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# Pressure in thermal scalar field theory to three-loop order

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In scalar field theory, with a $g^2\phi^4$ interaction, at temperature $T$ we compute the complete contribution to the pressure to order $g^4$. This involves a three-loop thermal diagram. We neglect the zero-temperature mass.

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## I. INTRODUCTION

This Brief Report is about scalar field theory with an interaction potential $g^2\phi^4$ in thermal equilibrium at temperature $T$. We assume that $T$ is much greater than the zero-temperature mass, which we consequently neglect. Contributions to the pressure of orders $T^4$, $T^4g^2$, and $T^4g^3$ are well known [1]. The purpose of this Brief Report is to calculate the contributions of order $T^4g^4$. This involves the three-loop diagram (h) of Fig. 1, whose computation is our main new result.

For completeness, we first discuss lower-loop diagrams. We employ the resummation method [2], as formulated for scalar field theory in [3]. Thus we use the "free" Lagrangian

$$\tfrac{1}{2}[(\partial\phi)^2-M^2\phi^2] \tag{1}$$

and the "interaction" Lagrangian

$$\tfrac{1}{2}M^2\phi^2-\mu^\epsilon g^2\phi^4+a\mu^\epsilon g^4\phi^4 \ . \tag{2}$$

Here

$$M^2=g^2T^{2-\epsilon}\mu^\epsilon \tag{3}$$

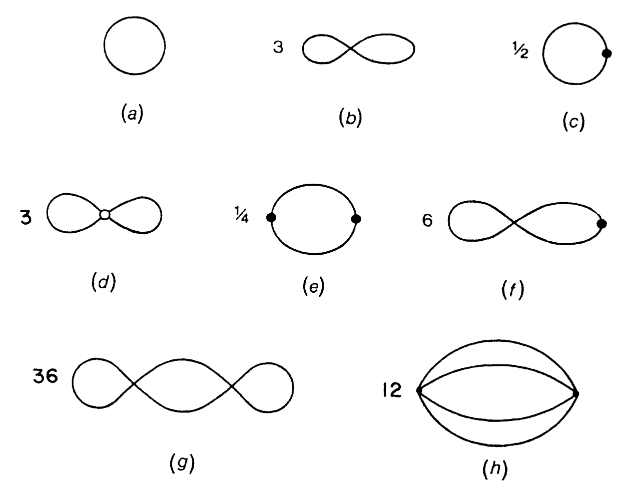


FIG. 1. Thermal graphs contributing to the pressure to the required order. The black dot denotes the $M^2$ "counterterm" in (2), and the open circle represents the ultraviolet divergence counterterm $a$ in (2). Combinatoric factors are shown in the diagram.

is the leading term in the thermal mass. (We employ dimensional regularization with space-time dimension $4-\epsilon$ and minimal subtraction with the unit of mass, $\mu$.) The insertion of the canceling $M^2$ terms in (1) and (2) constitutes the resummation [3]. The last term in (2) is the (zero-temperature) ultraviolet renormalization (to the order we require), with

$$a=\frac{9}{2\pi^2}\frac{1}{\epsilon} \ . \tag{4}$$

In the diagrams the $M^2$ "counterterm" in (2) is represented by a black dot and the counterterm (4) is represented by an open circle [in graph (d) of Fig. 1]. The propagators all have mass $M$.

We employ the imaginary-time formalism rather than the real-time one. We cast the imaginary-time formalism into a form, explained in Appendix A, which is quite close to the real-time one, but which avoids singular terms which would occur in graphs (e)–(g) of Fig. 1. The method we use would be applicable also for thermal Green's functions, in which case it would generate the analytically continued imaginary-time formalism.

