Limiting rotational period of neutron stars

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We seek an absolute limit on the rotational period for a neutron star as a function of its mass, based on the minimal constraints imposed by Einstein's theory of relativity, Le Chatelier's principle, causality, and a low-density equation of state, uncertainties in which can be evaluated as to their effect on the result. This establishes a limiting curve in the mass-period plane below which no pulsar that is a neutron star can lie. For example, the minimum possible Kepler period, which is an absolute limit on rotation below which mass shedding would occur, is 0.33 ms for a $M=1.442M_{\odot}$ neutron star (the mass of PSR1913+16). A still lower curve, based only on the structure of Einstein's equations, limits any star whatsoever to lie in the plane above it. Hypothetical stars such as strange stars, if the matter of which they are made is self-bound in bulk at a sufficiently large equilibrium energy density, can lie in the region above the general-relativistic forbidden region, and in the region forbidden to neutron stars.

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I. INTRODUCTION

Limits are frequently of interest as a means of distinguishing between alternative interpretations of an observed phenomenon. For example, Rhoades and Ruffini [1] derived an upper bound on the mass of a neutron star. It is $\sim 3.2M_{\odot}$. Several compact objects whose inferred mass is larger than this have been identified as candidates for moderate mass black holes on this basis [2,3]. In this paper we derive a lower limit on the rotational period of a gravitationally bound star as a function of its mass. Our purpose here is to provide a decisive means, based on rotational period and mass, of distinguishing between pulsars that *can* be neutron stars, or more generally, gravitationally bound stars, and pulsars that *cannot*.

It is timely that such a limit be established. Since the discovery of the first millisecond pulsar in 1982 [4], the discovery rate of fast pulsars has quickened, culminating in the recent observation of ten in a survey of the globular cluster 47Tucanae [5]. More than half of them are in binary systems, auguring well for mass determinations. These discoveries are interesting for a number of reasons, including the evolutionary history of globular clusters [6]. The particular significance for the subject of this paper of these and other globular cluster pulsars is that the interstellar dispersion to a cluster can be very accurately calibrated once one discovery is made. This facilitates the search for other fast pulsars within the same cluster, perhaps having still shorter periods, because the interstellar dispersion is a well-known factor that limits the sensitivity of searches for short period pulsars [7].

For a given mass pulsar, how short can the rotational period be if the pulsar is a neutron star, and what would be the significance for its nature if a shorter period were observed?

So far as is known all stars are bound by the gravitational interaction including neutron stars and hybrid stars, the form that neutron stars would take if the pressure in the core is sufficiently high to convert nucleons to quark matter. Other types of stars have been conceived but not identified. They are made of hypothetical matter that is stable and *self*-bound at high density. As such, they would not be subject to the limit on rotation of a gravitationally bound star, and could if self-bound at sufficiently high density, rotate at periods below that limit. We will return to this alternative later.

Several authors have previously investigated how fast neutron stars can rotate, based on particular theories of dense matter [8–10]. The minimum Kepler period for those theories, below which mass shedding would occur, is ~ 0.5 ms for $1.44M_{\odot}$ stars. However, given the uncertainty in the equation of state in the high-density domain of matter, no decisive conclusion could be reached if a pulsar with a shorter period were found, save that none of those theories is correct. *A decisive rotational limit for gravitationally bound (i.e., neutron) stars can be derived only from a set of constraints that do not exceed our certain knowledge and principles, or if some do, that the uncertainty in the outcome can be accurately gauged.*

In this paper we establish with reasonable accuracy, and without recourse to any but a minimal set of physical assumptions as enumerated in the next section, a region of the mass-period plane that is inaccessible to neutron stars. To do this we adopt a flexible ansatz for the equation of state and then by means of a modified Levenberg-Marquardt method [11] solve the nonlinear least-squares problem posed, namely, that of finding the minimum rotational period of neutron stars of given mass by varying the parameters of the ansatz. Included in the ansatz is the possibility that Le Chatelier's principle is stretched to its limit, which we discuss later. Then we adopt an even more flexible ansatz (but still satisfying the minimal constraints) and determine by how much the period can be lowered. The reduction is 1%. So we believe that within the numerical accuracies discussed later, we have established a region that is forbidden to neutron stars.

A neutron star is bound by the gravitational interaction. Above the density of approximately that of metallic

^{56}Fe (~ 7.86 g/cm^3), the pressure of cold catalyzed matter is positive, and in the interior of the star it is very large; the density is so high that the nucleons reside in the range of the repulsive interaction of their neighbors. This repulsion, and the high Fermi energy, would blow the star apart into its constituents if gravity were turned off. As is well known, there is a maximum mass compact star, the limiting star, for which all compact stars of greater mass are hydrostatically unstable. For any equation of state such a limit exists [12]. Compact stars rotating at the frequency at which centrifuge and gravity balance, the Kepler frequency, form a sequence that also terminates with a star of maximum mass for the sequence. From the radius-mass relation of gravitationally bound stars, we know that the limiting star has the highest Kepler frequency. This frequency is an upper bound on rotation frequency. Other instabilities may occur at lower frequency and set the effective bound, but the physics is less certain.

