BRIEF REPORTS

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Conditions for the existence of closed timelike curves in 2+1 gravity

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(Received 27 November 1991)

We consider open universe solutions to Einstein gravity in 2+1 dimensions, and conjecture that closed timelike curves do not exist surrounding a system of spinless particles with timelike total momentum. We prove this for a general two-particle system, and provide evidence for the conjecture in the case of N particles.

PACS number(s): 04.20.Jb, 95.30.Sf, 98.80.Cq

I. INTRODUCTION

The dynamics of point particles in 2+1 gravity has long been of interest. It provides a simple laboratory for studying solutions to Einstein's equations, since spacetime is locally flat in the absence of matter [1,2]. It also has physical relevance to the dynamics of cosmic strings in 3+1 dimensions if the strings are taken to be parallel and motion along the string axis is suppressed [3].

Throughout this Brief Report we consider only open universes, where the total mass $M<1/4G$. Then the metric arising from a point source of mass M and intrinsic spin J at rest at the origin is [2]

$$ds^2=(dT+4GJ\,d\Theta)^2-dr^2-(1-4GM)^2r^2d\Theta^2 . \quad (1)$$

Here $-\infty<T<\infty$, $0\leq r<\infty$, and $\Theta=0$ is identified with $\Theta=2\pi$. This clearly possesses closed timelike curves (CTC's) sufficiently close to the source. For example, by holding T and r fixed and letting Θ run from 0 to 2π, one obtains a CTC provided $r<4GJ/(1-4GM)$. Evidently, intrinsic spin gives rise to difficulties with causality. One then wonders whether orbital angular momentum gives rise to the same difficulties. In other words, can one have enough purely orbital angular momentum enclosed in a small enough radius to make a CTC?

This was recently answered affirmatively by Gott [4], who gave an exact solution representing the gravitational scattering of two spinless point particles, which possesses closed timelike curves encircling the two particles. However, Gott's solution has been shown to possess a spacelike total momentum, corresponding to a tachyonic center of mass [5]. Our conjecture is that this connection holds in general: Closed timelike curves exist surrounding a system of spinless particles only if their total momentum is spacelike.

We offer evidence for this conjecture as follows. In Sec. II we establish that the conjecture holds rigorously in the case of a system of two particles. Our proof is similar to a construction due to Waelbroeck [6]. In Sec. III we make an ansatz for a timelike configuration of N particles that seems optimal for constructing a CTC and show that in fact no CTC around the N particles exists. Section IV summarizes our conclusions.

II. TWO-PARTICLE CASE

Exact multiparticle solutions to 2+1 gravity may be constructed by cutting up a Minkowski spacetime and making identifications across the cuts [2]. For a free spinless particle of mass m traveling with a Lorentz boost L from rest, the spacetime may be constructed by identifying

$$x\leftrightarrow L\Omega L^{-1}(x-a)+a . \quad (2)$$

Here x is a spacetime coordinate and a is any point along the particle's world line. Ω is a spatial rotation matrix:

$$\Omega=\begin{bmatrix}1 & 0 & 0\\ 0 & \cos\alpha & -\sin\alpha\\ 0 & \sin\alpha & \cos\alpha\end{bmatrix} , \quad (3)$$

where $\alpha=8\pi Gm$ is the deficit angle of the resulting cone.

A two-particle effective spacetime can be constructed by sequentially carrying out the identification for each particle. If m_1 is located at $\mathbf{a}_1=\mathbf{v}_1t+\mathbf{a}_1^0$ and m_2 is located at $\mathbf{a}_2=\mathbf{v}_2t+\mathbf{a}_2^0$, the net identification is

$$x\leftrightarrow L_1\Omega_1L_1^{-1}L_2\Omega_2L_2^{-1}x+(1-L_1\Omega_1L_1^{-1})a_1 + L_1\Omega_1L_1^{-1}(1-L_2\Omega_2L_2^{-1})a_2 . \quad (4)$$