## II. LOWER-LOOP DIAGRAMS

We list, for completeness, the contributions from graphs (a)–(d) of Fig. 1 to the pressure

$$P=T\ln Z/V \ . \tag{5}$$

Graph (a) has two contributions:

$$P_a'=\int\frac{d^3\mathbf{p}}{(2\pi)^3}[T\ln N(\omega)+\omega] \tag{6}$$

and

$$P_a''=-\int\frac{d^{3-\epsilon}\mathbf{p}}{(2\pi)^{3-\epsilon}}\frac{\omega}{2} \ , \tag{7}$$

where

$$\omega=(\mathbf{p}^2+M^2)^{1/2} \tag{8}$$

and

$$N(\omega)=[\exp(\omega/T)-1]^{-1} \ . \tag{9}$$

Equation (6) has the known expansion [1]

$$P_a'=\frac{\pi^2}{90}T^4-\frac{1}{24}M^2T^2+\frac{1}{12\pi}M^3T+\frac{1}{8(2\pi)^2}M^4[\ln(g/4\pi)+C-\tfrac{3}{4}]+\cdots \tag{10}$$

 

($C$ is Euler's constant). Because of its implicit $T$ dependence, we must also keep (7), which gives, in dimensional regularization,

$$P_a'' = \frac{1}{2(4\pi)^2} g^4 T^4 \left[ \frac{1}{\epsilon} - 3\ln T + 2\ln\mu - \ln g + \frac{1}{2}\ln(4\pi) - \frac{C}{2} + \frac{3}{4} \right] . \quad (11)$$

Note that the ln$g$ terms cancel between (10) and (11).

The contributions from graphs (b) and (c) of Fig. 1 together may be written

$$P_b + P_c = -12g^2L^2 + M^4/(48g^2) , \quad (12)$$

where

$$L = \frac{1}{2}\frac{1}{(2\pi)^{3-\epsilon}} \int \frac{d^{3-\epsilon}\mathbf{p}}{2\omega} + \frac{1}{(2\pi)^3}\int \frac{d^3\mathbf{p}}{2\omega} N(\omega) - M^2/(24g^2) = -\frac{1}{8\pi} MT . \quad (13)$$

Thus

$$P_b + P_c = \tfrac{1}{48} M^2 T^2 - 3M^4/(4\pi)^2 . \quad (14)$$

Graphs (e)–(g) are each separately of order $g^4$, but the terms of this order cancel in their sum.

Graph (d) gives

$$P_d = -3ag^4\mu^\epsilon \left[ (2\pi)^{-3+\epsilon} \int \frac{d^{3-\epsilon}\mathbf{p}}{2\omega}[2N(\omega)+1] \right]^2 = -\tfrac{3}{2} g^4 T^4/(4\pi)^2 [\epsilon^{-1} - 2\ln T + \ln\mu + \ln\pi + C - 12\zeta'(2)/\pi^2] , \quad (15)$$

where we have used

$$\int_0^\infty dt\, t \ln t (e^t - 1)^{-1} = \zeta'(2) + (1-C)\pi^2/6 , \quad (16)$$

$\zeta$ being the Riemann zeta function.

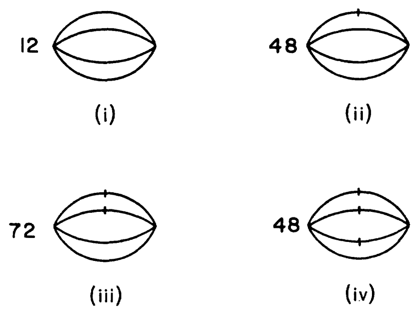


FIG. 2. Expansion of graph (h) of Fig. 1 in terms of Minkowski-space graphs, according to Appendix A. The cut lines represent (17). The other lines represent ordinary Feynman propagators.

## III. THREE-LOOP DIAGRAM

In this section we compute graph (h) of Fig. 1. The first step is to show that this thermal graph gives the graphs of Fig. 2. In these graphs an ordinary line denotes a zero-temperature, Minkowski-space propagator. A cut line carrying momentum $p$ denotes

$$\delta(p_0^2 - \omega^2) N(\omega) . \quad (17)$$

More precisely, the real parts of each of these graphs should be taken. This result is established in Appendix A.

To the requisite order, we may replace $M$ by zero in each of the graphs of Fig. 2, because they remain infrared convergent and the error is of less order than $g^4$. Then graphs (i) and (ii) of Fig. 2 give zero, since their zero-temperature subgraphs vanish in dimensional regularization.

