \Omega_0 e^{3(2-\gamma)\tau}$ and $\theta \approx \theta_0 e^{-3\tau}$, so that $\mu \approx \mu_0 e^{-3\gamma\tau}$. Thus as $\tau \to -\infty$, $\mu \to +\infty$ and by Eq. (2.6), the energy density $\hat{\mu}$ of the fluid also diverges. It follows from Eq. (2.9) that the clock time along the normal congruence between τ_1 and τ_2 is

$$\int_{\tau_1}^{\tau_2}\frac{3}{\theta(u)}du.$$

The asymptotic form of θ implies that the improper integral

$$\int_{-\infty}^{\tau}\frac{3}{\theta(u)}du$$

converges, and thus the big-bang singularity occurs at a finite time in the past. We take this as the origin of t, and then the clock time t since the big-bang (as measured along the normal congruence) is

$$t = \int_{-\infty}^{\tau}\frac{3}{\theta(u)}du. \quad (4.2)$$

At the end of Sec. III, we showed that there are two sets of orbits (class I and class II) which correspond to cosmological models with nonextreme tilt (i.e., $\lim_{\tau\to\pm\infty} v^2 < 1$). We shall refer to these models, whose evolution we now discuss, as *models of class I and class II.*

Asymptotic behavior

By the preceding discussion, the models of class I and class II evolve from a nontilted Kasner-like big-bang singularity at a finite time in the past. Since $q > 0$, Eqs. (4.1) and (4.2) imply that $t \to +\infty$ as $\tau \to +\infty$, so that the models expand indefinitely into the future. The models of class I approach a nontilted Milne-like future asymptotic state as $\tau \to +\infty$. Since the dimensionless shear of the normal congruence tends to zero as $\tau \to +\infty$, ($\Sigma_\pm \to 0$), and the tilt tends to zero, the dimensionless shear of the fluid congruence tends to zero [$\hat{\Sigma}_\pm \to 0$; see Eqs. (A3), (A4) and (A6)]. The models of class I thus isotropize at late times. Since the orbits of class I form a set of nonzero measure, *this behavior is typical for irrotational tilted Bianchi V models* with $\frac{2}{3} < \gamma < \frac{4}{3}$. However, since there are sets of orbits of nonzero measure for which the tilt becomes extreme, this behavior is not generic .

At late times, the models of class II approach a tilted asymptotic state which corresponds to flat spacetime as described by the equilibrium point $\tilde{M}$. It follows from Eq. (A3) that the dimensionless fluid shear does not tend to zero, and so the models of class II do not isotropize at late times. In addition, the fluid acceleration is dynamically significant at late times since the dimensionless acceleration scalar $\dot{U}$ does not tend to zero [see Eq. (A5)]. Since the orbits of class II form a set of non-zero measure, *this behavior is typical for irrotational tilted Bianchi V models with* $\frac{4}{3} < \gamma < 2$. However, since there are sets of orbits of nonzero measure for which the tilt becomes extreme, this behavior is not generic.

Intermediate evolution

One can also draw conclusions about the possible *intermediate evolution* of the tilted Bianchi V models. By this we mean the evolution for finite values of τ. The class I orbits are written symbolically $K_1 \to M$, where K_1 represents the initial state and M the final state. For simplicity we restrict our considerations to the LRS-invariant set $\Sigma_- = 0$. Then K_1 coincides with Q in Figs. 4, 7, and 8. One can identify various sequences of orbits which join Q to M, and pass through various saddle points:

(1) $Q \to F \to M$ (for $\frac{2}{3} < \gamma < \frac{4}{3}$, Figs. 7 and 8).

(2) $Q \to Q^- \to M^- \to M$ (for $\frac{2}{3} < \gamma < \frac{6}{5}$, Fig. 7).

(3) $Q \to \tilde{M} \to M$ (for $\frac{6}{5} < \gamma < \frac{4}{3}$, Fig. 8).

We shall refer to these sequences as *heteroclinic se-*

quences. For each such sequence there is a set of orbits of nonzero measure which are approximated by the sequence, and each orbit determines a one-parameter family of cosmological models.

