Linear-order stability of the gravitational plane-wave Cauchy horizons

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It is proven that the plane-wave spacetimes cannot focus so strongly that linearized gravitational waves of compact spatial support become singular on the Cauchy horizons associated with these spacetimes. Implications of this result for the collision of gravitational waves are discussed.

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I. INTRODUCTION

An important problem in general relativity is the task of understanding what happens when gravitational waves collide. Ideally, one would like to be able to answer such questions as the following: When gravitational waves collide, what is the outcome of their interaction? When do the waves simply scatter one another and when do the waves interact with one another resulting in something interesting such as a black hole? Unfortunately, solving this problem in all generality appears exceedingly difficult.

Over the past twenty years, insight into this problem has been gained through the study of the colliding plane-wave spacetimes [1]. These spacetimes represent two transversely infinite plane gravitational waves that collide and interact. The picture that emerges is one where each wave focuses the other leading (generically) to the formation of a future singularity [2]. However, the fact that the waves are exactly planar and transversely infinite leaves this work open to the criticism that the waves studied are "unrealistic." Does the behavior these spacetimes exhibit offer any indication of the behavior that one can expect from the collision of more "realistic" gravitational waves as would arise from astrophysical situations?

Yurtsever has shown that the singular behavior the colliding plane-wave spacetimes (with parallel linear polarizations) exhibit is not an artifact of the exact planarity and infinite transverse extent of the waves [3]. Yurtsever's argument is simple and worth summarizing here. Fix a globally hyperbolic linearly polarized colliding plane-wave spacetime (M,g_{ab}) (that is singular to the future.) Fix a Cauchy surface Σ therein and a future-directed timelike curve γ that "runs into" the future singularity (i.e., it enters the interaction region and is future incomplete without having a future end point.) Then, as Yurtsever has demonstrated, the set $C=\overline{I^{-}[\gamma]}\cap\Sigma$ is compact. Since any change of the initial data that the spacetime induces on $\Sigma-C$ does not affect the evolution on $D^{+}(C)$, we see that the singularity encountered by γ cannot be due to the infinite transverse size of the wave. In other words, two colliding gravitational waves that are planar over a sufficiently large region (almost-plane waves [4]) will develop the singularities seen in the colliding plane-wave spacetimes.

As a first step in a larger program, we attempt to gain further insight into the collision of gravitational waves through a linear analysis of the problem. Of course, if both waves are linearized there is no interaction between the waves. So, instead we treat one of the waves as "weak" and the other as "strong." The "weak" wave is modeled by a linearized gravitational wave evolving on a plane-wave spacetime which represents the "strong" wave. The interaction between the waves is simply the effect the plane-wave background has on the perturbation as it evolves: e.g., the perturbation can be focused by the plane wave. Further, we shall use only the linear approximation to describe the perturbation. Any higher-order perturbations this linear perturbation may generate shall be ignored. Thus, any behavior that occurs in the full theory that is not seen in this linear analysis is an indication that the behavior is due to the nonlinearities in the full theory. In particular, we compare the singular behavior such linearized waves have on the Cauchy horizons of the plane-wave spacetimes to the singular behavior found in the study of the colliding plane-wave and almost-plane-wave spacetimes.

For definiteness, fix a past-complete, globally hyperbolic, plane-wave spacetime (M,g_{ab}) that is a sandwich-wave spacetime (e.g., all nonzero curvature is confined between two null planes.) This spacetime is future extendible (as a plane-wave spacetime) with Cauchy horizon $\mathcal{H}$.

To begin, consider the case that is the analogue of the colliding plane-wave spacetimes. Here the linearized wave is plane symmetric (and is not traveling parallel to the wave of the background spacetime). This case has already been analyzed [5] and, as one might expect, the linearized waves are singular on $\mathcal{H}$. So, in this case, at linear order we see an indication for the development of the singularities that are present in the full theory.

