Adiabatic regularization of the quantum stress-energy tensor in curved spacetimes: Stochastic quantization method

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It is found that the adiabatic regularized quantum stress-energy tensor for matter in any background spacetime can be obtained in the framework of stochastic quantization. As an illustration, we investigate the modes of a massive scalar field with arbitrary curvature coupling to the inhomogeneous conformally flat spacetime. The difficulty of the mode-mixing behavior arising from the spacetime inhomogeneity is overcome and a simple algorithm is presented to evaluate the adiabatic regularized quantum stress-energy tensor. The expressions so obtained are useful in formulating the semiclassical theory of gravity and in the numerical study of the back-reaction effects of quantized fields in inhomogeneous spacetimes, such as the problems of homogenization in the early universe and the evaporation of a quantum black hole. It can be seen that our prescriptions may be extended to investigate more realistic models with fermions and gauge fields. This will be studied in future papers.

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I. INTRODUCTION

In the theory of quantum fields in curved spacetime [1] one treats the gravitational field as a classical background field, and the expectation values of some matter stress tensors are regarded as the sources of the generalized Einstein equation. The back-reaction problem is then described by the equation

$$G_{\mu\nu}+\Lambda g_{\mu\nu}+\lambda_1\,{}^{(1)}H_{\mu\nu}+\lambda_2\,{}^{(2)}H_{\mu\nu}=-8\pi G\langle T_{\mu\nu}\rangle\ , \tag{1.1}$$

where Λ is the cosmological constant, G is Newton's constant, and $G_{\mu\nu}$ the Einstein tensor. Two conserved tensors ${}^{(1)}H_{\mu\nu}$ and ${}^{(2)}H_{\mu\nu}$ are defined by

$${}^{(1)}H_{\mu\nu}=\frac{1}{\sqrt{-g}}\frac{\delta}{\delta g_{\mu\nu}}\int\sqrt{-g}\,R^2d^4x\ , \tag{1.2}$$

$${}^{(2)}H_{\mu\nu}=\frac{1}{\sqrt{-g}}\frac{\delta}{\delta g_{\mu\nu}}\int\sqrt{-g}\,R_{\mu\nu}R^{\mu\nu}d^4x\ . \tag{1.3}$$

Although $\langle T_{\mu\nu}\rangle$ is formally divergent, it can be made finite by the renormalization procedure, as had been proved for an arbitrary background metric by using the DeWitt–Schwinger formalism [2] and the point-splitting method [3]. The renormalizable requirement forces one to introduce the tensors $g_{\mu\nu}$, ${}^{(1)}H_{\mu\nu}$, and ${}^{(2)}H_{\mu\nu}$ in Eq. (1.1).

However, in this approach the finite part of the stress tensor is rather difficult to evaluate. On the other hand, the adiabatic regularization method [4,5], which can be used to find the finite parts of the quantum stress tensor, is a particularly efficient method in the numerical study of the dynamics of quantum fields in curved spacetime. The adiabatic regularization method has been developed for the cases of spatially flat Robertson-Walker, Bianchi type-I, Gowdy T^3, and closed Robertson-Walker universes [4–11].

As noted by Parker and Fulling in their original paper [4], the adiabatic regularization method is applicable to any spacetime which has a metric of sufficient symmetry to allow a decomposition of the quantized fields into modes. Therefore, when one wants to extend the method of adiabatic regularization to a more general spacetime with an inhomogeneity, because the mode solution of the curved-space field equation (Klein-Gordon equation for scalar fields) could not be separated when the inhomogeneity is introduced, one will immediately encounter the difficulty of mode-mixing behavior and thus the adiabatic approximation of mode functions cannot be straightforwardly obtained.

In a previous paper [12] we have shown a possible way to overcome the difficulty of mode-mixing behavior arising from the inhomogeneity of spacetime and gave a possible way to find an adiabatic approximation to the WKB solution for each mode function. However, our previous method can only apply to quantum fields in a spacetime with a small inhomogeneity which is required to have spatial reflection symmetry. In that paper we also have adopted the early-time approximation. Therefore the application of our previous results is rather restricted. For example, the problem of how a *large* spacetime inhomogeneity, if it exists in the early universe, will be damped in the early epoch to a small value which may form the seeds of the galaxies [13,14] in the present time, cannot be investigated. Also, the effect of back reaction in black-hole evaporation [1,15–18], known to have great significance in the final stage of a quantum black hole, can be numerically studied from Eq. (1.1) only if the quantum stress tensor in a generally inhomogeneous spacetime has been obtained. Therefore it appears worthwhile to find a method which enables one to evaluate the adiabatic regularized quantum stress tensor for matter in a general curved spacetime.

In this paper we will show that the adiabatic regularized quantum stress-energy tensor for matter in *any* background spacetime can be obtained with the help of the stochastic quantization method [19–25]. Although the prescription presented in this paper could be applied to a more general metric, for simplicity, we will in this paper first treat the modes of a massive scalar field with arbitrary curvature coupling to a spacetime which is conformally flat. In particular, we will present a simple scheme and use it to evaluate the adiabatic regularized quantum stress-energy tensor. Our results are useful in formulating the semiclassical theory of gravity and in the numerical study of the back-reaction effects of quantized fields in inhomogeneous spacetimes, such as the problems of inhomogeneity damping through quantum effects in the early universe and black-hole evaporation. Our prescription can be extended to investigate more realistic models with fermions and gauge fields. This will be studied in future work.

Our present work is one of a series of investigations about quantum fields in inhomogeneous spacetimes. In addition to the paper of Ref. [12], we have also studied particle creation and the Coleman-Weinberg mechanism in inhomogeneous spacetimes in recent papers [26,27].

This paper is organized as follows. In Sec. II we define the modes and calculate the associated classical stress-energy tensor. In Sec. III we first give a brief overview of the Parisi-Wu stochastic quantization method and then apply it to our model. It is found that the mode functions will be solutions of a mode-mixed differential-integral equation, which seems impossible to solve at first sight, just like using the canonical quantization method [12]. We thus give a physical argument of how to overcome this difficulty. Then, with this guide, we can, in Sec. IV, after a lengthy evaluation, obtain the equilibrium value (and thus the quantum expectation value) of bilinear product functions. Collecting these results, the adiabatic regularized quantum stress-energy tensor can be obtained. In particular, we have in this section presented a simple scheme which will enable one to change the tremendous work involved in the multi-integrations of stochastic time variables to just a few simple algebra calculations. We will detail this algorithm with an example. Section V presents a discussion of the results of this paper. We also mention some future interesting work.

Throughout this paper we use the metric signature $(+,-,-,-)$ and conventions $R^{\rho}_{\mu\nu\lambda}=+\Gamma^{\rho}_{\mu\nu,\lambda}-\cdots$ and $R_{\mu\lambda}=R^{\nu}_{\mu\nu\lambda}$. The units are such that $\hbar=c=1$.