The contribution from graph (iii) of Fig. 2 is

$$P_3 = 72g^4\mu^{2\epsilon}(2\pi)^{-6+2\epsilon} \int \frac{d^{3-\epsilon}\mathbf{p}}{2p} \int \frac{d^{3-\epsilon}\mathbf{q}}{2q} N(p)N(q)(4\pi)^{-2+\epsilon/2}\Gamma(\epsilon/2)\frac{[\Gamma(1-\epsilon/2)]^2}{\Gamma(2-\epsilon)} 4(1-\cos\theta)^{-\epsilon/2}(2pq)^{-\epsilon/2} . \quad (18)$$

Here the factor 4 comes from the four possibilities $p_0 = \pm p$, $q_0 = \pm q$. Because we take the real part, they all give the same contributions up to order $\epsilon$. A short calculation, using again (16), gives

$$P_3 = g^4(4\pi)^{-2} T^4 [\epsilon^{-1} - 3\ln T + 2\ln\mu + \tfrac{3}{2}\ln\pi + \tfrac{1}{2}(1+3C) - 18\zeta'(2)/\pi^2] . \quad (19)$$

In graph (iv) of Fig. 2, there are eight contributions corresponding to the eight possible signs for $p_0$, $q_0$, and $k_0$. We begin with the case when they are all positive. The contribution is

$$48g^4(2\pi)^{-9}\int \frac{d^3\mathbf{p}\,d^3\mathbf{q}\,d^3\mathbf{k}}{8pqk}[(\mathbf{p}+\mathbf{q}+\mathbf{k})^2 - (p+q+k)^2]^{-1} N(p)N(q)N(k)$$

$$= 3g^4(2\pi)^{-8}\int \frac{d^3\mathbf{p}\,d^3\mathbf{q}}{pq}\int \frac{dk}{|\mathbf{p}+\mathbf{q}|} \ln\left[ \frac{k|\mathbf{p}+\mathbf{q}| + \mathbf{p}\cdot\mathbf{q} - pq - qk - kp}{-k|\mathbf{p}+\mathbf{q}| + \mathbf{p}\cdot\mathbf{q} - pq - qk - kp} \right] N(p)N(q)N(k) . \quad (20)$$

We now change to a variable $y = |\mathbf{p}+\mathbf{q}|$, so that $d\cos\theta_{\mathbf{p},\mathbf{q}} = y\,dy/(pq)$. Then (20) gives

$$6g^4(2\pi)^{-6}\int dp\,dq\,dk\,N(p)N(q)N(k)\int_{|p-q|}^{p+q} dy \ln\left[ \frac{y^2 - p^2 - q^2 + 2ky - 2w}{y^2 - p^2 - q^2 - 2ky - 2w} \right] , \quad (21)$$

where $w=pq+qk+kp$. This gives

$$12g^4(2\pi)^{-6}\int dp\,dq\,dk\,N(p)N(q)N(k)[(p+q+k)\ln(p+q+k)+p\ln p+q\ln q+k\ln k -(q+k)\ln(q+k)-(k+p)\ln(k+p)-(p+q)\ln(p+q)]\ , \tag{22}$$

where we have everywhere taken real parts.

We must now add the other seven contributions in which

$$(p,q,k)\to(p,q,-k),(p,-q,k),(-p,q,k),(-p,-q,k),(-p,q,-k),(p,-q,-k),(-p,-q,-k)\ .$$

The result, again taking the real part, is

$$24g^4(2\pi)^{-6}\int dp\,dq\,dk\,N(p)N(q)N(k)[(k+p+q)\ln(k+p+q)-(p+q-k)\ln|p+q-k| -(p-q+k)\ln|p-q+k|-(-p+q+k)\ln|-p+q+k|]\ . \tag{23}$$

We have attempted to do these integrations analytically (see below) with little practical success, but numerically we find that (23) gives

$$P_4=24g^4T^4(2\pi)^{-6}\times 14.17=\tfrac{1}{2}g^4T^4(4\pi)^{-2}\times 1.74\ . \tag{24}$$