Yet, are these planar waves singular on the Cauchy horizon simply because they are transversely infinite? To answer this we take the extreme case where the linearized wave is spatially limited (and hence is transversely finite.) Is such a wave focused so strongly by the plane wave that it becomes singular on the Cauchy horizon? As a first step towards answering this question, one might attempt to use a form of Yurtsever's argument to show that if the perturbation were exactly planar over a sufficiently large region then it must be singular on $\mathcal{H}$. However, as $\overline{I^{-}[p]\cap\Sigma}$[20] is noncompact for all $p\in\mathcal{H}$ and all Cauchy

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surfaces Σ, Yurtsever's argument cannot be applied here. In fact, as we shall prove, all spatially limited perturbations are well behaved on $\mathcal{H}$.

Theorem 1. Fix a globally hyperbolic plane-wave spacetime (M, g_{ab}) that admits a plane-wave extension. Fix on M any smooth solution γ_{ab} of the linearized vacuum Einstein equation (linearized about g_{ab}) such that there exists some Cauchy surface Σ on which the pair $(\gamma_{ab}, \nabla_m \gamma_{ab})$ is zero outside some compact set C in Σ. Then there exists a perturbation $\tilde{\gamma}_{ab}$, related by gauge to γ_{ab}, that is smooth on all of M and is smoothly extendible to $\mathcal{H}$, the Cauchy horizon of Σ resulting form a plane-wave extension of (M, g_{ab}).

The condition that $(\gamma_{ab}, \nabla_m \gamma_{ab})$ is zero outside some compact set C of a Cauchy surface Σ captures the idea that γ_{ab} is "spatially limited." This interpretation is greatly reinforced by the fact that $\tilde{\gamma}_{ab}$ meets all Cauchy surfaces compactly. [In fact $\tilde{\gamma}_{ab}$ vanishes on $D(\Sigma - C)$.]

Theorem 1 suggests, though does not prove, that when a sufficiently weak gravitational wave collides with a (nearly) plane gravitational wave no singularities will develop. A result showing that the linear perturbation is globally bounded (in an appropriate sense) would greatly strengthen this argument as it would indicate that the perturbation does not become arbitrarily strong and hence for sufficiently weak waves it may be justified to ignore the nonlinearities of the full theory.

Our linear analysis of the collision of gravitational waves *seemingly* misses the most interesting features of gravitational wave collisions seen in the full theory of general relativity. Consider an almost-plane-wave spacetime that is planar over a sufficiently large region. As Yurtsever has shown, the full theory of general relativity predicts (generically) the formation of a curvature singularity. Yet, our linear-order analysis of such waves gives no indication for the development of any singularities: The perturbation is everywhere finite, even on the Cauchy horizon.

Whether the linear-order analysis really misses the singular behavior seen in the full theory of general relativity depends on one's approach to the linear-order analysis. First, if one views the perturbation γ_{ab} as the first derivative of a one-parameter family of metrics, then the singular behavior is not seen since this procedure involves taking the limit in which the waves become infinitesimally weak. For increasingly weaker waves, the transverse size of the region with exact planarity must become infinite in order for the development of a singularity to be guaranteed by the Yurtsever argument [3]. Second, if one views the perturbation γ_{ab} as a small but nonzero difference between the true metric and some background metric (and uses the linearized Einstein equation for its evolution) then the condition that γ_{ab} be finite is insufficient to indicate that no singularities will be present in the full theory. After all, if the perturbation becomes too large, then this approximation breaks down. Clearly, some criterion (stronger than merely being finite) for determining when this linear-order analysis gives answers that are "close" to those of full general relativity is necessary for a successful implementation of this interpretation.

In Sec. II, the plane-wave spacetimes are briefly reviewed and two technical propositions needed for the proof of Theorem 1 are established. In Sec. III, Theorem 1 is proved.

Our notation and conventions are those of Ref. [6]. In particular, all metrics are such that timelike vectors have negative norm. Further, all fields will be taken to be smooth, i.e., C^∞. (This condition can be relaxed, though we do not do so here.)