II. MODEL AND STRESS TENSOR

We consider the action describing a massive scalar field (ϕ) coupled arbitrarily (ξ) to the gravitational background

$$S=\int \mathcal{L} d^4x = \int \sqrt{-g}\left[\frac{1}{2}g^{\mu\nu}\phi_{,\mu}\phi_{,\nu}-\frac{1}{2}m^2\phi^2-\frac{\xi}{2}R\phi^2\right]d^4x \; , \tag{2.1}$$

where R is the Ricci scalar and $_{,\mu}$ denotes a derivative with respect to x^{μ}. The scalar wave equation, obtained by varying S with respect to ϕ, is

$$\Box\phi+m^2\phi+\xi R\phi=0 \; , \tag{2.2}$$

where $\Box\phi=g^{\mu\nu}\nabla_{\mu}\nabla_{\nu}\phi=g^{\mu\nu}\phi_{;\mu\nu}$. The stress tensor is defined by

$$\begin{aligned} T_{\mu\nu}&=\frac{2}{\sqrt{-g}}\frac{\delta S}{\delta g^{\mu\nu}} \\ &=(2\xi-1)\phi_{,\mu}\phi_{,\nu}+(\tfrac{1}{2}-2\xi)g_{\mu\nu}g^{ab}\phi_{,a}\phi_{,b} \\ &\quad -2\xi g_{\mu\nu}\phi\Box\phi+2\xi\phi\phi_{;\mu\nu}-\tfrac{1}{2}g_{\mu\nu}m^2\phi^2+\xi G_{\mu\nu}\phi^2 \; . \end{aligned} \tag{2.3}$$

Using the wave equation (2.2) the stress tensor can be written as

$$\begin{aligned} T_{\mu\nu}&=(2\xi-1)\phi_{,\mu}\phi_{,\nu}+(\tfrac{1}{2}-2\xi)g_{\mu\nu}g^{ab}\phi_{,a}\phi_{,b} \\ &\quad +2\xi\phi\phi_{;\mu\nu}-(\tfrac{1}{2}-2\xi)g_{\mu\nu}m^2\phi^2+\xi G_{\mu\nu}\phi^2 \\ &\quad +2\xi^2 g_{\mu\nu}R\phi^2 \; . \end{aligned} \tag{2.4}$$

We consider a model spacetime with an inhomogeneous conformally flat metric of the form

$$ds^2=C^2(x_0,\mathbf{x})(dx_0^2-d\mathbf{x}^2) \; . \tag{2.5}$$

After introducing a modified wave function

$$\phi=C^{-1}\chi \; , \tag{2.6}$$

the wave equation of Eq. (2.2) becomes

$$\ddot{\chi}-\nabla^2\chi+C^2[(\xi-\tfrac{1}{6})R+m^2]\chi=0 \; , \tag{2.7}$$

where the Riemann-Christoffel curvature is

$$R=6C^{-3}\left[C_{,00}-\sum_i C_{,ii}\right] . \tag{2.8}$$

Note that Eq. (2.7) cannot be solved by splitting it into separated ordinary differential equations, as the spacetime is inhomogeneous.

The stress tensors in this metric are given by

$$\begin{aligned} T_{00}&=[(6\xi-\tfrac{1}{2})C^{-4}(C_{,0})^2+(4\xi-\tfrac{1}{2})C^{-4}\sum_j (C_{,j})^2-(\tfrac{1}{2}-2\xi)m^2+\xi C^{-2}G_{00}+2\xi^2 R]\chi^2 \\ &\quad +(1-6\xi)C^{-3}C_{,0}\chi\chi_{,0}+(1-6\xi)C^{-3}\sum_j C_{,j}\chi\chi_{,j}-\tfrac{1}{2}C^{-2}(\chi_{,0})^2+2\xi C^{-2}\chi\chi_{,00}-(\tfrac{1}{2}-2\xi)C^{-2}\sum_j(\chi_{,j})^2 \; , \end{aligned} \tag{2.9}$$

$$\begin{aligned} T_{ii}&=[(4\xi-\tfrac{1}{2})C^{-4}(C_{,0})^2+(10\xi-1)C^{-4}(C_{,i})^2+(\tfrac{1}{2}-4\xi)C^{-4}\sum_j (C_{,j})^2+(\tfrac{1}{2}-2\xi)m^2+\xi C^{-2}G_{ii}-2\xi^2 R]\chi^2 \\ &\quad +(2\xi-\tfrac{1}{2})C^{-2}(\chi_{,0})^2+[(2-8\xi)C^{-3}C_{,i}+4\xi C^{-4}(C_{,i})^2]\chi\chi_{,i}-(6\xi-1)C^{-3}C_{,0}\chi\chi_{,0} \\ &\quad +(6\xi-1)C^{-3}\sum_j C_{,j}\chi\chi_{,j}+2\xi C^{-2}\chi\chi_{,ii}+(2\xi-1)C^{-2}(\chi_{,i})^2+(\tfrac{1}{2}-2\xi)C^{-2}\sum_j(\chi_{,j})^2 \; , \end{aligned} \tag{2.10}$$

where the Einstein tensor $G_{\mu\nu} \equiv R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R$ is

$$G_{00} = -2C^{-1}(C^{-1})_{,00} + \frac{C^{-2}}{2}[(C^2)_{,00} - \sum_j (C^2)_{,jj}] - 3C^{-1}[C_{,00} - \sum_j C_{,jj}] , \quad (2.11)$$

$$G_{ii} = -2C^{-1}(C^{-1})_{,ii} - \frac{C^{-2}}{2}[(C^2)_{,00} - \sum_j (C^2)_{,jj}] + 3C^{-1}[C_{,00} - \sum_j C_{,jj}] . \quad (2.12)$$

In the quantum theory $T_{\mu\nu}$ is divergent and needs to be renormalized.

III. STOCHASTIC QUANTIZATION

A. Overview

To evaluate the renormalized stress tensor we will adopt the stochastic quantization method. The Parisi-Wu stochastic quantization [19] for a field $\chi(x_\mu)$ with the Euclidean action $S[\chi]$ is formulated in terms of the Langevin equation

$$\frac{\partial}{\partial t}\chi(x_\mu, t) = -\frac{\delta S[\chi]}{\delta \chi(x_\mu, t)} + \eta(x_\mu, t) , \quad (3.1)$$

in which an extra variable, t, is a *fictitious* time variable (the Langevin time) and η is a random variable with a Gaussian distribution [19], thus

$$\langle \eta(x_\mu, t)\eta(y_\mu, \tilde{t})\rangle = 2\delta(t-\tilde{t})\delta^4(x_\mu - y_\mu) . \quad (3.2)$$

The central assertion of stochastic quantization is that in the limit $t \to \infty$ equilibrium is reached, and that the equal-time (t) correlation functions of χ become identical to the corresponding quantum Green functions [19–21], i.e.,

$$\lim_{t\to\infty} \langle \chi(x_\mu^{(1)}, t) \cdots \chi(x_\mu^{(n)}, t)\rangle = \langle \chi(x_\mu^{(1)}) \cdots \chi(x_\mu^{(n)})\rangle . \quad (3.3)$$

Note that the left-hand side contains a stochastic η average, whereas the right-hand side represents the standard field-theoretic vacuum expectation value. The stochastic quantization was originally invented to quantize a gauge theory without gauge fixing [19,21]. It has also been used to quantize a fermion system [22] and gravity theory [23]. Stochastic quantization has also been discussed for Minkowski spacetime [24]. More references can be found in the review article of Ref. [25].

B. Application: Difficulty and solution

Due to such successes it is natural to apply the stochastic quantization method to study a quantum field in curved spacetime. The work described below is just a preliminary result that can be obtained in this direction (see the discussions in Sec. V).

