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Polyhedral cosmic strings

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Quantum field theory in Möbius corners is discussed using the method of images. The vacuum average of the stress-energy tensor of a free field is derived and is shown to be a simple sum of straight cosmic-string expressions. It does not seem possible to set up a spin-$\frac{1}{2}$ theory easily.

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I. INTRODUCTION

A straight, ideal cosmic string can be thought of as a conical defect in space. A number of field-theoretic calculations have been performed in such a background which we do not wish to summarize here but simply refer to a short review article [1]. The attraction of this background is that it is locally flat and the effect of the defect can be accommodated by changed boundary (i.e., periodicity) conditions.

The defect is an angular one. The spatial metric can be written most naturally in cylindrical coordinates:

$$ds^2=dz^2+d\rho^2+A^2\rho^2 d\phi^2 \; , \tag{1}$$

where ϕ is a "physical" angle running from 0 to 2π. The deficit angle is $2\pi(1-A)$.

An object (a global monopole) with a defect in solid angle has been discussed by Barriola and Vilenkin [2] (see also Harari and Lousto [3], Mazzitelli and Lousto [4]). The metric is

$$ds^2=dr^2+A^2r^2(d\theta^2+\sin^2\theta\, d\phi^2) \; . \tag{2}$$

It is not locally flat, the curvature being proportional to $(1-A^2)/r^4$. (Actually, such a metric was earlier suggested by Sokolov and Starobinsky [5] as an example of a metric with a conical singularity.)

In the present work we wish to discuss another situation with a solid angle deficit that is locally flat. The price that must be paid is that there are string singularities. Further, the deficit can only take the values $4\pi(1-1/N)$ where N is an integer. The nontrivial values are thus much too large to be realistic. Nevertheless, the geometry is interesting if only because its simplicity makes field-theory calculations easy and, in some ways, obvious. It is in this spirit that the present calculation is presented.

II. THE BASIC CONSTRUCTION

The triangulation of a two-sphere obtained by its intersection with the symmetry planes of an inscribed regular polyhedron is classic, if not ancient. Consider such a triangulation applied to the constant r sections of the flat-space metric

$$ds^2=dr^2+r^2(d\theta'^2+\sin^2\theta'\, d\phi'^2) \tag{3}$$

and join the triangle vertices to the origin $r=0$ to give a set of "triangular cones." We allow r to extend to infinity.

If there are $2N$ triangles equivalent under the corresponding extended point group Γ', one obtains a division of the total solid angle into $2N$ portions of $2\pi/N$, one for each cone.

One can now either treat such a cone as a physical region of $\mathbb{R}^3$ and proceed to do (quantum) field theory in it with, say, Dirichlet or Neumann boundary conditions or one can think of the cone as analogous to the segment $0\le\phi'<\beta$ ($\beta=2\pi/N$) which, when its edges are identified, yields the straight cosmic string (1) with $A=\beta/2\pi$.

For this latter interpretation it is necessary to be more careful regarding the group action. Conventionally, the triangles are alternately shaded and unshaded. The pure rotations of Γ', i.e., Γ, take shaded to just shaded and unshaded to just unshaded regions. The remaining elements interchange shaded and unshaded triangles and correspond to single reflections (with possibly a rotation as well). Thus, if we use the full extended group to construct the Green's function (of scalar field theory say) by the image method, we will automatically obtain Dirichlet or Neumann boundary conditions depending on how we combine the various contributions. Therefore this is suited to our first interpretation of the triangular cone as a physical region. The idea of the triangular cone, as a *trihedral kaleidoscope,* is due to Möbius [6]. (See, e.g., Coxeter [7,8].)

To produce a *periodic* structure it is necessary to use the rotation part Γ only and to take the quadrilateral combination of a triangle and one of its contiguous reflections, say, as the fundamental domain on the two-sphere. This "quadrilateral cone" is the analogue of the segment in the straight cosmic-string case mentioned above. One expects the edges of the cone, which are just the axes of symmetry of the corresponding regular solid, to be stringlike.

If we adopt this attitude then the coordinates in (3), (r,θ',ϕ'), are unphysical and it is necessary to find a coordinate transformation (analogous to $\phi=2\pi\phi'/\beta$ for the cosmic string) that takes the quadrilateral cone to physical space, i.e., onto an $\mathbb{R}^3$. (Coordinates on this $\mathbb{R}^3$ will be our definition of "physical coordinates" although nothing physical can, of course, depend on any particular choice.)

A possible coordinate transformation is provided by the conformal transformation that takes a spherical triangle into an upper half plane (and so takes the quadrilateral into the whole plane). The calculations are, again, classic, the basic paper being that of Schwarz [9]. Of course, the field-theory calculations are most easily done in the metric (3).

A brief discussion of the wave equation in triangular cones was early given in the book by Pockels [10], as noted by Laporte [11] who further developed the mode analysis. The reader will find in this paper a useful summary of the situation. Keller [12] includes the corners in his list of domains for which the image method applies and there is mention of them, as "Möbius corners," in Terras and Swanson [13] in connection with potential problems. No doubt they occur elsewhere.

