Dependence of density perturbations on the coupling constant in a simple model of inflation

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In the standard inflationary scenario with inflaton potential $V(\Phi)=M^4-\frac{1}{4}\lambda\Phi^4$, the resulting density perturbations $\delta\rho/\rho$ are proportional to $\lambda^{1/2}$. Upper bounds on $\delta\rho/\rho$ require $\lambda \lesssim 10^{-13}$. Ratra has shown that an alternative treatment of reheating results in $\delta\rho/\rho \propto \lambda^{-1}$, so that an upper bound on $\delta\rho/\rho$ does not put an obvious upper bound on λ. We verify that $\delta\rho/\rho \propto \lambda^{-1}$ is indeed a possibility, but show that $\lambda \lesssim 10^{-13}$ is still required.

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The inflationary paradigm [1–4] explains many mysteries of large-scale cosmology. It also provides a source of density fluctuations, which act as the seeds for structure formation, and predicts that these fluctuations have a Harrison-Zel'dovich spectrum [5–8]. The main problem with the standard inflationary scenario is that it requires very small self-couplings of the inflaton field Φ in order to produce mass fluctuations with the correct amplitude of $\delta\rho/\rho \simeq 10^{-5}$ at the horizon crossing. This is because $\delta\rho/\rho \propto \lambda^{1/2}$, where λ is the quartic self-coupling of Φ. It turns out that $\delta\rho/\rho \lesssim 10^{-5}$ requires $\lambda \lesssim 10^{-13}$. Many models have been constructed which attempt to make such small couplings arise naturally.

However, Ratra argues that a very small coupling may not be necessary [9]. He finds that the dependence of $\delta\rho/\rho$ on λ is sensitively dependent on "reheating," that is, on how the transition from the inflationary era to the radiation-dominated era is modeled. In the standard inflationary scenario, the reheating transition takes place in few Hubble times. In Ratra's alternative scenario, reheating is instantaneous (which means, in practice, much less than a Hubble time). In this case Ratra finds that $\delta\rho/\rho$ is proportional to λ^{-1}, a dramatically different result. Since, as Ratra points out, the reheating process is quite complicated, involving nonequilibrium thermodynamics of a quantum field in curved space, we should be cautious about adopting a specific model of it unless we are convinced that its predictions are robust. It is therefore extremely important to check this point, and to see whether or not a small $\delta\rho/\rho$ can result from a coupling which is larger than $\lambda \simeq 10^{-13}$.

We have reanalyzed Ratra's results for the simple potential

$$V(\Phi)=M^4-\tfrac{1}{4}\lambda\Phi^4 , \tag{1}$$

where M is a constant, and $\Phi=0$ at the start of inflation. Of course, this potential is unbounded below and must be modified for $\Phi > \Phi_{\rm max}$, where $\Phi_{\rm max}=(4/\lambda)^{1/4}M$ and is defined via $V(\Phi_{\rm max})=0$. This potential was originally intended to mock up a Coleman-Weinberg potential in a gauge theory (in which case $\lambda \sim g^4$, where g is the gauge coupling). This possibility was subsequently discarded (since $\lambda \sim g^4$ is much too large), but the prediction for $\delta\rho/\rho$ from the potential of Eq. (1) was throughly analyzed in both the standard scenario and in Ratra's alternative scenario, and therefore provides a good test case. Ratra has also analyzed several other possible potentials, but we will not do so here. All of our results will apply strictly to the potential of Eq. (1); we will have nothing to say about Ratra's other models, although it would be interesting to compare his results for an exponential potential with those of, for example, Ref. [10].

Ratra's analysis includes a complete rederivation of the fluctuation amplitude and spectrum, making use of gauge-noninvariant variables followed by careful identification of the gauge-variant modes. However, the final result can (necessarily) be derived using the more standard gauge-invariant formalism of Bardeen [11]. In fact, we can simply use the final formula of Bardeen, Steinhardt, and Turner (BST) [8], without reference to its long derivation. Many other analyses have confirmed this formula, except for small differences in the overall normalization. These will not be relevant, however.

