Dirty black holes: Thermodynamics and horizon structure

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Considerable interest has recently been expressed in (static spherically symmetric) black holes in interaction with various classical matter fields (such as electromagnetic fields, dilaton fields, axion fields, Abelian Higgs fields, non–Abelian gauge fields, etc.). A common feature of these investigations that has not previously been remarked upon is that the Hawking temperature of such systems appears to be suppressed relative to that of a vacuum black hole of equal horizon area. That is, $kT_H \leq \hbar/(4\pi r_H) \equiv \hbar/\sqrt{4\pi A_H}$. This paper will argue that this suppression is generic. Specifically, it will be shown that $kT_H = (\hbar/4\pi r_H)\, e^{-\phi(r_H)}\left(1 - 8\pi G\, \rho_H\, r_H^2\right)$. Here $\phi(r_H)$ is an integral quantity, depending on the distribution of matter, that is guaranteed to be positive if the weak energy condition is satisfied. Several examples of this behavior will be discussed. Generalizations of this behavior to nonsymmetric nonstatic black holes are conjectured.

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I. INTRODUCTION

For a variety of reasons, considerable attention has recently been focused on static spherically symmetric black holes in interaction with various static spherically symmetric classical fields. For example, the system (gravity + electromagnetism + dilaton) has been discussed by Gibbons and Maeda [1], by Ichinose and Yamazaki [2, 3], and in an elegant paper by Garfinkle, Horowitz, and Strominger [4], this particular system currently being deemed to be of interest due to its tentative connection with low-energy string theory. The resulting charged dilatonic black holes were rapidly generalized by Shapere, Trivedi, and Wilczek [5] to the dyonic dilatonic black holes appropriate to the system (gravity + electromagnetism + dilaton + axion). The system (gravity + electromagnetism + axion) has been considered by Allen, Bowick, and Lahiri [6], by Campbell, Kaloper, and Olive [7], and by Lee and Weinberg [8]. The considerably simpler system of (gravity + axion) and the associated axionic black holes had previously been discussed by Bowick, Giddings, Harvey, Horowitz, and Strominger [9]. The system (gravity + electromagnetism + Abelian Higgs field) has been discussed by Dowker, Gregory, and Traschen [10] using Euclidean signature formalism. Colored black holes, arising in the system (gravity + non–Abelian gauge field), have been discussed by Galtsov and Ershov [11], by Straumann and Zhou [12], by Bizon [13], and by Bizon and Wald [14]. A variation on these themes: the system (gravity + axion + non–Abelian gauge field), has recently been considered by Lahiri [15]. For brevity, any black hole in interaction with nonzero classical matter fields will be referred to as "dirty".

A common feature of these various investigations is

that whenever the Hawking temperature of the resulting dirty black hole can be computed, the Hawking temperature (equivalently, the surface gravity) appears to be suppressed relative to that of a clean vacuum Schwarzschild black hole of equal horizon area (equivalently, of equal entropy). Specifically, the inequality

$$kT_H \leq \frac{\hbar}{4\pi r_H} \equiv \frac{\hbar}{\sqrt{4\pi A_H}} \tag{1.1}$$

appears to be satisfied.

I claim that this inequality is not an accident, but rather that this inequality is related to the *classical* nature of the fields interacting with the black hole. Indeed it shall be shown that, for a general spherically symmetric distribution of matter with a black hole at the center, the Hawking temperature is given by

$$kT_H = \frac{\hbar}{4\pi r_H}\, e^{-\phi(r_H)}\left(1 - 8\pi G\, \rho_H\, r_H^2\right). \tag{1.2}$$

Now r_H and ρ_H, the radius and matter density at the horizon, clearly depend only on conditions local to the horizon itself. In contrast, $\phi(r_H)$ is an integral quantity that depends on the distribution of matter all the way from $r = r_H$ to $r = \infty$. The remarkable feature of the analysis is that, if the matter surrounding the black hole satisfies the weak energy condition (WEC), which is certainly the case for classical matter, then the Einstein field equations imply that $\phi(r_H)$ is non–negative. The inequality $kT_H \leq \hbar/(4\pi r_H)$ follows immediately.

(*Warning:* Since semiclassical quantum effects are capable of violating the WEC, it follows that quantum physics may allow a violation of this inequality. On the other hand, violations of the WEC in the vicinity of the event horizon are quite likely to destabilize the horizon, disrupt the black hole, and lead to a traversable wormhole, thereby rendering moot the question of the Hawking temperature [16].)

A side effect of the investigation is the discovery of

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a particularly pleasant functional parametrization of the static spherically symmetric metric that permits a simple (formal) integration of the Einstein field equations in a form suitable for the direct application of the WEC.

