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# Properties of radiation near the black-hole horizon and the second law of thermodynamics

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By considering a gedanken experiment of adiabatically lowering a box containing matter with rest energy $E$ and entropy $S$ into a black hole, Bekenstein claimed that the necessary condition for the validity of the generalized second law of thermodynamics is $S/E \leq 2\pi R$, where $R$ is the effective radius of the box. Unruh and Wald claimed that this condition is not necessary but the acceleration radiation can guarantee the generalized second law. In this paper, we point out that the Unruh-Wald conclusion does not hold because Hawking radiation near the horizon is not thermal. Bekenstein's conclusion does not hold because the thin box approximation is not correct near the horizon. Neither Hawking radiation (or acceleration radiation) nor $S/E \leq 2\pi R$ can guarantee the second law. We have sufficient reasons to conjecture that gravitation can influence the matter equation of state. For radiation, the usual equation of state $\rho=\alpha T^4$ and $S=\frac{4}{3}\alpha T^3$ does not hold in the strong gravitation field, e.g., near the black-hole horizon. We derive the equation of state for radiation near the horizon and find that it is very different from the equation in flat spacetime. The second law of thermodynamics can be satisfied if we impose some restrictions on one parameter of the equation of state. As a corollary, we get an upper bound on $S/E$ which resembles Bekenstein's result.

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## I. INTRODUCTION

In general it is assumed that the entropy of a black hole is $\frac{1}{4}$ of the horizon's area ($c=\hbar=G=K_B=1$). When matter interacts with a black hole, the second law of thermodynamics means that the sum of the entropy of the matter and $\frac{1}{4}$ of the horizon's area never decreases (Bekenstein called it the generalized second law of thermodynamics [1]). To test this conjecture, Bekenstein and Unruh and Wald discussed the following gedanken experiment [2,3]: A box containing matter of rest energy $E$ and entropy $S$ is lowered adiabatically from infinity to a Schwarzschild black hole; after opening the box at some desired position and releasing the contents into the black hole, the empty box is pulled back to infinity.

In Bekenstein's discussion he did not consider the effect of Hawking radiation [4]. The box is dropped on the surface of the horizon; after opening the box and releasing its contents into the black hole, the empty box is pulled back to infinity (to cancel the effect of the box's weight). The increase of the horizon's area is $8\pi ER$ ($R$ is the effective radius of the box) which is the minimum [2]. Bekenstein claimed that the necessary condition (also sufficient for this gedanken experiment) for the validity of the second law of thermodynamics is

$$S/E \leq 2\pi R \ . \tag{1}$$

Unruh and Wald pointed out that the box cannot be dropped on the surface of the horizon if we consider the effect of Hawking radiation (or acceleration radiation [5]). There is a floating point and if we open the box and release the contents at that point the increase of entropy of the black hole takes the minimum, which is just the entropy of the thermal radiation with rest energy $E$ and volume $V$ (the volume of the box). The existence of Hawking radiation preserves the validity of the generalized second law because the thermal radiation is the state of matter and radiation which maximizes entropy at a fixed energy and volume [3].

In the argument of Unruh and Wald, they assumed that the Hawking radiation from infinity to the horizon is always thermal, so the pressure of radiation approaches infinity as the box approaches the horizon, and the box cannot be dropped on the horizon. But in general it is recognized that the particle concept cannot be used near the horizon and the most appropriate means to describe the radiation is the stress tensor [6,7]. From the stress tensor of black-hole radiation [6] we find that the radiation near the horizon is not thermal; its pressure and energy density are both finite and not very large on the horizon. So the argument of Unruh and Wald does not hold; we can drop the box on the horizon as long as the weight of the box is sufficiently large. Hawking radiation cannot guarantee the generalized second law.