In the next sections we enumerate the minimal constraints on the equation of state, describe the variational ansatz, present the results that define the region in the mass-period plane that is inaccessible to neutron stars, test the sensitivity of the results to such things as the variational ansatz, choice of low-density equation of state and even the assumption of the subluminal constraint. We also derive a limit on rotation period for any star, that follows from the structure of Einstein's equations alone. Of course it lies below the limit for neutron stars. We provide a discussion of the implication if a pulsar is found in the region forbidden to neutron stars but allowed by general relativity.

II. MINIMAL CONSTRAINTS ON THE EQUATION OF STATE

We adopt the following as the minimal principles and constraints.

(1) Einstein's general relativistic equations for stellar structure hold.

(2) The matter of the star satisfies $dp/d\rho \geq 0$ which is a necessary condition that a body is stable both as a whole, and also with respect to the spontaneous expansion or contraction of elementary regions away from equilibrium (Le Chatelier's principle).

(3) The high-density equation of state, whatever it is, matches continuously in energy and pressure to the low-density one of Baym, Pethick, and Sutherland [13].

(4) Causal constraint for a *perfect fluid;* a sound signal cannot propagate faster than the speed of light, $v(\epsilon) \equiv \sqrt{dp/d\epsilon} \leq 1$, which is the appropriate expression for sound signals also in general relativity [14].

(We denote pressure by p, energy density by ϵ, and baryon number density by ρ and use units $G = c = \hbar = 1$.) The first two constraints are most secure. The third is easily tested as to its effect on the result. The causal constraint as expressed, is frequently invoked, and most theories of dense matter obey it at least to densities expected in neutron stars, and some theories, such as the $\sigma - \omega$ nuclear field theory explicitly obey it at all densities. Strictly speaking, the above expression, even for a perfect fluid, is the signal speed only in a homogeneous body if v is dependent on ϵ, and in an inhomogeneous body only if v is independent of ϵ. Nevertheless for a perfect fluid it is a general causal constraint (see Appendix). It is not known how to impose a causal constraint on an imperfect fluid. However the effect on our results of lifting the constraint $\sqrt{dp/d\epsilon} \leq 1$ is easy to test. The results of these tests and others are given in Sec. V.

While it is widely assumed that PSR1913+16 with a mass $M = 1.442 \pm 0.003$ is a neutron star [15], for our purpose we cannot make this assumption. However we shall pay special attention to this mass, since it is close to the iron core mass (the Chandrasekhar mass) of presupernova stars, and to the compact objects that supernova appear to produce.

III. VARIATIONAL ANSATZ

In the "low-density" region, up to some fiducial density ρ_0 near nuclear density we use the equation of state of Baym, Pethick, and Sutherland [13] (BPS) for charge neutral nuclear matter, which has input also from the work of Baym, Bethe, and Pethick [16] and of Siemens [17]. We take this to be reasonably constrained by bulk properties of nuclear matter at the saturation density of nuclei. A tabulated entry in the BPS tables that is suitable for the fiducial density occurs at $\rho_0 = 0.1625$ fm^{-3}. Beyond the fiducial density we represent the unknown equation of state by a parametrized form. As an option we allow for the possibility of a first-order phase transition of the type where the pressure is a constant and the energy density increases linearly with baryon density over a certain range. This is the most extreme allowable form since it satisfies the *equality* in Le Chatelier's principle. Above the fiducial density or the phase transition region as the case may be, we adopt a "gamma law" form. At the density at which it becomes noncausal, if it does, we replace it by its causal limit. In summary the equation of state is continuous and defined by the following equations.

(a) Low density:

$$\epsilon(\rho) = \epsilon_{\mathrm{BPS}}(\rho), \quad p(\rho) = p_{\mathrm{BPS}}(\rho) \quad (\text{for } \rho < \rho_0) \; . \tag{1}$$

(b) Phase transition region:

$$\epsilon(\rho) = \frac{\rho}{\rho_0}(\epsilon_0 + p_0) - p_0, \quad p(\rho) = p_0 \quad (\text{for } \rho_0 \leq \rho \leq \rho_1 \equiv \rho_0 + \Delta) \; , \tag{2}$$

where $\epsilon_0 \equiv \epsilon_{\mathrm{BPS}}(\rho_0)$, $p_0 \equiv p_{\mathrm{BPS}}(\rho_0)$. This satisfies the condition that the baryon chemical potential $\mu(\rho) \equiv d\epsilon/d\rho = $ const in the interval Δ. This segment satisfies Le Chatelier's principle at its limit, namely, the equality in constraint number 2.

