Rigidly rotating cosmic strings

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The complete class of cosmic string solutions in flat space that undergo rigid rotation around a fixed axis is exhibited. It consists of spiral-type curves and plane cycloids and possesses three free parameters.

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I. INTRODUCTION

Exact solutions to the Nambu-Goto equations are of considerable interest in the theory of cosmic strings (for an introduction see Refs. [1,2]). The intersection behavior of networks in cosmologic scenarios determines the rate at which cosmic strings are responsible for the large-scale structure of the Universe [3]. In order to study the history of simple string configurations, several families of exact solutions were found. Let us mention the Kibble-Turok one-parameter family [4] of self-avoiding strings that was generalized to a two-parameter solution by Turok [5]. Extensions to three and five parameters were presented by Chen, DiCarlo, and Hotes [6] and DeLaney, Engle, and Scheick [7]. Among this class of string motions are some which describe rotating closed loops that retain their shape during time evolution.

Such solutions illustrate the fact that strings must not be considered to consist of "atoms." A rubberband forming a circle may rotate stationary (i.e., along its own direction) and thus provide, in fact, a static configuration, the elastic force precisely balancing the outward directed centrifugal force. In the case of relativistic (cosmic) strings, i.e., strings obeying the Nambu-Goto equations of motion, this is not possible. The motion of a piece of string along its direction ("longitudinal motion") is completely unobservable and thus a mere coordinate effect that cannot produce any force. On the other hand, the rigidly rotating strings obtained in the families mentioned above have cusps which provide enough amount of "transverse motion" to ensure a fixed shape. In some sense, these cups act as "atoms."

In this paper we exhibit the class of all strings that undergo rigid motion in flat space around a fixed axis, i.e., whose time evolution is a time-dependent SO(3) matrix applied to some initial configuration. The final solution is labeled by three parameters called ω, ν, and K. ω is the frequency of the rotation. For $K\neq 0$, one obtains spiral-type curves without cusps but with infinite extension in the direction of the axis. $K=0$ gives a class of cycloids in the plane which have been discussed previously by Burden [8]. The parameter ν controls the separation angle between two cusps and thus determines whether the string is closed.

The computation is shown in some detail in Sec. III. After discussing some properties of the solutions in Sec. IV, we close the paper by mentioning a simple characterization of a slightly enlarged family of strings.

II. THE ANSATZ

The coordinates on the world sheet we begin with are denoted by $(t,\rho), t\equiv x^0$ being the time coordinate of the Minkowski frame used, and ρ is unconstrained. Hence, the string history is described by $\mathbf{x}\equiv\mathbf{x}(t,\rho)$, and it extremizes the Nambu-Goto action

$$S_{\rm NG}=\int dt\, d\rho\, \mathcal{L}_{\rm NG} \tag{2.1}$$

with the density

$$\mathcal{L}_{\rm NG}=\sqrt{(\mathbf{x}'\dot{\mathbf{x}})^2+\mathbf{x}'^2(1-\dot{\mathbf{x}}^2)}\ . \tag{2.2}$$

A prime stands for ∂_ρ, an overdot for ∂_t. The basic ansatz for a rigidly rotating string is then provided by

$$\mathbf{x}(t,\rho)=R(t)\mathbf{x}_0(\rho) \tag{2.3}$$

with

$$R(t)=\begin{bmatrix} \cos\omega t & \sin\omega t & 0\\ -\sin\omega t & \cos\omega t & 0\\ 0 & 0 & 1\end{bmatrix}. \tag{2.4}$$

Without loss of generality, we have chosen the rotation axis to coincide with the spatial three-direction. Clearly, the notion "rigid" makes use of an observer sitting in a Minkowski frame that defines the coordinate system $x^\mu\equiv(t,\mathbf{x})$. Since we have not constrained ρ, (2.3) is, up to obvious symmetry, the most general ansatz. ω is the angular frequency of the rotation.