One thinks of the evolution of the cosmological model as being described by a point in state space moving along an orbit of the DE. The velocity of the point is given by the vector field on the right-hand side of the DE. Near an equilibrium point the velocity will be close to zero, and the moving point will slow down and linger near the equilibrium point. However, if the equilibrium point is a saddle point, the moving point will eventually be repelled (unless it happens to be on the stable manifold). Thus, during a certain time interval, the evolution of the cosmological model will be approximated by the self-similar model which corresponds to the equilibrium (saddle) point. For this reason, we shall refer to the saddle points in a heteroclinic sequence as *intermediate asymptotes.*

Consider the sequence (1). A cosmological model whose orbit is approximated by this sequence starts in a nontilted Kasner-like state corresponding to the equilibrium point Q. Then the orbit follows the orbit $Q \to F$, and approaches the FRW equilibrium point F, which means that the model isotropizes, while approximating a Bianchi I model. Finally the orbit follows the separatrix $F \to M$, so that the model is approximated by the FRW model with negative spatial curvature. The model is tilted ($v < 0$) throughout the evolution, but the tilt remains small ($| v | << 1$).

Consider the sequence (2). A cosmological model whose orbit is approximated by this sequence, starts in a nontilted Kasner-like state Q. As the orbit approaches Q^-, the tilt becomes close to extreme ($v \approx -1$). As the orbit approaches M^-, the tilt remains near extreme and the spatial curvature becomes dynamically significant ($A \approx 1$). Finally, as the orbit approaches M the model isotropizes and the tilt tends to zero. At no time during the evolution is the matter density dynamically significant (i.e., $\Omega \approx 0$). It is of interest to keep track of the fluid shear, as described by $\hat{\Sigma}_+$ [see Eqs. (A3), (A4), and (A6)]. Note that $\hat{\Sigma}_- = 0$. It follows from Eqs. (A3) and (A6) that near Q, $\hat{\Sigma}_+ \approx 1$, near Q^-, $\hat{\Sigma}_+ \approx -\frac{1}{2}(3\gamma - 4)$, near M^-, $\hat{\Sigma}_+ \approx -\frac{1}{2}(3\gamma - 2)$, and near M, $\hat{\Sigma}_+ \approx 0$.

One can analyze the sequence (3) in a similar way.

V. MATHEMATICAL ASPECTS OF THE PROBLEM

In this section we comment on some mathematical aspects of the problem, mainly from a dynamical systems perspective.

A DE with a constraint

In the formulation of the EFE's for the orthogonal Bianchi models of class A using expansion-normalized variables (see WH), one obtains a DE

$$\frac{d\mathbf{X}}{d\tau} = \mathbf{F}(\mathbf{X}), \qquad \mathbf{X} \in \mathbb{R}^n,$$

where the vector field $\mathbf{F}$ is analytic. The present situation is different in that the basic variables are also required to satisfy an *algebraic constraint* $G(\mathbf{X}) = 0$, which defines an invariant subset. Since this constraint cannot be eliminated globally, one has to describe the evolution by using a DE in $\mathbb{R}^n$, with the physical state space being the above invariant subset. This subset is a hypersurface with boundary, which is smooth except at two points on the boundary. Note that $\nabla \mathbf{G} = \mathbf{0}$ and $G(\mathbf{X}) = 0$ only at the points $(\Sigma_+, \Sigma_-, A, v) = (0, \pm 1, 0, 0)$.

In the LRS case ($\Sigma_- = 0$), this invariant subset is a smooth hypersurface (a two-manifold with boundary) which cannot be projected globally onto any of the coordinate planes (see Fig. 4). In Ref. [4], when discussing the LRS case, Collins and Ellis essentially eliminate our variable A, and use Σ_+ and v as variables. The result of this elimination is that in their diagrams the FRW equilibrium point F, the Milne equilibrium point M and the nonsingular orbit $F \to M$, which describes the FRW model with negative spatial curvature, are superimposed.