(c) Parametrized region:

$$\begin{aligned} \epsilon(\rho) &= \frac{A}{\gamma - 1}(u^{\gamma} - u) + u\epsilon_1 + (1-u)(A - p_0) \; , \\ p(\rho) &= A(u^{\gamma} - 1) + p_0 \quad (\text{for } \rho_1 < \rho < \rho_s) \; , \end{aligned} \tag{3}$$

where $u = \rho/\rho_1$ and $\epsilon_1 \equiv (\rho_1/\rho_0)(\epsilon_0 + p_0) - p_0$ if $\Delta \neq 0$

while $u = \rho/\rho_0$ and $\epsilon_1 \equiv \epsilon_0$ if $\Delta \equiv 0$. Constraint 2 requires $A\gamma \geq 0$. If the equation of state, Eq. (3), becomes noncausal at ρ_s defined by $(dp/d\epsilon)_{\rho_s} = 1$, at which density we denote the energy density and pressure by ϵ_s, p_s, we replace it thereafter by the following.

(d) Causal limit:

$$\epsilon(\rho) = \epsilon_s - p_s + p(\rho) ,$$
$$p(\rho) = \frac{1}{2}\left\{ p_s - \epsilon_s + (p_s + \epsilon_s)\left[\frac{\rho}{\rho_s}\right]^2 \right\} \quad (\text{for } \rho \geq \rho_s) . \tag{4}$$

(We note in passing that Le Chatelier's principle at its extreme $dp/d\rho = 0$ or, equivalently, $p =$ const, corresponds also to the mixed phase of a system undergoing a first-order phase transition when it is a *one-component* system. This is a special and extreme example of a first-order phase transition. In multicomponent systems with a first-order phase transition, the pressure, in general, is not a constant in the mixed phase but rather an increasing function of density [18]. Nevertheless we shall refer to this special region, $p =$ const, as a condensate or phase transition region. Since compact baryon stars in equilibrium have particles carrying both baryon or electric charge or both, a first-order phase transition region, if it exists in the star, will almost certainly be of the multicomponent type for which $dp/d\rho > 0$.)

IV. LIMITING VALUE OF ROTATIONAL PERIOD AS A FUNCTION OF MASS FOR NEUTRON STARS

The maximum mass star $M_{\max}$ in the sequence of gravitationally bound compact stars belonging to a given equation of state, has the minimum Kepler period since the mass is the largest and the radius is the smallest because of the gravitational attraction. Therefore we can incorporate the minimal constraints into a variational search over A, γ, and Δ (when the option for a phase transition is exercised) for the lower limit on the rotational period of a neutron star as a function of maximum mass $M_{\max}$ by minimizing the function $f(M,P) = w_1(M - M_{\max})^2 + w_2 P^2$ where w_1, w_2 are weights. We use a modified Levenberg-Marquardt method [11] for finding the minimum of a nonlinear function of its arguments. The function has its minimum when $M = M_{\max}$ and P is the least possible Kepler period for that mass and, of course, $M_{\max}$ and P depend nonlinearly on the parameters mentioned above, through the solution of Einstein's equations. From numerical solutions for relativistic *rotating* stars it is found that the Kepler period of the maximum mass star of a given equation of state can be found to better than 10% from the mass and radius of the *nonrotating* maximum mass star, which is a solution of the Oppenheimer-Volkoff equations. The empirical relation is given by a numerical factor times the classical relation that balances gravity and centrifuge [8,19,20], $\Omega_K \approx 0.625(M/R^3)^{1/2}$, or

$$P_K \approx 10.1\left[\frac{R^3}{M}\right]^{1/2} = 0.0276\left[\frac{(R/\text{km})^3}{M/M_\odot}\right]^{1/2} \text{ ms} . \tag{5}$$

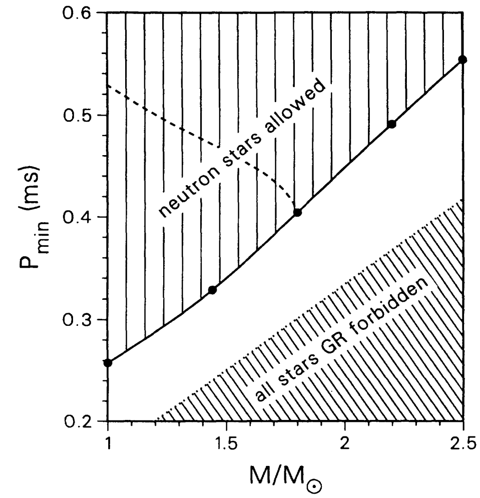

FIG. 1. The minimum rotational period of neutron stars is denoted by the solid curve. Calculated points are shown by dots. Periods of a sequence of stars whose maximum mass is $1.8M_\odot$ are shown by the dashed line. The region forbidden by structure of general relativity [modulo the approximate formula Eq. (5)] is represented by the diagonal shaded region.