Also of note is the fact that the matter fields at the horizon (as measured by a fiducial observer — a FIDO) are constrained to satisfy the boundary condition $\rho_H = \tau_H$ if the horizon is to be "canonical" in a sense to be described below. This boundary condition is in fact equivalent to demanding that the energy density measured by a freely falling observer (FFO) remain integrable as the observer crosses the horizon.

Several examples are discussed in detail: The Reissner–Nordström geometry and a "thin shell" example are particularly instructive elementary examples. The dyonic dilatonic black holes and their ilk are decidedly nontrivial examples.

Finally a conjecture is formulated as to a possible generalization of these results to spherically asymmetric nonstatic dirty black holes.

Units: Adopt units where $c \equiv 1$, but all other quantities retain their usual dimensionalities, so that in particular $G \equiv \ell_P/m_P \equiv \hbar/m_P^2 \equiv \ell_P^2/\hbar$.

II. METRIC

A. Functional form

The spacetime metric generated by any static spherically symmetric distribution of matter may (without loss of generality) be cast into the form

$$ds^2 = -g_{tt}\, dt^2 + g_{rr}\, dr^2 + r^2(d\theta^2 + \sin^2\theta\, d\varphi^2). \quad (2.1)$$

This form corresponds to the adoption of Schwarzschild coordinates. While one can relatively easily adopt the brute force approach of inserting this metric into the curvature computation formalism and "turning the crank," the resulting expression for the Einstein tensor is not as illuminating as it might otherwise be.

There is an art to further specifying the functional form of g_{tt} and g_{rr} in such a manner as to keep computations (and their interpretations) simple. For instance, to discuss traversable wormholes Morris and Thorne found the choices $g_{tt} = \exp[2\phi(r)]$, $g_{rr} = [1 - b(r)/r]^{-1}$ to be particularly advantageous [16]. For the discussion currently at hand I propose

$$g_{tt} = e^{-2\phi(r)}\left(1 - \frac{b(r)}{r}\right), \qquad g_{rr} = \left(1 - \frac{b(r)}{r}\right)^{-1}. \quad (2.2)$$

That is,

$$ds^2 = -e^{-2\phi(r)}[1 - b(r)/r]dt^2 + \frac{dr^2}{1 - b(r)/r}$$
$$+ r^2(d\theta^2 + \sin^2\theta\, d\varphi^2). \quad (2.3)$$

Following Morris and Thorne, the function $b(r)$ will be referred to as the "shape function." The shape function may be thought of as specifying the shape of the spa-

tial slices. On the other hand, $\phi(r)$ might best be interpreted as a sort of "anomalous redshift" that describes how far the total gravitational redshift deviates from that implied by the shape function. As will subsequently be seen the Einstein field equations have a particularly nice form when written in terms of these functions.

B. Putative horizons

For now, explore the meaning of the metric in the form (2.3) without yet applying the field equations. First, applying boundary conditions at spatial infinity permits one to set $\phi(\infty) = 0$ without loss of generality. Once this normalization of the asymptotic time coordinate is adopted one may interpret $b(\infty)$ in terms of the asymptotic mass $b(\infty) = 2GM$. (Naturally one is assuming an asymptotically flat geometry.)

The metric (2.3) has putative horizons at values of r satisfying $b(r_H) = r_H$. Only the outermost horizon is of immediate interest and comments will be restricted to that case. Now for the outermost horizon one has $\forall r > r_H$ that $b(r) < r$, consequently $b'(r_H) \leq 1$. The case $b'(r_H) = 1$ is anomalous and will be discussed separately. Assuming then that $b'(r_H) < 1$ the behavior of the metric near the putative horizon is

$$ds^2 \approx -e^{-2\phi(r_H)}\left(\frac{r - r_H}{r_H}\right)[1 - b'(r_H)]dt^2$$
$$+ \frac{1}{1 - b'(r_H)}\frac{r_H}{(r - r_H)}dr^2$$
$$+ r_H^2(d\theta^2 + \sin^2\theta\, d\varphi^2). \quad (2.4)$$

Thus the putative horizon is seen to possess all the usual properties of a Schwarzschild horizon provided that $e^{-2\phi(r)}$ is positive and of finite slope at $r = r_H$, corresponding to $|\phi(r_H)|$ and $|\phi'(r_H)|$ being finite.