The argument of Bekenstein is tenable only in the thin-box approximation $a\,dx/dy \ll x$ where $x=\sqrt{-g_{00}}$ is the redshift factor, $y$ is the proper distance from the box to the horizon and $a$ is the side length of box in the radial direction of the black hole. The minimum increase of the horizon's area is [1] $8\pi EY$, where $Y$ is the proper distance from the box's center of mass to the horizon when the box is on the horizon; only in the thin-box approximation do we have $Y \simeq a/2 \simeq R$, and Bekenstein's argument holds. But near the horizon $a\,dx/dy \simeq a/4M \simeq x$ ($M$ is the mass of the black hole) so the thin-box approximation is not correct. The gravitational field near the horizon is so strong that the distribution of matter in the box is not homogeneous. The distribution depends on the equation of state of the matter. If the contents of the box is thermal radiation and from infinity to the horizon the equation of state is always

$$\rho=\alpha T^4, \quad s=\tfrac{4}{3}\alpha T^3, \quad \alpha=\text{const} , \tag{2}$$

where $\rho$ is the energy density, $s$ is the entropy density and $T$ is the local temperature, then we can prove that $Y\rightarrow 0$ as the box approaches the horizon. Hence Bekenstein's claim does not hold; even if Eq. (1) is correct, it cannot guarantee the validity of the generalized second law.

Hawking radiation near the horizon is not thermal, which leads us to conjecture that the gravitational field can influence the equation of state of matter, so Eq. (2) might not be correct when the box is near the horizon. In fact this is necessary because we can reasonably demand that the entropy and rest energy of the radiation are constants as the box is slowly lowering, and this is inconsistent with Eq. (2) (the proof is left in Sec. III). In this paper we study the equation of state of radiation near the horizon and find that it is very different from Eq. (2). The second law of thermodynamics can be satisfied if we impose some restrictions on one parameter of the equation of state.

In Sec. II we first point out that the work done by the weight of radiation in the box on the outside is equal to its rest energy as the box is adiabatically lowered onto the horizon from infinity if Eq. (2) is also correct near the horizon; i.e., the radiation in the box takes an infinite redshift and its energy seen at infinity is zero. Second, the work to overcome the pressure of Hawking radiation is finite and not very large, so Hawking radiation cannot prevent the box from lowering onto the horizon. After the radiation in the box is released into the black hole, the above conclusions lead to the result that the increase of entropy of the black-hole and Hawking radiation is smaller than the entropy of radiation in the box; this violates the second law of thermodynamics. There are only two alternatives: one is that Eq. (2) is correct and the box cannot be dropped onto the horizon's surface; i.e., there is a minimum distance from the bottom of the box to the horizon which is not zero; the other is that the box can be dropped onto the horizon and Eq. (2) is not correct near the horizon.

In Sec. III for sufficient reasons we suggest that the ordinary equation of state of radiation is not correct near the horizon, which means that gravitation can influence the equation of state of matter. We have derived the equation of state of radiation near the horizon which is very different from Eq. (2). We find that if the parameter $K_0$ of the equation of state satisfies some condition, the box can be dropped on the horizon and the second law of thermodynamics is satisfied. As a corollary we get an upper bound on $S/E$ similar to Eq. (1).

We summarize and conclude in Sec. IV.

## II. ORDINARY EQUATION OF STATE OF RADIATION AND THE SECOND LAW OF THERMODYNAMICS

### A. The gedanken experiment

A Schwarzschild black hole (SBH) with mass $M$ is put in the center of a spherical thin cavity with sufficiently large radius $r_0$, negligible mass, and perfect reflectability. Now let the SBH be in thermal equilibrium with the Hawking radiation in the cavity. We fill a small box of volume $aA$ ($a$ is the height of the box and $A$ is its bottom area) with thermal radiation of temperature $T_r$ at infinity (the box is adiabatic), slowly lower the box through a small hole on the cavity to the black hole, and open the door at the bottom of the box to release the radiation when the proper distance from the bottom of box to the horizon is $l$, then slowly draw back the box (with the door open) to infinity and a cycle is completed.