Developments of the mode problem from various points of view are contained in papers by Poole [14], Hodgkinson [15], Meyer [16], Stiefel [17], Altmann [18], Altmann and Bradley [19], and Huber [20].

III. TRIANGULATIONS OF THE TWO-SPHERE

The classic account of the triangulations of S^2 and their relation to the finite-dimensional subgroups of SO(3) [or SU(2)] is that of Klein [21] and its continuation in the work of Klein and Fricke [22]. Discussions in English are less numerous but that by Forsyth [23] is useful and he also treats the conformal mapping by Schwarz triangle (automorphic) functions. Further informative references are the works by Ford [24], Cayley [25], Hurwitz and Courant [26], and Caratheodory [27]. A more recent source of information and of further references is the paper by Coxeter [7,8]. A summary of Schwarz's theory can also be found in Darboux's treatise [28].

There are five basic cases corresponding to the cyclic C_n, dihedral D_n, tetrahedral T, octahedral O, and icosohedral Y point groups, but the first two are more or less the same and correspond to the straight cosmic string.

The angles of the fundamental triangles are written π/ν_i $(i=1,2,3)$, where (ν_1,ν_2,ν_3) equals (2,3,3) for T, (2,3,4) for O, and (2,3,5) for Y. The ν_i are related by

$$\sum_i \frac{1}{\nu_i}=1+\frac{2}{N}$$

and, if there are n_i symmetry axes of the ν_i-fold type,

$$\sum_i n_i(\nu_i-1)=N-1 \ .$$

Further, $2(\sum_i n_i-1)=N$.

In our situation, we start from the metric (3) where θ' and ϕ' are standard angular coordinates on the unit sphere. The usual stereographic projection onto the equatorial plane yields the Cayley-Klein parameter $w=u+iv$ which undergoes a linear fractional transformation (a homography or Möbius transformation) when the sphere is rotated. For a projection from the north pole, $w=\cot(\theta'/2)\exp(i\phi')=(x'+iy')/(r-z')$ where x',y',z' are the Cartesian coordinates of a general point $(r^2=x'^2+y'^2+z'^2)$. The metric (3) becomes

$$ds^2=dr^2+4r^2\frac{dw\,dw^*}{(1+|w|^2)^2} \ . \tag{4}$$

The conformal transformations, $w\rightarrow\zeta$ that map one w triangle onto the upperhalf ζ plane are (e.g., Forsyth, 1893)

$$\begin{aligned}
1-\zeta&=\frac{(w^4+2iw^2\sqrt{3}+1)^3}{(w^4-2iw^2\sqrt{3}+1)^3} \quad \text{for T} \ ,\\
\zeta&=\frac{(w^8+14w^4+1)^3}{(w^{12}-33w^8-33w^4+1)^2} \quad \text{for O} \ ,\\
\zeta&=\frac{(w^{20}-22w^{15}+494w^{10}+228w^5+1)^3}{[w^{30}+1+522w^5(w^{20}-1)-10\,005w^{10}(w^{10}+1)]^2} \quad \text{for Y} \ .
\end{aligned} \tag{5}$$

The metric (4) is written equivalently

$$ds^2=dr^2+4r^2\left|\frac{dw}{d\zeta}\right|^2\frac{d\zeta\,d\zeta^*}{[1+|w(\zeta)|^2]^2} \ , \tag{6}$$

where w is an algebraic function of ζ by (5). Equation (6) explicitly shows the singularities which conventionally have been chosen at $\zeta=0$, $\zeta=1$, and $\zeta=\infty$. These positions can be altered by homographies applied to ζ or to w (Cayley [25]).

We now think of (r,ζ,ζ^*) as coordinates in one to one correspondence with the points of *physical* space. To emphasize this, angular variables θ and ϕ could be arbitrarily introduced by projecting the ζ plane onto a unit sphere by, say,

$$\zeta=\cot(\theta/2)e^{i\phi} \ . \tag{7}$$

The entire ζ plane is covered by the ranges $0\le\theta<\pi, 0\le\phi<2\pi$. The singularities lie at the north and south poles and at one point on the equator of this ζ sphere.

There are three (cosmic) strings corresponding to the edges of the fundamental quadrilateral (two vertices of which are identified). The "strength" of a string will be determined by its associated ν parameter, i.e., by its angular deficit. The strings meet at the origin, $r=0$.

IV. THE CYCLIC GROUP C_n AND THE COORDINATE QUESTION

It is useful to discuss this elementary case as an example of our general approach.

The triangulation of the unit sphere is a division into n lunes running from pole to pole, which are the only fixed points. The z' axis is taken as the (polar) rotation axis and the stereographic projection onto the (x',y') plane is a set of n wedges of angle $2\pi/n$. The standard conformal transformation is

$$w^n=\zeta$$

which opens out each wedge into the whole ζ plane. The singularity (the string) in $\mathbb{R}^3$ lies on the rotation axis.