A gradientlike DE

A *gradient* DE, that is, one of the form

$$\frac{d\mathbf{x}}{dt} = \nabla V,$$

where V is a C^1 function, generates a particularly simple dynamical system, since the ω-limit and α-limit set of any orbit contains only equilibrium points (see, for example, Ref. [9]). This implies that there is no recurrent behavior of any sort, that is, no periodic orbits or recurrent orbits. In addition, homoclinic orbits and heteroclinic cycles are excluded (see Ref. [10] for this terminology). The reason for this simplicity is that the potential function V is a monotone (strictly increasing) function along nonsingular orbits, since $\frac{dV}{dt} = \nabla V \cdot \nabla V$.

The cosmological DE (2.11) is not a gradient DE, except when it is restricted to the Bianchi I invariant subset $A = v = 0$. Nevertheless, the DE satisfies all the above properties of a gradient DE, because Σ_- is a monotone function along generic orbits, and the orbits in the invariant set $\Sigma_- = 0$ are sufficiently simple (see Figs. 5–11).

Structural stability

It can be argued that, when describing a physical situation mathematically using a DE, the DE should be *structurally stable.* Intuitively this means that a sufficiently small change of the DE does not change the qualitative behavior of the solutions. We refer to Ref. 11 for a simple introduction to this topic.

The autonomous DE's that are associated with the EFE's exhibit two features which imply that they are not structurally stable. First, they admit *saddle connections,* that is, orbits which are positively asymptotic and negatively asymptotic to distinct saddle points [12]. In the present situation saddle connections occur in the heteroclinic sequences, examples of which were given in Sec. IV.

The second structurally unstable property of the DE's associated with the EFE's is that they admit *nonisolated* equilibrium points, the most important being the Kasner circle $\mathcal{K}$ (see WH). In the present situation we have the circles $\mathcal{K}, \mathcal{K}^{\pm}$, the lines $\mathcal{C}^{\pm}_{\pm}, \mathcal{H}$ and $\mathcal{F}$, and the disc $\mathcal{D}$. Nonisolated equilibrium points also occur in problems in population dynamics and chemical kinetics [13], but have not been studied extensively in the dynamical systems literature. An exception is the book by Aulbach [13].

The implication of this structural instability of the DE's that are associated with the EFE's is not clear, and is probably worthy of further study. In any case, it should be noted that a DE that is structurally unstable with respect to general perturbations, may be structurally stable if the DE and the perturbations possess some symmetry property [14].

Bifurcations

The analysis in Sec. III shows that as the equation of state parameter γ varies, two types of bifurcations occur (see Ref. [15] for a brief introduction to this concept). Bifurcations are usually associated with a transfer of stability.

Transcritical bifurcations:

(1) At $\gamma = \frac{6}{5}$, the equilibrium point $\tilde{M}$ passes through M^- and enters the physical state space. $\tilde{M}$ loses stability, and M^- gains stability (compare Figs. 7 and 8).

(2) At $\gamma = \frac{4}{3}$, the equilibrium point $\tilde{M}$ passes through M, leading to a gain of stability by $\tilde{M}$ and a loss of stability by M (compare Figs. 8–10).

Line bifurcations:

(1) The line $\mathcal{F}$ of equilibrium points appears at $\gamma = \frac{2}{3}$, and leads to an exchange of stability between F and $\tilde{M}$ (compare Figs. 5–7).

(2) The lines $\mathcal{C}^{\pm}_{\pm}$ join the circle $\mathcal{K}$ to the circles $\mathcal{K}^{\pm}$ (see Fig. 3), for $\frac{2}{3} \leq \gamma \leq 2$, and lead to an exchange of stability between the circles $\mathcal{K}$ and $\mathcal{K}^{\pm}$, with $\mathcal{K}$ gaining stability as γ increases.

Line bifurcations occur in other analyses of the EFE's [16], and in population dynamics, but to the best of our knowledge, have not been studied systematically in the literature.

VI. CONCLUSION

We have given a complete analysis of the qualitative evolution of tilted irrotational Bianchi V cosmological models. However, our results only give a glimpse into the evolution of the class of all tilted Bianchi cosmologies, for which the state space is eight-dimensional (class A models) or seven-dimensional (class B models). We refer to Rosquist and Jantzen [19] and Bogoyavlensky [20] for a derivation of a DE which governs the evolution of the general class of tilted Bianchi models. We note, however, that their choice of basic variables differs from ours.