The temperature of Hawking radiation (HR) is $T_{\rm BH}=1/8\pi M$. We assume that $T_r \gg T_{\rm BH}$ and call HR+SBH the cold sink. The increase of the cold sink's energy in the above cycle is

$$\epsilon=E_r-W_\infty , \tag{3}$$

where $W_\infty$ is the work done at infinity and $E_r$ is the rest energy of the radiation in the box:

$$E_r=\alpha T_r^4 aA, \quad \alpha=\text{const} . \tag{4}$$

By virtue of the no-hair theorem (NHT) the increase of entropy of the cold sink is

$$\Delta S=\epsilon/T_{\rm BH} . \tag{5}$$

*Proof.* $E$ and $S$ are the energy and entropy of the cold sink; then NHT leads to

$$E=E(M,r_0), \quad S=S(M,r_0) ,$$

or

$$E=E(S,r_0)$$

so

$$\delta E=\left(\frac{\partial E}{\partial S}\right)_{r_0=\text{const}}\delta S=T_{\rm BH}\delta S$$

and

$$\Delta S=\epsilon/T_{\rm BH} \quad (\Delta S=\delta S, \quad \epsilon=\delta E) . \ \square$$

The second law of thermodynamics demands that

$$\Delta S\geq S_r \tag{6}$$

where $S_r$ is the entropy of the radiation in the box:

$$S_r=\tfrac{4}{3}\alpha T_r^3 aA . \tag{7}$$

$W_\infty$ can be written as

$$W_\infty=W_1-W_2 , \tag{8}$$

where $W_1$ is the work done by the weight of the radiation in the box and $W_2$ is the work to overcome the buoyancy of HR.

### B. Calculation of $W_1$

Let the proper distance form the bottom of the box to the horizon be $l$; then the work done by the weight of the radiation in the box on the outside is [3]

$$W_1 = E_r - E \ , \tag{9}$$

$$E = A \int_l^{l+a} \rho(y) x(y) dy \ , \tag{10}$$

where $y$ is the proper distance from the point in the box to the horizon, $\rho(y)$ is the energy density of radiation in the box, and $x(y)$ is the redshift factor

$$x(y) = \sqrt{-g_{\infty}(y)} = [1 - 2M/r(y)]^{1/2} \tag{11}$$

where $r(y)$ is the coordinate distance corresponding to the proper distance $y$.

The entropy of the radiation in the box is invariant as the box is lowering because the process is adiabotic:

$$A \int_l^{l+a} s(y) dy = S_r \ , \tag{12}$$

where $S(y)$ is the entropy density. Thermal equilibrium requires that [8]

$$T(y) x(y) = \text{const} \equiv T_0(l) \ . \tag{13}$$

Assuming the usual state equation (2) is always correct, we insert Eq. (13) into (2):

$$\rho(y) = \alpha T_0^4 / x^4(y) \ , \tag{14}$$

$$s(y) = \tfrac{4}{3} \alpha T_0^3 / x^3(y) \ . \tag{15}$$

From Eqs. (12) and (15) we get

$$S_r = \tfrac{4}{3} \alpha T_0^3 A \int_l^{l+a} \frac{dy}{x^3} \ . \tag{16}$$

From Eqs. (10) and (14) we obtain

$$E = \alpha T_0^4 A \int_l^{l+a} \frac{dy}{x^3} \ . \tag{17}$$

Then from (16) and (17) we have

$$E = \tfrac{3}{4} T_0 S_r \ , \tag{18}$$

where $T_0(l)$ is determined by Eq. (16). When $y \ll r_H = 2M$ we have

$$x(y) \simeq y/4M \ . \tag{19}$$

Inserting (19) into (16) we get ($l \ll a \ll r_H$)

$$S_r \simeq \tfrac{4}{3} \alpha T_0^3 A \, 32 M^3 l^{-2} \ .$$

From this and Eq. (7),

$$T_0(l) \simeq \left[ \frac{a}{32M} \right]^{1/3} T_r \left[ \frac{l}{M} \right]^{2/3} \ . \tag{20}$$

Inserting (20) into (18) and (9) we obtain

$$E(l) \simeq E_r \left[ \frac{a l^2}{32 M^3} \right]^{1/3} \ , \tag{21}$$

$$W_1 \simeq E_r \left[ 1 - \left[ \frac{1}{32} \right]^{1/3} \frac{1}{M} a^{1/3} l^{2/3} \right] , \quad l \ll a \ll r_H \ . \tag{22}$$

We find that $E \to 0$ and $W_1 \to E_r$ as $l \to 0$ (i.e., the box is on the horizon).