We now begin our investigation. The Langevin equation associated with Eq. (2.7) is

$$\frac{\partial}{\partial t}\chi = \chi_{,00} + \sum_j \chi_{,jj} - A\chi + \eta , \quad (3.4)$$

where

$$A \equiv C^2[(\xi - \tfrac{1}{6})R_E + m^2] . \quad (3.5)$$

The value of R_E is defined in Eq. (2.8) except that the $_{,0}$ here [and also in Eq. (3.4) and hereafter] is now a derivative with respect to Euclidean time. To solve the Langevin equation of Eqs. (3.4) we expand the fields χ and η and the function A by

$$\chi(x_\mu, t) = \int d^4k\, \Psi_k(t) e^{ikx} , \quad (3.6)$$

$$\eta(x_\mu, t) = \int d^4k\, \eta_k(t) e^{ikx} , \quad (3.7)$$

$$A(x_\mu) = \int d^4k\, A_k e^{ikx} . \quad (3.8)$$

Then the mode function Ψ_k will be a solution of the mode-mixed differential-integral equation

$$\frac{\partial}{\partial t}\Psi_k = -k^2\Psi_k + \eta_k - (2\pi)^{-3/2}\int d^4l\, A_l \Psi_{k-l} , \quad k^2 \equiv \sum_\mu k_\mu^2 . \quad (3.9)$$

The above equation exhibits the mixing between the different modes as the space-time inhomogeneity is introduced, and solving it is a horrible task. We will try to solve it in some approximations, which will, nevertheless, enable us to evaluate the *exact* form of adiabatic regularized quantum stress-energy tensor.

To get an idea to solve Eq. (3.9) one may consult the following argument. It is known that the infinity is coming from the large values of k. Therefore, to obtain the divergences in $\langle T_{\mu\nu}\rangle$ it is sufficient to calculate the bilinear product functions χ^2, $(\chi_{,0})^2$, $\chi\chi_{,ii}$, and $(\chi_{,i})^2$ [see Eqs. (2.9) and (2.10)] to some order in k_μ. We also know that large k corresponds to the short-wavelength modes, and short-wavelength modes only probe the local behavior of the background spacetime. Therefore it is natural to guess that the ultraviolet divergence may be made softer by taking into account the terms with higher-order spacetime derivatives. With this guide we can in the next section find the regularized quantum stress tensor.

We note that the above idea has been used by Bunch and Parker [28] to study the renormalization of interacting fields in curved spacetime.

IV. STOCHASTIC METHOD IN EVALUATING THE ADIABATIC REGULARIZED STRESS TENSOR

A. Method

To proceed, let us write Eq. (3.4) as

$$\frac{\partial}{\partial t}\chi = \chi_{,00} + \sum_j \chi_{,jj} - M^2\chi - \sum_{a_0,\ldots,a_3} \frac{x_0^{a_0} \cdots x_3^{a_3}}{a_0! \cdots a_3!} \tilde{A}_{,a_0a_1a_2a_3}\chi + \eta , \quad (4.1)$$

where the Taylor expansion of A about the origin is defined by

$$\widetilde{A}_{,a_0a_1a_2a_3} \equiv \left. \frac{\partial^{(a_0+a_1+a_2+a_3)}A}{\partial x_0^{a_0}\partial x_1^{a_1}\partial x_2^{a_2}\partial x_3^{a_3}} \right|_{x_0=x_1=x_2=x_3=0} , \quad (4.2)$$

and

$$M^2 \equiv \widetilde{A}_{,0000} . \quad (4.3)$$

The summations in Eq. (4.1) are such that $a_0+a_1+a_2+a_3>0$ with a_i a nonnegative integer. Not that in general relativity we do not have a fixed frame, and any position may be regarded as the original point $x_\mu=(0,0,0,0)$. Thus the results obtained below can be applied to all positions in a spacetime, and not just a special original point.

In terms of the mode functions defined in Eqs. (3.6) and (3.7), Eq. (4.1) becomes

$$\frac{\partial}{\partial t}\Psi_k = -(k^2+M^2)\Psi_k - \sum_{a_0,\ldots,a_3} \frac{(-i)^{a_0+a_1+a_2+a_3}}{a_0! \cdots a_3!} \widetilde{A}_{,a_0a_1a_2a_3} \times \frac{\partial^{(a_0+a_1+a_2+a_3)}\Psi_k}{\partial k_0^{a_0}\partial k_1^{a_1}\partial k_2^{a_2}\partial k_3^{a_3}} + \eta_k . \quad (4.4)$$

The above equation can be written in a differential-integral form

$$\Psi_k(t) = \int_0^t d\tau\, e^{(\tau-t)(k^2+M^2)} \left[\eta_k - \sum_{a_0,\ldots,a_3} \frac{(-i)^{a_0+a_1+a_2+a_3}}{a_0! \cdots a_3!} \widetilde{A}_{,a_0a_1a_2a_3} \frac{\partial^{(a_0+a_1+a_2+a_3)}\Psi_k}{\partial k_0^{a_0}\partial k_1^{a_1}\partial k_2^{a_2}\partial k_3^{a_3}} \right] , \quad (4.5)$$

which may be solved by iteration. From such an *approximate* solution we can obtain the equilibrium values of any bilinear field products with the help of Eq. (3.2), and the *exact* form of the adiabatic regularized quantum stress tensors are found according to Eq. (3.3).

As the solution will involve multi-integration of stochastic time, for convenience, we will, from now on, adopt the symbolical notations defined below:

$$GG \cdots GF(t) \equiv \int_0^t d\tau_1 e^{(\tau_1-t)(k^2+M^2)} \int_0^{\tau_1} d\tau_2 e^{(\tau_2-\tau_1)(k^2+M^2)} \cdots \int_0^{\tau_{n-1}} d\tau_n e^{(\tau_n-\tau_{n-1})(k^2+M^2)} F(\tau_n) , \quad (4.6)$$

$$\partial_\mu\partial_\nu \cdots F \equiv \frac{\partial\partial \cdots}{\partial k_\mu \partial k_\nu \cdots} F , \quad (4.7)$$

$$A_{\mu\nu\cdots} \equiv \left. \frac{\partial\partial \cdots}{\partial x_\mu \partial x_\nu \cdots} A \right|_{x_0=x_1=x_2=x_3=0} , \quad (4.8)$$

where F is an arbitrary function of stochastic time and k. Notice that $\widetilde{A}_{,a_0a_1a_2a_3}$ defined in Eq. (4.2) is different from $A_{\mu\nu}\cdots$ defined in Eq. (4.8). ($A_{\mu\nu}\cdots$ is not a function of stochastic time t or momentum k.) In the following, as we have used the shorthand notation, the variable k is occasionally neglected. Thus, without confusion, η usually stands for $\eta_k(t)$ defined in Eq. (3.7). A summation shall always be taken on the same index in an equation.