The metric (6) will be a complicated function of the new colatitude θ [in (7)]. In order to regain the "natural" metric (1) a coordinate transformation from θ back to θ' using $[\cot(\theta'/2)]^n=\cot(\theta/2)$ is necessary and then ρ and z can be introduced conventionally. The result of all this is just to rescale the azimuth ϕ' to ϕ and to leave the colatitude θ' alone. This is what one does immediately, of course, but the general case is not so clear and the conformal transformation seems to be the only systematic way of obtaining "physical" coordinates.

The remaining problem in the general case is the interpretation of the coordinates. That is to say, what is the physical significance of the singularities? In the cyclic case one can talk about lensing effects, for example. Is there anything similar for the other cases? Is it possible to produce a source for the singularities and does anything special happen at the origin where the three strings meet?

V. QUANTUM FIELD THEORY

We abandon the question of physical significance and return to the metric (3). The point (θ',ϕ') on the unit sphere is denoted by q and the elements of Γ' by γ. The action of Γ' on the unit sphere can be extended in the obvious way to $\mathbb{R}^3$ by $\gamma\mathbf{r}=\gamma(r,q)=(r,\gamma q)$. The triangular cone $\mathcal{C}'$ can be taken as the fundamental domain of Γ' acting on all of three-space:

$$\mathcal{C}'=\mathbb{R}^3/\Gamma'\approx\mathbb{R}^+\times S^2/\Gamma' .$$

The quadrilateral fundamental domain is $\mathcal{C}=\mathbb{R}^3/\Gamma$.

In the case of free scalar field theory, or quantum mechanics, the Green's function, or propagator, in the corner will be given an image sum of standard, Minkowski Green's functions, or of standard Euclidean propagators. Typically,

$$G(r,q,t;r',q',t')=\sum_{\gamma\in\Gamma'}a(\gamma)G_0(r,q,t;r',\gamma q',t') . \quad (8)$$

The phase factor $a(\gamma)$, in the simplest cases, are either all equal to unity (giving Neumann boundary conditions) or equal to one when γ is a rotation and minus one otherwise (giving Dirichlet conditions).

As mentioned earlier, to produce periodic conditions the summation in (8) should be restricted to the group Γ of pure rotations. The phase factors are then all unity. [For complex fields, we might allow the $a(\gamma)$ to be more general phases. This will be discussed later.]

The coordinates (r,q) on the left-hand side of (8) are, of course, restricted to the fundamental domains $\mathcal{C}'$ or $\mathcal{C}$ as the case may be.

It is often considered interesting to calculate vacuum averages of various operators, particularly $\langle\phi^2\rangle$ and $\langle T_{\mu\nu}\rangle$, as evidence of a Casimir effect. For example,

$$\langle\phi^2\rangle=-i\lim_{x'\to x}G(x;x') ,$$

where x stands for (r,q,t). Equation (8) shows that this will diverge and we remove the offending term, $\gamma=\mathrm{id}$, from the sum as our renormalization. Then

$$\langle\phi^2\rangle_{\mathrm{ren}}=-i\sum_{\gamma\neq\mathrm{id}}a(\gamma)G_0(r,q,t;r,\gamma q,t) . \quad (9)$$

The unit element of a group is denoted by id or, sometimes, by E.

For a massless field $G_0(x,x')=-(i/4\pi^2)(1/\lambda^2)$, as a distribution, where $\lambda^2=(t-t')^2-|\mathbf{r}-\mathbf{r}'|^2$. Therefore

$$\langle\phi^2(\mathbf{r})\rangle_{\mathrm{ren}}=\frac{1}{4\pi^2r^2}\sum_{\gamma\neq\mathrm{id}}a(\gamma)\frac{1}{d(q,\gamma q)^2} , \quad (10)$$

where $d(q,q')$ is the Euclidean distance between $(1,q)$ and $(1,q')$.

For complex fields the equation corresponding to (10) is

$$\langle|\phi|^2(\mathbf{r})\rangle_{\mathrm{ren}}=\frac{1}{2\pi^2r^2}\sum_{\gamma\neq\mathrm{id}}\mathrm{Re}[a(\gamma)]\frac{1}{d(q,\gamma q)^2} . \quad (11)$$