### C. Calculation of $W_2$

When the distance from the bottom of the box to the horizon is $l$, the work done on the cold sink on the outside to overcome the buoyance of HR is [3]

$$W_2 = A \int_l^L (P_1 x_1 - P_2 x_2) dl \ , \tag{23}$$

where $x_1$ and $P_1$ are the redshift factor and the pressure of HR at the bottom of the box, respectively, $x_2$ and $P_2$ are the redshift factor and the pressure of HR at the top of the box, respectively, and $L$ is the proper distance from the wall of the cavity to the horizon. Through a simple variable transformation, Eq. (23) can be written as

$$W_2 = A \int_l^{l+a} P(y) x(y) dy \tag{24}$$

which means that $W_2$ depends only on the values of $Px$ in $l \sim l+a$. When the box approaches the horizon $W_2$ depends only on the values of $Px$ near the horizon.

Unruh and Wald believed that HR near the horizon is thermal [3]; then

$$P(y) = \tfrac{1}{3} \alpha T^4(y) = \tfrac{1}{3} \alpha T_{\rm BH}^4 / x^4(y)$$

and on the horizon ($y=0$) we have $x(0)=0$ and $P(0) \to \infty$. Inserting it into Eq. (24) it is found that $W_2 \to \infty$ as $l \to 0$ which means that the box cannot be dropped on the horizon and there is a floating point outside the horizon. But as mentioned above the best method to describe the radiation near the horizon is the stress tensor from which we will see that HR near the horizon is not thermal. For the SBH the stress tensor (of a conformal coupled scalar field) is [6]

$$\begin{aligned} T^t{}_t &= -3 P_0 (f - h) \ , \\ T^r{}_r &= P_0 (f + h) \ , \\ T^\theta{}_\theta &= T^\phi{}_\phi = P_0 f \ , \end{aligned} \tag{25}$$

where $P_0 = \tfrac{1}{3} \alpha T_{\rm BH}^4$ and

$$f(r) = \frac{1 - (4 - 6M/r)^2 (2M/r)^6}{(1 - 2M/r)^2} \ , \tag{26}$$

$$h(r) = 24 \left[ \frac{2M}{r} \right]^6 \ . \tag{27}$$

From this we find that the radiation near the horizon ($r \to 2M$) is not thermal and its radial pressure is

$$P(r) = T^r{}_r \overset{r \to r_H}{\simeq} 36 P_0 \tag{28}$$

which is finite and not very large. So in the situation $T_r \gg T_{\rm BH}$ the box can be dropped on the horizon; i.e., there is no floating point outside the horizon. From Eqs. (28) and (24) we have

$$\begin{aligned} W_2 &\simeq A P(2M) \int_l^{l+a} x(y) dy \, (l \to 0) \\ &\simeq 36 A P_0 \int_l^{l+a} \frac{y}{4M} dy \\ &\simeq \alpha T_{\rm BH}^4 a A (3a + 6l)/2M \quad (l \ll a \ll r_H) \end{aligned} \tag{29}$$

$$\overset{l\to 0}{\simeq} \tfrac{3}{2}\alpha T_{\rm BH}^4 a A \frac{a}{M} = 12\pi\alpha T_{\rm BH}^5 a^2 A \ . \qquad (30)$$

Comparing Eqs. (22) and (29) we find that $W_2 \ll W_1$ if $T_r \gg T_{\rm BH}$ and $l \ll a \ll r_H$.