Denoting by $\overline{\Psi}^{(n)}$ a solution of Eq. (4.5) which contains spacetime derivatives of order not larger than n, we can, after a little analysis, obtain the following results:

$$\overline{\Psi}^{(n)} = \Psi^{(0)} + \Psi^{(1)} + \cdots + \Psi^{(n)} , \quad (4.9)$$

with

$$\Psi^{(0)} \equiv G\eta , \quad (4.10)$$

$$\Psi^{(1)} \equiv iA_\mu G\partial_\mu(G\eta) , \quad (4.11)$$

$$\Psi^{(2)} \equiv -A_\mu G\partial_\mu[A_\nu G\partial_\nu(G\eta)] + \tfrac{1}{2}A_{\mu\nu}G\partial_\mu\partial_\nu(G\eta) , \quad (4.12)$$

$$\Psi^{(3)} \equiv -iA_\mu G\partial_\mu\{A_\nu G\partial_\nu[A_\lambda G\partial_\lambda(G\eta)]\} + \frac{i}{2}A_\mu G\partial_\mu[A_{\nu\lambda}G\partial_\nu\partial_\lambda(G\eta)] + \frac{i}{2}A_{\mu\nu}G\partial_\mu\partial_\nu[A_\lambda G\partial_\lambda(G\eta)] - \frac{i}{6}A_{\mu\nu\lambda}\partial_\mu\partial_\nu\partial_\lambda(G\eta) , \quad (4.13)$$

$$\begin{aligned}\Psi^{(4)} \equiv{} & A_\mu G\partial_\mu[A_\nu G\partial_\nu\{A_\lambda G\partial_\lambda[A_\delta G\partial_\delta(G\eta)]\}] - \tfrac{1}{2}A_\mu G\partial_\mu\{A_\nu G\partial_\nu[A_{\lambda\delta}G\partial_\lambda\partial_\delta(G\eta)]\} \\ & - \tfrac{1}{2}A_\mu G\partial_\mu\{A_{\nu\lambda}G\partial_\nu\partial_\lambda[A_\delta G\partial_\delta(G\eta)]\} - \tfrac{1}{2}A_{\mu\nu}G\partial_\mu\partial_\nu\{A_\lambda G\partial_\lambda[A_\delta G\partial_\delta(G\eta)]\} \\ & + \tfrac{1}{4}A_{\mu\nu}G\partial_\mu\partial_\nu[A_{\lambda\delta}G\partial_\lambda\partial_\delta(G\eta)] + \tfrac{1}{6}A_\mu G\partial_\mu[A_{\nu\lambda\delta}G\partial_\nu\partial_\lambda\partial_\delta(G\eta)] \\ & + \tfrac{1}{6}A_{\mu\nu\lambda}G\partial_\mu\partial_\nu\partial_\lambda[A_\delta G\partial_\delta(G\eta)] - \tfrac{1}{24}A_{\mu\nu\nu\lambda}G\partial_\mu\partial_\nu\partial_\lambda\partial_\delta(G\eta) .\end{aligned} \quad (4.14)$$

We will separately calculate each term in the following expansion:

$$\langle \Psi_k \Psi_{\tilde{k}} \rangle = \langle \Psi^{(0)}\tilde{\Psi}^{(0)} \rangle + \langle \Psi^{(1)}\tilde{\Psi}^{(1)} \rangle + 2\langle \Psi^{(2)}\tilde{\Psi}^{(0)} \rangle + \langle \Psi^{(2)}\tilde{\Psi}^{(2)} \rangle + 2\langle \Psi^{(3)}\tilde{\Psi}^{(1)} \rangle + 2\langle \Psi^{(4)}\tilde{\Psi}^{(0)} \rangle + \text{terms with higher-order spacetime derivatives} , \quad (4.15)$$

where

$$\tilde{\Psi}^{(n)} \equiv \Psi^{(n)}_{\tilde{k}} .$$

In Eq. (4.15) we have neglected the terms containing an odd number of derivatives, as they will become zero after performing the k_μ and $\tilde{k}_\mu$ integrations (see the discussion of step 3 in Sec. IV B). Now, because each term in the above equation contains multi-integration of the stochastic time and several derivatives with respect to k_μ, a tremendous amount of work seems to be needed to work them out. Thus we will present a simple algorithm to calculate them.

B. Simple algorithm

Step 1. From the following relations:

$$\langle \chi\chi \rangle = \int d^4k\, d^4\tilde{k} \langle \Psi_k(t)\Psi_{\tilde{k}}(t) \rangle e^{i(k+\tilde{k})x} , \quad (4.16)$$

$$\langle \chi\chi_{,\mu} \rangle = \tfrac{1}{2}\langle \chi^2 \rangle_{,\mu}, \quad \mu=0,1,2,3 , \quad (4.17)$$

$$\langle \chi\chi_{,\mu\mu} \rangle = \int d^4k\, d^4\tilde{k}\tilde{k}_\mu \tilde{k}_\mu \langle \Psi_k(t)\Psi_{\tilde{k}}(t) \rangle e^{i(k+\tilde{k})x} , \quad (4.18)$$

$$\langle \chi_{,\mu}\chi_{,\mu} \rangle = \int d^4k\, d^4\tilde{k}\, k_\mu \tilde{k}_\mu \langle \Psi_k(t)\Psi_{\tilde{k}}(t) \rangle e^{i(k+\tilde{k})x} , \quad (4.19)$$

we see that, despite the several types of bilinear field products which appear in the stress tensor [see Eqs. (2.9) and (2.10)], it is, in fact, only necessary to evaluate $\langle \Psi_k \Psi_{\tilde{k}} \rangle$.

Step 2. From Eq. (4.16) we see that, because of the k and $\tilde{k}$ integrations we can use integration by parts for each derivative ∂_μ in Eqs. (4.11)–(4.14) when performing the stochastic η and $\tilde{\eta}$ averages. In addition, the surface terms will become zero in the limit $t\to\infty$.

Step 3. Using step 2 and defining $T_i \equiv \tau_i - \tau_{i-1}$ it is seen that the functions to be evaluated have a general form

$$\begin{aligned}
&\langle T_1^{a_1} G T_2^{a_2} G \cdots T_n^{a_n} G \eta \tilde{T}_1^{\tilde{a}_1} \tilde{G} \tilde{T}_2^{\tilde{a}_2} \tilde{G} \cdots \tilde{T}_m^{\tilde{a}_m} \tilde{G} \tilde{\eta} \rangle \\
&\equiv \Big\langle \int_0^t d\tau_1 \int_0^{\tau_1} d\tau_2 \cdots \int_0^{\tau_{n-1}} d\tau_n (\tau_1 - t)^{a_1} (\tau_2-\tau_1)^{a_2} \cdots (\tau_n - \tau_{n-1})^{a_n} e^{(\tau_n - t)(k^2+M^2)} \eta_k(\tau_n) \\
&\quad \times \int_0^t d\tilde{\tau}_1 \int_0^{\tilde{\tau}_1} d\tilde{\tau}_2 \cdots \int_0^{\tilde{\tau}_{m-1}} d\tilde{\tau}_m (\tilde{\tau}_1 - t)^{\tilde{a}_1} (\tilde{\tau}_2-\tilde{\tau}_1)^{\tilde{a}_2} \cdots (\tilde{\tau}_m - \tilde{\tau}_{m-1})^{\tilde{a}_m} \\
&\quad \times e^{(\tilde{\tau}_m - t)(\tilde{k}^2+M^2)} \eta_{\tilde{k}}(\tilde{\tau}_m) \Big\rangle \\
&= \Big\langle \int_0^t d\tau_n \cdots \int_{\tau_3}^t d\tau_2 \int_{\tau_2}^t d\tau_1 (\tau_1 - t)^{a_1} (\tau_2-\tau_1)^{a_2} \cdots (\tau_n - \tau_{n-1})^{a_n} e^{(\tau_n - t)(k^2+M^2)} \eta_k(\tau_n) \\
&\quad \times \int_0^t d\tilde{\tau}_m \cdots \int_{\tilde{\tau}_3}^t d\tilde{\tau}_2 \int_{\tilde{\tau}_2}^t d\tilde{\tau}_1 (\tilde{\tau}_1 - t)^{\tilde{a}_1} (\tilde{\tau}_2-\tilde{\tau}_1)^{\tilde{a}_2} \cdots \\
&\quad \times (\tilde{\tau}_m - \tilde{\tau}_{m-1})^{\tilde{a}_m} e^{(\tilde{\tau}_m - t)(\tilde{k}^2+M^2)} \eta_{\tilde{k}}(\tilde{\tau}_m) \Big\rangle \\
&= \int_0^t d\tau_n \cdots \int_{\tau_3}^t d\tau_2 \int_{\tau_3}^t d\tau_2 \int_{\tau_2}^t d\tau_1 (\tau_1 - t)^{a_1} (\tau_2-\tau_1)^{a_2} \cdots (\tau_n - \tau_{n-1})^{a_n} \\
&\quad \times \int_{\tau_n}^t d\tilde{\tau}_{m-1} \cdots \int_{\tilde{\tau}_3}^t d\tilde{\tau}_2 \int_{\tilde{\tau}_2}^t d\tilde{\tau}_1 (\tilde{\tau}_1 - t)^{\tilde{a}_1} (\tilde{\tau}_2-\tilde{\tau}_1)^{\tilde{a}_2} \cdots (\tilde{\tau}_m - \tilde{\tau}_{m-1})^{\tilde{a}_m} \\
&\quad \times e^{2(\tau_n - t)(k^2+M^2)} 32\pi^4 \delta^4(k_\mu + \tilde{k}_\mu) . \qquad (4.20)
\end{aligned}$$