The putative horizon at $r_H = b(r_H)$ will be said to be of *canonical type* if

$$b'(r_H) < 1, \qquad |\phi(r_H)| < \infty, \qquad |\phi'(r_H)| < \infty. \quad (2.5)$$

Noncanonical horizons are of interest in their own right. On the one hand, if $b'(r_H) = 1$ one may Taylor expand:

$$b(r) = b(r_H) + b'(r_H)(r - r_H) + \frac{b''(r_H)}{2}(r - r_H)^2 + \cdots$$
$$= r + \frac{\gamma_2}{2r_H}(r - r_H)^2 + \cdots. \quad (2.6)$$

This allows the simple expansion $(1 - b/r) = \frac{1}{2}\gamma_2(r - r_H)^2/r_H^2 + \cdots$, thus indicating that in this case g_{rr} does not change sign at the horizon (provided that $\gamma_2 \neq 0$). This behavior is an indication of the merging of an inner and an outer horizon. In fact, the horizon of an extreme $Q = M$ Reissner–Nordström black hole is precisely of this type [with $\phi(r) \equiv 0$]. If $\gamma_2 = 0$ then one must go to higher order in the Taylor series expansion. If the first nonzero term is of order n, that is if $b(r) - r = \frac{1}{n!}\gamma_n(r - r_H)^n/r_H^n + \cdots$, then one may easily convince oneself that one is dealing with a n-fold merging of n degenerate horizons.

On the other hand, even if $b'(r_H) < 1$, one may still obtain noncanonical horizon structure due to the behavior of $\phi(r)$ near the putative horizon. For instance, take $\phi(r) = +\frac{1}{2}\ln(\frac{r-r_H}{r_H}) + f(r)$, where $f(r)$ is smooth and finite at the putative horizon. In this case the behavior of the metric near the putative horizon is

$$ds^2 \approx -e^{-2f(r_H)}[1 - b'(r_H)]dt^2$$
$$+\frac{1}{1 - b'(r_H)}\left(\frac{r_H}{r - r_H}\right)dr^2$$
$$+r_H^2(d\theta^2 + \sin^2\theta\, d\varphi^2). \tag{2.7}$$

Thus g_{tt} remains nonzero on the putative horizon, so that the putative horizon is not in fact a horizon at all, but rather is the throat of a traversable wormhole [16].

Finally, one should consider the possibility that the "anomalous redshift" might diverge in a region where the "shape function" is still well behaved. Specifically, consider the possibility that $\phi(r) \to +\infty$ as $r \to r_H$, while $b(r) \to r_0 \equiv 2Gm_0 < r_H$. Such a horizon is certainly noncanonical. Analysis of the Einstein field equations (see below) indicates that this case corresponds to a divergence in the stress–energy density as the horizon is approached.

Further discussion of noncanonical horizons will be postponed, and henceforth all horizons are taken to be of canonical type.

III. HAWKING TEMPERATURE

A. Surface gravity

The Hawking temperature of a black hole is given in terms of its surface gravity by $kT_H = (\hbar/2\pi)\kappa$. Now in general for a spherically symmetric system the surface gravity can be computed via

$$\kappa = \lim_{r \to r_H}\left\{\frac{1}{2}\frac{\partial_r g_{tt}}{\sqrt{g_{tt}g_{rr}}}\right\}. \tag{3.1}$$

(This result holds independently of whether or not one chooses to normalize the $g_{\theta\theta}$ and $g_{\varphi\varphi}$ components of the metric by adopting Schwarzschild coordinates.) For the choice of functional form described in (2.3) this implies

$$\kappa = \lim_{r \to r_H}\left\{\frac{1}{2}e^\phi\frac{\partial}{\partial r}\left[e^{-2\phi}\left(1 - \frac{b(r)}{r}\right)\right]\right\}$$
$$= \lim_{r \to r_H}\left\{\frac{1}{2}e^{-\phi}\left[-2\phi'(r)\left(1 - \frac{b(r)}{r}\right)\right.\right.$$
$$\left.\left.+\frac{b(r)}{r^2} - \frac{b'(r)}{r}\right]\right\}. \tag{3.2}$$

Now for a canonical horizon $|\phi(r_H)|$ and $|\phi'(r_H)|$ are both finite so that

$$\kappa = \frac{1}{2r_H}e^{-\phi(r_H)}[1 - b'(r_H)]. \tag{3.3}$$

At this stage of course, this formula is largely *definition*. This formula receives its physical significance only after

$b'(r_H)$ and $\phi(r_H)$ are related to the distribution of matter by imposing the Einstein field equations. Note that the derivation of the formula for the surface gravity continues to make perfectly good sense for degenerate horizons [i.e., $b'(r_H) = 1$], merely asserting in this case that $\kappa = 0$.