II. THE PLANE-WAVE SPACETIMES

The plane-wave spacetimes arise as a special case of the PP-wave spacetimes for which we now establish a few elementary results (PP is short for plane fronted with parallel rays.)

Definition 1. A PP-wave spacetime (M, g_{ab}) is a vacuum spacetime for which there exists a nonzero null vector field k^a such that $\nabla_a k_b = 0$, $M \approx \mathbf{R}^4$ and the wave surfaces (the integral surfaces of k_a) are geodesically complete.

An immediate consequence of this definition is that $C_{abcd} k^d = 0$, i.e., the Weyl tensor is type N and k^a is the principal null direction. Thus, if the spacetime is not everywhere flat (i.e., not Minkowski spacetime) k^a is unique up to a multiplicative constant. Since $dk = 0$ and M is simply connected, the wave surfaces are conveniently parametrized by U where

$$k_a = (dU)_a \ . \tag{2.1}$$

Two important properties of these surfaces are that they are geodetic and that for a geodesic that is not in a wave surface U is a good affine parameter. Both statements are easily proved as follows: Fix any geodesic with affine parameter λ and tangent vector t^a and consider

$$\frac{d^2 U}{d\lambda^2} = t^a \nabla_a (t^b \nabla_b U) = t^a t^b \nabla_a k_b = 0 \ , \tag{2.2}$$

where the second equality follows from t^a being geodetic and the definition of U. Thus, if the geodesic is initially in a wave surface ($dU/d\lambda = 0$ at p) then it remains in that surface [$U(\lambda) = U(p)$ for all λ]. If the geodesic is not in a wave surface ($dU/d\lambda \neq 0$ at p) then $dU/d\lambda$ is nonzero and constant along the geodesic. Furthermore, the wave surfaces are flat in the sense that parallel propagation of any vector (in or out of the surface) around a closed path in a wave surface returns the vector to its initial state. This follows from the fact that $C_{abc}{}^d x^a y^b = 0$ for any two vectors x^a and y^b in the wave surface. This property makes the following definition meaningful.

Definition 2. A plane-wave spacetime is a PP-wave spacetime such that $C_{abc}{}^d$ is constant over each wave surface.

The plane-wave spacetimes admit scalar fields (coordinates) (U, V, X^α) so that

$$g_{ab} = -(dU)_{(a}(dV)_{b)} + q_{\alpha\beta}(dX^\alpha)_a (dX^\beta)_b + h_{\alpha\beta}(U) X^\alpha X^\beta k_a k_b \ , \tag{2.3}$$

where $h_{\alpha\beta}$ is constant on each wave surface, $h_{[\alpha\beta]} = 0$, $h_\alpha{}^\alpha = 0$, $q_{\alpha\beta}$ is constant, positive definite, and is used, with its inverse $q^{\alpha\beta}$, to raise and lower lower-case Greek indices which serve to label the two X^α coordinates [7]. The requirement that the wave surfaces be geodesically complete in turn requires that the fields V and X^α take on all possible values. U may be bounded from above or below.

The plane-wave spacetimes admit a five-dimensional symmetry group whose orbits are the wave surfaces. For any constant κ and any solution $K^\alpha(U)$ of the differential equation

$$(K^\alpha)''-h^\alpha{}_\beta K^\beta=0 , \tag{2.4}$$

where primes denote derivatives by U, the vector field

$$\xi^a=K^\alpha(\partial/\partial X^\alpha)^a-(X^\alpha K'_\alpha-\kappa)k^a \tag{2.5}$$

is a Killing vector field. [Since there are four linearly independent solutions of Eq. (2.4), this gives five linearly independent Killing vector fields.] Furthermore, all null geodesics in these spacetimes whose tangent vectors are not parallel to k^a are equivalent under the action of the isometry group [8].