Restricting attention for the moment to the periodic case, the sum over Γ in (10) can be recast into a sum over the fixed points (or axes). Denote a typical symmetry axis (=two fixed points or vertices) by $\mathbf{k}$. Let the associated parameter be $\nu_{\mathbf{k}}$ (one of the ν_i) and the corresponding generator $A_{\mathbf{k}}$. Simple geometry gives

$$rd(q,A_{\mathbf{k}}^m q)=2\sin(m\pi/\nu_{\mathbf{k}})\rho(\mathbf{r},\mathbf{k}) ,$$

where $\rho(\mathbf{r},\mathbf{k})$ is the perpendicular Euclidean distance from $\mathbf{r}$ to the axis $\mathbf{k}$, so that (10) becomes (we drop the "ren" subscript)

$$\begin{aligned}\langle\phi^2(\mathbf{r})\rangle&=\frac{1}{16\pi^2}\sum_{\mathbf{k}}\left[\sum_{m=1}^{\nu_{\mathbf{k}}-1}\csc^2\left(\frac{m\pi}{\nu_{\mathbf{k}}}\right)\right]\frac{1}{\rho(\mathbf{r},\mathbf{k})^2}\\&=\frac{1}{48\pi^2}\sum_{\mathbf{k}}\frac{\nu_{\mathbf{k}}^2-1}{\rho(\mathbf{r},\mathbf{k})^2}=\sum_{\mathbf{k}}\langle\phi^2(\mathbf{r})\rangle_{\nu_{\mathbf{k}},\mathbf{k}} . \quad (12)\end{aligned}$$

This is just the sum of the $\langle\phi^2(\mathbf{r})\rangle$'s for each of the singularity axes taken separately.

A similar result holds for other vacuum averages. This can be shown quite generally, and easily, by writing the sum over Γ in (8) as one over the axes $\mathbf{k}$ and the associated cyclic operations:

$$\begin{aligned}G(\mathbf{r},t;\mathbf{r}',t')&=G_0(\mathbf{r},t;\mathbf{r}',t')+\sum_{\mathbf{k}}\sum_{m=1}^{\nu_{\mathbf{k}}-1}G_0(\mathbf{r},t;A_{\mathbf{k}}^m\mathbf{r}',t')\\&=G_0(\mathbf{r},t;\mathbf{r}',t')+\sum_{\mathbf{k}}G_{\mathbf{k}}(\mathbf{r},t;\mathbf{r}',t') . \quad (13)\end{aligned}$$

$G_0+G_{\mathbf{k}}$ is the Green's function for a singularity axis $\mathbf{k}$. Invariance under the general rotation R says that

$$G_0(R\mathbf{r},t;A_{R\mathbf{k}}R\mathbf{r}',t')=G_0(\mathbf{r},t;A_{\mathbf{k}}\mathbf{r}',t')\ ,$$

where $A_{R\mathbf{k}}=RA_{\mathbf{k}}R^{-1}$ is a generator about the axis $R\mathbf{k}$.

All renormalized vacuum averages then have a similar structure. That of the stress-energy tensor is given by

$$\langle T_{\alpha\beta}(\mathbf{r})\rangle=\sum_{\mathbf{k}}\langle T_{\alpha\beta}(\mathbf{r})\rangle_{\nu_{\mathbf{k}},\mathbf{k}}\ , \tag{14}$$

where $\langle T_{\alpha\beta}(\mathbf{r})\rangle_{\nu,\mathbf{k}}$ is the vacuum average around a cosmic string of angle $2\pi/\nu$ along the axis $\mathbf{k}$. This expression allows for a simple numerical evaluation since the individual vacuum averages are already known. We must note, of course, that the tensor indices in (14) refer to coordinates on the $\mathbb{R}^4$ covering space so that the standard result for a single, straight string along the z axis must be transformed to strictly Minkowskian coordinates in $\mathbb{R}^4$ and then rotated from $\mathbf{z}$ to $\mathbf{k}$. Performing these operations, the general form of $\langle T^{\alpha\beta}(\mathbf{r})\rangle$ for a string lying along the axis $\mathbf{k}$ is found to be

$$\langle T^{\alpha\beta}(\mathbf{r})\rangle_{\nu,\mathbf{k}}=\frac{C_\nu}{16\pi^2\rho^6}(\rho^2\eta^{\alpha\beta}+4v^\alpha v^\beta)\ , \tag{15}$$

where C_ν is a constant depending on the deficit angle and the vector $v^\alpha=(0,\mathbf{k}\wedge\mathbf{r})$. This expression is for conformally invariant, massless fields.

In Fig. 1 we show a contour plot for a periodic conformal scalar. We have chosen the octahedral group for illustrative purposes and have pictured $\langle T^{00}(\mathbf{r})\rangle$ for $r=1$ over the equatorial plane of the stereographic projection of the triangulation, which is also shown. Figure 2 shows a relief plot of the same quantity.

The plots for the Dirichlet and Neumann boundary

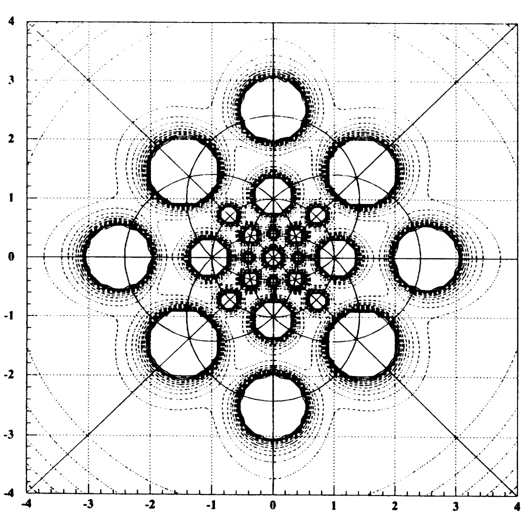

FIG. 1. Contour plot of the vacuum average of the energy density of a massless, conformally coupled scalar field for periodic boundary conditions in an octahedral Möbius corner. The section is for constant radius $r=1$ and the horizontal axes are those of the equatorial plane of the stereographic projection which is also depicted.