The results are summarized in Fig. 1. We show the minimum Kepler periods for star sequences that are subject to the above constraints as a function of the corresponding maximum (limiting) mass star. Neutron stars cannot lie below the solid curve of this figure within the very small limits discussed in detail below. We also show Kepler periods for the sequence whose maximum mass is $1.8M_\odot$. In Sec. VI we derive a region that is forbidden by the structure of general relativity, and it is also shown in the figure.

Although the pulsar 1913+16 with mass $1.442M_\odot$ is thought to be a neutron star we do not know this as a fact. Much less do we know that this star, if a neutron star, is at the mass limit. If it is a neutron star at the mass limit, then the periods of gravitationally bound stars for which our certain knowledge is limited to that enumerated above cannot be less than 0.33 m when the equation of state does not have a first-order phase transition. If it does, the period can be further reduced by 0.01 ms, which falls within the uncertainty of Eq. (5). In any case, for a neutron star with canonical mass of $\sim 1.44M_\odot$, apparently preferred by the creation process, the lowest possible stable period of rotation cannot be less than ~ 0.33 ms.

The bottom two curves shown in Fig. 2 are the equations of state *with* and *without* a phase transition region which yield a $1.442M_\odot$ star with the lowest possible rotational period for a neutron star of that mass. The radii of the corresponding static stars are 6.19 and 6.32 km, respectively, and the central energy densities are 26.1 and 26.0 times normal nuclear saturation density ($\epsilon_s \approx 2.5 \times 10^{14}$ g/cm^3 $= 0.71$ fm^{-4} $= 140$ MeV/fm^3). The high density required of neutron star models (see Fig. 3) if they are to withstand short rotational periods has been cited as reason enough to suspect that short period pulsars, if they exist, are not neutron stars [21,22].

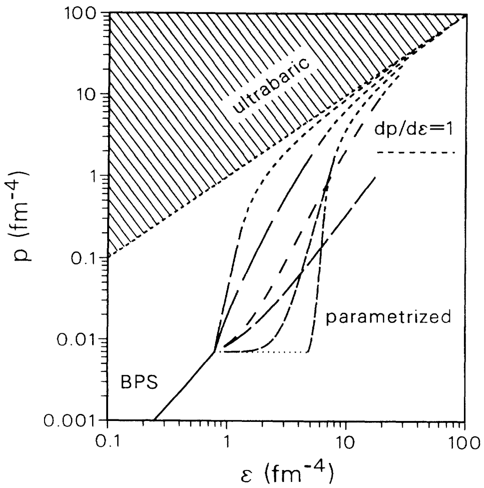

FIG. 2. Bottom two equations of state produce minimum rotational period for a $1.442M_\odot$ neutron star, one with condensate (p = const region) and the other without. Others illustrate the flexibility of the simplest parametrization, Eqs. (2)–(4). The three segments of each are (a) BPS, (b) parametrized region, (c) causal limit $dp/d\epsilon = 1$. Ultrabaric region has $p > \epsilon$. Its boundary is $p = \epsilon$. This region is inaccessible from below when $dp/d\epsilon \leq 1$.

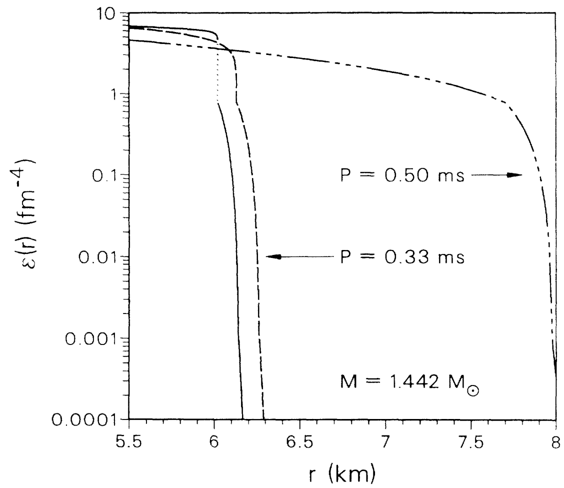

FIG. 4. Density profiles of edge ($R \sim 6.2$ km) of the two limiting mass spherical stars corresponding to the bottom two equations of state of Fig. 2 each with mass $1.442M_\odot$, and Kepler periods for rotating stars of 0.32 and 0.33 ms for the solid and dashed cases, respectively. Phase transition region (p = const) corresponds to the discontinuity in density (dotted). Other profile ($R \sim 8$ km) for same limiting mass but Kepler period 0.5 ms.