The remainder of this section is devoted to two technical propositions concerning the plane-wave spacetimes. In what follows, it is convenient to drop the Greek indices and use the matrix notation: instead of writing $A^\alpha{}_\beta$, write A; instead of writing $A^\alpha{}_\gamma B^\gamma{}_\beta$ write AB; instead of writing $\delta^\alpha{}_\beta$ write $\mathbb{1}$; further define the transpose A^T of A via $(A^T)^\alpha{}_\beta=A_\beta{}^\alpha$.

Set

$$a=\inf_M(U),\quad b=\sup_M(U) , \tag{2.6}$$

each possibly being infinite. For each $x\in(a,b)$, denote by A_x [i.e., $(A_x)^\alpha{}_\beta$] that unique (matrix) solution of the differential equation

$$A_x''-hA_x=0 \tag{2.7}$$

subject to the condition that

$$A_x(x)=0,\quad A_x'(x)=\mathbb{1} . \tag{2.8}$$

Proposition 1. A necessary and sufficient condition that a plane-wave spacetime (M,g_{ab}) be globally hyperbolic is that $A_x(y)$, defined by Eqs. (2.7)–(2.8), can be nondegenerate for all $x,y\in(a,b)$ except where $x=y$.

Proof: The necessity of this condition was shown by Penrose [8]. Here we establish its sufficiency. Since all plane-wave spacetimes satisfy strong casuality, (M,g_{ab}) is global hyperbolic iff the set

$$J^+[p]\cap J^-[q] \tag{2.9}$$

is compact for all $p,q\in M$. To show that this is indeed the case, fix any two points $p,q\in M$. In the case $U(p)\geq U(q)$ is not difficult to see that the region described by Eq. (2.9), if nonempty, is simply diffeomorphic to a closed interval. In the case $U(p)<U(q)$, it follows from the symmetries of the plane-wave spacetimes that we can take

$$X^\alpha(p)=X^\alpha(q)=0 . \tag{2.10}$$

[To see this, for each $p\in M$ denote by $\mathcal{G}_p$ the two-dimensional isometry group generated by the Killing vector fields $\xi^a{}_\beta$ given by Eq. (2.5) with $(A_p)^\alpha{}_\beta$ in place of K^α and $X_\beta(p)$ in place of κ. Noting that $\mathcal{G}_p$ leaves p fixed and is transitive over all wave surfaces, except $U=U(p)$, we use $\mathcal{G}_p$ to fix $X^\alpha(q)=0$ and then use $\mathcal{G}_q$ to fix $X^\alpha(p)=0$.] A series of straightforward calculations then reveals that $J^+[p]$ is given by those (U,V,X^α) that satisfy

$$V\geq X^T(A_p'A_p^{-1})X+V(p) \tag{2.11}$$

and $U\geq U(p)$, while $J^-[q]$ is given by those (U,V,X^α) that satisfy

$$V\leq X^T(A_q'A_q^{-1})X+V(q) \tag{2.12}$$

and $U\leq U(q)$. Thus, the region described by Eq. (2.9) is given by those (U,V,X^α) that satisfy Eqs. (2.11) and (2.12) and $U(p)\leq U\leq(q)$. (To simplify the notation, for each $p\in M$ we have identified A_p with $A_{U(p)}\circ U$.) To show that this region is indeed compact, we first introduce a number of quantities and point out a few of their properties.

Introduce the quantity

$$\Omega_{pq}=(A_p^T)'A_q-A_p^TA_q' . \tag{2.13}$$

It follows from Eq. (2.7) that Ω_{pq} is constant, as can easily be verified by taking its derivative. Evaluating Eq. (2.13) at p then q we find that

$$\Omega_{pq}=A_q(p)=-A_p^T(q) . \tag{2.14}$$

This shows that Ω_{pq} is nondegenerate and establishes a useful reciprocity relationship. Setting

$$S=A_p'A_p^{-1}-A_q'A_q^{-1} , \tag{2.15}$$

from Eqs. (2.11) and (2.12) we find that

$$X^TSX\leq V(q)-V(p) . \tag{2.16}$$

That S is symmetric follows from the fact that $\Omega_{pp}=0$ for any p. Again using the fact that $\Omega_{pp}=0$, it is straightforward to show that

