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Logarithmic pion-mass singularities in the radiative corrections to $\tau \rightarrow \pi \nu_\tau$

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We study the logarithmic pion-mass singularities in the radiative corrections, to order α, to the decay $\tau \rightarrow \pi \nu_\tau$. We show that such contributions are independent of strong-interaction effects. We verify that the total decay rate is in agreement with the Kinoshita-Lee-Nauenberg theorem on the cancellation of mass singularities.

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I. INTRODUCTION

The decay $\tau \rightarrow \pi \nu_\tau$ ($\tau_{\pi 2}$ for short) can be viewed as the cross of the decays $\pi \rightarrow e\nu_e, \mu\nu_\mu$ (π_{l2} for short). One could expect, then, that the results of former calculations of the radiative corrections to π_{l2} almost apply mutatis mutandis to $\tau_{\pi 2}$ through analytic continuation. Effects due to strong interaction are present in both decays. A way to handle these effects is by means of the modified pion decay constant, which contains some contributions coming from the pion structure. For π_{l2} this is done in Refs. [1,2], and for $\tau_{\pi 2}$ in Ref. [3]. There it is shown that the largest contributions come from the logarithmic lepton-mass singularities (LMS's) $\ln(m_l/m_\pi)$ in π_{l2}, and from the logarithmic pion-mass singularities (PMS's) $ln(m_\pi/m_\tau)$ in $\tau_{\pi 2}$. Marciano and Sirlin [4] have shown that the coefficient of the LMS is independent of strong interactions. Then, one is motivated to look for the analogous result in $\tau_{\pi 2}$. In this paper we show that the coefficient of the PMS terms in the radiative corrections, to first order in the fine-structure constant, to the $\tau_{\pi 2}$ decay is also independent of strong interactions. However, the proof requires an extra ingredient than in the π_{l2} case: we have to assume the fulfillment of the PCAC (partial conservation of axial-vector current) hypothesis in order to eliminate two pion form factors in the virtual corrections not present in the bremsstrahlung part.

The paper is constructed as follows. In Sec. II, we compute the structure-dependent PMS in the bremsstrahlung corrections, since the procedure serves as a guide when we treat the virtual corrections. Section III is devoted to computing the structure-dependent PMS in the virtual corrections. In Sec. IV, we combine both results in order to establish the desired result, and discuss the independent model PMS within the context of the Kinoshita-Lee-Nauenberg (KLN) theorem on the cancellation of mass singularities [5]. Following Ref. [3], we work within the general framework of quantum electrodynamics and the $V-A$ theory.

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II. BREMSSTRAHLUNG CORRECTIONS

The structure-dependent contributions in the bremsstrahlung corrections come from the emission of a real photon from any hadronic line, as depicted in Fig. 1. The total bremsstrahlung amplitude, denoted by M_B, is given by [3]

$$M_B = M_0 \left[\frac{p\cdot\varepsilon}{p\cdot k} - \frac{l\cdot\varepsilon}{l\cdot k} \right] + \frac{eGf_\pi}{\sqrt{2}} \bar{u}_\nu \not{p} (1-\gamma_5) \frac{k\cdot\varepsilon}{l\cdot k} u_\tau$$
$$+ \frac{eG}{\sqrt{2}} \bar{u}_\nu \gamma_\mu (1-\gamma_5) u_\tau \varepsilon_\lambda(k) \frac{T_B^{\mu\lambda}(p,k)}{p\cdot k} . \quad (1)$$

The first line in Eq. (1) corresponds to the model-independent contributions, which we will denote by M_B^i. The second line is the structure-model-dependent contribution, which will be denoted by M_B^d. The tensor $T_B^{\mu\lambda}(p,k)$ is constructed under the assumption of Lorentz covariance in terms of form factors:

$$T_B^{\mu\lambda}(p,k) = H_1 g^{\mu\lambda} + H_2 k^\mu p^\lambda + H_3 p^\mu p^\lambda + H_4 p^\mu k^\lambda + H_5 k^\mu k^\lambda + iH_6 \varepsilon^{\mu\lambda\alpha\beta} k_\alpha p_\beta , \quad (2)$$

where $H_i = H_i(k^2, s)$, with $s=(p+k)^2$. After contraction with $\varepsilon_\lambda(k)$, the photon polarization four-vector, we can eliminate the H_4 and H_5 terms in Eq. (2). Furthermore, gauge invariance tells us that $H_3=0$, and $H_1 = -H_2 p\cdot k$. Then, $T_B^{\mu\lambda}(p,k)$ is finally written as

$$T_B^{\mu\lambda}(p,k) = H_2 (k^\mu p^\lambda - p\cdot k g^{\mu\lambda}) + iH_6 \varepsilon^{\mu\lambda\alpha\beta} k_\alpha p_\beta . \quad (3)$$

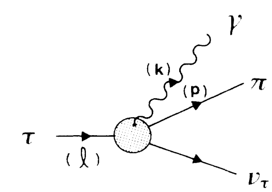

FIG. 1. Diagram involving the strong interaction in the bremsstrahlung correction to $\tau \rightarrow \pi \nu_\tau$. The photon leg is attached to any internal hadronic line.