One may now insert (2.3) into the Euler-Lagrange equations following from (2.2). However, some manipulations show that one obtains the identical equations of motion by first inserting the ansatz (2.3) into the action and only then varying with respect to $\mathbf{x}_0(\rho)$. Let us skip this check and use (2.3) to obtain the reduced density for the action

$$\mathcal{L}\equiv\mathcal{L}_{\rm NG}(\text{ansatz})$$
$$=\sqrt{\mathbf{y}'^2+c'^2-\omega^2[(\mathbf{y}'\mathbf{y})^2+\mathbf{y}^2c'^2]}\ , \tag{2.5}$$

where we have set

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$$\mathbf{x}_0(\rho)=\begin{pmatrix}\mathbf{y}(\rho)\\ c(\rho)\end{pmatrix} . \tag{2.6}$$

In order not to introduce a new notation, we have decided to let $\mathbf{y}(\rho)$ denote the two-component vector as appearing in (2.6). This provides a variational problem whose Euler-Lagrange equations are

$$\partial_\rho\left[\frac{\mathbf{y}'-\omega^2(\mathbf{y}'\mathbf{y}')\mathbf{y}}{\mathcal{L}}\right]+\frac{\omega^2(\mathbf{y}'\mathbf{y})\mathbf{y}'+\omega^2c'^2\mathbf{y}}{\mathcal{L}}=0 , \tag{2.7}$$

$$\partial_\rho\left[\frac{c'-\omega^2c'\mathbf{y}^2}{\mathcal{L}}\right]=0 . \tag{2.8}$$

III. SOLUTION OF THE EQUATIONS

Equation (2.8) is integrated to give

$$c'\frac{1-\omega^2\mathbf{y}^2}{\mathcal{L}}=K \tag{3.1}$$

for a constant K. Here, $\mathcal{L}$ is just an abbreviation for the expression (2.5). Taking the square in order to solve for c', and reinserting the result in $\mathcal{L}$, we find

$$\mathcal{L}^2=\frac{[\mathbf{y}'^2-\omega^2(\mathbf{y}'\mathbf{y})^2](1-\omega^2\mathbf{y}^2)}{1-\omega^2\mathbf{y}^2-K^2} . \tag{3.2}$$

Now we use the freedom that is left to specify the coordinate ρ. The action due to (2.5) is easily seen to be invariant under $\rho\to f(\rho)$. From (2.6), we see that $\mathbf{y}(\rho)$ is the projection of $\mathbf{x}_0(\rho)$ into the plane orthogonal to the axis [in our coordinates the (12) plane]. Hence, we can choose ρ to be the arclength along this curve. The condition which fixes ρ (up to an additive constant) is

$$\mathbf{y}'^2=1 . \tag{3.3}$$

For notational ease, we set $\omega=1$ and reintroduce generic ω in the end. With these assumptions, and using the abbreviation

$$\mathcal{D}=\mathbf{y}^2 , \tag{3.4}$$

we multiply (2.7) by $\mathbf{y}$ to obtain

$$[1-\tfrac{1}{4}\mathcal{D}'^2+c'^2(1-\mathcal{D})][-1+\tfrac{1}{2}\mathcal{D}''(1-\mathcal{D})+c'^2\mathcal{D}] =-\tfrac{1}{8}\mathcal{D}'(1-\mathcal{D})[\mathcal{D}'\mathcal{D}''+2\mathcal{D}'c'^2+4c'c''(\mathcal{D}-1)] . \tag{3.5}$$

The relation (3.2) becomes

$$c'^2=\frac{K^2}{4}\frac{4-\mathcal{D}'^2}{(1-\mathcal{D})(1-\mathcal{D}-K^2)} . \tag{3.6}$$