B. Euclidean signature techniques

Another way of calculating the Hawking temperature is via the periodicity of the Euclidean signature analytic continuation of the manifold [17]. Proceed by making the formal substitution $t \to -it$ to yield

$$ds_E^2 = +e^{-2\phi(r)}[1 - b(r)/r]dt^2 + \frac{dr^2}{1 - b(r)/r}$$
$$+r^2(d\theta^2 + \sin^2\theta\, d\varphi^2). \tag{3.4}$$

As is usual, discard the entire $r < r_H$ region, retaining only the (analytic continuation of) that region that was outside the outermost horizon (i.e., $r \geq r_H$). Taylor series expand the metric in the region $r \approx r_H$. Provided that the horizon is canonical one may write $(1 - b/r) \equiv (r - b)/r \approx (r - r_H)r_H^{-1}[1 - b'(r_H)]$ to give

$$ds_E^2 \approx +e^{-2\phi(r_H)}[1 - b'(r_H)]\left(\frac{r - r_H}{r_H}\right)dt^2$$
$$+\frac{1}{1 - b'(r_H)}\left(\frac{r_H}{r - r_H}\right)dr^2$$
$$+r_H^2(d\theta^2 + \sin^2\theta\, d\varphi^2). \tag{3.5}$$

Construct a new radial variable ϱ by taking

$$d\varrho = \frac{1}{\sqrt{1 - b'(r_H)}}\sqrt{\frac{r_H}{r - r_H}}dr$$
$$= \frac{2}{\sqrt{1 - b'(r_H)}}d[\sqrt{r_H(r - r_H)}]. \tag{3.6}$$

Then $r_H(r - r_H) = \frac{1}{4}[1 - b'(r_H)]\varrho^2$, and the Euclidean signature metric may be written as

$$ds_E^2 \approx +e^{-2\phi(r_H)}[1 - b'(r_H)]^2\frac{1}{4r_H^2}(\varrho^2\,dt^2) + d\varrho^2$$
$$+r_H^2(d\theta^2 + \sin^2\theta\, d\varphi^2). \tag{3.7}$$

Now the (ϱ, t) plane is a smooth two-dimensional manifold if and only if t is interpreted as an angular variable with period

$$\beta = 2\pi\, 2r_H\, e^{\phi(r_H)}\,[1 - b'(r_H)]^{-1}. \tag{3.8}$$

Invoking the usual incantations [17], this periodicity in imaginary (Euclidean) time is interpreted as evidence of a thermal bath of temperature $kT = \hbar/\beta$, so that the Hawking temperature is identified as

$$kT_H = \frac{\hbar}{4\pi r_H}e^{-\phi(r_H)}[1 - b'(r_H)]. \tag{3.9}$$

This is the same result as was obtained by direct calculation of the surface gravity, though this formulation has the advantage of (1) shedding further illumination on the subtleties associated with noncanonical horizons, and (2)

verifying the relationship between Hawking temperature and surface gravity.

IV. EINSTEIN FIELD EQUATIONS

A. Formal solution

The Einstein tensor corresponding to (2.3) can be obtained by the standard simple but tedious computation. Choose an orthonormal basis attached to the (t, r, θ, φ) coordinate system (i.e., choose a fiducial observer basis — a FIDO basis):

$$G_{\hat{t}\hat{t}} = \frac{b'}{r^2}, \tag{4.1}$$

$$G_{\hat{r}\hat{r}} = -\frac{2}{r}\left(1 - \frac{b}{r}\right)\phi' - \frac{b'}{r^2}. \tag{4.2}$$

Whereas the forms of $G_{\hat{t}\hat{t}}$ and $G_{\hat{r}\hat{r}}$ are quite pleasing, the form of $G_{\hat{\theta}\hat{\theta}} \equiv G_{\hat{\varphi}\hat{\varphi}}$ is quite horrible. Fortunately one will not need to use $G_{\hat{\theta}\hat{\theta}}$ or $G_{\hat{\varphi}\hat{\varphi}}$ explicitly. For completeness note

$$G_{\hat{\theta}\hat{\theta}} = G_{\hat{\varphi}\hat{\varphi}} = \left[1 - \frac{b}{r}\right]\left[-\phi'' + \phi'\left(\phi' - \frac{1}{r}\right)\right]$$
$$- \frac{3}{2}\phi'\left(\frac{b}{r^2} - \frac{b'}{r}\right) - \frac{1}{2}\frac{b''}{r}. \tag{4.3}$$

All other components of the Einstein tensor are zero. To minimize computation use the results of Morris and Thorne [16] with the substitution $\phi_{\text{Morris-Thorne}} = -\phi_{\text{here}} + \frac{1}{2}\ln(1 - b/r)$.