Finally, we combine the contributions from (10), (11), (14), (15), (19), and (24) to get

$$P=T^4\left[\frac{\pi^2}{90}-\frac{1}{48}g^2+\frac{g^3}{12\pi}+\frac{1}{2}\frac{g^4}{(4\pi)^2}\{-3\ln(T/\mu)-\tfrac{1}{2}\ln\pi-2\ln 2+\tfrac{1}{2}C-5+1.74\}\right]\ . \tag{25}$$

Note that this contains the renormalization-group invariant (to the required order)

$$\bar{g}^2(T)=g^2\left[1+9\frac{g^2}{2\pi^2}\ln(T/\mu)\right]\ . \tag{26}$$

Last, we mention an attempt (which relates to work in [4]) to do some of the integrations in (23) analytically. Consider the integral

$$J(b)=\tfrac{1}{2}\int_0^\infty dt\frac{t\ln(t^2-b^2)}{e^t-1}\ . \tag{27}$$

In terms of this, (23) may be written

$$144T^2g^4(2\pi)^{-6}\mathrm{Re}\int dp\,dq\,N(p)N(q)[J\{(p+q)/T\}-J\{(p-q)/T\}]\ . \tag{28}$$

We find, for $J$ (see Appendix B),

$$J(b)=-\tfrac{1}{4}b^2[\ln(b/2\pi i)-\tfrac{3}{2}]+i\pi b[\ln\Gamma(b/2\pi i)-\tfrac{1}{2}\ln(2\pi)]+2\pi^2\zeta'(-1,b/2\pi i)+\tfrac{1}{6}\pi^2\ln(2\pi)\ , \tag{29}$$

where in the last line $\zeta$ is the generalized Riemann zeta function and the derivative is with respect to the first argument. We may, without loss of generality, define the branch of the functions in (27) by taking the limit where $b$ approaches the real axis from the upper-half plane. This choice has been assumed in (29). However, this does not seem to be a very practical way to evaluate (23).

In conclusion, we note that the total constant term in the curly brackets in (25) is $-4.93$, and this may be replaced by $-\frac{1}{2}\pi^2$ to quite good accuracy. In this approximation, (25) and (26) give

$$P\simeq T^4\left[\frac{\pi^2}{90}-\frac{\bar{g}^2(T)}{48}+\frac{g^3}{12\pi}-\frac{g^4}{64}\right]\ . \tag{30}$$

This simple form might be a useful approximation to the pressure at three-loop order.

## ACKNOWLEDGMENTS

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## APPENDIX A

Consider the contribution to the pressure $P$ from graph (h) of Fig. 1. It is

$$12g^4(2\pi)^{-9}T^3\int d^3\mathbf{p}\,d^3\mathbf{q}\,d^3\mathbf{r}\,d^3\mathbf{k}\,\delta^3(\mathbf{p}+\mathbf{q}+\mathbf{r}+\mathbf{k})\Sigma_{n_p,n_q,n_r,n_k}\delta_{n_p+n_q+n_r+n_k,0}(p^2q^2r^2k^2)^{-1}\ , \tag{A1}$$

where

$$p^2=\mathbf{p}^2-p_0^2=\mathbf{p}^2+(2\pi T n_p)^2,\quad \text{etc.}$$

We write

$$T^{-1}\delta_{n_p+n_q+n_r+n_k,0}=(p_0+q_0+r_0+k_0)^{-1}[N(p_0+q_0+r_0+k_0)]^{-1}$$
$$=(p_0+q_0+r_0+k_0)^{-1}[N(p_0)N(q_0)N(r_0)N(k_0)]^{-1}$$
$$\times[1+N(p_0)+\cdots+N(p_0)N(q_0)+\cdots+N(p_0)N(q_0)N(r_0)+\cdots]\ . \quad \text{(A2)}$$

Now we use

$$T\Sigma_{n_p}=(2\pi i)^{-1}\int_C dp_0 N(p_0),\ \ \text{etc.}\ , \quad \text{(A3)}$$

where $C$ is a contour surrounding the poles of $N$ but no other poles, and note that (A2) has no poles. Then (A1) may be evaluated in terms of the poles in the Feynman denominators. This gives (writing now $p=|\mathbf{p}|$, etc.)