To obtain this equation a relation

$$\langle \eta_k(t)\eta_{\tilde{k}}(\tau) \rangle = 32\pi^4 \delta^4(k+\tilde{k})\delta(t-\tau) , \quad (4.21)$$

which is a consequence of Eqs. (3.2) and (3.7), has been used. Thus Eq. (4.16) becomes

$$\langle \chi^2 \rangle = 32\pi^4 \int d^4k \langle \Psi_k(t)\Psi_k(t) \rangle . \quad (4.22)$$

This equation tells us that the terms in $\langle \Psi_k(t)\tilde{\Psi}_k(t) \rangle$ with an odd number of k_μ will not contribute to $\langle \chi^2 \rangle$.

Step 4. Despite the several useful properties that have been obtained it is still very complicated to perform the calculation directly from Eq. (4.20). Thus let us find a useful form and its properties. We define $D_i \equiv \tau_i - t$ and find that

$$\langle D_1^{a_1} G D_2^{a_2} G \cdots D_n^{a_n} G \eta \tilde{D}_1^{\tilde{a}_1} \tilde{G} \tilde{D}_2^{\tilde{a}_2} \tilde{G} \cdots \tilde{D}_m^{\tilde{a}_m} \tilde{G} \tilde{\eta} \rangle$$

$$\equiv \Big\langle \int_0^t d\tau_1 \int_0^{\tau_1} d\tau_2 \cdots \int_0^{\tau_{n-1}} d\tau_n (\tau_1 - t)^{a_1} (\tau_2 - t)^{a_2} \cdots (\tau_n - t)^{a_n} e^{(\tau_n - t)(k^2+M^2)} \eta_k(\tau_n)$$

$$\times \int_0^t d\tilde{\tau}_1 \int_0^{\tilde{\tau}_1} d\tilde{\tau}_2 \cdots \int_0^{\tilde{\tau}_{m-1}} d\tilde{\tau}_m (\tilde{\tau}_1 - t)^{\tilde{a}_1} (\tilde{\tau}_2 - t)^{\tilde{a}_2} \cdots (\tilde{\tau}_m - t)^{\tilde{a}_m}$$

$$\times e^{(\tilde{\tau}_m - t)(\tilde{k}^2+M^2)} \eta_{\tilde{k}}(\tilde{\tau}_m) \Big\rangle$$

$$= \Big\langle \int_0^t d\tau_n \cdots \int_{\tau_3}^t d\tau_2 \int_{\tau_2}^t d\tau_1 (\tau_1 - t)^{a_1} (\tau_2 - t)^{a_2} \cdots (\tau_n - t)^{a_n} e^{(\tau_n - t)(k^2+M^2)} \eta_k(\tau_n)$$

$$\times \int_0^t d\tilde{\tau}_m \cdots \int_{\tilde{\tau}_3}^t d\tilde{\tau}_2 \int_{\tilde{\tau}_2}^t d\tilde{\tau}_1 (\tilde{\tau}_1 - t)^{\tilde{a}_1} (\tilde{\tau}_2 - t)^{\tilde{a}_2} \cdots (\tilde{\tau}_m - t)^{\tilde{a}_m} e^{(\tilde{\tau}_m - t)(\tilde{k}^2+M^2)} \eta_{\tilde{k}}(\tilde{\tau}_m) \Big\rangle$$

$$= \int_0^t d\tau_n \cdots \int_{\tau_3}^t d\tau_2 \int_{\tau_2}^t d\tau_1 (\tau_1 - t)^{a_1} (\tau_2 - t)^{a_2} \cdots (\tau_n - t)^{a_n}$$

$$\times \int_{\tau_n}^t d\tilde{\tau}_{m-1} \cdots \int_{\tilde{\tau}_3}^t d\tilde{\tau}_2 \int_{\tilde{\tau}_2}^t d\tilde{\tau}_1 (\tilde{\tau}_1 - t)^{\tilde{a}_1} (\tilde{\tau}_2 - t)^{\tilde{a}_2} \cdots (\tilde{\tau}_m - t)^{\tilde{a}_m}$$

$$\times e^{2(\tau_n - t)(k^2+M^2)} 32\pi^4 \delta^4(k_\mu + \tilde{k}_\mu) \ . \tag{4.23}$$

From this relation one can easily derive a *reduction formula* and an *integration formula*

$$\langle D_1^{a_1} G D_2^{a_2} G \cdots D_n^{a_n} G \eta \tilde{D}_1^{\tilde{a}_1} \tilde{G} \tilde{D}_2^{\tilde{a}_2} \tilde{G} \cdots \tilde{D}_m^{\tilde{a}_m} \tilde{G} \tilde{\eta} \rangle = \frac{-1}{a_1+1} \langle D_2^{a_1+1+a_2} G \cdots D_n^{a_n} G \eta \tilde{D}_1^{\tilde{a}_1} \tilde{G} \tilde{D}_2^{\tilde{a}_2} \tilde{G} \cdots \tilde{D}_m^{\tilde{a}_m} \tilde{G} \tilde{\eta} \rangle$$

$$= [(a_1+1)(\tilde{a}_1+1)]^{-1}$$

$$\times \langle D_2^{(a_1+1+a_2} G \cdots D_n^{a_n} G \eta \tilde{D}_2^{\tilde{a}_1+1+\tilde{a}_2} \tilde{G} \cdots \tilde{D}_m^{\tilde{a}_m} \tilde{G} \tilde{\eta} \rangle \ , \tag{4.24}$$

$$\langle D^a G \eta \tilde{D}^{\,\tilde{a}} \tilde{G} \tilde{\eta} \rangle \underset{t\to\infty}{\longrightarrow} (-1)(-2) \cdots [-(a+\tilde{a})][2(k^2+M^2)]^{-(a+\tilde{a}+1)} 32\pi^4 \delta^4(k_\mu + \tilde{k}_\mu) \ . \tag{4.25}$$