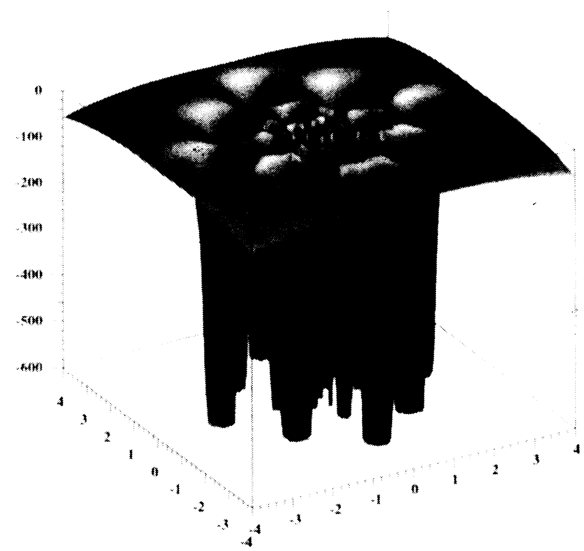

FIG. 2. Relief plot of Fig. 1.

conditions are generally similar. We give in Fig. 3 a contour plot of the tetrahedron Dirichlet case.

VI. PHASE FACTORS

In quantum mechanics, or in the theory of a complex field, one is allowed to choose the phase factors in (8) to be a unitary representation of the ramified covering group of $\mathcal{O}'(\mathcal{O})$, which is $\Gamma'(\Gamma)$. The character tables indicate nontrivial one-dimensional representations in the T and O cases. To establish some other notation, we check these facts in a standard way. Consider the presentation $A_1^{\nu_1}=A_2^{\nu_2}=A_3^{\nu_3}=A_1A_2A_3=E$ where A_i generates rotations through $2\pi/\nu_i$ about the vertex of angle π/ν_i. The Abelian nature of the representation $\{a(\gamma)\}$ implies that

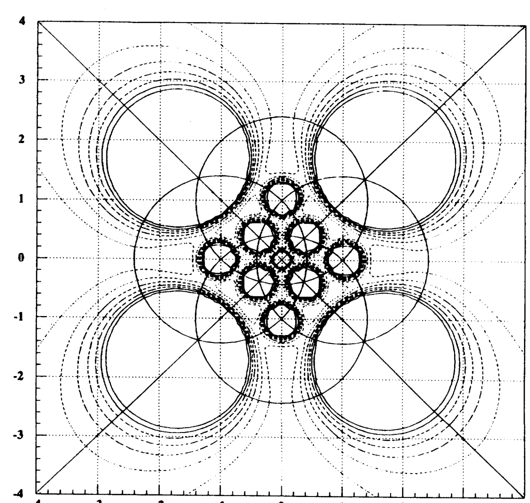

FIG. 3. Contour plot of the vacuum average of the energy density of a massless, conformally coupled scalar for Dirichlet boundary conditions in a tetrahedral Möbius corner.

$$a(\gamma)=e^{2\pi i \mathbf{p}\cdot\mathbf{q}} \tag{16}$$

where $q_i=s_i/\nu_i$ and p_i is the number of times the generator A_i occurs in the word presentation of γ. The final relation translates into $\sum q_i\in\mathbb{Z}$ with $0\le q_i<1$.

Analyzing the cases yields the complex representations $\mathbf{s}=(0,2,1)$ or $(0,1,2)$ for T and the real one $(1,0,2)$ for O. [Equivalent statements are $H_1(\mathbb{R}^3/T;\mathbb{Z})\approx\mathbb{Z}_3$ and $H_1(\mathbb{R}^3/\mathrm{O};\mathbb{Z})\approx\mathbb{Z}_2$. Also $H_1(\mathbb{R}^3/\mathrm{Y};\mathbb{Z})\approx 0$.]

For T, one interpretation of (0,2,1), in the covering space picture, would be of 4 strings of two units of flux running from the origin to the four vertices and four strings of unit flux to the four face centers (which are the vertices of the counter tetrahedron). Such a flux distribution has no effect on the quantum mechanics on the covering space so that the Green's function, or propagator, G_0, in (8) is unaffected.

Nothing physical is altered by adding three flux units along any of the $\nu=3$ rotation axes and we have the equivalent interpretation as four U(1) Aharonov-Bohm flux tubes along these axes all carrying either one unit or two units of flux corresponding, respectively, to the two cases (0,2,1) and (0,1,2).