The Einstein field equations are

$$G_{\alpha\beta} = 8\pi G\, T_{\alpha\beta} = 8\pi\, \frac{\ell_P^2}{\hbar}\, T_{\alpha\beta}. \tag{4.4}$$

In the FIDO orthonormal basis used above, the nonzero components of the stress–energy tensor are

$$T_{\hat{t}\hat{t}} = \rho, \qquad T_{\hat{r}\hat{r}} = -\tau, \qquad T_{\hat{\theta}\hat{\theta}} = T_{\hat{\varphi}\hat{\varphi}} = p. \tag{4.5}$$

The first two Einstein equations are then simply rearranged to give

$$b' = 8\pi G\, \rho\, r^2, \tag{4.6}$$

$$\phi' = -\frac{8\pi G}{2}\frac{(\rho - \tau)r}{1 - b/r}. \tag{4.7}$$

Instead of imposing the third Einstein equation $G_{\hat{\theta}\hat{\theta}} = G_{\hat{\varphi}\hat{\varphi}} = 8\pi\, G\, p$, observe that (as is usual) this equation is redundant with the imposition of the conservation of stress energy. Thus one may take the third equation to be

$$\tau' = (\rho - \tau)[-\phi' + \tfrac{1}{2}\{\ln(1 - b/r)\}'] - 2(p + \tau)/r. \tag{4.8}$$

Taking ρ and τ to be primary, one may formally integrate the Einstein equations, and then substitute this into the conservation of stress–energy to determine p. Specifically,

$$b(r) = r_H + 8\pi G\int_{r_H}^r \rho\tilde{r}^2 d\tilde{r}, \tag{4.9}$$

$$\phi(r) = \frac{8\pi G}{2}\int_r^\infty \frac{(\rho - \tau)\tilde{r}}{(1 - b/\tilde{r})}d\tilde{r}, \tag{4.10}$$

$$p(r) = \frac{r}{2}\left[\frac{(\rho - \tau)}{2(1 - b/r)}\left\{\frac{b - 8\pi G\,\tau\,r^3}{r^2}\right\} - \tau'\right] - \tau. \tag{4.11}$$

Inserting these results into the formula for the Hawking temperature now yields the promised result

$$kT_H = \frac{\hbar}{4\pi r_H}\,\exp\left(-\frac{8\pi G}{2}\int_{r_H}^\infty \frac{(\rho - \tau)r}{(1 - b/r)}dr\right)$$
$$\times \left(1 - 8\pi G\,\rho_H\,r_H^2\right). \tag{4.12}$$

The Hawking temperature is seen to depend both on data local to the event horizon (r_H, ρ_H) and on a "redshift" factor whose computation requires knowledge of $\rho(r)$ and $\tau(r)$ all the way from the horizon to spatial infinity.

Once the problem has been cast in this form the role of the weak energy condition is manifest. WEC implies that $\rho - \tau \geq 0$ and that $\rho \geq 0$. Consequently $\forall r$, $\phi(r) \geq 0$. Also $b'(r_H) \geq 0$. Thus adopting the WEC allows one to assert the promised inequality

$$kT_H \leq \frac{\hbar}{4\pi r_H} \equiv \frac{\hbar}{\sqrt{4\pi A_H}}. \tag{4.13}$$

B. Convergence issues

Several points regarding these formulas are worth mentioning. First, the condition $b'(r_H) \leq 1$ which is automatically satisfied by the outermost putative horizon (regardless of whether or not it be canonical) implies, via the Einstein field equations, a constraint on ρ_H, viz., $\rho_H < 1/(8\pi G r_H^2) \equiv \hbar/(8\pi \ell_P^2 r_H^2)$. This constraint has the nice feature of guaranteeing that the Hawking temperature is non-negative. Turning to questions of convergence of the various integrals encountered, note that

$$2GM = r_H + 2G\int_{r_H}^\infty 4\pi\rho r^2 dr, \tag{4.14}$$

so that this integral is guaranteed to converge by the assumed asymptotic flatness of the spacetime. The only questionable integral is that for $\phi(r_H)$. Specifically, its convergence properties near the putative horizon are somewhat subtle. Assuming $b'(r_H) < 1$ one may write this integral as

$$\phi(r_H) \equiv \frac{8\pi G}{2}\int_{r_H}^\infty \frac{(\rho - \tau)r}{1 - b/r}dr,$$
$$\approx (\text{finite}) + \frac{8\pi G r_H^2}{2[1 - b'(r_H)]}\int_{r_H}^{(1+\epsilon)r_H}\frac{\rho - \tau}{r - r_H}dr. \tag{4.15}$$

This integral converges provided that $(\rho - \tau) \leq k(r - r_H)^\alpha$ as $r \to r_H$ for some arbitrary constant k and some constant $\alpha > 0$. In particular this implies that $\rho_H = \tau_H$ is a necessary condition for the existence of a canonical horizon. It should come as no great surprise then to observe that all "reasonable" classical field solutions satisfy this

boundary condition. Indeed, this boundary condition is equivalent to requiring the energy density measured by a freely falling observer (FFO) to remain integrable as one crosses the horizon.