Note that the functions of T_i and D_i used, respectively, in Eqs. (4.20) and (4.23) have a relation

$$\sum_{i=1}^{j} T_i = D_j \ . \tag{4.26}$$

This relation implies the following *derivative formula:*

$$(D_1^{a_1} G D_2^{a_2} G \cdots D_n^{a_n} G)_{,\mu} = 2k_\mu (T_1 D_1^{a_1} G D_2^{a_2} G \cdots D_n^{a_n} G + D_1^{a_1} G T_2 D_2^{a_2} G \cdots D_n^{a_n} G + \cdots$$

$$+ D_1^{a_1} G D_2^{a_2} G \cdots T_n D_n^{a_n} G)$$

$$= 2k_\mu D_1^{a_1} G D_2^{a_2} G \cdots D_n^{(a_n+1)} G \ . \tag{4.27}$$

C. Example

Using the reduction formula, integration formula, and derivative formula we could simplify our calculation. Let us see this in the following example in which the calculations are performed step by step:

$$\langle G\partial_\mu [G \partial_\nu \partial_\lambda (G\eta)] \tilde{G} \partial_\delta (\tilde{G}\tilde{\eta}) \rangle = \langle (G_{,\mu} G)_{,\nu,\lambda} G \eta (\tilde{G})_{,\delta} \tilde{G} \tilde{\eta} \rangle$$

$$= \langle (2k_\mu DGG)_{,\nu,\lambda} G \eta 2\tilde{k}_\delta \tilde{D} \tilde{G} \tilde{G} \tilde{\eta} \rangle$$

$$= \langle (2\delta_{\mu\nu} DGG + 4k_\mu k_\nu DGDG)_{,\lambda} G \eta 2\tilde{k}_\delta \tilde{D} \tilde{G} \tilde{G} \tilde{\eta} \rangle$$

$$= \langle (4\delta_{\mu\nu} k_\lambda DGDG + 4\delta_{\mu\lambda} k_\nu DGDG + 4\delta_{\nu\lambda} k_\mu DGDG + 8k_\mu k_\nu k_\lambda DGD^2G) G \eta 2\tilde{k}_\delta \tilde{D} \tilde{G} \tilde{G} \tilde{\eta} \rangle$$

$$= (8\delta_{\mu\nu} k_\lambda k_\delta + 8\delta_{\mu\lambda} k_\nu k_\delta + 8\delta_{\nu\lambda} k_\mu k_\delta) \langle DGDGG \eta \tilde{D} \tilde{G} \tilde{G} \tilde{\eta} \rangle$$

$$+ 16 k_\mu k_\nu k_\lambda k_\delta \langle DGD^2 GG \eta \tilde{D} \tilde{G} \tilde{G} \tilde{\eta} \rangle$$

$$= -\tfrac{1}{2}(\delta_{\mu\nu} k_\lambda k_\delta + \delta_{\mu\lambda} k_\nu k_\delta + \delta_{\nu\lambda} k_\mu k_\delta) \langle D^4 G \eta \tilde{D}^{\,2} \tilde{G} \tilde{\eta} \rangle - \tfrac{4}{5} k_\mu k_\nu k_\lambda k_\delta \langle D^5 G \eta \tilde{D}^{\,2} \tilde{G} \tilde{\eta} \rangle$$

$$= 32\pi^4 \delta^4(k_\mu + \tilde{k}_\mu) \{ -360(\delta_{\mu\nu} k_\lambda k_\delta + \delta_{\mu\lambda} k_\nu k_\delta + \delta_{\nu\lambda} k_\mu k_\delta) [2(k^2+M^2)]^{-7}$$

$$+ 4032 k_\mu k_\nu k_\lambda k_\delta [2(k^2+M^2)]^{-8} \} \ . \tag{4.28}$$

D. Results

We can now evaluate each term in Eq. (4.15) by just some simple algebra. The results are as follows.

$$\langle \Psi^{(0)}\tilde{\Psi}^{(0)}\rangle = 32\pi^4\delta^4(k_\mu+\tilde{k}_\mu)[2(k^2+M^2)]^{-1} , \tag{4.29}$$

$$\langle \Psi^{(1)}\tilde{\Psi}^{(1)}\rangle = 32\pi^4\delta^4(k_\mu+\tilde{k}_\mu)\{-6A_\mu A_\nu[2(k^2+M^2)]^{-5}(4k_\mu k_\nu)\} , \tag{4.30}$$

$$\begin{aligned}\langle \Psi^{(2)}\tilde{\Psi}^{(0)}\rangle = 32\pi^4\delta^4(k_\mu+\tilde{k}_\mu)(&A_\mu A_\nu\{[2(k^2+M^2)]^{-4}(2\delta_{\mu\nu})-3[2(k^2+M^2)]^{-5}(4k_\mu k_\nu)\}\\ &-\tfrac{1}{2}A_{\mu\nu}\{[2(k^2+M^2)]^{-3}(2\delta_{\mu\nu})-2[2(k^2+M^2)]^{-4}(4k_\mu k_\nu)\}) ,\end{aligned} \tag{4.31}$$

$$\begin{aligned}\langle \Psi^{(2)}\tilde{\Psi}^{(2)}\rangle = 32\pi^4\delta^4(k_\mu+\tilde{k}_\mu)(&A_\mu A_\nu A_\lambda A_\delta\{20[2(k^2+M^2)]^{-7}(4\delta_{\mu\nu}\delta_{\lambda\delta})-105[2(k^2+M^2)]^{-8}(8\delta_{\mu\nu}k_\lambda k_\delta+8\delta_{\lambda\delta}k_\mu k_\nu)\\ &\quad+630[2(k^2+M^2)]^{-9}(16k_\mu k_\nu k_\lambda k_\delta)\}\\ &-\tfrac{1}{2}(A_\mu A_\nu A_{\lambda\delta}+A_{\mu\nu}A_\lambda A_\delta)\{10[2(k^2+M^2)]^{-6}(4\delta_{\mu\nu}\delta_{\lambda\delta})-40[2(k^2+M^2)]^{-7}(8\delta_{\mu\nu}k_\lambda k_\delta)\\ &\quad-45[2(k^2+M^2)]^{-7}(8\delta_{\lambda\delta}k_\mu k_\nu)\\ &\quad+210[2(k^2+M^2)]^{-8}(16k_\mu k_\nu k_\lambda k_\delta)\}\\ &+\tfrac{1}{4}(A_{\mu\nu}A_{\lambda\delta})\{6[2(k^2+M^2)]^{-5}(4\delta_{\mu\nu}\delta_{\lambda\delta})-20[2(k^2+M^2)]^{-6}(8\delta_{\mu\nu}k_\lambda k_\delta+8\delta_{\lambda\delta}k_\mu k_\nu)\\ &\quad+80[2(k^2+M^2)]^{-7}(16k_\mu k_\nu k_\lambda k_\delta)\}) ,\end{aligned} \tag{4.32}$$