It is thus natural to ask whether one can hope to extend the analysis of the present paper. The first step is to find all the equilibrium points of the DE. The equilibrium points correspond to transitively self-similar solutions of the EFE's [21], that is, solutions which admit a four-parameter similarity group acting transitively on spacetime. So the problem becomes finding all Bianchi models which are transitively self-similar. This problem has been solved in the orthogonal (not tilted) case [22] but not in the tilted case, although some particular tilted solutions are known [23]. The nontilted equilibrium points will, however, play an important role in the qualitative analysis of the tilted models, since they lie on the boundary of the tilted physical state space. For example, in the present paper, we have shown the existence of an instability of the Kasner solutions, that is associated with the presence of tilt. This instability is generated mathematically by the lines of equilibrium points $\mathcal{C}^{\pm}_{\pm}$ that branch out of the Kasner circle. Thus, even if the tilted equilibrium point problem cannot be solved completely, it may be possible to study the stability of the nontilted equilibrium points in the full tilted state space, and thereby gain insight into the evolution of the tilted models.

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APPENDIX A: PROPERTIES OF THE FLUID CONGRUENCE

In a tilted Bianchi cosmology, there are two preferred timelike congruences, the normal congruence $\mathbf{n}$ and the fluid congruence $\mathbf{u}$, and hence two types of preferred orthonormal frame. It is convenient, as we have done, to perform the mathematical analysis in a *geometrical frame* $\{\mathbf{e}_a\}$, which is adapted to the normal congruence ($\mathbf{e}_0 = \mathbf{n}$). For some aspects of the interpretation, however, it is necessary to work in a *physical frame* $\{\hat{\mathbf{e}}_a\}$, which is adapted to the fluid ($\hat{\mathbf{e}}_0 = \mathbf{u}$). In the present situation the two frames are related by a Lorentz transformation in the two-space spanned by $\mathbf{e}_0$ and $\mathbf{e}_1$:

$$\begin{aligned} \hat{\mathbf{e}}_0 &= \cosh\beta\mathbf{e}_0 + \sinh\beta\mathbf{e}_1 \\ \hat{\mathbf{e}}_1 &= \sinh\beta\mathbf{e}_0 + \cosh\beta\mathbf{e}_1 . \end{aligned} \tag{A1}$$

We now express the kinematical quantities of the fluid in terms of Σ_+, Σ_-, A and v. Let $\hat{\theta}, \hat{\sigma}_{\alpha\beta}$ and $\dot{u}_\alpha$ denote the expansion scalar, the shear tensor and the acceleration vector of the fluid congruence relative to the physical frame $\{\hat{\mathbf{e}}_a\}$. It follows that $\hat{\sigma}_{\alpha\beta}$ is diagonal, and that $\dot{u}_\alpha = \dot{u}_1\delta^1_\alpha$. As with the normal congruence [see Eqs. (2.2) and (2.7)], we define the dimensionless shear variables

$$\hat{\Sigma}_+ = \frac{\hat{\sigma}_+}{\hat{\theta}}, \qquad \hat{\Sigma}_- = \frac{\hat{\sigma}_-}{\hat{\theta}}.$$

We also define the dimensionless acceleration variable by

$$\dot{U} = \frac{\dot{u}_1}{\hat{\theta}}.$$

It follows from Eqs. (A1) by using the commutators, Eq. (2.3), and the evolution equation for $\dot{v} \equiv \mathbf{e}_0(v)$, that

$$\hat{\theta} = \frac{B}{\sqrt{1-v^2}}\theta, \tag{A2}$$

$$1 + \hat{\Sigma}_+ = \frac{1}{B}(1 + \Sigma_+ - vA), \tag{A3}$$

$$\hat{\Sigma}_- = \frac{1}{B}\Sigma_-, \tag{A4}$$

$$\dot{U} = (\gamma - 1)v, \tag{A5}$$

where

$$B = \frac{3 - 2vA + v^2(2\Sigma_+ - 1)}{3[1 - (\gamma - 1)v^2]}. \tag{A6}$$

The relation between the fluid density $\hat{\mu}$ and the normal density μ is

$$\hat{\mu} = \frac{(1 - v^2)}{1 + (\gamma - 1)v^2}\mu, \tag{A7}$$

as follows from Eqs. (2.3) and (2.6). Observe that the transition between the geometrical frame is well defined if and only if $v^2 < 1$ (i.e., β is finite). In the limit $v^2 \to 1$, the normal congruence becomes null.