The energy density profiles of the edges of the two optimally configured spherical neutron stars as regards the stability of their rotating counterparts to mass shedding at fast rotation ($P \sim 0.33$ ms) are shown in Fig. 4. The notable features of these star profiles is the nearly uniform density interior, and the rapid drop in density very close to the edge, characteristic of an equation of state that is stiff at high density and very soft at intermediate as has been found necessary for stability to fast rotation [10,23]. The fiducial density is at $\epsilon \approx 0.7$ fm^{-4} and the

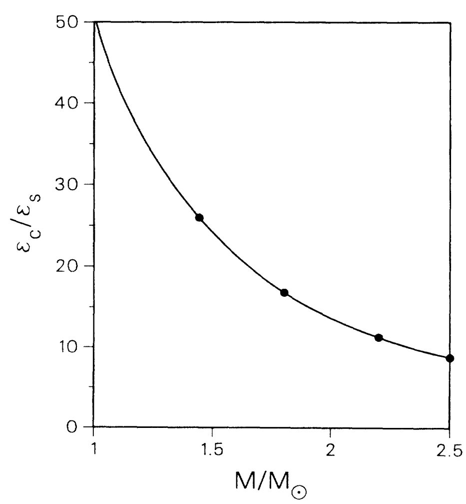

FIG. 3. Central energy density in units of normal nuclear saturation density for the nonrotating neutron stars whose rotating counterparts have the shortest possible Kepler period, namely, those corresponding to Fig. 1.

bulge in both profiles below this density correspond to neutron star matter below the density of nuclear matter, which is represented by the BPS equation of state. It should be noted that the subnuclear crust, which is magnified in this figure by the suppressed origin, is actually very thin, ~ 0.2 km. (The radii of these stars as quoted above is determined more than five decades below the lowest density shown in Fig. 4.) Also for comparison we show the profile of a limiting star of the same mass whose Kepler period is 0.5 ms. This is a more conventional neutron star model and it is evident that to attain the limiting period of 0.33 ms, even though only slightly smaller, the structure of the star, and the underlying equation of state must be radically different. The central energy density of the spherical star with the 0.5-ms Kepler period for the rotating counterpart, is $18.6\epsilon_s$.

It may be of some interest to comment that the Oppenheimer-Snyder limiting mass star, corresponding to an ideal neutron gas, has $M = 0.72M_\odot$, $R = 9.6$ km, and for the rotating counterpart a Kepler period of $P = 0.98$ ms and lies far above the scale of Fig. 1. The Rhoades-Pietronero-Ruffini maximum mass star has $M = 3.2M_\odot$, $R = 13.7$ km, and the rotating counterpart has Kepler period $P = 0.79$ ms and lies about 0.1 ms above the projection of our limiting curve.

V. TEST OF SENSITIVITY OF RESULTS TO DETAILS

Here we give the evidence concerning the accuracy with which the limiting curve is determined which separates the period-mass plane into a region that is accessible to neutron stars and one that is not. Our tests are performed for a canonical mass of $1.442M_\odot$ since this is likely to be near the mass of most neutron stars that are actually produced in supernova, it being rather commonly accepted and understood why the favored mass lies in a rather narrow range (around the Chandrasekhar limit which establishes the mass of the iron core of the progenitor star).

The sensitivity of our results to the choice of "low-density" equation of state was tested by setting the pressure to zero at and below the fiducial density. The corresponding period is reduced thereby by a small amount, 0.014 ms. This extreme softening permits the greatest compaction at the surface and therefore the greatest stability to mass shedding, so no other low-density equation of state having the same pressure and energy at saturation could give a lower value for the limiting period. We remark here that some early calculations [24] of purely neutron matter which neglect the penalty of the high isospin asymmetry imposed by the nuclear force actually predict bound neutron matter at a density near normal nuclear density. However, the Fermi energy contributes only about half the value of the symmetry coefficient in the empirical mass formula, the remainder arising from the isosopin part of the nuclear force, which is neglected in the work that finds neutron matter to be bound. If neutron matter were bound, then $p=0$ at its equilibrium density, and defines the edge of the star which would occur near nuclear density ($\sim 2.5\ 10^{14}$ g/cm^3), as in the above test, instead of at the density of iron (~ 7.86 g/cm^3) as for the lowest energy state of cold catalyzed matter [12]. So our test embraces the extreme case of bound neutron matter. However, there is no reason to expect neutron matter to be bound. Nuclei of Z very different from A become increasingly less bound with increasing $(A-Z)^2$, which gives rise to the well-known "valley of stability."

We have already stated the result that the shortest period of a $1.442M_\odot$ star can be reduced by 0.01 ms below 0.33 ms if the equation state is allowed to have a region of constant pressure as a function of density; such an equation of state satisfies Le Chatelier's principle at its limit (the equality in the second constraint). This is true whether we take the density of the onset of the constant pressure region to lie near nuclear density or above. By the test of the preceding paragraph, the maximum reduction in period that can be gained by having the onset of the region of constant pressure occur at a lower density than the fiducial one is <0.014 ms.