To see this, consider a freely falling observer who starts falling from spatial infinity with initial velocity zero. Let V^μ denote the four–velocity of the FFO, and let K^μ denote the timelike Killing vector. That is, $K^\mu = (1,0,0,0)$; $K_\mu = (-g_{tt},0,0,0)$. Then the inner product $K^\mu V_\mu$ is conserved along geodesics, so that $V_t = 1$, $V^t = -g^{tt} = -1/g_{tt}$. Since the four–velocity must be normalized ($\|V\| = -1$), one may solve for the radial component to find (outside the outermost horizon)

$$V^\mu = \left(\frac{1}{g_{tt}}, \sqrt{\frac{1}{g_{rr}}} \sqrt{\frac{1}{g_{tt}} - 1}, 0, 0 \right). \qquad (4.16)$$

In the FIDO basis

$$V^{\hat\mu} = \left(\sqrt{\frac{1}{g_{tt}}}, \sqrt{\frac{1}{g_{tt}} - 1}, 0, 0 \right). \qquad (4.17)$$

So the energy density measured by a FFO is $\rho_{\text{FFO}} \equiv T_{\hat\mu\hat\nu} V^{\hat\mu} V^{\hat\nu} = \rho/g_{tt} + (-\tau)(g_{tt}^{-1} - 1) = \tau + (\rho - \tau)/g_{tt}$. Finally, inserting the functional form for g_{tt} one sees

$$\rho_{\text{FFO}} = \tau + \frac{(\rho - \tau)}{e^{-2\phi}(1 - b/r)} \approx \frac{e^{+2\phi}(\rho - \tau)r_H}{(1 - b')(r - r_H)}. \qquad (4.18)$$

So that the boundary condition $(\rho - \tau) \le k(r - r_H)^\alpha$, $\alpha > 0$, required to keep $\phi(r_H)$ finite, implies the integrability of ρ_{FFO}. Conversely, the integrability of ρ_{FFO} implies either (1) the finiteness of $\phi(r_H)$ (canonical horizon), or (2) $\phi(r) \to -\infty$ (corresponding to a traversable wormhole).

V. EXAMPLES

A. Reissner–Nordström

For the Reissner–Nordström geometry the symmetries of the situation together with the form of the electromagnetic stress–energy tensor implies

$$\rho = \tau = p = E^2/8\pi. \qquad (5.1)$$

This automatically gives $\phi(r) = 0$, $\forall r$. The electromagnetic field equations imply $E = Q/r^2$, so that

$$kT_H^{\text{RN}} = \frac{\hbar}{4\pi r_H} \left(1 - \frac{GQ^2}{r_H^2} \right). \qquad (5.2)$$

This is an unusual, though correct formula for the Hawking temperature of a Reissner–Nordström black hole. To see this note that explicit solution of the Einstein–Maxwell field equations gives $g_{tt} = (g_{rr})^{-1} = 1 - (2GM/r) + (GQ^2/r^2)$, whence $\kappa = \frac{1}{2} \lim_{r \to r_H} \partial_r g_{tt} = \frac{1}{2}(\{2GM/r_H^2\} - \{2GQ^2/r_H^3\}) = (1/2r_H)(\{2GM/r_H\} - \{2GQ^2/r_H^2\}) = (1/2r_H)(1 - \{GQ^2/r_H^2\})$, which is the above result.

B. Thin shell geometry

Consider a thin spherical shell of matter of density ρ_S, radius r_S, and thickness $(\delta r)_S$, which surrounds a vacuum black hole of Schwarzschild radius r_H. The mass of this thin shell is $m_S = 4\pi\rho_S r_S^2(\delta r)_S$, and the asymptotic total mass satisfies $2GM = r_H + 2Gm_S$. The shape function exhibits a step function discontinuity: $b(r) = r_H + \Theta(r - r_S)2Gm_S$. Direct integration of $\phi'(r)$ is not an appropriate way of calculating $\phi(r_H)$ due to the discontinuity in $b(r)$. Rather it is more appropriate to solve for $\phi(r_H)$ by using the continuity of g_{tt} to develop matching conditions. Everywhere except at the shell itself both ρ and τ are zero, so $\phi(r)$ is piecewise constant. Applying boundary conditions at the horizon and at spatial infinity gives $\phi(r) = \phi(r_H)\Theta(r_S - r)$. The matching conditions are thus