This may be inserted into (3.5) to eliminate c. Several cancellations occur, and one gets

$$8\mathcal{D}''(\mathcal{D}-1)^2(\mathcal{D}+K^2-1) =(\mathcal{D}'^2-4)[4(\mathcal{D}-1)^2+K^2(\mathcal{D}'^2-4)] \tag{3.7}$$

as a differential equation for $\mathcal{D}$, the squared distance to the rotation axis, alone. Using $\mathcal{D}$ and $\mathcal{D}'$ as variables (and the relation $\mathcal{D}''=\mathcal{D}'d\mathcal{D}'/d\mathcal{D}$), an integration yields

$$\mathcal{D}'^2=4\frac{(\mathcal{D}-1)[(\mathcal{D}-1)\gamma+1+\gamma K^2]+\gamma K^2}{\mathcal{D}-1+\gamma K^2} \tag{3.8}$$

with γ a constant. From now on it is convenient to use

$$z=\mathcal{D}-1 \tag{3.9}$$

and a new coordinate $\Theta\equiv\Theta(\rho)$, such that

$$\gamma\left[\frac{d\rho}{d\Theta}\right]^2=-(z+\gamma K^2) . \tag{3.10}$$

[This just turns (2.5) into $|z|$ if the prime is read as ∂_Θ.] Hence, (3.8) becomes

$$\left[\frac{d\mathcal{D}}{d\Theta}\right]^2\equiv\left[\frac{dz}{d\Theta}\right]^2 =-4\left[z^2+z\left[\frac{1}{\gamma}+K^2\right]+K^2\right] , \tag{3.11}$$

which may be solved by elementary integration to give

$$z\equiv\mathcal{D}-1=A\cos^2\Theta+B . \tag{3.12}$$

The constants A and B are written as

$$A=-\operatorname{sgn}(\gamma)\left[\left[\frac{1}{\gamma}+K^2\right]^2-4K^2\right]^{1/2} , \tag{3.13}$$

$$B=-\frac{1}{2}\left[\frac{1}{\gamma}+K^2\right] +\frac{1}{2}\operatorname{sgn}(\gamma)\left[\left[\frac{1}{\gamma}+K^2\right]^2-4K^2\right]^{1/2} , \tag{3.14}$$

where sgn denotes the sign. Some freedom in the signs has been used to achieve the convenient limit $A\to-1/\gamma$, $B\to0$ for $K\to0$. The integration constant in solving (3.11) is omitted since it generates just a trivial shift $\Theta\to\Theta-\Theta_0$.

Having exploited (3.5), we insert the result into (3.6) which gives us

$$c=\pm K\Theta \tag{3.15}$$

with both signs possible. This result together with (3.12), which indicates the periodicity of $\mathbf{y}^2$, shows that the possible solutions to our problem are of spiral-type (for $K\neq0$) or restricted to a plane (for $K=0$).

However, we have only solved part of (2.7) so far. Rewriting the equations, in terms of the coordinate Θ by means of (3.10) and inserting (3.12) and (3.15), we obtain

$$(A\cos^2\Theta+B)\frac{d^2\mathbf{y}}{d\Theta^2}+2A\sin\Theta\cos\Theta\frac{d\mathbf{y}}{d\Theta} +[\cos^2\Theta(A^2+2AB+A^2K^2)-AB+K^2B]\mathbf{y}=0 , \tag{3.16}$$

which is to be solved with (3.12) as constraint for $\mathbf{y}^2$. Luckily, it turns out that this puts no further restriction to the constants, and the final solution may be written (modulo trivial rotations in the 12-plane) as

$$\mathbf{y}=\sqrt{B+1}\begin{bmatrix} a\cos\Theta\cos\nu\Theta+\sin\Theta\sin\nu\Theta \\ -a\cos\Theta\sin\nu\Theta+\sin\Theta\cos\nu\Theta \end{bmatrix}, \quad (3.17)$$

where we have set

$$\gamma=\frac{1}{1-\nu^2} \quad (3.18)$$

and

$$a=\frac{1}{2\nu}(A+1+\nu^2-K^2)\equiv\frac{1}{\nu}(A+B+1) . \quad (3.19)$$

In the limit when K approaches zero, $a\to\nu$. In the final presentation of the solution, we reinsert arbitrary ω (by $\mathbf{x}\to\mathbf{x}/\omega$). Hence,

$$\mathbf{x}(t,\Theta)=\frac{1}{\omega}R(t)\mathbf{x}_0(\Theta) \quad (3.20)$$

with $\mathbf{x}_0(\Theta)$ as in (2.6) consisting of (3.17) and (3.15), and $R(t)$ given by (2.4). The parameters are ω, ν, and K, the further constants defined by