The effect of a specific choice of fiducial density ρ_0 in the range 0.1484–0.1814 fm^{-3} near nuclear saturation amounts to an uncertainty in minimum period of only ± 0.001 ms for our canonical neutron star.

The parametrization of the equation of state used above, and the flexibility to reach the limit allowed by Le Chatelier's principle through the inclusion of a phase transition with constant pressure, as exemplified by the extremes illustrated in Figs. 2 and 4 evidently represents a very flexible family within which we search for a minimum period. To test this we also expanded the variational ansatz with two additional powers of density, to find whether our limit can be significantly lowered. The expanded ansatz is

$$\begin{aligned}\epsilon(\rho)=&\frac{A}{\gamma-1}(u^{\gamma}-u)+\frac{B}{\gamma-2}(u^{\gamma-1}-u)\\&+\frac{C}{\gamma-3}(u^{\gamma-2}-u)\\&+u\epsilon_0+(1-u)(A+B+C-p_0)\,,\\p(\rho)=&A(u^{\gamma}-1)+B(u^{\gamma-1}-1)+C(u^{\gamma-2}-1)\\&+p_0\,,\end{aligned}\tag{6}$$

where $B>0$, $C>0$ are additional parameters that are varied to minimize the period at given mass. (The restriction to positive values assures that condition 2 is always satisfied when $\gamma>2$.) The period could be reduced by only 0.003 ms for a $1.442M_\odot$ star, which is a very small reduction (1%) for an ansatz with two additional parameters.

It is believed that a signal in any medium cannot propagate faster than the speed of light. Nevertheless the principle of causality, which is the usual phrase employed to state this belief, does not follow from Lorentz invariance [25]. Although searches for tachyons have so far proved fruitless, the principle is not proven thereby. Moreover, it is not known what condition the equation of a state of an *imperfect* fluid should satisfy so that superluminal signals cannot propagate in any region of a body made of it. Therefore we test the effect on the minimum rotational period if the causal constraint is lifted. The period can be reduced thereby by 0.037 ms. This is the largest of the uncertainties.

All of the above quoted uncertainties are small and lie within the accuracy of the means of estimating the Kepler period, Eq. (5), save the relaxation of the "causal" constraint, which is slightly larger. We conclude that the solid line of Fig. 1 marks the lower boundary for neutron stars rather accurately. Moreover, as pointed out in Sec. VII, there are gravitational wave instabilities that are expected to occur at periods larger than the Kepler period, so the latter is an absolute bound.

We have not questioned the validity of general relativity as concerns the structure of compact stars. The value of M/R for the limiting mass stars in various models of the equation of state is generally found to lie in the range 0.2–0.3 and for our minimally constrained models that are designed to minimize the rotational period $M/R\approx 0.34$. Solar system phenomena (such as the advance of the perihelion of planets) test general relativity only in the weak-field limit $M/R<2\times 10^{-6}$. However recent experimental results derived from a decade of accurate timing observations on a binary neutron star system [26,15], test the theory to $M/R\approx 0.2$ with the conclusion that correct theories of gravity are very tightly constrained and that Einstein's theory lies at the center of the constrained region. It appears that there is little ground for questioning the appropriateness of the usual equations for relativistic stellar structure, and in any case, we do not know how to assess an error to constraint 1, if the correct theory is even a small variant of the usual one.

VI. GENERAL RELATIVISTIC LIMIT ON ROTATION

It may also be of interest to note a limit that can be derived from the least number of constraints. Assuming only that Einstein's equations of stellar structure hold, then $M/R < \frac{4}{9}$ for any static star [27]. Using the approximate relation Eq. (5) we obtain

$$P \approx 10.1M \left[\frac{M}{R}\right]^{3/2} > \frac{273}{8}M$$

$$= 0.167 \frac{M}{M_\odot} \text{ ms} \quad \text{(any star)} , \quad (7)$$

which for PSR1913+16 is $P > 0.24$ ms. Since the limit on M/R follows from the structure of Einstein's equations the above limit on period applies both to neutron stars as well as to hypothetical stars made of self-bound matter. The forbidden region is marked in Fig. 1.

VII. DISCUSSION AND ALTERNATIVES

The purpose of this paper, as stated earlier, is to provide a decisive means of distinguishing pulsars that *can* be (but are not necessarily) gravitationally bound compact stars such as neutron or hybrid stars (quark-core neutron stars), from those that *cannot.* The limit that we have obtained for the rotational period as a function of star mass is necessarily more severe than would be obeyed by real stars because there is no physical principle that requires the equation of state, whatever it is, to minimize the rotation period. Moreover, there are unstable pulsation modes of a rotating star associated with gravitational radiation reaction [28–30] which occur at larger periods than the Kepler period [31–35]. If a pulsar has a rotation period smaller than any of the periods for these critical modes, it will spin down by gravitational radiation until its period approaches that of the *largest* of the unstable modes. However, the physics that enters the estimate of these instabilities is far less certain than that which determines the Kepler period, so we use this absolute lower bound of a uniformly rotating star. Therefore if a pulsar with rotation period and mass falling below our limiting curve is found, it actually lies even further below the limit established by nature for neutron stars.