The BST formula for $\delta\rho/\rho$ for a perturbation with wave number k which first crossed out of the horizon at time t_c and then reentered during the matter-dominated era is

$$\frac{\delta\rho}{\rho} \simeq \frac{1}{5\pi}\frac{H^2}{\dot{\Phi}(t_c)} . \tag{2}$$

Here H is the Hubble parameter during inflation, related to M via $H=(8\pi/3)^{1/2}M^2/M_{\rm Pl}$, where $M_{\rm Pl}$ is the Planck mass. The field $\Phi(t)$ is treated as a classical, spatially uniform, background field; quantum fluctuations in Φ are what ultimately result in the density fluctuations of Eq. (2).

Clearly, to compute $\delta\rho/\rho$ we need to compute $\dot{\Phi}(t_c)$. To do so, we use the equation of motion

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$$\ddot{\Phi}+3H\dot{\Phi}-\lambda\Phi^3=0 \tag{3}$$

which follows from the potential of Eq. (1). This equation is easy to solve in the slow-rollover approximation, where we neglect $\ddot{\Phi}$. When this approximation is valid we find

$$\Phi(t)=\left[\Phi_*^{-2}+\tfrac{2}{3}\lambda H^{-1}(t_*-t)\right]^{-1/2} . \tag{4}$$

Here t_* is the time when inflation ends, and Φ_* is the value of Φ at this time: $\Phi_*=\Phi(t_*)$. At the moment we will leave Φ_* as a free parameter, but of course we must have $\Phi_*\le\Phi_{\max}$. The slow-rollover approximation breaks down when $\ddot{\Phi}\simeq 3H\dot{\Phi}$; using Eq. (4), this occurs when $\Phi\simeq\Phi_{\rm SR}$, where

$$\Phi_{\rm SR}=\left[\frac{3}{\lambda}\right]^{1/2} H . \tag{5}$$

Thus we must also have $\Phi_*\le\Phi_{\rm SR}$. Using Eq. (5) we can rewrite Eq. (4) as

$$\Phi(t)=\left[\Phi_*^{-2}+2\Phi_{\rm SR}^{-2}H(t_*-t)\right]^{-1/2} . \tag{6}$$

Then we can use $3H\dot{\Phi}=\lambda\Phi^3$, valid during the slow-rolling epoch, to compute $\dot{\Phi}(t_c)$. The factor of $H(t_*-t_c)$ which appears is related to k and M via

$$\Delta\beta\equiv H(t_*-t_c)$$
$$\simeq 69+\ln(k_U/k)+\ln(M/M_{\rm Pl}) , \tag{7}$$

where k_U is the wave number of the present Hubble radius ($2\pi/k_U\simeq 10^{28}$ cm), and we have implicitly assumed a reheating temperature of order M. (This is not essential, and was done only to simplify the formula.) We ultimately find

$$\frac{\delta\rho}{\rho}\simeq\frac{3H^3}{5\pi\lambda}\left[\Phi_*^{-2}+2(\Delta\beta)\Phi_{\rm SR}^{-2}\right]^{3/2} . \tag{8}$$

This is the key equation from which we will be able to understand the difference between the standard scenario and the alternative scenario.

In the standard scenario, inflation ends when the slow-rollover approximation breaks down: once Φ exceeds $\Phi_{\rm SR}$, the field moves rapidly to the minimum of the potential. Thus, in the standard scenario, we have $\Phi_*\simeq\Phi_{\rm SR}$. Since $\Delta\beta\gg 1$, Eq. (8) implies

$$\frac{\delta\rho}{\rho}\simeq\frac{1}{5\pi}\left[\frac{8}{3}\right]^{1/2}(\Delta\beta)^{3/2}\lambda^{1/2} \quad \text{[standard scenario]} . \tag{9}$$

This is the usual result; in particular, we see that $\delta\rho/\rho$ is proportional to $\lambda^{1/2}$, and that $\delta\rho/\rho\lesssim 10^{-5}$ for $\Delta\beta\gtrsim 45$ requires $\lambda\lesssim 10^{-13}$.