### D. $\Delta S$ and the second law of thermodynamics

Insert (22) and (29) into (8) and (3) we obtain $(l \ll a \ll r_H)$

$$\epsilon = \left[\frac{1}{32}\right]^{1/3} E_r M^{-1} a^{1/3} l^{2/3} + \tfrac{3}{2}\alpha T_{\rm BH}^4 a A \frac{a+2l}{M} \qquad (31)$$

as $l \to 0$

$$\epsilon \to \frac{3}{2}\alpha T_{\rm BH}^4 a^2 A/M \ . \qquad (32)$$

By (5) and (32), the increase of entropy of the cold sink is

$$\Delta S = \frac{3}{2}\alpha T_{\rm BH}^3 a^2 A/M \ll S_r \quad (l=0) \qquad (33)$$

which violates the second law of thermodynamics. This indicates whether the box cannot be dropped on the horizon; i.e., there is a positive minimum of $l$, or the equation of state (2) is not correct for the radiation in the box near the horizon. From the above arguments we know that the pressure of HR is finite and not very large everywhere from infinity to the horizon; i.e., the minimum of $l$ is zero. We will see in the next section that we have sufficient reasons to infer that Eq. (2) is not correct when the box is near the horizon.

## III. EQUATION OF STATE OF RADIATION NEAR THE HORIZON AND THE SECOND LAW OF THERMODYNAMICS

In Sec. II, we have assumed that the radiation in the box is always described by the equation of state

$$\rho = \alpha T^4, \quad s = \tfrac{4}{3}\alpha T^3 \qquad (34)$$

which results in $\Delta S < S_r$ as the box is dropped on the horizon, so the second law of thermodynamics is violated. This probably means that Eq. (34) is not correct when the box is near the horizon. There are two reasons to support this point.

(i) Hawking radiation near the horizon is not thermal, which enlightens us that gravitation can affect the equation of state of matter.

(ii) As the box is slowly lowering to the horizon we can reasonably require that the rest energy and entropy of radiation in the box be invariant,

$$A\int_l^{l+a} \rho\, dy = E_r \ , \qquad (35)$$

$$A\int_l^{l+a} s\, dy = S_r \ , \qquad (36)$$

and (35), (36), and (34) are inconsistent near the horizon.

*Proof.* (34), (36) and $T = T_0/x$ lead to $T_0 \sim l^{2/3}$ $(l\to 0)$. The left-hand side of (35) is

$$A\alpha T_0^4 \int_l^{l+a} x^{-4} dy \sim l^{8/3}\int_l^{l+a} y^{-4} dy$$
$$\sim l^{8/3} l^{-3} \sim l^{-1/3} \overset{l\to 0}{\longrightarrow} \infty \ .$$

The right-hand side of (35) is $E_r$, so they conflict with each other.□

These two reasons sufficiently indicate that gravitation can affect the equation of state of matter, and near the horizon the equation of state of radiation in the box is not (34). Let us now investigate the equation of state of radiation near the horizon.

We assume that $s$ and $\rho$ of the radiation in the box satisfy the equation for any $l$:

$$s = C\rho x \ , \qquad (37)$$

where $x = x(y)$ is the redshift factor and $C = C(l)$ is a parameter depending on the position of the box and is constant through the box. Later we will see that Eq. (37) returns to the usual relation of $s$ and $\rho$ for the radiation in flat spacetime when the box is far from the horizon. By virtue of the law of thermodynamics and fluid mechanics we have [3]

$$\rho + p = sT \qquad (38)$$

$$\frac{d(xp)}{dy} = -\rho\frac{dx}{dy} \qquad (39)$$

The thermal equilibrium indicates

$$Tx = T_0, \quad T_0 = T_0(l) \ . \qquad (40)$$

Insert (37) and (40) into (38) we get

$$p = (CT_0 - 1)\rho \ . \qquad (41)$$

From (39) and (41) we have

$$\rho = \rho_0 x^{\xi} \ ,$$
$$\rho_0 = \rho_0(l), \quad \xi = \xi(l) = CT_0/(1 - CT_0) \ , \qquad (42)$$

which is correct for any $l$; $\rho_0$ and $\xi$ are determined by the constraints (35) and (36).