Results analogous to (12), (13), and (14) hold with the appropriate modifications of the individual cosmic-string contributions.

Figure 4 shows a relief plot of $\langle T^{00}(\mathbf{r})\rangle$ for the (0,2,1) tetrahedron case.

The above discussion can be generalized by taking the field to belong to a representation of some non-Abelian internal symmetry group G. Then the $a(\gamma)$ will be elements of $\mathrm{Hom}_G(\Gamma,G)$ and we can use our knowledge of the representations of Γ to construct this homomorphism. Since the vacuum averages involve a trace over the internal indices, expressions such as (10) will contain the G character $\chi(\gamma)$.

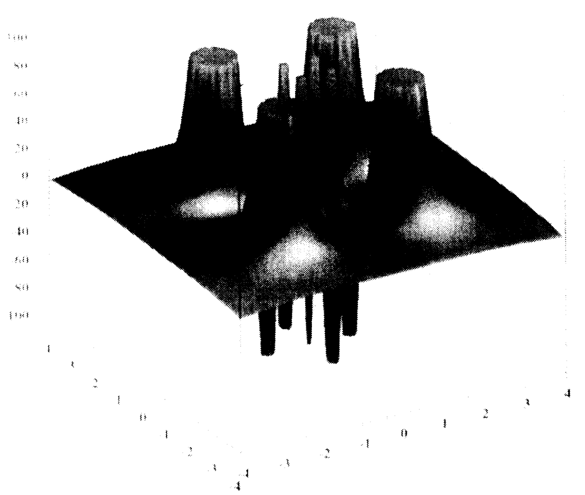

FIG. 4. Relief plot of the vacuum average of the energy density of a massless, conformally coupled complex scalar with twisted boundary conditions in a tetrahedral corner.

A simple example would be ϕ in the fundamental representation of $G=\mathrm{U}(2)$ and $a(\gamma)$ the 2×2 irreducible representation of O usually denoted by E. G has been chosen to be U(2) rather than SU(2) to allow the use of the irrep E, which is not unimodular.

A glance at the character tables shows that certain classes of O will disappear from sums such as (10). The result is the same as that for the unimodular representation $E\oplus E^*$ in the T case. Thus, so far as certain vacuum averages are concerned, the gravitational effect of a singular string can be removed by a suitably contrived arrangement of (non-Abelian) internal symmetry fluxes.

VII. OTHER FIELDS

For simplicity, we will discuss the periodic case first, and indicate the modifications needed for the extended group Γ' later.

A. The electromagnetic field

For brevity we use the $\phi=H-i\mathrm{E}$ for formalism employed elsewhere in a similar context. Using Cartesian axes in the covering $\mathbb{R}^3$ space, the relevant vector Green's function G_0 is a 3×3 matrix and rotational invariance now reads

$$G_0(R\mathbf{r},t;A_{R\mathbf{k}}R\mathbf{r}',t')=D(R)G_0(\mathbf{r},t;A_{\mathbf{k}}\mathbf{r}',t')D(R^{-1}) , \tag{17}$$

where $D(R)$ is a spin-one representation of SO(3).

It is possible to extend this formalism to the spin-j case and we will imagine this to be done simply by taking the matrices to be $(2j+1)\times(2j+1)$ ones.

The projected Green's function on $T\times\mathbb{R}^3/\Gamma$ is

$$\begin{aligned} G(\mathbf{r},t;\mathbf{r}',t') &= \sum_{\gamma\in\Gamma} a(\gamma)G_0(\mathbf{r},t;\gamma\mathbf{r}',t')D(\gamma) \\ &= \sum_{\gamma\in\Gamma} a(\gamma)D(\gamma)G_0(\gamma^{-1}\mathbf{r},t;\mathbf{r}',t') , \end{aligned} \tag{18}$$

where the $D(\gamma)$ factor arises from a rotation between a local dreibein system (invariant under Γ) and the globally Cartesian one. [Cf. Banach and Dowker [29] equation (A5).] The U(1) factors, $a(\gamma)$, have been retained for generality.

The summation can again be written over the singularity axes, as in (13), but where $G_{\mathbf{k}}$ is this time given by

$$G_{\mathbf{k}}(\mathbf{r},t;\mathbf{r}',t')=\sum_{m=1}^{\nu_{\mathbf{k}}-1} a(A_{\mathbf{k}}^m)G_0(\mathbf{r},t;A_{\mathbf{k}}^m\mathbf{r}',t')D(A_{\mathbf{k}}^m) . \tag{19}$$

$G_0+G_{\mathbf{k}}$ is the spin-j Green's function for a singularity axis $\mathbf{k}$ in Cartesian coordinates and with respect to a globally Cartesian dreibein system.

In the electromagnetic spin-one case, the vacuum average of the energy density is proportional to $\mathrm{Tr}G$ and we see from (18) and (19) that it has the same form as in (14). The same statement, i.e., (14) with (15), holds for the complete stress-energy tensor.