The contribution of M_B^d to the total transition probability arises from its interference with M_B^i, and its square $|M_B^d|^2$. First, we will consider the interference term. The calculation can be simplified if one observes that

$$p_\mu T_B^{\mu\lambda}(p,k)=0$$

and

$$p_\lambda T_B^{\mu\lambda}(p,k)=H_2(p^2k^\mu-p\cdot kp^\mu)\ .$$

Summing over photon and lepton polarizations and integrating over the neutrino momentum, photon-pion angle, and pion energy, we obtain, for the transition probability, the result

$$\Delta P_B^d=-P_0\frac{\alpha}{2\pi}\frac{1}{f_\pi}\ln\mu\int_0^1 H_2^*(0,x)dx+\cdots\ , \tag{4}$$

where the ellipsis means that no PMS terms are discarded, P_0 is the uncorrected decay rate, and $\mu=m_\pi/m_\tau$. Equation (4) tells us that only the form factor H_2 gives a PMS. The absence of contributions from H_6 can be traced back to the fact that a factor $(p\cdot k)^2$ arises from the product of H_6 with the M_B^i amplitude, canceling the factors $p\cdot k$ in the denominators of M_B^i and M_B^d.

Combining Eq. (4) with the independent-model bremsstrahlung contributions we obtain, for the bremsstrahlung corrections to the transition probability [6],

$$\Delta P_B=P_0\frac{\alpha}{\pi}\left[2\left[\frac{1+\mu^2}{1-\mu^2}\ln\mu+1\right]\left[\ln\frac{\lambda}{m_\tau}-\ln(1-\mu^2)-\frac{1}{2}\ln\mu+\frac{3}{4}\right]-\frac{\mu^2(10-7\mu^2)}{2(1-\mu^2)^2}\ln\mu\right.$$
$$\left.+\frac{2(1+\mu^2)}{1-\mu^2}L(1-\mu^2)+\frac{15-21\mu^2}{8(1-\mu^2)}-\frac{1}{2f_\pi}\ln\mu\int_0^1 H_2^*(0,x)dx+\cdots\right]\ , \tag{5}$$

where λ is a small photon mass used to handle the infrared divergence.

The contribution from $|M_B^d|^2$ can be similarly computed. Denoting by $\Delta P_B'$ the corresponding transition probability, the result is

$$\Delta P_B'=-P_0\frac{\alpha}{2\pi}\frac{1}{f_\pi^2}\ln\mu\times\int_0^1[|H_2(0,x)|^2+|H_6(0,x)|^2](1-x)dx+\cdots\ . \tag{6}$$

This contribution can be analyzed analogously as in the π_{l2} case. As a first approximation we can neglect the x dependence of the two form factors, $H_i(0,x)\simeq H_i(0,0)$, $i=1,2$. Then, Eq. (6) reduces to

$$\Delta P_B'\simeq-P_0\frac{\alpha}{4\pi}\frac{1}{f_\pi^2}[|H_2(0,0)|^2+|H_6(0,0)|^2]\ln\mu\ . \tag{7}$$

Taking the values $|H_2|=0.011$, $H_6=0.014$, and $f_\pi=131.14$ MeV, from Ref. [7], we estimate that $\Delta P_B'\simeq(4\times10^{-6})P_0$. Comparing this number with the percentage value of the model-independent radiative corrections, when the neutrino mass is zero in Ref. [3], we observe that $\Delta P_B'$ is quite small, and hence can be ignored.

III. VIRTUAL CORRECTIONS

The virtual model-dependent radiative corrections come from the diagram in Fig. 2. It has the amplitude

$$M_V^d=-\frac{\alpha G}{4\sqrt{2}\pi^3 i}\int\frac{d^4k}{k^2}\bar{u}_\nu\gamma_\mu(1-\gamma_5)\frac{2l_\lambda-\not{k}\gamma_\lambda}{k^2-2l\cdot k}u_\tau\times\frac{T_V^{\mu\lambda}(p,k)}{k^2-2p\cdot k}\ , \tag{8}$$

where the tensor $T_V^{\mu\lambda}(p,k)$ contains all the structure dependence (the Born term has been separated out and it was included in the virtual model-independent amplitude). The resulting expression for $T_V^{\mu\lambda}(p,k)$ is exactly the same as in Eq. (2). Since terms k^2 do not contribute to the PMS, we can replace $H_i(k^2,s)$ with $H_i(0,s)$, $s=(p+k)^2$. Unlike the π_{l2} case, we have to consider the contribution of all six form factors. However, we have to reduce $T_V^{\mu\lambda}$ to just two form factors H_2 and H_6 as in Eq. (3). To this end we proceed as follows. The tensor $T_V^{\mu\lambda}(p,k)$ is the Fourier transform