$$g_{tt}(r_S^+) = 1 - 2GM/r_S, \qquad (5.3)$$

$$g_{tt}(r_S^-) = e^{-2\phi(r_H)}(1 - r_H/r_S). \qquad (5.4)$$

One immediately obtains

$$e^{-2\phi(r_H)} = \frac{1 - 2GM/r_S}{1 - r_H/r_S} = 1 - \frac{2Gm_S/r_S}{1 - r_H/r_S}. \qquad (5.5)$$

Finally, noting that $\rho = 0$ on the horizon, one sees that the Hawking temperature is suppressed by

$$kT_H = \frac{\hbar}{4\pi r_H} \sqrt{1 - \frac{2Gm_S/r_S}{(1 - r_H/r_S)}}. \qquad (5.6)$$

Physically, this suppression of the Hawking temperature may be attributed to the fact that the shell introduces an extra gravitational redshift that decreases the energy of the Hawking photons on their way out to spatial infinity.

C. Charged dilatonic black hole

As a decidedly nontrivial example consider geometry and fields surrounding a charged dilatonic black hole [1, 4]. The calculation about to be exhibited is a rather obtuse way of calculating the Hawking temperature, depending as it does on delicate cancellations among r_H, ρ_H, and ϕ_H. The only virtue of this computation is that it illustrates general features of the formalism. (Units: For this section only set $G \equiv 1$.)

Consider then a solution to the combined (gravity + electromagnetism + dilaton) equations of motion. The Lagrangian is

$$\mathcal{L} = \sqrt{-g}\left\{ -R/8\pi + 2(\nabla\Phi)^2 + F^2/4\pi \right\}. \qquad (5.7)$$

(Warning: $\Phi \ne \phi$) In Schwarzschild coordinates the solution corresponding to an electric monopole is

$$ds^2 = -\left(1 - \frac{2M}{a + \sqrt{r^2 + a^2}} \right) dt^2$$
$$+ \left(1 - \frac{2M}{a + \sqrt{r^2 + a^2}} \right)^{-1} \frac{r^2}{r^2 + a^2} dr^2$$
$$+ r^2(d\theta^2 + \sin^2\theta \, d\varphi^2), \qquad (5.8)$$

$$F_{\hat{t}\hat{r}} = Q/r^2, \tag{5.9}$$

$$e^{2\Phi} = 1 - \frac{Q^2}{M(a + \sqrt{r^2 + a^2})}. \tag{5.10}$$

Here one has used the freedom to make an overall shift in Φ to set $\Phi(\infty) = 0$. The parameter a is defined by $a \equiv Q^2/2M$. In terms of the formalism developed in this paper,

$$1 - \frac{b}{r} = \left(1 + \frac{a^2}{r^2}\right)\left(1 - \frac{2M}{a + \sqrt{r^2 + a^2}}\right), \tag{5.11}$$

$$e^{-2\phi(r)} = \left(1 + \frac{a^2}{r^2}\right)^{-1} = \frac{r^2}{r^2 + a^2}. \tag{5.12}$$

The horizon occurs at $2M = a + \sqrt{r_H^2 + a^2}$, that is, $r_H^2 + a^2 = (2M - a)^2$, so that the surface gravity is

$$\kappa = \frac{1}{2r_H} \frac{r_H}{\sqrt{r_H^2 + a^2}} \left(1 - 8\pi \rho_H r_H^2\right)$$

$$= \frac{1}{2(2M - a)} \left(1 - 8\pi \rho_H r_H^2\right). \tag{5.13}$$

To calculate ρ_H one evaluates the nonzero components of the stress-energy tensor:

$$\rho = \frac{1}{8\pi}e^{-2\Phi}E^2 + \|\nabla\Phi\|^2, \tag{5.14}$$

$$\tau = \frac{1}{8\pi}e^{-2\Phi}E^2 - \|\nabla\Phi\|^2, \tag{5.15}$$

$$p = \frac{1}{8\pi}e^{-2\Phi}E^2 - \|\nabla\Phi\|^2. \tag{5.16}$$

As one approaches the event horizon it is easy to verify that $\|\nabla\Phi\| \to 0$, while $E \to Q/r_H^2$, so that $\rho \to \frac{1}{8\pi}(1 - \{Q^2/2M^2\})(Q^2/r_H^4)$. Thus

$$8\pi \rho_H r_H^2 = \left(1 - \frac{Q^2}{2M^2}\right)\frac{Q^2}{r_H^2}$$

$$= \frac{M - a}{M}\frac{Q^2}{2M(2M - 2a)} = \frac{a}{2M}. \tag{5.17}$$

Combining this considerable morass yields the simple result

$$\kappa = \frac{1}{4M}. \tag{5.18}$$

As previously mentioned, this calculation is a particularly obtuse manner in which to compute the surface gravity. This computation is of interest only insofar as it illustrates general principles and serves as a check on the formalism. The inequality $\kappa < 1/(2r_H)$, which previously appeared to be just a random accident of the calculation, is now seen to be intrinsically related to the fact that classical fields satisfy the WEC.