$$24g^4(2\pi)^{-6}\int\frac{d^3\mathbf{p}\,d^3\mathbf{q}\,d^3\mathbf{r}\,d^3\mathbf{k}}{16pqrk}\delta^3(\mathbf{p}+\mathbf{q}+\mathbf{r}+\mathbf{k})$$
$$\times((p+q+r+k)^{-1}\{N(p)N(q)N(r)+\cdots+N(p)N(q)+\cdots+N(p)+\cdots+1\}$$
$$+(p+q+r-k)^{-1}\{N(k)[N(p)N(q)+N(q)N(r)+N(r)N(p)+N(p)+N(q)+N(r)+1]$$
$$-N(p)N(q)N(r)\}+\cdots$$
$$+(p+q-r-k)^{-1}\{N(r)N(k)[N(p)+N(q)+1]-N(p)N(q)[N(r)+N(k)+1]\}+\cdots)\ , \quad \text{(A4)}$$

where we have used $N(-k)=-N(k)-1$, etc., and each term appears twice, hence providing a factor or 2. Note that the rule for determining the numerators in (A4) is as follows: For any positive (negative) product of $N$'s in the numerator, delete the corresponding momenta in the denominator, and then only positive (negative) terms will remain in the denominator.

It is not difficult to see that the graphs of Fig. 2 represent the terms in (A4). The denominators in (A4) are just the denominators of nonrelativistic perturbation theory encountered in evaluating the $T$-independent parts (the parts without the cut lines) of the graphs.

The apparent singularities at $p+q+k-r=0$, $p+q-r-k=0$, etc. in (A4) are removed when the graphs of Fig. 2 are combined. Therefore it is correct to take the denominators in individual terms in (A4) each to be principal values. This is equivalent to taking the real parts.

In this example the graphs in Fig. 2 are almost the same as the corresponding real time graphs for the pressure, except that the graph with four cut lines (which would be purely imaginary anyway) is omitted.

Now take graphs (e)–(g) of Fig. 1. For simplicity, we concentrate on (e). The above method leads to the contribution

$$\tfrac{1}{2}M^4(2\pi)^{-3}\int\frac{d^3\mathbf{p}\,d^3\mathbf{q}}{4pq}\left[\frac{N(p)+N(q)+1}{p+q}-\frac{N(p)-N(q)}{p-q}\right]\delta^3(\mathbf{p}+\mathbf{q})=\tfrac{1}{2}M^4(2\pi)^{-3}\int\frac{d^3\mathbf{p}}{4p^2}\left[\frac{2N(p)+1}{2p}-N'(p)\right]\ .$$

There is no ambiguity in the second term, which arises from integrals containing double poles, such as

$$(2\pi i)^{-1}\int_C dp_0\frac{N(p_0)-N(p)}{(p_0-p)^2}\ .$$

This version of the analytically continued imaginary-time formalism is expressed in terms of (zero-temperature) forward-scattering amplitudes [5]. We have used it in other contexts [6].

## APPENDIX B

In order to derive (29), we differentiate (27) and obtain (for $\mathrm{Im}\,b>0$)

$$\frac{dJ}{db^2}=\frac{1}{4}\left[\frac{i\pi}{b}-\ln(b/2\pi i)+\psi(b/2\pi i)\right]\ , \quad \text{(B1)}$$

where $\psi$ is the Euler psi function. Using the relation [4]

$$\int y\psi(y)dy=\tfrac{1}{2}(y^2-y)+y[\ln\Gamma(y)-\tfrac{1}{2}\ln(2\pi)]-\zeta'(-1,y)+\text{const}\ , \quad \text{(B2)}$$

together with the boundary condition [see (16)]

$$J(0)=\zeta'(2)+\tfrac{1}{6}(1-C)\pi^2=2\pi^2[\zeta'(-1,0)+\tfrac{1}{12}\ln(2\pi)]\ , \quad \text{(B3)}$$

(29) is easily deduced.


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