$$\begin{aligned}\langle \Psi^{(3)}\tilde{\Psi}^{(1)}\rangle = 32\pi^4\delta^4(k_\mu+\tilde{k}_\mu)(&A_\mu A_\nu A_\lambda A_\delta\{-84[2(k^2+M^2)]^{-8}(8\delta_{\mu\nu}k_\lambda k_\delta)-63[2(k^2+M^2)]^{-8}(8\delta_{\mu\lambda}k_\nu k_\delta+8\delta_{\lambda\nu}k_\mu k_\delta)\\ &\quad+420[2(k^2+M^2)]^{-9}(16k_\mu k_\nu k_\lambda k_\delta)\}\\ &-\tfrac{1}{2}(A_\mu A_{\nu\lambda}A_\delta)\{-45[2(k^2+M^2)]^{-7}(8\delta_{\mu\nu}k_\lambda k_\delta+8\delta_{\mu\lambda}k_\nu k_\delta+8\delta_{\lambda\delta}k_\mu k_\nu)\\ &\quad+252[2(k^2+M^2)]^{-8}(16k_\mu k_\nu k_\lambda k_\delta)\}\\ &-\tfrac{1}{2}(A_{\mu\nu}A_\lambda A_\delta)\{-45[2(k^2+M^2)]^{-7}(8\delta_{\mu\nu}k_\lambda k_\delta)\\ &\quad-30[2(k^2+M^2)]^{-7}(8\delta_{\mu\lambda}k_\nu k_\delta+8\delta_{\nu\lambda}k_\mu k_\delta)\\ &\quad+168[2(k^2+M^2)]^{-8}(16k_\mu k_\nu k_\lambda k_\delta)\}\\ &+\tfrac{1}{6}(A_{\mu\nu\lambda}A_\delta)\{-20[2(k^2+M^2)]^{-6}(8\delta_{\mu\nu}k_\lambda k_\delta+8\delta_{\mu\lambda}k_\nu k_\delta+8\delta_{\nu\lambda}k_\mu k_\delta)\\ &\quad+90[2(k^2+M^2)]^{-7}(16k_\mu k_\nu k_\lambda k_\delta)\}) ,\end{aligned} \tag{4.33}$$

$$\begin{aligned}\langle \Psi^{(4)}\tilde{\Psi}^{(0)}\rangle = 32\pi^4\delta^4(k_\mu+\tilde{k}_\mu)(&A_\mu A_\nu A_\lambda A_\delta\{4[2(k^2+M^2)]^{-7}(4\delta_{\mu\nu}\delta_{\lambda\delta})+3[2(k^2+M^2)]^{-7}(4\delta_{\mu\lambda}\delta_{\nu\delta}+8\delta_{\mu\delta}\delta_{\nu\lambda})\\ &\quad-24[2(k^2+M^2)]^{-8}(8\delta_{\mu\nu}k_\lambda k_\delta)\\ &\quad-18[2(k^2+M^2)]^{-8}(8\delta_{\mu\lambda}k_\nu k_\delta+8\delta_{\nu\lambda}k_\mu k_\delta)\\ &\quad-15[2(k^2+M^2)]^{-8}(8\delta_{\mu\delta}k_\nu k_\lambda+8\delta_{\nu\delta}k_\mu k_\lambda+8\delta_{\lambda\delta}k_\mu k_\nu)\\ &\quad+105[2(k^2+M^2)]^{-9}(16k_\mu k_\nu k_\lambda k_\delta)\}\\ &-\tfrac{1}{2}(A_\mu A_\nu A_{\lambda\delta})\{4[2(k^2+M^2)]^{-6}(4\delta_{\mu\nu}\delta_{\lambda\delta})\\ &\quad+3[2(k^2+M^2)]^{-6}(4\delta_{\mu\lambda}\delta_{\nu\delta}+8\delta_{\mu\delta}\delta_{\mu\lambda})-20[2(k^2+M^2)]^{-7}(8\delta_{\mu\nu}k_\lambda k_\delta)\\ &\quad-15[2(k^2+M^2)]^{-7}(8\delta_{\mu\lambda}k_\nu k_\delta+8\delta_{\nu\lambda}k_\mu k_\delta+8\delta_{\mu\delta}k_\nu k_\lambda\\ &\qquad+8\delta_{\nu\delta}k_\mu k_\lambda+8\delta_{\lambda\delta}k_\mu k_\nu)\\ &\quad+90[2(k^2+M^2)]^{-8}(16k_\mu k_\nu k_\lambda k_\delta)\}\\ &-\tfrac{1}{2}(A_\mu A_{\nu\lambda}A_\delta)\{3[2(k^2+M^2)]^{-6}(4\delta_{\mu\nu}\delta_{\lambda\delta}+4\delta_{\mu\lambda}\delta_{\nu\delta}+8\delta_{\mu\delta}\delta_{\nu\lambda})\\ &\quad-15[2(k^2+M^2)]^{-7}(8\delta_{\mu\nu}k_\lambda k_\delta+8\delta_{\mu\lambda}k_\nu k_\delta+8\delta_{\nu\lambda}k_\mu k_\delta)\\ &\quad-12[2(k^2+M^2)]^{-7}(8\delta_{\mu\delta}k_\nu k_\lambda+8\delta_{\nu\delta}k_\mu k_\lambda+8\delta_{\lambda\delta}k_\mu k_\nu)\\ &\quad+72[2(k^2+M^2)]^{-8}(16k_\mu k_\nu k_\lambda k_\delta)\}\end{aligned}$$

$$-\tfrac{1}{2}(A_{\mu\nu}A_\lambda A_\delta)\{3[2(k^2+M^2)]^{-6}(4\delta_{\mu\nu}\delta_{\lambda\delta})+2[2(k^2+M^2)]^{-6}(4\delta_{\mu\lambda}\delta_{\nu\delta}+8\delta_{\mu\delta}\delta_{\nu\lambda})$$
$$-15[2(k^2+M^2)]^{-7}(8\delta_{\mu\nu}k_\lambda k_\delta)$$
$$-10[2(k^2+M^2)]^{-7}(8\delta_{\mu\lambda}k_\nu k_\delta+8\delta_{\nu\lambda}k_\mu k_\delta)$$
$$-8[2(k^2+M^2)]^{-7}(8\delta_{\mu\delta}k_\nu k_\lambda+8\delta_{\nu\delta}k_\mu k_\lambda+8\delta_{\lambda\delta}k_\mu k_\nu)$$
$$+48[2(k^2+M^2)]^{-8}(16k_\mu k_\nu k_\lambda k_\delta)\}$$
$$+\tfrac{1}{4}(A_{\mu\nu}A_{\lambda\delta})\{3[2(k^2+M^2)]^{-5}(4\delta_{\mu\nu}\delta_{\lambda\delta})+2[2(k^2+M^2)]^{-5}(4\delta_{\mu\lambda}\delta_{\nu\delta}+8\delta_{\mu\delta}\delta_{\nu\lambda})$$
$$-12[2(k^2+M^2)]^{-6}(8\delta_{\mu\nu}k_\lambda k_\delta)$$
$$-8[2(k^2+M^2)]^{-6}(8\delta_{\mu\lambda}k_\nu k_\delta+8\delta_{\nu\lambda}k_\mu k_\delta$$
$$+8\delta_{\mu\delta}k_\nu k_\lambda+8\delta_{\nu\delta}k_\mu k_\lambda+8\delta_{\lambda\delta}k_\mu k_\nu)$$
$$+40[2(k^2+M^2)]^{-7}(16k_\mu k_\nu k_\lambda k_\delta)\}$$
$$+\tfrac{1}{6}(A_\mu A_{\nu\lambda\delta})\{3[2(k^2+M^2)]^{-5}(4\delta_{\mu\nu}\delta_{\lambda\delta}+4\delta_{\mu\lambda}\delta_{\nu\delta}+8\delta_{\mu\delta}\delta_{\mu\lambda})$$
$$-12[2(k^2+M^2)]^{-6}(8\delta_{\mu\nu}k_\lambda k_\delta+8\delta_{\mu\lambda}k_\nu k_\delta+8\delta_{\nu\lambda}k_\mu k_\delta$$
$$+8\delta_{\mu\delta}k_\nu k_\lambda+8\delta_{\nu\delta}k_\mu k_\lambda+8\delta_{\lambda\delta}k_\mu k_\nu)$$
$$+60[2(k^2+M^2)]^{-7}(16k_\mu k_\nu k_\lambda k_\delta)\}$$
$$+\tfrac{1}{6}(A_{\mu\nu\lambda}A_\delta)\{2[2(k^2+M^2)]^{-5}(4\delta_{\mu\nu}\delta_{\lambda\delta}+4\delta_{\mu\lambda}\delta_{\nu\delta}+8\delta_{\mu\delta}\delta_{\nu\lambda})$$
$$-8[2(k^2+M^2)]^{-6}(8\delta_{\mu\nu}k_\lambda k_\delta+8\delta_{\mu\lambda}k_\nu k_\delta+8\delta_{\nu\lambda}k_\mu k_\delta)$$
$$-6[2(k^2+M^2)]^{-6}(8\delta_{\mu\delta}k_\nu k_\lambda+8\delta_{\nu\delta}k_\mu k_\lambda+8\delta_{\lambda\delta}k_\mu k_\nu)$$
$$+30[2(k^2+M^2)]^{-7}(16k_\mu k_\nu k_\lambda k_\delta)\}$$
$$-\tfrac{1}{24}A_{\mu\nu\lambda\delta}\{2[2(k^2+M^2)]^{-4}(4\delta_{\mu\nu}\delta_{\lambda\delta}+4\delta_{\mu\lambda}\delta_{\nu\delta}+8\delta_{\mu\delta}\delta_{\nu\lambda})$$
$$-6[2(k^2+M^2)]^{-5}(8\delta_{\mu\nu}k_\lambda k_\delta+8\delta_{\mu\lambda}k_\nu k_\delta+8\delta_{\nu\lambda}k_\mu k_\delta$$
$$+8\delta_{\mu\delta}k_\nu k_\lambda+8\delta_{\nu\delta}k_\mu k_\lambda+8\delta_{\lambda\delta}k_\mu k_\nu)$$
$$+24[2(k^2+M^2)]^{-6}(16k_\mu k_\nu k_\lambda k_\delta)\}) \ . \quad (4.34)$$