Ratra, however, suggests that Φ_* should not be identified with $\Phi_{\rm SR}$. Instead, he proposes that Φ_* may be much less than Φ_{SR}. Strictly within the context of the potential of Eq. (1), this is not possible. However, we can consider a modified potential, one which drops quickly to zero for $\Phi>\Phi_*$. In this case, inflation would end when Φ reaches Φ_*. This is the scenario that Ratra refers to as "rapid reheating." If $\Phi_*\ll(\Delta\beta)^{-1/2}\Phi_{\rm SR}$, then Eq. (8) yields

$$\frac{\delta\rho}{\rho}\simeq\frac{3}{5\pi\lambda}\left[\frac{H}{\Phi_*}\right]^3 \quad \text{[alternative scenario]} . \tag{10}$$

We see that now $\delta\rho/\rho$ is proportional to λ^{-1}, confirming Ratra's result.

Let us now examine what limits, if any, can be placed on λ in the alternative scenario. Since we have a new free parameter Φ_* it would seem that we could increase λ yet keep $\delta\rho/\rho$ fixed by simultaneously decreasing Φ_* This is correct, but only as long as we remain within the range of validity of Eq. (10), $\Phi_*\ll(\Delta\beta)^{-1/2}\Phi_{\rm SR}$. From Eq. (5), however, we see that $\Phi_{\rm SR}$ decreases as λ increases, so larger values of λ put tighter constraints on the allowed values of Φ_* in the alternative scenario. To get a global overview, let us start from Eq. (8), which is always valid. Consider keeping λ fixed, and varying Φ_* in order to minimize $\delta\rho/\rho$. It is clear from Eq. (8) that minimizing $\delta\rho/\rho$ with λ fixed requires *maximizing* Φ_*. But the maximum value of Φ_* is $\Phi_{\rm SR}$, and $\Phi_*=\Phi_{\rm SR}$ just results in the standard scenario. This implies that, for a given value of λ, the smallest possible $\delta\rho/\rho$ is achieved in the standard scenario. Thus, achieving the same value of $\delta\rho/\rho$ in the alternative scenario requires a *smaller* value of λ than is needed in the standard scenario. For example, to get $\delta\rho/\rho\simeq 10^{-5}$ with $\Delta\beta\simeq 60$ requires $\lambda\simeq 4\times 10^{-14}$ in the standard scenario. In the alternative scenario with $\Phi_*=\frac{1}{10}(\Delta\beta)^{-1/2}\Phi_{\rm SR}$, we find that $\lambda\simeq 3\times 10^{-19}$ is required. More generally, it is easy to check that $\Phi_*=10^{-\nu}(\Delta\beta)^{-1/2}\Phi_{\rm SR}$ requires $\lambda\simeq 3\times 10^{-13-6\nu}$ for $\nu\gtrsim 1$. Thus we conclude that, while it is possible to arrange a potential for which $\delta\rho/\rho\propto\lambda^{-1}$, the upper limit on λ actually *decreases*, which is the opposite of the desired goal.

Also, we see that getting $\delta\rho/\rho\propto\lambda^{-1}$ does not really depend on how much time it takes for reheating to occur, but rather on when inflation ends. The important point is whether Φ_* is larger or smaller than $(\Delta\beta)^{-1/2}\Phi_{\rm SR}$. If $\Phi_*\gg(\Delta\beta)^{-1/2}\Phi_{\rm SR}$, then inflation ends due to the increasing acceleration of Φ in a smooth potential; this is the standard scenario. If $\Phi_*\ll(\Delta\beta)^{-1/2}\Phi_{\rm SR}$, then inflation ends due to Φ crossing a sudden, sharp feature in the potential; this is the alternative scenario. We feel that the two scenarios would be more aptly named "late turn off" and "early turn off," corresponding to whether inflation ends after or before Φ reaches $(\Delta\beta)^{-1/2}\Phi_{\rm SR}$, rather than "slow reheating" and "fast reheating." As we have seen, whether $\delta\rho/\rho$ is proportional to $\lambda^{1/2}$ or λ^{-1} does not actually depend on the speed of reheating, but rather on the value of the field when inflation ends.

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