There is a region in the period-mass plane below that allowed to neutron stars and above that forbidden by general relativity (Fig. 1). What could be the general nature of a star that might fall in this intermediate region? First we need to be more precise about what is meant by a "neutron" star. Of course qualitatively we mean a star whose constituents by mass are mostly baryons, if not mostly neutrons (with possibly a quark matter core). However the essential feature of an equation of state that purports to describe neutron star matter is its behavior near the saturation density of nuclear matter, ρ_s, a density prescribed by the range of the nuclear force. There the energy density should be approximately $\rho_s m_n$ (where m_n = neutron mass). It could be, though it is unlikely, that neutron matter is self-bound. If so, the binding per nucleon must be less than that of nuclear matter because we know that the energy of a many-nucleon system is lower for isospin symmetric systems, shifted somewhat by the Coulomb energy. The valley of stability tells us this. So if neutron star matter is self-bound the density at which this occurs is near nuclear density and the energy density there is $\approx \rho_s(m_n - b_n)$ where b_n is the neutron matter binding energy per nucleon, bounded as just remarked by $b_n \ll b_{nm} \approx 16$ MeV. The correction to the energy density is therefore negligible. However, if bound, the pressure vanishes at that density. (On the nuclear scale the temperature is low so that we may take $T=0$.) For a star, zero pressure marks its edge because vanishing pressure can support no overlaying matter against the gravitational attraction exerted on it. The energy profile of the star would be similar to the dashed curve of Fig. 4, except that the subnuclear crust, the bulge below $\epsilon \approx 0.7$ fm^{-4} and outside $r \approx 6.13$ km, would be absent. In Sec. V we found that such a truncation would lead to a slightly lower minimum period, about 0.014 ms less at $M = 1.442 M_\odot$. If neutron matter is bound, the mass and rotational limit are only slightly modified. The binding per nucleon is less than 16 MeV, and therefore is small compared to the gravitational binding of a neutron star. This can be explicitly computed for any equation of state, or estimated by $E_g \approx \frac{3}{5}M^2/R$, and is of the order of 100 MeV per nucleon. So we may say of a neutron star that it is gravitationally bound.

The only general category of stars that could have smaller rotation periods than neutron stars is the category of stars that are made of hypothetical matter that is stable and self-bound in bulk at sufficiently high-equilibrium energy density [22,23]. The connection of the limiting rotational period of a star composed of matter that is *self*-bound ($p=0$) at the equilibrium energy density ϵ_e is trivial to establish. Since the density (and pressure) decrease monotonically from the center to the edge of a star, and the edge occurs when $p=0$, at which point the energy density is ϵ_e, it follows that the energy density in the interior, and the average in particular, satisfies $\bar{\epsilon} \geq \epsilon_e$. The equality holds only when gravity is negligible, meaning that the mass is small. The average density also satisfies the identity $M \equiv (4\pi/3)R^3\bar{\epsilon}$. Using this relation in Eq. (5) we have $P_K = 1.6(3\pi/\bar{\epsilon})^{1/2} < 1.6(3\pi/\epsilon_e)^{1/2}$. Observing also the limit imposed on M/R by general relativity, Eq. (7), we can write

$$1.21 \left[\frac{\epsilon_0}{\epsilon_e}\right]^{1/2} > \frac{P_K}{\text{ms}} > 0.167 \frac{M}{M_\odot} \quad \text{(self-bound star)} , \quad (8)$$

where $\epsilon_0 \approx 140$ MeV/fm^3 is the equilibrium energy density of normal nuclear matter. Obviously if ϵ_e is large enough, a few times nuclear density, the period of a self-bound star can lie in the region prohibited to neutron stars. Its energy profile is unlike those of Fig. 4; instead the star has a sharp edge at which the energy density falls from the *high* equilibrium value of the bound matter to zero in a strong interaction length. Such a star has a small radius for its mass as compared to a gravitationally bound star. This is what permits it to rotate rapidly without shedding matter at its equator. We note in passing the limit placed by the general relativistic inequality, $M/R < \frac{4}{9}$, on the equilibrium density of self-bound matter by a star of mass M made of it:

$$\epsilon_e \leq \bar{\epsilon} < \left[\frac{4}{9}\right]^3 \frac{3}{4\pi}\frac{1}{M^2} = 52 \left[\frac{M_\odot}{M}\right]^2 \epsilon_0 \ . \tag{9}$$

In the event that self-bound matter exists in bulk then objects from microscopic nuggets to stars could exit, limited in mass only by gravitational collapse [36,37]. Gravity plays the minor role of preventing self-bound stars from fissioning into smaller bound fragments except by the addition of sufficient energy to compensate the creation of additional surface area.