We now derive an inequality which implies that θ and $\hat{\theta}$ have the same sign.

Proposition A.1: If $\Omega > 0$ and $v^2 < 1$ then

$$(1 + \Sigma_+)^2 - A^2 > 0.$$

Proof: Suppose that $(1 + \Sigma_+)^2 = A^2$. Since $A \geq 0$, $1 + \Sigma_+ = A$. It follows by eliminating A and Σ_+ from Eq. (2.13) using Eq. (2.12), that

$$[1 + (\gamma - 1)v^2]\Sigma_-^2 + (1 - v)[1 - (\gamma - 1)v]\Omega = 0,$$

a contradiction. Thus $(1 + \Sigma_+)^2 - A^2$ does not change sign on the subset defined by $\Omega > 0$ and $v^2 < 1$ and since it is positive at the equilibrium point F, it is positive on the whole set. □

Comment: The geometrical interpretation of this inequality is given in Ref. [17].

Proposition A.2: For nonvacuum models ($\Omega > 0$) with nonextreme tilt ($v^2 < 1$), $\hat{\theta} > 0$ if and only if $\theta > 0$.

Proof: We can rearrange equation (A6) to obtain

$$B = \frac{(1 - v)[3(1 + v) - 2vA] + 2v^2[(1 + \Sigma_+) - A]}{3[1 - (\gamma - 1)v^2]}.$$

Proposition A.1 implies that $A < 1 + \Sigma_+$, and since $v^2 < 1$ and $0 \leq A < 1$, it follows that $B > 0$. The desired conclusion is an immediate consequence of Eq. (A.2). □

APPENDIX B: EIGENVALUES OF THE LINEARIZATION OF THE DE AT THE EQUILIBRIUM POINTS

We list the eigenvalues of the linearization of the DE at its equilibrium points, which are given in Sec. III.

$F: -\frac{3}{2}(2 - \gamma), -\frac{3}{2}(2 - \gamma), \frac{1}{2}(3\gamma - 2)$ (eliminate v)

$M: -2, -(3\gamma - 2), (3\gamma - 4)$ (eliminate Σ_+)

$\mathcal{F}: -2, 0, -2$ (eliminate Σ_+)

$\mathcal{K}: 0, 3(2 - \gamma), 2\Sigma_+ + (3\gamma - 4)$ (eliminate A)

$\tilde{M}: -2, -\frac{2 - \gamma}{\gamma - 1}, -\frac{(3\gamma - 4)(5\gamma - 6)}{(\gamma - 1)(9\gamma - 10)}$ (eliminate Σ_+)

$\mathcal{C}_\pm^\pm: 0, 3(2 - \gamma), 0$ (eliminate A)

$M^+: -2, 0, 2, (\gamma < 2)$ (eliminate Σ_+)

$M^-: -2, -4, -\frac{2(5\gamma - 6)}{2 - \gamma}, (0 < \gamma < 2)$ (eliminate Σ_+)

$\mathcal{K}^\pm: 0, 2(1 + \Sigma_+), -\frac{2}{2 - \gamma}[2\Sigma_+ + (3\gamma - 4)]$ (eliminate A)

$\mathcal{H}: 0, -2(1 + \Sigma_+), 2(1 - 2\Sigma_+)$ (eliminate A)

APPENDIX C: ASYMPTOTIC BEHAVIOR AS $\tau \to +\infty$

Proposition C.1: $\lim_{\tau \to +\infty} \Sigma_- = 0$, for all nonsingular orbits with $\Omega > 0$.

Proof: If $v = 0$, the result is known to be true, since untilted nonvacuum models of Bianchi type I or V are known to isotropize, except for the Bianchi I models with $\gamma = 2$, which correspond to the equilibrium points (singular orbits) in the disc $\mathcal{D}$. This is the reason for excluding singular orbits in the statement of the proposition.