$$T_V^{\mu\lambda}=\int d^4x\langle\pi(p)|T[J_{(\gamma)}^\lambda(x)J_{(w)}^\mu(0)]|0\rangle e^{ik\cdot x}\ , \tag{9}$$

where $J_{(\gamma)}^\lambda$ and $J_{(w)}^\mu$ are the electromagnetic and weak hadronic currents, respectively. By using translation invariance, Eq. (9) can also be written in the form

$$T_V^{\mu\lambda}(p,k)=\int d^4x\langle\pi(p)|T[J_{(\gamma)}^\lambda(0)J_{(w)}^\mu(x)]|0\rangle e^{i(k-p)\cdot x}\ . \tag{10}$$

By contracting with p_μ, we obtain

$$p_\mu T_V^{\mu\lambda}(p,k)=i\int d^4x\langle\pi(p)|\delta(x_0)[J_{(w)}^0(x),J_{(\gamma)}^\lambda(0)]+T[\partial_\mu J_{(w)}^\mu(x)J_{(\gamma)}^\lambda(0)]|0\rangle e^{i(k-p)\cdot x}\ . \tag{11}$$

The current algebra relation

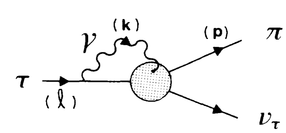

FIG. 2. Diagram involving the strong interaction in the virtual corrections to $\tau\rightarrow\pi\nu_\tau$.

$$[J^0_{(w)}(x), J^\lambda_{(\gamma)}(0)] = J^\lambda_{(w)}(x)\delta^3(\mathbf{x})$$

and the PCAC hypothesis

$$\partial_\mu J^\mu_{(w)}(x) = \partial_\mu A^\mu(x) = f_\pi m_\pi^2 \Phi(x) ,$$

where Φ is the pion field, allow us to write

$$p_\mu T_V^{\mu\lambda}(p,k) = i\langle \pi(p)|J^\lambda_{(w)}(0)|0\rangle + if_\pi m_\pi^2 \int d^4x \langle \pi(p)|T[\Phi(x)J^\lambda_{(\gamma)}(0)]|0\rangle \times e^{i(k-p)\cdot x} . \quad (12)$$

The first term in Eq. (12) is the Born term, already considered in the M_V^i amplitude. Thus, we get, finally

$$p_\mu T_V^{\mu\lambda}(p,k) = if_\pi m_\pi^2 T_V^\lambda(p,k) , \quad (13)$$

with an obvious definition for $T_V^\lambda(p,k)$. As terms proportional to m_π^2 do not give contributions to the PMS, we can safely write

$$p_\mu T_V^{\mu\lambda}(p,k) = O(m_\pi^2) . \quad (14)$$

Using Eqs. (2) and (14), we obtain the relations

$$H_1 + H_2 p\cdot k + H_3 p^2 = 0 \quad (15a)$$

and

$$H_4 p^2 + H_5 p\cdot k = 0 . \quad (15b)$$

We first consider the contribution of H_4 and H_5 in Eq. (8):

$$M^d_{V(H_4,H_5)} = \frac{\alpha G}{4\sqrt{2}\pi^3 i}\bar{u}_\nu \gamma^\mu (1-\gamma_5) u_\tau \times \int \frac{d^4k}{k^2(k^2-2p\cdot k)} \times [p_\mu H_4(0,s) - k_\mu H_5(0,s)] .$$

After Feynman parametrization, and shifting variable $k' = k - px$, we get

$$M^d_{V(H_4,H_5)} = \frac{\alpha G}{4\sqrt{2}\pi^3 i}\bar{u}_\nu \not{p}(1-\gamma_5) u_\tau \times \int_0^1 dx \int \frac{d^4k'}{(k'^2 - m_\pi^2 x^2)^2} \times [H_4(0,s) + xH_5(0,s)] .$$

Using Eq. (15b), it is easy to show that

$$\int \frac{d^4k'}{(k'^2 - m_\pi^2 x^2)^2}[H_4(0,s) + xH_5(0,s)] = 0 ,$$

i.e., the combined contribution of H_4 and H_5 vanishes. Then, for our purpose, we are left with the tensor

$$T_V^{\mu\lambda}(p,k) = H_1 g^{\mu\lambda} + H_2 k^\mu p^\lambda + H_3 p^\mu p^\lambda + iH_6 \varepsilon^{\mu\lambda\alpha\beta} k_\alpha p_\beta , \quad (16)$$

which, after applying gauge invariance, reduces to

$$T_V^{\mu\lambda}(p,k) = H_2(k^\mu p^\lambda - p\cdot k g^{\mu\lambda}) + iH_6 \varepsilon^{\mu\lambda\alpha\beta} k_\alpha p_\beta . \quad (17)$$