$$A_p^TSA_q=\Omega_{pq} . \tag{2.17}$$

Setting

$$\tilde{S}=(\det A_p\det A_q)^{1/2}S , \tag{2.18}$$

$$\tilde{X}=X/(\det A_p\det A_q)^{1/4} , \tag{2.19}$$

Eq. (2.16) becomes

$$\tilde{X}^T\tilde{S}\tilde{X}\leq V(q)-V(p) . \tag{2.20}$$

The advantage of Eq. (2.20) over Eq. (2.16) is that the quantity $\tilde{S}$ is continuous (while S is not) as can be seen as follows: Although $A_p'A_p^{-1}$ is singular at $U(p)$, the combination $A_p'A_p^{-1}(\det A_p)^{1/2}$ is continuous when we take its value at $U(p)$ to be the identity. Further, by Eqs. (2.17) and (2.18),

$$\det\tilde{S}=\det\Omega_{pq}>0 . \tag{2.21}$$

Since $\tilde{S}$ is positive definite at p, this shows that $\tilde{S}$ must be positive definite everywhere.

We are now in a position to argue that the set given by

Eq. (2.9) is compact. Recall that for this set to be compact, those (U,V,X^α) satisfying Eqs. (2.11) and (2.12) and $U(p)\le U\le U(q)$ must be a closed bounded set. It is clear that this set is closed. That it is bounded follows by the following series of arguments. First, U is already bounded $[U(p)\le U\le U(q)]$. Second, since $\tilde S$ is positive definite and continuous and the fact that the allowed U's form a compact set implies that the $\tilde X$'s satisfying Eq. (2.20) form a bounded set. Hence, by Eq. (2.19) the X's satisfying Eq. (2.16) also form a bounded set. Finally, that V is bounded follows from Eqs. (2.11) and (2.12) by using the boundedness of the $\tilde X$'s, the continuity of the combination $A_p' A_p^{-1}(\det A_p)^{1/2}$, and the fact that the allowed U's form a compact set. Hence, the set given by Eq. (2.9) is compact.□

A nice geometrical condition equivalent to the condition in Proposition 1 can be obtained by using the fact that

$$R_{abcd}=-h_{\alpha\beta}(k\wedge dX^\alpha)_{ab}(k\wedge dX^\beta)_{cd} \tag{2.22}$$

and the equivalence of all null goedesics not parallel to k^a under the five-dimensional isometry group. With these facts it is not difficult to show that the condition in Proposition 1 is equivalent to the condition that no null geodesic in the spacetime possesses a pair of conjugate points.

Having established a necessary and sufficient condition for the global hyperbolicity of a plane-wave spacetime, we now prove the following.

Proposition 2. Fix a globally hyperbolic plane-wave spacetime (M,g_{ab}) that is plane-wave extendible to the future. [By a plane-wave extension we mean an isometric embedding θ: $M\to M'$ such that (M',g'_{ab}) is also a plane-wave spacetime.] For any wave surface $\mathcal{N}$ in M, there exists a plane-wave extension (M',g'_{ab}) of (M,g_{ab}) so that $(I^+[\theta(\mathcal{N})],g'_{ab})$ is globally hyperbolic.

Proof: Choosing k^a to be past directed, U increases to the future. Extend the spacetime (M,g_{ab}) by simply extending $h_{\alpha\beta}$ continuously from (a,b) to $[b,c)$ for some $c>b$ [where a and b are defined by Eq. (2.6)]. Using this extended $h_{\alpha\beta}$ and Eqs. (2.7) and (2.8), extend $A_x(y)$ to all $x,y\in(a,c)$.