Substituting these results into Eq. (4.15) and using the relations Eqs. (4.16)–(4.19) we can, from Eqs. (2.9) and (2.10), evaluate the adiabatic regularized quantum stress-energy tensors.

V. REMARKS, DISCUSSIONS, AND FUTURE RESEARCH

In this paper we have used the Parisi-Wu stochastic method[19] to quantize a scalar field in an inhomogeneous spacetime. Our purpose was not just to present an alternative method to find some results which have been obtained in other approaches. In fact, we showed that the stochastic quantization method may be used to find the regularized stress-energy tensor in an inhomogeneous spacetime, which is unable to be gotten by any other quantization method (at least in the present time).

Our prescription and results need further remarks and discussion as stated in the following.

(1) From the above results we see that the ultraviolet divergence is softer in the terms containing more spacetime derivatives. Thus to regularize the divergence appearing in the quantum stress tensor we only need to subtract the terms with small powers of spacetime derivatives, i.e., we only need to find the approximate solution of Eq. (4.5) and this will be sufficient to find the exact form of the adiabatic regularized stress-energy tensor.

(2) Note that $\langle \Psi^{(0)}\tilde{\Psi}^{(0)}\rangle$ contains terms of order k^{-2}, $\langle \Psi^{(1)}\tilde{\Psi}^{(1)}\rangle$ contains terms of order k^{-8}, $\langle \Psi^{(2)}\tilde{\Psi}^{(0)}\rangle$ contains terms of order k^{-6} and k^{-8}, $\langle \Psi^{(2)}\tilde{\Psi}^{(2)}\rangle$ contains terms of order k^{-10}, k^{-12}, and k^{-14}, $\langle \Psi^{(3)}\tilde{\Psi}^{(1)}\rangle$ contains terms of order k^{-10}, k^{-12}, and k^{-14} while $\langle \Psi^{(4)}\tilde{\Psi}^{(0)}\rangle$ contains terms of order k^{-8}, k^{-10}, k^{-12}, and k^{-14}. Therefore in the evaluation of the regularized stress tensor [see Eqs. (4.16)–(4.19)] not all terms will appear ultraviolet divergent. However, our calculations have contained all terms up to fourth spacetime derivatives which may contribute a finite value. The reason is similar to that discussed in the original paper of Parker, Fulling, and Hu [4,5], in which it has been shown that the

original mode sum can be renormalized by subtracting from it the quantity calculated to fourth adiabatic order which contains a finite term. In other words, the explicitly finite terms must also be subtracted from the quantum stress tensor in the renormalization process. We leave for the future a more detailed discussion and an explicit proof. In fact, one can check that our result will produce that evaluated by Bunch [7] who explicitly presents some adiabatic regularized bilinear field products in a Robertson-Walker universe. (Notice that to compare our result with Ref. [7] we must integrate the variable k_0.)

Note that, in our method, as the adiabatic regularized stress tensor involves not only a derivative with respect to time but also to space, we may call this method the "*generalized adiabatic regularization method,*" and thus the subtracted terms involve all that to the fourth spacetime derivative.

(3) Our results can also be applied to n-dimensional theory. Simply replace $32\pi^4\delta^4(k+\tilde{k})$ in the above results by $2(2\pi)^n\delta^n(k+\tilde{k})$. Then, a check of our results in two dimensions (cf. Ref. [7]) is very simple.

(4) Our final results can be applied to Minkowski space by simply adding a negative sign to any term carrying a double derivative with respect to the Euclidean time. (There are no terms with odd derivatives with respect to the Euclidean time according to the discussion in Sec. IV.) It is worthwhile to investigate the stochastic quantization for a field theory in curved Minkowski spacetime [24].

(5) In the stochastic quantization method one can obtain adiabatic regularized bilinear field products in any curved spacetime, as we have found, but it is unable to find the WKB solution of mode functions, even in a homogeneous spacetime. On the other hand, in the canonical quantization method one can obtain the WKB solution of mode equations in a homogeneous spacetime but cannot get that in a generally inhomogeneous spacetime. We *conjecture* that there may exist a method of directly obtaining the adiabatic regularized bilinear field products without knowing the WKB solution of mode function in any curved spacetime, within the framework of canonical quantization method or path integral quantization. It would be interesting to find such a method.

(6) A remarkable result of the stochastic quantization method is that the gauge-invariant observables can be computed without fixing the gauge [19,21]. Such an advantage is expected to survive for a gauge field theory in curved spacetime. This is the next area we plan to study. We also plan to investigate a theory containing a fermion field [22].

(7) It is well known that quantum fields have a profound influence on the dynamical behavior of the early universe (e.g., the avoidance of the initial cosmological singularity [10,29–31], the isotropization of an anisotropic universe [6,32,33], and the inflationary universe scenario [14,34]. As we have presented a method to find the adiabatic regularized stress-energy tensor, the above studies may be extended to an inhomogeneous spacetime. This is a more realistic cosmological problem, because not only anisotropy but also inhomogeneity will exist in the early universe [13,14].

Finally, to study the back-reaction problem it is necessary to numerically solve the Einstein equation Eq. (1.1) and mode equation (3.9) [or Eq. (4.4)] with $\partial/\partial t\Psi=0$ and $\langle\eta\rangle=0$. However, if the generalized adiabatic regularization method proposed here is to be useful in the study of the semiclassical theory of gravity, there must be a clear algorithm [9] to get the finite term by subtracting certain terms of the solution obtained by the numerically calculated exact mode function. This is an important field of investigation.

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