Substituting Eq. (17) into Eq. (8), and neglecting some terms proportional to k^2 and $p\cdot k$, we arrive at

$$M_V^d = -\frac{\alpha G}{4\sqrt{2}\pi^3 i}\int \frac{d^4k}{k^2(k^2-2l\cdot k)(k^2-2p\cdot k)} \times \bar{u}_\nu [2l\cdot p(H_2+H_6)\not{k} + (k^2-2l\cdot k)H_6 \not{p}](1-\gamma_5)u_\tau . \quad (18)$$

Proceeding as usual, we can extract the $\ln\mu$ term from Eq. (18). In the road we find that the contribution of H_6 cancels, leaving the result

$$M_V^d = -\frac{\alpha G}{4\sqrt{2}\pi}\bar{u}_\nu \not{p}(1-\gamma_5)u_\tau \ln\mu \int_0^1 H_2(0,x)dx + \cdots . \quad (19)$$

The contribution to the transition probability is given by the interference of M_V^d and the uncorrected decay amplitude. Then, after summing over lepton spins, and performing the integrations over the neutrino momentum and pion momentum, one gets

$$\Delta P_V^d = P_0 \frac{\alpha}{2\pi}\frac{1}{f_\pi}\ln\mu \int_0^1 H_2^*(0,x)dx + \cdots , \quad (20)$$

for the transition probability induced by the structure-dependent form factors. Combining Eq. (20) with the contribution from the model-independent virtual corrections, we obtain, for the virtual corrections to the transition probability [8],

$$\Delta P_V = P_0 \frac{\alpha}{\pi}\left[2\left[\frac{1+\mu^2}{1-\mu^2}\ln\mu + 1\right][\ln(m_\tau/\lambda) + \tfrac{1}{2}\ln\mu - \tfrac{3}{4}] + \frac{\mu^2}{1-\mu^2}\ln\mu + \frac{1}{2} + \frac{1}{2f_\pi}\ln\mu \int_0^1 H_2^*(0,x)dx + \cdots\right] . \quad (21)$$

IV. RESULT AND DISCUSSION

The result is established adding Eqs. (5) and (21): all the structure-dependent PMS's cancel in the total decay probability. The only surviving model-independent PMS's are given by

$$P = P_0\left[1 + \frac{\alpha}{\pi}B(\mu)\right] , \quad (22)$$

where $P_0 = G^2 f_\pi^2 m_\tau^3 (1-\mu^2)^2/16\pi$, and [9]

$$B(\mu) = -2\left[\frac{1+\mu^2}{1-\mu^2}\ln\mu + 1\right]\ln(1-\mu^2) + \frac{1}{2} - \frac{8-5\mu^2}{2(1-\mu^2)^2}\mu^2\ln\mu + \frac{2(1+\mu^2)}{1-\mu^2}L(1-\mu^2) + \frac{15-21\mu^2}{8(1-\mu^2)} . \quad (23)$$

In the limit $m_\pi=0$, Eq. (23) reduces to

$$B(\mu)\simeq P_0\left[1+\frac{\alpha}{\pi}\left[\tfrac{19}{8}+2L(1)\right]\right] \tag{24}$$

and $P_0=G^2f_\pi^2m_\tau^3/16\pi$. Then, the total rate is logarithmically convergent, in agreement with the KLN theorem. We note that the partial decay rate Eq. (15) of Ref. [3], where hard bremsstrahlung has been excluded, still contains PMS's. When all the photons are taken into account, as we have done in Eq. (5) above, the PMS's cancel out exactly.

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[9] Our result for the $B(\mu)$ function differs from the analogous result in π_{l2}. See Eq. (22b) in Ref. [1] above. One can check that, in the limit $\mu\to0$, his $B(\mu)$ still contains a term $\ln\mu$.

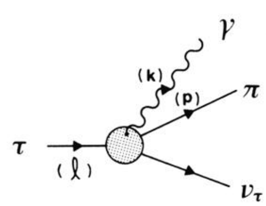

FIG. 1. Diagram involving the strong interaction in the bremsstrahlung correction to $\tau \rightarrow \pi \nu_\tau$. The photon leg is attached to any internal hadronic line.

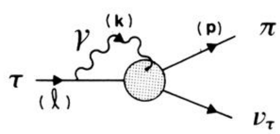

FIG. 2. Diagram involving the strong interaction in the virtual corrections to $\tau \rightarrow \pi \nu_\tau$.