$$A^2=(1-\nu^2+K^2)^2-4K^2 \quad (3.21)$$

with both signs possible, but equivalent [a convenient sign choice has been made in (3.13)],

$$B=-\frac{1}{2}(1-\nu^2+K^2+A) , \quad (3.22)$$

and a from (3.19). For $\nu\to 0$, a vanishes. Equation (3.21) and the positivity of $B+1$ [which appears under the square root in (3.17)] require some additional inequalities for the independent parameters ν and K. The parameter space consists of two pieces, namely,

$$\nu\geq 1+|K| \quad (3.23)$$

and

$$0\leq\nu\leq 1-|K| \quad (3.24)$$

and the intersection point of these two regions ($\nu=1$, $K=0$) must be excluded.

IV. DISCUSSION OF THE SOLUTIONS

The curves in the plane defined by (3.17) are hypocycloids for $\nu<1$ and epicycloids for $\nu>1$. If $K\neq 0$, (3.15) causes the spiral-type shape of the string, and there exists a well-defined tangent (no cusp). In this case, the string is not closed but of infinite extension in the direction of the rotation axis.

In the limit $K\to 0$, (3.15) collapses to cycloids in the plane (recall $a\to\nu$, $B\to 0$). These planar solutions are included in the class discussed by Burden [8]. At any $\Theta=\pi/2+p\pi$ (p integer), the string possesses a cusp of local behavior (using $\varepsilon\equiv\Theta-\pi/2$ small):

$$\mathbf{y}=\begin{bmatrix} \nu^3\varepsilon^3 \\ 1+\frac{\varepsilon^2}{2}(1+\nu^2) \end{bmatrix}+O(\varepsilon^4) . \quad (4.1)$$

The angle between two adjacent cusps as measured from the axis is $\pi(\nu-1)$; hence, the string closes only if

$$m\pi|\nu-1|=2\pi n , \quad (4.2)$$

where m and n are two integers (without common divisor), i.e., if ν is rational. In this case, m is the member of cusps and n is the number of times the string winds around the axis until it closes. A complete coordinate interval is given by $0\leq\Theta\leq m\pi$. The two possible signs of $\nu-1$ in (4.2) decide whether the curve is an epicycloid ($\nu>1$) or a hypocycloid ($\nu<1$). Figure 1 shows the solution for $m=3$, $n=1$, $\nu=\frac{5}{3}>1$, and in Fig. 2 the string for $m=7$, $n=1$, $\nu=\frac{5}{7}<1$ is depicted. $m=n=1$, $\nu=3$ gives a cardioid.

The usual gauge for cosmic strings is achieved by introducing another coordinate σ:

$$\Theta=\frac{\omega}{\nu^2-1}(\sigma-\nu t) . \quad (4.3)$$

When expressed as $\mathbf{x}\equiv\mathbf{x}(t,\sigma)$, the solution obeys

$$(\partial_t\mathbf{x})(\partial_\sigma\mathbf{x})=0 , \quad (4.4)$$

$$(\partial_t\mathbf{x})^2+(\partial_\sigma\mathbf{x})^2=1 . \quad (4.5)$$

Some manipulations using the addition theorems for sin and cos reveal that

$$\mathbf{x}(t,\sigma)=\tfrac{1}{2}[\mathbf{f}(\sigma+t)+\mathbf{g}(\sigma-t)] \quad (4.6)$$

with

$$\mathbf{f}(\sigma)=\begin{bmatrix} \frac{\sqrt{B+1}}{\omega}(a+1)\cos\frac{\omega\sigma}{\nu+1} \\ -\frac{\sqrt{B+1}}{\omega}(a+1)\sin\frac{\omega\sigma}{\nu+1} \\ -K\frac{\sigma}{\nu+1} \end{bmatrix}, \quad (4.7)$$

$$\mathbf{g}(\sigma)=\begin{bmatrix} \frac{\sqrt{B+1}}{\omega}(a-1)\cos\frac{\omega\sigma}{\nu-1} \\ -\frac{\sqrt{B+1}}{\omega}(a-1)\sin\frac{\omega\sigma}{\nu-1} \\ K\frac{\sigma}{\nu-1} \end{bmatrix}, \quad (4.8)$$