By Proposition 1, $A_y(x)$ is nondegenerate for all $x,y\in(a,b)$ except where $x=y$. In fact, $A_b(x)$ is nondegenerate for all $x\in(a,b)$. [Equivalently, by Eq. (2.14), $A_x(b)$ is nondegenerate for all $x\in(a,b)$.] We can see this as follows. For any $a<u<v<b$, a straightforward calculation shows that, on (u,b),

$$(A_u^{-1}A_v)'=-(A_u^TA_u)^{-1}\Omega_{uv} , \tag{2.23}$$

from which it follows that, on (u,b),

$$A_v(x)=A_u(x)\left[\int_x^v(A_u^TA_u)^{-1}dU\right]A_v(u) . \tag{2.24}$$

Suppose that $A_b(x)$ is degenerate for some $x\in(a,b)$, i.e., that there exists a vector ζ such that $[A_b(x)]\zeta=0$. Then, in order that $[A_v(x)]\zeta\to 0$ as $v\to b$, Eq. (2.24) demands that $[A_b(u)]\zeta=0$ for all $u\in(a,x)$. The uniqueness of the solutions to Eq. (2.7) then implies that $[A_b(u)]\zeta=0$ for all u. However, this is incompatible with the fact that $A_b'(b)=\mathbb{1}$. Hence, $A_b(x)$ is nondegenerate for all $x\in(a,b)$.

Set $a'=U(\mathcal{N})$. Since $A_{a'}$ is nondegenerate on $(a',b]$, by continuity there then exists $b'\in(b,c]$ such that $A_{a'}$ is nondegenerate on (a',b'). Using Eq. (2.24), with $u=a'$ and $v=y$, it follows that $A_y(x)$ is nondegenerate for all $x,y\in(a',b')$ except where $x=y$. Thus, taking M' to be the region $a<U<b'$, g'_{ab} to be the metric given by Eq. (2.3) with the extended $h_{\alpha\beta}$, and θ the natural embedding of M into M', then by Proposition 1 it follows that $(I^+[\theta(\mathcal{N})],g'_{ab})$ is globally hyperbolic.□

III. STABILITY PROOF

To begin the proof of Theorem 1 construct the perturbation $\tilde\gamma_{ab}$ from γ_{ab} by setting

$$\tilde\gamma_{ab}=\gamma_{ab}+\pounds_v g_{ab} , \tag{3.1}$$

where v^a is that unique vector field that satisfies

$$\nabla_m\nabla^m v^a=-\nabla_b(\gamma^{ab}-\tfrac{1}{2}\gamma_m{}^m g^{ab}) \tag{3.2}$$

subject to the condition that the pair $(v^a,\nabla_m v^a)$ is zero on Σ. This change in gauge is motivated by the fact that the perturbation $\tilde\gamma_{ab}$ satisfies the linear, diagonal, second-order hyperbolic equation

$$\nabla_m\nabla^m\tilde\gamma_{ab}+2R_a{}^m{}_b{}^n\tilde\gamma_{mn}=0 . \tag{3.3}$$

Thus, by the global evolution theorem for such equations [6,9], $\tilde\gamma_{ab}$ is determined by the pair $(\tilde\gamma_{ab},\nabla_m\tilde\gamma_{ab})$ evaluated on any Cauchy surface. In fact, we note the following.

Proposition 3. $\tilde\gamma_{ab}=0$ on $D(\Sigma-C)$.