Now let us show that Eq. (37) returns to the usual relation in flat spacetime $l \to \infty$. Insert (42) and (37) into (35) and (36) we find that $[x(\infty) \to 1]$

$$E_r = \rho_0 a A, \quad S_r = C\rho_0 a A$$

so

$$C(l\to\infty) = S_r/E_r = \tfrac{4}{3}/T_r \qquad (43)$$

which is just $C$ in flat spacetime. It is not difficult to prove that (34) is a good approximation of the equation of state of radiation when $l \gg r_H$.

Let us look again at the situation $l \to 0$. Inserting (42) and (37) into (35) and using $x(y) \simeq y/4M$ $(y \ll r_H)$ we find

$$E_r \simeq \frac{1}{\xi+1} a A\rho_0 \left[\frac{a}{4M}\right]^{\xi} ,$$

$$S_r \simeq \frac{1}{\xi+2} a A C\rho_0 \left[\frac{a}{4M}\right]^{\xi+1}$$

from which we get

$$\zeta \simeq \frac{2-CT_{\rm BH}\gamma}{CT_{\rm BH}\gamma-1} \quad (l \to 0) , \tag{44}$$

where

$$\gamma \equiv \tfrac{3}{2}\pi a T_r , \tag{45}$$

$$\rho_0 \simeq \frac{aT_r^4}{(\gamma'-1)(a/4M)(2-\gamma')/(\gamma'-1)} \tag{46}$$

and

$$\gamma' \equiv \gamma C T_{\rm BH} . \tag{47}$$

Define

$$k \equiv 1/CT_{\rm BH} ,$$
$$k_0 \equiv K(l \to 0) = 1/T_{\rm BH} C(l \to 0) . \tag{48}$$

Equation (44) can be written as

$$\zeta(l \to 0) \simeq \frac{2k_0-\gamma}{\gamma-k_0} . \tag{49}$$

From (42) and (49) we have

$$T_0(l \to 0) \simeq (2k_0-\gamma)T_{\rm BH} . \tag{50}$$

Equations (37), (42), (44), (46), and (50) have determined the equation of state of radiation in the box as $l \to 0$ and $k_0$ is a parameter depending on $a$ and $T_r$. Now let us make some remarks.

### A. The second law of thermodynamics

First let us examine whether or not the second law of thermodynamics is satisfied. From (10), (37), and (36) we have

$$E = \frac{1}{c} S_r = k_0 T_{\rm BH} S_r \tag{51}$$

where we have used (48) in the last stage. Inserting (51) into (3), (5), and (9),

$$\Delta S = k_0 S_r + \frac{W_2}{T_{\rm BH}} . \tag{52}$$

The second law of thermodynamics demands that

$$\Delta S \ge S_r . \tag{53}$$

From (52), (53), and $W_2/T_{\rm BH} \ll S_r$ (so disregard the term $W_2/T_{\rm BH}$) we have

$$k_0 \ge 1 , \tag{54}$$

which is the condition of the validity of the second law of thermodynamics.

### B. A restriction on the value of $k_0$

In the situation $T_r \gg T_{\rm BH}$ we have

$$T_0(l) \ge T_{\rm BH} \quad \forall l \ge 0 \tag{55}$$

and

$$T_0(l) = T_{\rm BH} \quad \text{only if } l=0 .$$

This is because

$$\Delta S = \frac{\epsilon}{T_{\rm BH}} \simeq \frac{E}{T_{\rm BH}}, \quad \frac{\partial E}{\partial l} > 0 \text{ or } \frac{\partial(\Delta S)}{\partial l} > 0 ;$$

this means that $\Delta S$ takes the minimum as $l=0$. If $T_0(l=0) < T_{\rm BH}$ and $T_0(l \to \infty) > T_{\rm BH}$, there must exist a position $l=l_0>0$ where $T_0(l_0)=T_{\rm BH}$, and this means that the process of opening the box to release the radiation at $l_0$ is reversible and $\Delta S(l=l_0)$ is the minimum, which is in conflict with the above conclusion. So $T_0 \ge T_{\rm BH}$. Although it has been proved only in the case $l \to 0$, it is very easy to verify that (55) holds for any $l$.