Here Ω_1 is a rotation through $\alpha_1=8\pi Gm_1$, Ω_2 is a rotation through $\alpha_2=8\pi Gm_2$, and L_1,L_2 are Lorentz boosts through the velocities $\mathbf{v}_1, \mathbf{v}_2$ respectively. Now suppose we impose the condition that we are in the center-of-mass frame. This means that the net identification can be written [2]

$$x \leftrightarrow \Omega(x-b)+b+c \ . \tag{5}$$

Here Ω is a pure (not boosted) rotation through the total deficit angle β, b is a purely spatial vector giving the location of the center of mass, and c is a purely temporal vector specifying a shift in time corresponding to the angular momentum:

$$c^0=-8\pi GJ \ . \tag{6}$$

Requiring the composite system to be in the center-of-mass frame implies three conditions on the masses and velocities:

$$\sin\frac{\alpha_1}{2}\sinh\phi_1=\sin\frac{\alpha_2}{2}\sinh\phi_2 \ ,$$
$$\frac{\mathbf{v}_2}{|\mathbf{v}_2|}=-\Omega_{\beta/2}^{-1}\frac{\mathbf{v}_1}{|\mathbf{v}_1|} \ , \tag{7}$$
$$\beta=2(\gamma_1+\gamma_2)$$

where

$$\tan\gamma_1=\tan\frac{\alpha_1}{2}\cosh\phi_1, \quad \tan\gamma_2=\tan\frac{\alpha_2}{2}\cosh\phi_2 \ ,$$

and ϕ_1,ϕ_2 are the rapidities corresponding to $|\mathbf{v}_1|,|\mathbf{v}_2|$. The first condition states that the two particles have momenta equal in magnitude, the second that the momenta are directed back to back on the cone ($\Omega_{\beta/2}$ is a rotation through half the deficit angle). The third condition fixes the deficit angle. Note that the universe is open, with $\beta<2\pi$, for arbitrary ϕ_1 and ϕ_2, as long as $\alpha_1,\alpha_2<\pi$.

At $t=0$ we take m_1 to be at the origin and traveling in the $+x$ direction, and we take m_2 to be located on the y axis, with $a_2^0=(0,a_{2y}^0)$ (see Fig. 1). Then the location of the center of mass and total angular momentum are fixed, as are the distances of closest approach R_1,R_2 of m_1,m_2 to the center of mass. For these quantities we obtain

$$R_1=\frac{1}{2}\left[2\cos^2\frac{\alpha_1}{2}-\sin\alpha_1\cosh\phi_1\cot\frac{\beta}{2}\right]|a_{2y}^0| \ ,$$
$$R_2=\frac{1}{2}\frac{\sin\alpha_1\cosh\phi_1}{\sin(\beta/2)}|a_{2y}^0| \ , \tag{8}$$
$$8\pi GJ=\sin\alpha_1\sinh\phi_1 a_{2y}^0 \ .$$

Note that

$$|8\pi GJ|=2\sin\frac{\beta}{2}\tanh\phi_1 R_2 \ . \tag{9}$$

We now have enough information to show that there are no CTC's in this spacetime. Any curve encircling m_1 and m_2 must also enclose the center of mass. The length of such a curve is bounded below by the length of the shortest curve encircling just m_2 and the center of mass. This minimal curve has a length of $2R_2$ if $\beta<\pi$ or $2R_2\sin(\beta/2)$ if $\beta>\pi$ (in which case it is shorter to cut across the cone). But in both cases, the length is greater than the time shift $|8\pi GJ|$ resulting from the angular momentum, and there can be no CTC.

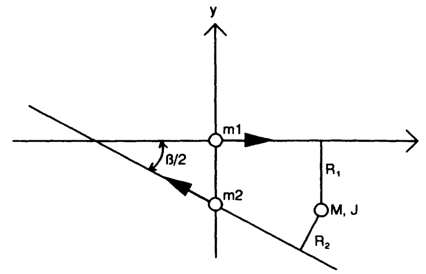

FIG. 1. Two-body spacetime at $t=0$.

Finally, consider the limit $\phi_1\rightarrow\infty$. In this limit, $R_1\approx R_2$ and $\beta\rightarrow 2\pi$. We see that the system possesses a closed lightlike curve just as space become cylindrical, as was noted in [7,8] for the case $m_1=m_2$.

III. N-PARTICLE CASE

For a general system of N particles, the calculations become too cumbersome to permit an exhaustive analysis similar to the one given above for two particles. We propose the following ansatz as an attempt at constructing a timelike system possessing as much angular momentum as possible enclosed in a given space.