Proof: First, we show that $(\tilde\gamma_{ab},\nabla_m\tilde\gamma_{ab})$ is zero on $\Sigma-C$. That $\tilde\gamma_{ab}$ is zero on $\Sigma-C$ is an immediate consequence of Eq. (3.1) and the conditions that γ_{ab} and $\nabla_m v^a$ are zero on $\Sigma-C$. That $\nabla_m\tilde\gamma_{ab}$ is zero on $\Sigma-C$ follows from Eq. (3.1), the condition that $\nabla_m\gamma_{ab}$ is zero on $\Sigma-C$, and the fact that $\nabla_a\nabla_b v_c=0$ on $\Sigma-C$. [For any vector field v^a such that $(v^a,\nabla_m v^a)$ is zero on a surface S with unit-timelike normal n_a, it can be shown that $\nabla_a\nabla_b v_c=-n_a n_b\nabla_m\nabla^m v_c$ on S. By Eq. (3.2) and the condition that $\nabla_m\gamma_{ab}=0$ on $\Sigma-C$ it follows that $\nabla_a\nabla_b v_c=0$ on $\Sigma-C$.] Having shown that $(\tilde\gamma_{ab},\nabla_m\tilde\gamma_{ab})$ is zero on $\Sigma-C$, that $\tilde\gamma_{ab}$ is zero on $D(\Sigma-C)$ follows from Eq. (3.3) and the uniqueness theorem for hyperbolic equations: $\tilde\gamma_{ab}=0$ is a solution of Eq. (3.3) that meets the condition that $(\tilde\gamma_{ab},\nabla_m\tilde\gamma_{ab})$ be zero on $\Sigma-C$ so, by uniqueness, $\gamma_{ab}=0$ is the solution on $D(\Sigma-C)$.□

Choose the time orientation of the spacetime (M,g_{ab}) so that its plane-wave extension is in the future. Fix any wave surface $\mathcal{N}$ such that $C\subset I^+[\mathcal{N}]$. That such a surface exists follows form the compactness of C. Construct a plane-wave extension (M',g'_{ab}) of (M,g_{ab}) such that $(I^+[\theta(\mathcal{N})],g'_{ab})$ is globally hyperbolic, where $\theta:M\to M'$ is the embedding associated with this extension. (See Proposition 2 of Sec. II.) Fix a Cauchy surface Σ' of $(I^+[\theta(\mathcal{N})],g'_{ab})$ such that $\theta(C)\subset D^+(\Sigma')$. Again, that such a surface exists follows from the compactness of C.

Proposition 4. $C'=\mathrm{supp}[(\theta^*\tilde\gamma)_{ab}]\cap[\Sigma'\cap\theta(M)]$ is compact.

Proof: Since $\mathrm{supp}[(\theta^*\tilde{\gamma})_{ab}]$ is a closed subset of $J^+[\theta(C)]\cup J^-[\theta(C)]$ and $\theta(C)\subset D^+(\Sigma')$ the set C' is a closed subset of $J^-[\theta(C)]\cap[\Sigma'\cap\theta(M)]=J^-[\theta(C)]\cap\Sigma'$. But, this last set is compact as $\theta(C)$ is a compact subset of $D(\Sigma')$. (See, e.g., Proposition 6.6.6 of Ref. [9].) Hence, C' being a closed subset of a compact set is itself compact.□

Consider the initial data that $(\theta^*\tilde{\gamma})_{ab}$ induces on the open subset $\Sigma'\cap\theta(M)$ of the Cauchy surface Σ'. By Proposition 4, this is zero outside the compact set C'. Extend these initial data smoothly, but otherwise arbitrarily, to all of Σ'. [Note that taking the initial data to be zero on $\Sigma'\cap\theta(M)^c$ is such an extension.] Appealing once again to the global evolution theorem for linear, diagonal, second-order hyperbolic equations [6,9] we learn that there exists a unique evolution γ'_{ab} to all of $D(\Sigma')$ of our extended initial data. Since γ'_{ab} on $D(\Sigma')\cap\theta(M)$ is determined by its initial data on $\Sigma'\cap\theta(M)$, by uniqueness, $\gamma'_{ab}=(\theta^*\tilde{\gamma})_{ab}$ on $D(\Sigma')\cap\theta(M)$. Thus, since $\mathcal{H}=\partial\theta(M)\subset D(\Sigma')$, $(\theta^*\tilde{\gamma})_{ab}$ (and its derivatives) is smoothly extendible to $\mathcal{H}$. In particular, the extension is simply γ'_{ab} (and its derivatives) evaluated on $\mathcal{H}$.

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