VI. DISCUSSION

For an arbitrary static spherically symmetric black hole this note has established a general formula for the Hawking temperature in terms of the energy density and radial tension. Adopting Schwarzschild coordinates, and writing

$$b(r) = r_H + 8\pi G \int_{r_H}^{r} \rho \tilde{r}^2 d\tilde{r}, \tag{6.1}$$

one finds that

$$kT_H = \frac{\hbar}{4\pi r_H} \exp\left(-\frac{8\pi G}{2}\int_{r_H}^{\infty}\frac{(\rho - \tau)r}{(1 - b/r)}dr\right)$$

$$\times \left(1 - 8\pi G \rho_H r_H^2\right). \tag{6.2}$$

Generalizations of this result to axisymmetric spacetimes (for instance, to Kerr–Newman black holes embedded in an axisymmetric cloud of matter) would clearly be of interest. Generalizations to arbitrary event horizons are probably unmanageable. On the one hand, the dominant energy condition (DEC) guarantees the constancy of the surface gravity (and hence the constancy of the Hawking temperature) over the surface of an arbitrary stationary event horizon. Furthermore, one might conceivably hope to generalize the factor $4\pi r_H$ to $\sqrt{4\pi A_H}$. On the other hand, there is no particular reason to believe that ρ_H is constant over the event horizon, nor is it clear how to generalize the notion of $\phi(r_H)$. (Presumably in terms of some line integral from the horizon to spatial infinity?)

If the central result of this paper is supplemented by the weak energy condition one may further assert (for static spherically symmetric dirty black holes) the general inequality

$$kT_H \le \frac{\hbar}{4\pi r_H}. \tag{6.3}$$

This inequality may be somewhat strengthened if one explicitly separates out the electromagnetic contribution to the stress energy. Note that $\rho_H \ge (\rho_{em})_H \equiv E^2/8\pi \equiv Q^2/(8\pi r_H^4)$. Thus for electrically charged static spherically symmetric dirty black holes,

$$kT_H \le \frac{\hbar}{4\pi r_H}\left(1 - \frac{GQ^2}{r_H^2}\right). \tag{6.4}$$

(Generalization to magnetic charge and the dyonic case is trivial.) The possibility of further generalizing these inequalities is more promising. I will restrain myself to a single one.
Conjecture:

For a stationary dirty black hole in interaction with matter fields satisfying the dominant energy condition,

$$kT_H \le \frac{\hbar}{\sqrt{4\pi A_H}}. \tag{6.5}$$

Notes: (1) It should be noted that this inequality is satisfied by the Kerr–Newman geometry. (2) The restrictions "stationary" and "dominant energy condition" cannot be dispensed with as they are required merely in order to guarantee the existence of a constant Hawking temperature. (3) With regard to this conjectured inequality, it should be pointed out that a weaker inequality that requires stronger hypotheses can be derived from the "four laws of black hole mechanics" [18]. Restricting the results of that paper to the case of zero rotation, one observes the equality (S_H = entropy = $\frac{1}{4}kA_H/\ell_P^2$)

$$M = \int_{r_H}^{\infty} (2T^\mu_{\ \nu} - T\delta^\mu_{\ \nu})K^\nu d\Sigma_\mu + 2T_H S_H. \qquad (6.6)$$

By invoking the strong energy condition, the integral can be made positive, in which case one obtains the inequality

$$kT_H \le \frac{M}{2S_H/k} \equiv \frac{2M\ell_P^2}{A_H}. \qquad (6.7)$$

When restricted to spherical symmetry this reduces to

$$kT_H \le \frac{2GM}{r_H}\frac{\hbar}{4\pi r_H}. \qquad (6.8)$$

Which is clearly weaker than the inequalities considered above.

In summary, this paper has exhibited a general formalism for calculating the surface gravity and Hawking temperature of spherically symmetric static dirty black holes. The formalism serves to tie together a number of otherwise seemingly accidental results scattered throughout the literature. Clear directions for future research are indicated.

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