We now consider the case $v \neq 0$, which implies, by Eq. (2.12), that $A \neq 0$. The evolution equation for Σ_- is

$$\Sigma_-' = -(2 - q)\Sigma_-. \tag{C1}$$

It follows from Eq. (2.14) and the assumption $\Omega > 0$ that $q < 2$. Thus Σ_- is monotone, and hence $\lim_{\tau \to +\infty} \Sigma_- = L$ (exists). Second, by the LaSalle invariance principle [18] the ω-limit set of a nonsingular orbit Γ with $\Omega > 0$ must satisfy

$$\omega(\Gamma) \subset \{\Sigma_- = 0\} \cup \{q = 2\}.$$

Suppose that $L \neq 0$. Then $\omega(\Gamma) \subset \{q = 2\}$ and hence $\lim_{\tau \to +\infty} q = 2$. It follows from Eq. (2.14) that $\lim_{\tau \to +\infty} A = 0$ and $\lim_{\tau \to +\infty}(\Sigma_+^2 + \Sigma_-^2) = 1$, which implies that $\lim_{\tau \to +\infty} \Sigma_+ = M > -1$. Since $q + 2\Sigma_+ = (q - 2) + 2(\Sigma_+ + 1)$, the evolution equation $A' = (q + 2\Sigma_+)A$, implies that if τ is sufficiently large, then $A' > 0$, contradicting $\lim_{\tau \to +\infty} A = 0$. We conclude that $L = 0$. □

This proposition shows that the ω-limit set of any nonsingular orbit with $\Omega > 0$ is contained in the invariant set $\Sigma_- = 0$. More information can be obtained if we consider the cases $\frac{2}{3} < \gamma < 2$ and $0 < \gamma < \frac{2}{3}$ separately.

Proposition C.2: If $\frac{2}{3} < \gamma < 2$, then for all orbits Γ with $\Omega > 0$, $v^2 < 1$ and $A > 0$, the ω-limit set $\omega(\Gamma)$ is one of the equilibrium points M, $\tilde{M}$, and M^-.

Proof: By Proposition C.1, $\omega(\Gamma) \subset S \equiv \{\Sigma_- = 0\}$. The set S contains the isolated equilibrium points $\{T, T^+, Q, Q^-, F, M^+\}$, the line $\mathcal{H}$ of equilibrium points, and the isolated equilibrium points $\{M, M^-, \tilde{M}\}$ ($\tilde{M} \in S$ if and only if $\frac{6}{5} < \gamma < 2$). It can be shown, by studying the eigenvalues, and using the DE restricted to appropriate invariant sets, that the stable manifolds (attracting sets) of the first set of equilibrium points and of the line $\mathcal{H}$, satisfy at least one of $\Omega = 0$, $v^2 = 1$, $A = 0$. Further, orbits with $\Sigma_- = 0$ and hence orbits with Σ_- sufficiently close to zero, are repelled by these equilibrium points (see Figs. 7–10). Thus the desired conclusion follows. □

Comment: It follows that the class of orbits referred to in Proposition C.2 satisfies

$$\lim_{\tau\to+\infty} \Sigma_- = \lim_{\tau\to+\infty} \Sigma_+ = 0, \qquad \lim_{\tau\to+\infty} A = 1.$$

The different possibilities in Figs. 7–10 are distinguished by the limiting value of v.

Proposition C.3: If $0 < \gamma < \frac{2}{3}$, then for all orbits Γ with $\Omega > 0$, $v^2 < 1$, the ω-limit set $\omega(\Gamma)$ is the equilibrium point F.

Proof: The proof is similar to that of Proposition C.2. Because of the exchange of stability between F and M that occurs at $\gamma = \frac{2}{3}$, all orbits are attracted to F. □

Comment: It follows that the class of orbits referred to in Proposition C.3 satisfies

$$\lim_{\tau\to+\infty} \Sigma_- = \lim_{\tau\to+\infty} \Sigma_+ = \lim_{\tau\to+\infty} A = \lim_{\tau\to+\infty} v = 0.$$

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