B. The spin-$\frac{1}{2}$ field

There appears to be an obstruction to setting up a spinor field on a fundamental domain $\mathcal{C}$. The reason is that the image method for deriving the Green's function around a single, straight cosmic string, i.e., in the cyclic $\mathbb{Z}_\nu$ case, does not work if ν is even. This problem can be circumvented for the straight string by artificially introducing a U(1) flux through the string so as to give an extra minus sign when the string is encircled. In the T, O, and Y cases this is not possible because of the relation, $A_1A_2A_3=E$, between the generators.

It is possible to put spinors around the general straight cosmic string but the Green's function can only be written as an image sum when the angle is $2\pi/\nu$ with ν odd. It is not clear whether spinors can be set up easily on $\mathcal{C}$ without using images.

VIII. THE EXTENDED GROUPS

The extended groups are the complete symmetry groups of the regular solids. In the Schönflies notation, they are denoted by T_d, O_h, and Y_h.

They can be generated by the planes of symmetry, in particular by reflections in three (concurrent) planes (e.g., Coxeter [8] Sec. 5.4) and, as mentioned before, the elements fall into two sets depending on whether they contain an even or an odd number of reflections. Even, or proper, elements can be written as $\gamma\in\Gamma$ and odd, or improper, elements as $\gamma\sigma$, where σ is a reflection in a symmetry plane of the regular solid.

The image sum (18) then becomes

$$G(\mathbf{r},t;\mathbf{r}',t')=\sum_{\gamma\in\Gamma} a(\gamma)[G_0(\mathbf{r},t;\gamma\mathbf{r}',t') +a(\sigma)G_0(\mathbf{r},t;\gamma\sigma\mathbf{r}',t')\Sigma]D(\gamma)\ , \tag{20}$$

where $\Sigma(\sigma)$ is the action on the field induced by σ.

Actually, because a reflection mixes left and right, it is necessary to extend the $(2j+1)(2j+1)$ formalism to a $2(2j+1)2(2j+1)$ Dirac one. For neutral fields we can write $\psi=\phi\oplus C\phi^*$ where C is the charge conjugation matrix. The $D(\gamma)$ in (20) should thus be taken as a direct sum representation:

$$D(\gamma)=D(\gamma)\oplus D(\gamma)=D(\gamma)\otimes\mathbf{1}\ .$$

Choosing $\phi=H-iE$ corresponds to setting C equal to the unit matrix.

The reflection $\sigma_{\mathbf{t}}$ in the plane with normal $\mathbf{t}$, can be written as a rotation through π about the axis $\mathbf{t}$ combined with the parity inversion $\iota:\mathbf{r}\to-\mathbf{r}$, i.e., $\sigma_{\mathbf{t}}=R_{\mathbf{t}}(\pi)\iota$. For arbitrary spin, we have the reflective action

$$[R_{\mathbf{t}}(\pi)P\psi](\mathbf{r},t)=\Sigma(\sigma_{\mathbf{t}})\psi(\sigma_{\mathbf{t}}\mathbf{r},t)\ ,$$

where P is the usual parity action

$$(P\psi)(\mathbf{r},t)=\mathbf{1}\otimes\sigma^1\psi(-\mathbf{r},t)\ ,$$

σ^1 being the standard Pauli matrix. Thus the representation $\Sigma(\sigma_{\mathbf{t}})$ is given by

$$\Sigma(\sigma_{\mathbf{t}})=D[R_{\mathbf{t}}(\pi)]\otimes\sigma^1\ .$$

Explicitly for spin-one, in the Cartesian basis ($C=\mathbf{1}$),

$$D[R_{\mathbf{t}}(\pi)]=2\mathbf{t}\otimes\mathbf{t}-\mathbf{1}\ .$$

The action on $\phi=\mathbf{H}=-i\mathbf{E}$ corresponds to complex conjugation together with a reversal in sign of the parallel component while the sign of the normal part is unchanged. Thus $a(\sigma)$ should be set equal to -1 in order to give the correct (perfect) boundary conditions in $\mathcal{C}'$ for the electromagnetic field.

The reflection term in $G(\mathbf{r},t;\mathbf{r}',t')$ disappears trivially from $\mathrm{Tr}G$. Therefore the vacuum average of the electromagnetic stress tensor is the same in the periodic and reflective cases.

For the cyclic wedge this agrees with known, and old results (e.g., Deutschand Candelas [30]). In this simple geometry the result also holds for the conformally coupled scalar field because the contribution to the average coming from single reflection terms vanishes. This also agrees with old results. There is, however, a difference in the polyhedral cases for the scalar field.

For the octahedron and icosahedron, the extended groups can be composed as $\Gamma'=\Gamma\cup\Gamma\iota$. The inversion ι can be expressed in terms of the generating reflections $\sigma_1,\sigma_2,\sigma_3$, in the sides of a fundamental triangle, as

$$\iota=(\sigma_1\sigma_2\sigma_3)^{h/2}\ ,$$

where h is the Coxeter-Killing number connected with the order of the group by $|\Gamma'|=h(h+2)$.