From (50) and (55) we obtain

$$k_0 \ge \tfrac{1}{2}(\gamma+1) . \tag{56}$$

### C. An upper bound on $S/E$

According to the condition that $\rho(y \to 0)$ is finite we find

$$\zeta(l=0) \ge 0 . \tag{57}$$

From (49) and (57) we have

$$\tfrac{1}{2}\gamma \le k_0 \le \gamma$$

and from (56) furthermore

$$\tfrac{1}{2}(\gamma+1) \le k_0 \le \gamma ; \tag{58}$$

hence,

$$\tfrac{1}{2}(\gamma+1) \le \gamma, \quad \text{i.e.,} \quad \gamma \ge 1 . \tag{59}$$

From this and (45) and $S_r/E_r = \frac{4}{3}/T_r$ we have

$$S_r/E_r \le 2\pi a . \tag{60}$$

Assume that the box is a cube; with volume $a \times b \times c$, then

$$S_r/E_r \le 2\pi \min(a,b,c) . \tag{61}$$

As the thermal radiation is the state of matter and radiation which maximizes entropy with a fixed rest energy and volume [9], for any system of ordinary matter and radiation with rest energy $E$ and volume $a \times b \times c$ we have

$$S/E \le 2\pi \min(a,b,c) \tag{62}$$

which is similar to Eq. (1) given by Bekenstein.

### D. Necessary and sufficient conditions for $\Delta S = S_r$

From the discussion of 2 we know that the necessary condition for $\Delta S = S_r$ is

$$l=0 \quad \text{and} \quad T_0 = T_{\rm BH}$$

and from (50) and (52) (neglect $W_2/T_{\rm BH}$) we obtain the necessary and sufficient condition for $\Delta S = S_r$:

$$l=0, \quad k_0=1, \quad \text{and} \quad \gamma=1 . \tag{63}$$

### E. Discussion about the energy condition [9]

It is not difficult to verify that the equation of state obtained above satisfies the weak and the principal energy conditions. The strong energy condition demands

$$\rho+p\geq 0\ , \tag{64}$$

$$\rho+3p\geq 0\ . \tag{65}$$

It is easy to see that (64) is obeyed. From (41), (48), and (50) we get the condition for the validity of (65):

$$k_0\geq \tfrac{3}{4}\gamma\ . \tag{66}$$

Hence the range of $k_0$ can be obtained from (58) and (66):

$$\max[\tfrac{3}{4}\gamma,\tfrac{1}{2}(\gamma+1)]\leq k_0\leq\gamma\ , \tag{67}$$

where

$$\gamma\geq 1$$

and

$$\max[\tfrac{3}{4}\gamma,\tfrac{1}{2}(\gamma+1)]=\begin{cases}\tfrac{3}{4}\gamma & (\gamma\geq 2)\ ,\\ \tfrac{1}{2}(\gamma+1) & (\gamma<2)\ .\end{cases} \tag{68}$$

## IV. CONCLUSION

Gravitation can affect the equation of state of matter. The usual equation of state of radiation $\rho=\alpha T^4$, $s=\frac{4}{3}\alpha T^3$ is correct only if the gravitation is weak. It is not correct if the gravitation is strong, e.g., when the radiation is near the horizon of a black hole. We have derived the equation of state of radiation near the horizon, which is very different from the usual equation in flat spacetime, and find that the second law of thermodynamics is satisfied if we add some restrictions on the parameter $k_0$.

In our discussion we have not used the conjecture that the entropy of a black hole is $\frac{1}{4}$ of the area of its horizon. We derived the formula $\Delta S=\epsilon/T_{\rm BH}$ from the no-hair theorem, so our claim is more general and also holds if we consider the corrections of the back reaction of Hawking radiation to the entropy of black hole [10].

As a corollary we obtain an upper bound on $S/E$ similar to Bekenstein's, which has been proved in statistical physics [2,11]. This indicates that our argument is reasonable.


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