The most likely candidate for stars made of matter that is self-bound at high density are strange stars, hypothetical stars composed of almost equal numbers of up, down, and strange quarks. Strange quark matter was hypothesized *to be the absolute ground state of the strong interaction* independently by Bodmer [38] and Witten [39,40]. As it turns out, the hypothesis is quite plausible on the basis of scale arguments, and very difficult to either prove or disprove. Several recent accounts of the subject can be found in Refs. [22,41]. Another candidate in the category of self-bound stars are the so-called Q stars [42,43]. These have been discussed as exotic solutions of *effective* nuclear field theories. While exotic solutions of *fundamental* theories, such as the sphaleron [44] of the classical Weinberg-Salam theory of the electroweak interaction, must be taken seriously, the status of an exotic solution of an *effective* theory is much more tenuous. Moreover, the properties of hypothetical Q stars, whether dilute, very large and of many solar masses, or dense, small and of solar mass, are not delimited by any known physics. In contrast, the argument for the possible existence of strange stars depends on simple physics and the QCD energy scale which place the energy per nucleon of strange matter *close* to the nucleon mass, the only uncertainty being whether a fraction of a nucleon mass above or below [39, 22].

VIII. SUMMARY

A neutron star cannot have a period for uniform rotation that lies by more than a few percent below the solid curve of Fig. 1. For a $1.442M_\odot$ neutron star this means that the period must exceed $0.33^{+0.03}_{-0.04}$ ms where the upper error is that of Eq. (5) and the lower is the largest of the test errors and corresponds to lifting the causal constraint appropriate for a perfect fluid (where 0.037 is rounded to 0.04). This curve refers to the mass-shedding limit (Kepler period) and gravitational-wave instabilities will actually set a more stringent lower bound that lies *above* our curve. Therefore the most conservative bound for the region excluded to neutron stars is the one adopted here. If a pulsar is found with mass and period that place it below our critical curve, it must be a different kind of object and it appears that the only alternative for breaking the limit on gravitationally bound stars is a state of matter that is self-bound in bulk at an energy density larger by a factor of 5 to 10 than normal nuclear matter. The most plausible candidate to date, in our opinion, is strange quark matter.

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APPENDIX

From the derivation for sound propagation in relativistic fluid dynamics for a perfect fluid [45], one finds that $v(\epsilon) \equiv \sqrt{dp/d\epsilon}$ must be a constant in space-time to represent the velocity of sound propagation. Otherwise it does not factor out of a derivative and yield the wave equation. At first sight, even for a perfect fluid, it might appear that $v(\epsilon) \leq 1$ is a causal constraint only for homogeneous bodies, or for any body provided that v is a constant independent of ϵ for the density range found in the body. However for perfect fluids, $v(\epsilon) \leq 1$ is a causal constraint in any case, as the following reasoning indicates: Imagine that in some range of ϵ, the opposite condition holds, namely, that $v(\epsilon) > 1$. Make a uniform body at such a density and one has a medium which carries superluminal signals. Then in some Lorentz frame in which a signal is seen to leave point A and arrive later at B, there are other frames in which it arrives at B before it has left A. Alternately consider, as an example of an inhomogeneous body, a neutron star whose density range includes the range in which $v(\epsilon) > 1$. Between any two points in that region, A and B, that are sufficiently close that $dp/d\epsilon$ is essentially constant, a signal propagates according to the usual wave equation with speed v. Over the large central region of a 1.4 solar mass neutron star the relative variation in energy density is very small over meter distances in the radial direction $[(\Delta\epsilon/\epsilon_{\text{central}})/\Delta R < 10^{-6}/\text{meter}]$. If the relative energy change is so small over macroscopic distances, then a suitably small relative change in $dp/d\epsilon$ is assured. Take a perpendicular direction to the radial one and the variation is even less. No matter the distance, causality is violated in the sense of superluminal signals if $v > 1$. One could arrange a number of devices to propagate a superluminal signal over larger distances once one can do so over a small one (say by attaching optic fibers, or constructing relay stations, or drilling holes to A and B from opposite directions, etc.). So $\sqrt{dp/d\epsilon} \leq 1$ is in fact required to ensure nonsuperluminal transmission of signals in any body made of a perfect fluid, whether or not the above derivative is a constant. Whether or not a practical device can be constructed so as to take preemptive action based on a superluminal signal is a separate question.

The above discussion does not pertain to imperfect fluids, and for this reason we have tested the effect of the constraint on the limiting period of rotation of a neutron star.

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