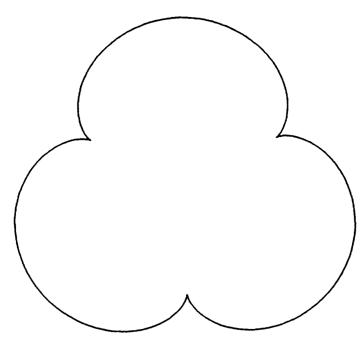

FIG. 1. The shape of the string for $K=0$, $\nu=\frac{5}{3}$ is that of an epicycloid.

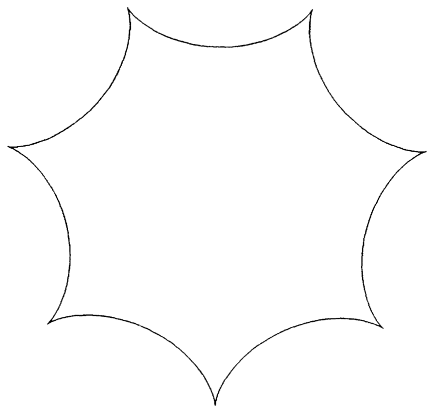

FIG. 2. The parameter values $K=0$, $\nu=\frac{5}{7}$ lead to a hypocycloid.

and

$$\mathbf{f}'(\sigma)^2=\mathbf{g}'(\sigma)^2=1 \ . \tag{4.9}$$

One easily checks that $K=0$, $\nu=2$ [hence, $m=2$, $n=1$ in (4.2)] gives the string contained in the family found by Kibble and Turok [4] (there it belongs to the parameter value $\alpha=1$).

As is well known, the coordinate σ as defined by (4.4) and (4.5) measures the local energy density along the string (divided by the string tension μ). This enables one to show that the total energy between two adjacent cusps in the limit $K=0$ is

$$E=\mu\pi\left|\frac{\nu^2-1}{\omega}\right| \ . \tag{4.10}$$

Moreover, $\mathbf{x}^2=1/\omega^2$ for any cusp. As a consequence, cusps move at the speed of light. This is a well-known fact, and it fits into the result that the size of the string is determined by ω.

The limiting case $K=0$, $\nu\rightarrow 1$ may be studied by setting $n=1$, $m\rightarrow\infty$ (infinitely many cusps) in (4.2). The total energy of the string approaches in this limit $4\pi\mu/|\omega|$. Such a string may be considered as consisting of atoms (= cusps) and behaves essentially nonrelativistic (like a rubberband as mentioned in the Introduction). The tension μ effectively balances the centrifugal force.

Let us finally give a characterization of a slightly larger class of exact string solutions than (4.6). Performing an arbitrary Lorentz transformation to (4.6) in the spatial three-direction (boost), an additional parameter is generated. Explicit computation reveals some simplification, and in the end one arrives at the remarkably simple result: Any cosmic string which undergoes rigid rotation in some Minkowski frame that is boosted in the three-direction is of the form (up to trivial symmetry)

$$\mathbf{x}(t,\sigma)=\tfrac{1}{2}[\mathbf{h}_1(\sigma+t)+\mathbf{h}_2(\sigma-t)] \ , \tag{4.11}$$

where $\mathbf{h}_1$ and $\mathbf{h}_2$ are both of the type

$$\mathbf{h}_i(\sigma)=\begin{pmatrix} \dfrac{C_i}{\Omega_i}\cos(\Omega_i\sigma) \\ \dfrac{C_i}{\Omega_i}\sin(\Omega_i\sigma) \\ D_i\sigma \end{pmatrix} \tag{4.12}$$

with

$$2C_i^2+D_i^2=1 \tag{4.13}$$

(no sum). This specifies a family of solution with four parameters, say C_1, C_2, Ω_1, and Ω_2, that contains (4.6) as a subclass.

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