We choose to work in the center-of-mass frame. The construction consists of N particles with equal masses $\alpha=8\pi Gm$, and equal velocities $\tanh\phi$, uniformly spaced around a circle of radius R about the center of mass, with their velocities tangent to the circle. In terms of the coordinates on the effective spinning cone with metric

$$ds^2=(dT+4GJ\,d\Theta)^2-dr^2-(1-4GM)^2r^2d\Theta^2 \ , \tag{10}$$

the particles are assigned the initial coordinates

$$T_k=0 \ ,$$
$$\Theta_k=\frac{2\pi}{N}(k-1) \ , \tag{11}$$
$$r_k=R \ ,$$

for $k=1,2,\ldots,N$. In terms of the flat coordinates $t\equiv T+4GJ\Theta$, $\theta\equiv(1-4GM)\Theta$, with metric $ds^2=dt^2-dr^2-r^2d\theta^2$, the equivalent coordinate assignments are

$$t_k=8\pi GJ(k-1)/N \ ,$$
$$\theta_k=2\pi(1-4GM)(k-1)/N \ , \tag{12}$$
$$r_k=R$$

(see Fig. 2). The effective net identification is then

$$x\leftrightarrow L_1\Omega_1L_1^{-1}\cdots L_N\Omega_NL_N^{-1}x+(1-L_1\Omega_1L_1^{-1})a_1+L_1\Omega_1L_1^{-1}(1-L_2\Omega_2L_2^{-1})a_2$$
$$+\cdots+L_1\Omega_1L_1^{-1}\cdots L_{N-1}\Omega_{N-1}L_{N-1}^{-1}(1-L_N\Omega_NL_N^{-1})a_N \ . \tag{13}$$

Explicitly evaluating this shows that the center of mass is indeed at rest at the origin, with a total deficit angle β, where

$$\beta=2\pi-2N(\gamma_1-\gamma_2)\ ,$$
$$\tan\gamma_1=\tan\frac{\pi}{N}\cosh\psi\ ,$$
$$\tan\gamma_2=\tan\frac{\alpha}{2}\cosh\phi\ , \tag{14}$$
$$\sin\frac{\alpha}{2}\sinh\phi=\sin\frac{\pi}{N}\sinh\psi\ .$$

The total angular momentum is calculated to be

$$8\pi GJ=2RN\sin\frac{2\pi-\beta}{2N}\tanh\psi\ . \tag{15}$$

However, the shortest path enclosing all N particles is easily seen to be a polygon of total length $2RN\sin(2\pi-\beta)/2N$. Since this is always less than $8\pi GJ$, no CTC exists surrounding the N particles.

In the limit $\psi\to\infty$, this configuration possesses a closed lightlike curve. But as in the two-body case, this is also the limit in which space becomes cylindrical, with $\beta\to 2\pi$.

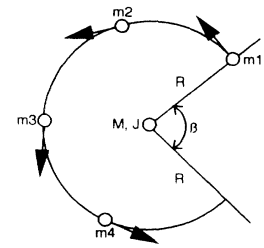

FIG. 2. $N=4$ body spacetime in flat coordinates at $t_k=(8\pi GJ/N)(k-1)$.

IV. CONCLUSIONS

To summarize, we conjecture that closed timelike curves do not exist surrounding a system of spinless particles with timelike total momentum. If the conjecture is true, the only possible occurrence of a CTC in an open timelike universe of spinless particles is around some subsystem having a spacelike momentum. This possibility has been ruled out by the work of Carroll, Farhi, and Guth [8,9], who have shown that such subsystems do not exist—a spacelike subsystem (constructed out of timelike spinless particles) in conjunction with other timelike spinless sources cannot form an open timelike universe. Hence the conjecture implies that CTC's do not exist anywhere in an open timelike universe.

ACKNOWLEDGMENTS

The author wishes to thank Roman Jackiw for advice during this project, Miguel Ortiz for helpful discussions, and Eddie Farhi and Alan Guth for discussing their work on the CTC problem. This material is based on work supported under the National Science Foundation. This work was supported in part by funds provided by the U.S. Department of Energy (D.O.E.) under Contract No. DE-AC02-76ER03069.

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