In general, h is the period of the product of the reflection generators of a finite reflection group. When $h/2$ is even, $(\sigma_1\sigma_2\sigma_3)^{h/2}$ is a half-turn. The generators A_i introduced earlier are relaxed to the σ_i by $A_1=\sigma_3\sigma_2$, etc.

In the cases O_h and Y_h, (20) can be replaced by

$$G(\mathbf{r},t;\mathbf{r}',t')=\sum_{\gamma\in\Gamma} a(\gamma)[G_0(\mathbf{r},t;\gamma\mathbf{r}',t')D(\gamma)\otimes\mathbf{1}\pm G_0(\mathbf{r},t;-\gamma\mathbf{r}',t')D(\gamma)\otimes\sigma^2]\ . \tag{21}$$

Incidentally, a light ray sent into a tetrahedral corner will emerge, after six reflections, in a different direction whereas, in the octahedral and icosahedral cases, it will come out simply reversed (and translated).

IX. CONCLUSION AND COMMENTS

The method of images has been used to set up noninteracting field theories in Möbius corners. The physical system corresponds to a combination of three concurrent ideal cosmic strings. Some Casimir effects have been calculated and, although somewhat academic, the expressions are, we feel, sufficiently attractive to warrant exposure.

We still have to address the questions raised in Sec. IV, particularly the product of a source for the metric.

In the cyclic case C_n one can introduce m-fold coverings of the sphere with a lune of angle $2\pi m/n$ as fundamental domain ($m \in \mathbb{Z}$). Letting m tend to infinity one gets an infinitely sheeted covering (cf. Sommerfeld [31]) and a wedge of arbitrary angle can be treated.

For the other cases it is only possible to find a finite number of finitely sheeted coverings (Schwarz [9]). It is not clear whether this means that one cannot find a way of treating an arbitrary solid angle deficit. The corresponding conformal transformation in terms of hypergeometric functions is standard, but the problem is the construction of the Green's function on the multisheeted Riemann surface. Unless this can be done, and this is an open problem, there is little point in analyzing the possible observational significance of the metric (6), except as a mathematical exercise. Serebryanyi [32] gives some interesting generalities on the converging space method.

At a calculational level the group theoretical analysis of the mode problem and the Clebsch-Gordan series is interesting and has been looked at in some depth by the chemists, e.g., Damhus, Harnung, and Schaffer [33].

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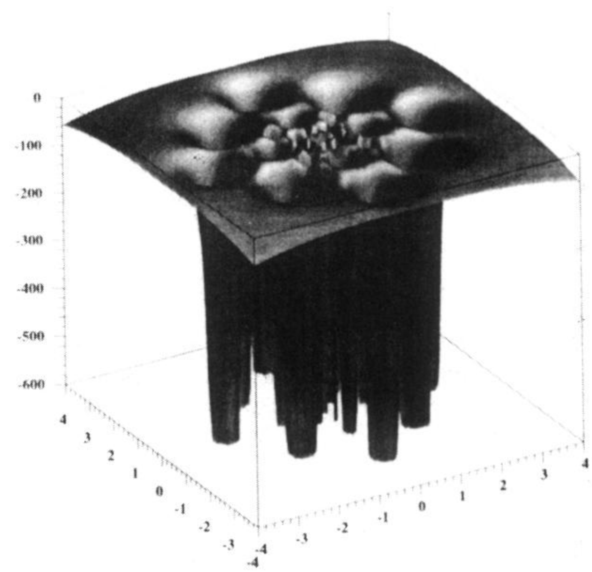

FIG. 2. Relief plot of Fig. 1.

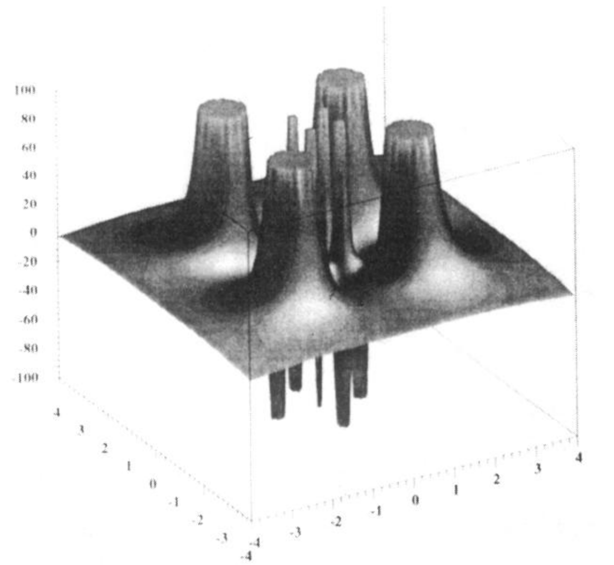

FIG. 4. Relief plot of the vacuum average of the energy density of a massless, conformally coupled complex scalar with twisted boundary conditions in a tetrahedral corner.